DIOPHANTINE APPROXIMATION
AND THE THEORY OF HOLOMORPHIC CURVES

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In the last few years, due to the works of Osgood [O1,2], Lang [L1,2,3], Vojta [V1,2,3,4] and others, there appear to be evidences that the Theory of Diophantine Approximation and The Theory of Holomorphic curves (Nevanlinna Theory) may be somehow related. Currently, the relationship between the two theories is still on a formal level even though the resemblance of many of the corresponding results is quite striking. Vojta has come up with a dictionary for translating results from one theory into the other. Again the dictionary is essentially formal in nature and seems somewhat artificial at this point, it is perhaps worthwhile to begin a systematic investigation. Recently, I began to study the Theory of Diophantine Approximations, with the motivation of formulating the theory so that it parallels the theory of curves. These notes is a (very) partial survey of some of the results in diophantine approximations and the corresponding results in Nevanlinna Theory.

(I) Diophantine Approximation

The theory of diophantine equations is the study of solutions of polynomials over number fields. Typically, results in diophantine equations come in the form of certain finiteness statements; for instance statements asserting that certain equations have only a finite number of rational or integral solutions. We begin with a simple example.

Example 1 Consider the algebraic variety $X^2 + Y^2 = 3Z^2$ in P^2, we claim that there is no rational (integral) points (points with rational (integral) coordinates; on projective spaces a rational point is also an integral point) on this variety. To see this, suppose $P = [x, y, z]$ be a rational point on the variety with $x, y, z \in Z$ and $gcd(x, y, z) = 1$.

Then $x^2 + y^2 = 0$ (mod 3) so that $x = y = 0$ (mod 3). Thus x^2 and y^2 are divisible by 9 and it follows that z is divisible by 3, contradicting the assumption that $gcd(x, y, z) = 1$. This example illustrates one of the fundamental tools in diophantine equation:

"To show that a variety has no rational point, it is sufficient to show that the homogenous defining equation has no non-zero solutions mod p for one prime p"

The converse to this statement, the so called *"Hasse Principle"* is not valid in general. The following example, due to Selmer:

$$3X^2 + 4Y^2 + 5Z^3 = 0$$

has no rational points and yet for any prime p, the corresponding equation mod p admits non-trivial solutions.

Example 2 The algebraic set $y^2 = x^3 + 17$ in A^2 has many rational points, for example (-2, 3), (-1, 4), (2, 5), (4, 9), (8, 23), (43, 282), (52, 375), (5234, 378661) are integral points; (-8/9, 109/27); (137/64, 2651/512) are rational points (unlike the projective varieties, a rational point on an affine variety may not be integral). In fact $V(Q)$ is infinite. If we homogenize the equation (replace x by X/Z, y by Y/Z), we get

$$Y^2 Z = X^3 + 17Z^3$$

this defines a variety in P^2. It has one point at infinity: $[0, 1, 0]$. The rational points are given by $\{(x, y) \in A^2(Q) | y^2 = x^3 + 17\} \cup \{[0, 1, 0]\}$. It can be shown that the line connecting any two Q-rational points intersects the variety again in a Q-rational point. In this way one can show that there are infinitely many Q-rational points. The variety is an example of an elliptic curve. There are two fundamental theorems concerning elliptic curves: The Mordell-Weil theorem asserts that the set of rational points on an elliptic curve is finitely generated. The Siegel theorem asserts that the set of integral points on an elliptic curve is finite. In this example there are exactly 16 integral points, consisting of the eight points listed above and their negative (negative in the

sense of the group law of an elliptic curve). For further discussions concerning elliptic curves we refer the readers to [**Sil**].

Example 3 Consider the equation

$$x^3 - 2y^3 = n$$

where n is any fixed integer. We claim that such an equation has only a finite number of integral solutions. First we reduce the problem to an estimate.

The left hand side of the equation can be factorized as

$$(x - \sqrt[3]{2}y)(x - \theta\sqrt[3]{2}y)(x - \theta^2\sqrt[3]{2}y)$$

where θ is the primitive cube root of unity. Dividing the equation by y^3 we get

$$\left(\frac{x}{y} - \sqrt[3]{2}\right)\left(\frac{x}{y} - \theta\sqrt[3]{2}\right)\left(\frac{x}{y} - \theta^2\sqrt[3]{2}\right) = \frac{n}{y^3}.$$

Since θ is non-real, the absolute value of the second and third terms on the left above are clearly bounded away from zero and we can choose the lower bound to be independent of x and y (take the smaller of the distances to the real axis from $\theta\sqrt[3]{2}$ and $\theta^2\sqrt[3]{2}$ for instance), so that

(1) $$\left|\frac{x}{y} - \sqrt[3]{2}\right| \le \frac{C}{|y|^3}$$

for some constant C independent of x and y. The problem is reduced to the problem of approximating irrational numbers by rationals. We shall see shortly that the inequality above can have only finitely many rational solutions.

First we recall a classical result of Liouville (cf. [**Schm 2**]).

Theorem 1 (Liouville 1851) *Let α be an algebraic number of degree $d \ge 2$ over $\mathbf{Q}$; i.e., $[\mathbf{Q}(\alpha) : \mathbf{Q}] = d$. Then there is a constant $C > 0$ (depending on α) so that for any rational number p/q (p, q integers and $q > 0$),*

(2) $$\left|\frac{p}{q} - \alpha\right| \ge \frac{C}{q^d}.$$

Proof. Let $f(X) \in Z[X]$ be the minimal polynomial (of degree d) of α. For any rational number p/q, clearly $q^d f(p/q)$ is a non-zero integer (non-zero because f has no rational roots). Thus we get a lower bound for $|f(p/q)|$:

(3) $$|f(p/q)| \geq 1/q^d$$

On the other hand, if $|p/q - \alpha| \leq 1$ then we can estimate $|f(p/q)|$ from above:

$$|f(p/q)| = |f(p/q) - f(\alpha)| = |f'(c)| \, |p/q - \alpha| \leq C'|p/q - \alpha|$$

where $C' = \sup_{|x-\alpha| \leq 1} |f'(x)|$. Combining with (3) we get

$$|p/q - \alpha| > C/q^d$$

where $C = 1/C'$, as claimed. If $p/q - \alpha| > 1$ then the theorem is trivially verified by taking $c = 1$ for instance. QED

Remark 1 The assumption that α be algebraic is crucial. In fact, this theorem is used by Liouville to construct transcendental numbers. For example, let

$$\alpha = \sum_{1 \leq n < \infty} 2^{-n!}, \, p_k = 2^{k!} \sum_{1 \leq n \leq k} 2^{-n!}, q_k = 2^{k!}$$

then

$$\left| \frac{p_k}{q_k} - \alpha \right| = \sum_{k+1 \leq n < \infty} 2^{-n!} < 2q_k^{-k-1} < cq_k^{-d}$$

for any given c and d, for all k sufficiently large. Thus α cannot be algebraic by the theorem. For more details concerning *Liouville numbers* and criterion of transcendence see Mahler [**Ma**] and Gelfond [**Ge**].

Remark 2 Liouville's theorem implies the following statement. Let α be an algebraic number of degree $d \geq 2$, then for any $\epsilon > 0$ there are at most finitely many rational numbers p/q (p, q integers and $q > 0$) such that

(4)
$$\left|\frac{p}{q} - \alpha\right| \leq \frac{1}{q^{d+\varepsilon}}$$

Suppose otherwise, then there are rational numbers p/q with arbitrary large q satisfying (4). Such rational numbers clearly violates (2) because for any $c > 0$, $q^{-(d+\varepsilon)} < cq^{-d}$ for q sufficiently large.

Returning to the example, the number $\alpha = \sqrt[3]{3}$ is algebraic of degree $d = 3$. Comparing inequality (1) with (2) we see that Liouville's theorem is almost but not quite strong enough to guarantee the finiteness of integral solutions of the equation in example 3. Before we recount the history of the improvements of Liouville's theorem, we should mention that for algebraic numbers of degree $d = 2$, Liouville's estimate is essentially sharp. This is a consequence of a very well-known result of Dirichlet ([**Schm2**]):

Theorem 2 (Dirichlet 1842) *For any irrational number α there exists infinitely many rational numbers p/q (p, q integers and $q > 0$) such that*

$$\left|\frac{p}{q} - \alpha\right| \leq \frac{1}{q^2}.$$

Remark 3 It follows that there are infinitely many rationals p/q with p and q relatively prime and satisfy the estimate above.

Remark 4 Dirichlet's theorem holds for any irrational number, algebraic or transcendental.

Thus for algebraic number α of degree 2, the exponent in (4) cannot be improved to 2. For algebraic numbers of higher degree the exponent $d + \varepsilon$ was improved to

$$1 + \frac{1}{2}d \quad \text{(Thue, 1901)}$$
$$2\sqrt{d} + \varepsilon \quad \text{(Siegel, 1921)}$$
$$\sqrt{2d} + \varepsilon \quad \text{(Dyson, also Gelfond, 1947)}$$

and finally to $2 + \varepsilon$ by Roth (1955). Roth was awarded the Fields medal for this achievement.

Theorem 3 (Roth 1955) *Let α be an algebraic number of degree $d \geq 2$. Then for any $\varepsilon > 0$ the inequality*

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{q^{2+\varepsilon}}$$

holds with the exception of finitely many rationals p/q where p, q are integers and $q > 0$.

Remark 5 In the case where the degree of α is 2, Liouville's theorem is stronger than Roth's theorem.

    Lang (**L1**]) conjectured that perhaps the estimate in Theorem 3 can be improved to

$$\left| \frac{p}{q} - \alpha \right| \geq \frac{1}{q^2 \log^{1+\varepsilon} q}.$$

The conjecture is still open at this time (the corresponding statement of this conjecture in Nevanlinna Theory is due to Wong [**2**], see also [**S-W**]). However, this estimate had been verified for some special numbers. There is also the theorem of Khinchin that this estimate holds for all but a set of numbers of zero Lebesgue measure (cf. Khinchin [**Kh**]). More precisely:

Theorem 4 (Khinchin) *Let φ be a positive continuous function on the positive real line such that $x\varphi(x)$ is non-increasing. Then for almost all (i.e., except on a set of zero Lebesgue measure) irrational number α, the inequality*

$$\left| \frac{p}{q} - \alpha \right| \geq \frac{\varphi(q)}{q}$$

holds for all but a finite number of solutions in integers p, q ($q > 0$) if and only if the integral

$$\int_c^\infty \varphi(x)\,dx$$

converges for some positive constant c.

Using his improvement of Thue's theorem, Siegel proved the following finiteness theorem:

Theorem 5 (Siegel) *On an affine curve (over any number field) of positive genus there can only be a finite number of integral points.*
In the projective case, Mordell proved that

Theorem 6 (Mordell) *The set of rational points on an elliptic curve (i.e., a curve of genus one) is a finitely generated abelian group.*

Mordell also made the famous conjecture (solved by Falting in the affirmative, cf. *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73**, 183):

Mordell's Conjecture *There are only finitely many rational points on a curve of genus > 1.*

Remark 5 Unlike the affine case, there is no distinction between integral and rational points on a complete curve.
Falting's original proof of the Mordell's conjecture is geometric and did not use the theorem of Thue-Siegel-Roth. Vojta (1988) proved the Mordell conjecture over function fields and, more recently Falting gave a proof of the general case of the Mordel conjecture, using the Thue-Siegel-Roth's theorem. So far (essentially) all the known results in diophantine equations are consequences of the Thue-Siegel-Roth's theorem.
The extension of Roth's theorem to approximation of p-adic numbers by algebraic numbers, handling several valuations at the same time, is due to Ridout ([**Ri**]) and Mahler ([**Ma**]). First we recall the product formula of Artin-Whaples. Let k be a number field and v a valuation on k. Denote by k_v the completion of k with respect to v and by $n_v = [k_v : Q_v]$ the local degree. Define an absolute value associated to an archimedean valuation v by

$$\|x\|_v = |x| \qquad \text{if } K_v = R$$
$$\|x\|_v = |x| \qquad \text{if } K_v = C$$

If v is non-archimedean then v is an extension of p-adic valuation on Q for some prime p, the absolute value is defined so that

$$\|x\|_v = |x|_p^{n_v}$$

if $x \in Q - \{0\}$. With these conventions, there exists a complete set M_k of inequivalent valuations on k such that the product formula is satisfied with multiplicity one; i.e.,

(Artin-Whaples) $$\prod_{v \in M_k} \|x\|_v = 1$$

for all $x \in k - \{0\}$. Extend $\| \ \|_v$ to the algebraic closure $\bar{k}_v$ of k_v.

Let k be a field of characteristic zero, denote by $k[X]$ and $k(X)$ the polynomial ring and the rational function field over k respectively. Fix an *irreducible* polynomial $p(X)$ in $k(X)$, define the order at p of a rational function $r(X)$ in $k(X)$ to be α if $r = p^\alpha s/t$ where s and t are polynomials relatively prime to p. A p-adic valuation on $k(X)$ is defined by

$$|r|_p = e^{-(\text{ord}\, r)(\deg p)}.$$

For $r(X)$ in $k(X)$, there exists polynomials f and g in $k[X]$, where g is not the zero element, such that $r = f/g$. Define a valuation on $k(X)$ by

$$|r|_\infty = |f/g|_\infty = e^{\deg f - \deg g}.$$

Denote by $k(X)_\infty$ the completion of $k(X)$ with respect to the valuation $| \ |_\infty$ and $k(X)_p$ the completion of $k(X)$ with respect to $| \ |_p$. The Artin-Whaple Product Formula is satisfied for $k(X)$ (and also its finite algebraic extension).

We now give an analytic interpretation of the product formula. Consider the special case of $k = C$, the complex number field, there is also the field $\mathfrak{M}$ of meromorphic functions defined on a domain G in the Riemann sphere CP^1. If $G = CP^1$ then $\mathfrak{M} = C(X) =$ the field of rational functions in one variable. Fix a point $z_0 \neq \infty$ in G and for a function $f \in \mathfrak{M}$, we may write

$$f(z) = (z - z_0)^{\text{ord}_{z_0} f} g(z)$$

where g is meromorphic with $g(z_0) \neq 0$ or ∞. For a positive constant c, a valuation is defined by

$$|f|_{z_0} = c^{\operatorname{ord}_{z_0} f}.$$

Thus $\operatorname{ord}_{z_0} f > 0$ if z_0 is a zero of f and $\operatorname{ord}_{z_0} f < 0$ if z_0 is a pole. If $z_0 = \infty$, we may write

$$f(z) = z^{-\operatorname{ord}_\infty f} g(z)$$

where g is meromorphic with $g(z_0) \neq 0$ or ∞. A valuation is defined by setting

$$|f|_\infty = c^{\operatorname{ord}_\infty f}.$$

In the case of $G = \boldsymbol{CP}^1$, the valuation $|f|_{z_0}$ coincides with $|f|_p$ where $p(z) = z - z_0$ is an irreducible polynomial and $|f|_\infty = |f|_{p_\infty}$. In this case it is clear that

$$\prod_{w \in \boldsymbol{CP}^1} |f|_w = c^{\# \text{ of zeros} - \# \text{ of poles}}$$

and the Product Formula for $C(X)$ is equivalent to the following well-known theorem in complex analysis:

"For a rational function on $\boldsymbol{CP}^1$ the number of zeros and the number of poles are equal"

It is understood that the numbers of zeros and poles are counted with multiplicities.

The generalization of the above statement to meromorphic functions is the Argument Principle:

Argument Principle *Let f be a function meromorphic on a domain G containing the closed disk $\bar{\Delta}_r$ or radius r. Assume that there are no zeros nor poles on the boundary $\partial \Delta_r$ of the disk, then*

$$n(0, r) - n(\infty, r) = \frac{1}{2\pi i} \int_{\partial \Delta_r} \frac{f'}{f} dz$$

where $n(0,r)$ and $n(\infty,r)$ are respectively the number of zeros and poles of f inside Δ_r.

If f is a rational function then there are only finitely many zeros and poles of f. We may choose sufficiently large r so that all zeros and poles of f, with the exception of the point at infinity, are inside the open disk Δ_r. Then

$$\mathrm{I}(f) = -\frac{1}{2\pi i} \int_{\partial \Delta_r} \frac{f'}{f} dz = \mathrm{ord}_\infty f$$

and

$$e^{I(f)} = |f|_\infty.$$

With this interpretation, it is clear that we recover the Product Formula. Another way of relating zeros and poles of meromorphic functions is through Jensen's Formula which will be discussed in the next section.

Roth's Theorem can be restated as follows:

Theorem 7 *Let k be a number field (a finite algebraic extension of $\mathbf{Q}$), and $\{a_v \in \bar{\mathbf{Q}} \mid v \in S\}$ where $\bar{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$ and S is a finite set of valuations on k containing all the archimedean valuations. Then for any positive real numbers c and ε, the inequality*

$$\prod_{v \in S} \min\{1, \|x - a_v\|_v\} \geq c\, H(x)^{-(2+\varepsilon)}$$

holds for all but finitely many x in k. Here H is the (multiplicative) height.

The analogue in function fields of Liouville's theorem is due to Mahler ([**Ma1**]). He also showed that Liouville's theorem cannot be improved if the characteristic of the field of constant k is positive.

Theorem 8 (Mahler) *Let $\alpha = \alpha(X)$ be an element of $k(X)_\infty$ algebraic, of degree $d \geq 2$, over $k(X)$. Then there exists a constant C such that*

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{C}{|q|^d}$$

for any polynomials p and q (q not the zero element) $\in k[X]$. If the characteristic of k is positive, the exponent d cannot be improved.

The p-adic case is due to Uchiyama ([U]):

Theorem 9 (Uchiyama) *Let $\alpha = \alpha(X)$ be an element of $k(X)_p$ algebraic, of degree $d \geq 2$, over $k(X)$. Then there exists a constant C such that*

$$\left| \alpha - \frac{r}{s} \right| \geq \frac{C}{(\max\{|r|_p; |s|_p\})^d}$$

for any polynomials r and s (q not the zero element) $\in k[X]$. If the characteristic of k is positive, the exponent d cannot be improved.

However, if the characteristic of k is *zero*, the function field analogue of Roth's theorem is valid.

Theorem 10 (Uchiyama) *Assume that char $k = 0$. Then*

(i)*Let $\alpha = \alpha(X)$ be an element of $k(X)_\infty$ algebraic, of degree $d \geq 2$, over $k(X)$. Then for any $\varepsilon > 0$, there exists a constant C such that*

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{C}{|q|^{2+\varepsilon}}$$

for all but a finite number of pairs of polynomials p and q (q not the zero element) $\in k[X]$.

(ii)*Let $\alpha = \alpha(X)$ be an element of $k(X)_p$ algebraic, of degree $d \geq 2$, over $k(X)$. Then for any $\varepsilon > 0$, there exists a constant C such that*

$$\left| \alpha - \frac{r}{s} \right| \geq \frac{C}{(\max\{|r|_p; |s|_p\})^{2+\varepsilon}}$$

for all but a finite number of pairs of polynomials r and s (q not the zero element) $\in k(X)$.

In the positive characteristic case, Armitage ([Ar]) found a condition for which Roth's theorem holds.

Theorem 11 (Armitage) *Assume that char $k > 0$. Then the conclusions of Uchiyama's theorem hold for those algebraic α which does not lie in a cyclic extension of $k(X)$.*

Remark Armitage actually proved the theorem for fields more general than function fields. But the conditions on these fields are somewhat technical to state at this point. One of these conditions is that the Artin-Whaples product formula holds. In the language of Nevanlinna theory, this means that the First Main Theorem holds).

Roth's theorem is extended to simultaneous approximations by W. M. Schmidt ([**Schm1,2**]) and later by Schlickewei ([**Schl**]) to the non-archimedean case. First we recall some terminologies. For a linear form L of $(n + 1)$-variables with algebraic coefficients, (we shall also identified L with a hyperplane of P^n), the Weil function $\lambda_{v;L}$ is defined by

$$\lambda_{v;L}(x) = \frac{1}{[k:Q]} \log \frac{(n+1)\|L\|_v \max_{0 \leq j \leq n}\{\|x_j\|_v\}}{\|L(x)\|_v}.$$

where for a linear form $L(x) = \sum_{0 \leq j \leq n} a_j x^j, \|L\|_v = \max_{0 \leq i \leq n} \{\|a_i\|_v\}$. Hence $\lambda_{v,L}(x) \geq 0$.

Given a hyperplane L of P^n and a point $x \in P^n(k)$ but $x \notin L$, the *proximity* and *counting* functions are defined by

$$m(x, L) = \sum_{v \in S} \lambda_{v,L}(x); \quad N(x, L) = \sum_{v \notin S} \lambda_{v,L}(x).$$

Note that both the proximity and the counting functions are ≥ 0.

By the definition of height we have the analogue of the First Main Theorem in Nevanlinna Theory ([**V1**], [**R-W**]:

Theorem 12 (First Main Theorem) *If L is a linear form and $L(x) \neq 0$, then*

$$m(x, L) + N(x, L) = h(x) + \frac{1}{[k:Q]} \log \prod_{v \in M_k} \frac{(n+1)\|L\|_v}{\|L(x)\|_v}$$

where $h(x)$ is the logarithmic (additive) height.

The Theorem below (cf. Schmidt [**Schm1**] Theorem 2) is an analogue of the Second Main Theorem of Nevanlinna Theory for holomorphic curves. We shall use the same notation for a linear form and the hyperplane it defines. The following version of the subspace theorem is due to Schlickelwei [**Schl**] (see also Schmidt [**Schm1**], Theorem 3) and [**Schm2**]). The formulation below is due to Vojta ([**Vo1**] Theorem 2.2.4).

Theorem 13 (Subspace Theorem) *Let $\{L_{v,i} \mid v \in S, 1 \leq i \leq n+1\}$ be linear forms in n-variables with algebraic coefficients. Assume that for each fixed $v \in S$ (a finite set of valuations on $\boldsymbol{k}$ containing all archimedean valuations) the $n + 1$ linear forms $L_{v,1}, \ldots, L_{v,n+1}$ are linearly independent. Then for any $\varepsilon > 0$ there exists a finite set $\mathcal{J}$ of hyperplanes of $\boldsymbol{k}^{n+1}$ such that the inequality*

$$\prod_{v \in S} \prod_{1 \leq i \leq n+1} \frac{1}{\|L_{v,i}(x)\|_v} \leq \{size\ (x)\}^{\varepsilon}$$

holds for all S-integral points $x = (x_0, \ldots, x_n) \in \mathcal{O}_{S^{n+1}} - \cup_{L \in \mathcal{J}} L$. Here

$$size\ (x) = \max_{v \in S} \max_{0 \leq j \leq n} \{\|x_j\|_{v.}\}.$$

It is more convenient to formulate the Subspace Theorem projectively and express the estimate in terms of height rather than size.

Theorem 14 *Let $\{L_{v,i} \mid v \in S, 1 \leq i \leq q\}$ be linear forms of $(n + 1)$-variables (or hyperplanes in $\boldsymbol{P}^n$) with algebraic coefficients. Assume that for each fixed $v \in S$, the hyperplanes $L_{v,1}, \ldots, L_{v,q}$ are in general position. Then for any $\varepsilon > 0$ there exists a finite set $\mathcal{J}$ of hyperplanes of $\boldsymbol{P}^n(\boldsymbol{k})$ such that the inequality*

$$\sum_{1 \leq i \leq q} \sum_{v \in S} \lambda_{v,L_{v,i}}(x) \leq (n + 1 + \varepsilon)\ h(x)$$

holds for all points $x \in \boldsymbol{P}^n(\boldsymbol{k}) - \cup_{L \in \mathcal{J}} L$.

The Subspace Theorem of Schmidt can be extended to the case of hyperplanes in *sub-general position* by using the Nochka weight (cf. [**No**]). This extension is due to Ru and Wong [**R-W**]. First we recall the definition of sub-general position due to Chen (cf. [**Ch**]).

Definition Let V be a vector space over F (a field of characteristic 0) of dimension (over F) $k + 1$. Denote by V^* the dual of V. For $1 \leq k \leq n < q$, a collection of non-zero vectors $A = \{v_1, \ldots, v_q\}$ in V^* is said to be in *n-subgeneral position* iff the linear span of any (distinct) $n + 1$ elements of A is V^*. If $n = k$ the concept coincides with the usual concept of *general position*.

Remarks (*i*) It is clear that $\{v_1, \ldots, v_q\}$ is in n-subgeneral position iff $\{\alpha_1 v_1, \ldots, \alpha_q v_q\}$ is in n-subgeneral position where each α_j is a unit of F (i.e., $\alpha_j \in F - \{o\}$). Denote by $P(V^*)$ the projective space of V^*. Then the elements of $P(V^*)$ are identified as hyperplanes of the projective space $P(V)$. A collection of hyperplanes $\{a_j \in P(V^*) \mid 1 \leq j \leq q\}$ is said to be in *n-subgeneral position* iff $\{v_1, \ldots, v_q\}$ is in *n-subgeneral position* where $v_j \in V^*$ satisfies $P(v_j) = a_j$. For $n = k$ this concept agrees with the usual concept of hyperplanes in general position.

(*ii*) If $A = \{v_1, \ldots, v_q\}$ is in n-subgeneral position then it is also in m-subgeneral position for all $m \geq n$ provided that $m < q$.

(*iii*) Let $(b_j \in P(W^*) \mid 1 \leq j \leq q\}$ be hyperplanes in general position, where W is a vector space over F of dimension $n + 1$. Let V be a vector subspace of W of dimension $k + 1$; then $A = \{a_j = b_j \cap P(V) \mid 1 \leq j \leq q\}$ is a set of hyperplanes in $P(V)$, not necessarily in general position but is in n-subgeneral position.

Lemma (Nochka-Chen) *Let $A = \{v_1, \ldots, v_q\}$ be a set of vectors in V^* in n-subgeneral position. Then there exists a function, called the Nochka weight associated to $A, \omega : A \to R$ and a constant θ with the following properties:*

(*i*) $\dfrac{k+1}{2n - k + 1} \leq \theta \leq \dfrac{k+1}{n+1},$

(*ii*) $0 \leq \omega(a) \leq \theta$ *for* $a \in A$,

(*iii*) $\sum_{a \in A} \omega(a) = k + 1 + \theta(\#A - 2n + k + 1),$

(*iv*) *for any subset B of A with $\# B \leq n + 1$, $\sum_{a \in A} \omega(a) \leq$* $d(B) =$ *dimension of the linear space spanned by elements of B.*

The generalization of The Subspace Theorem to the case of hyperplanes in sub-general position takes the following form (cf. [**R-W**]):

Theorem 15 *Let* $\{L_{v,i} \mid v \in S, 1 \leq i \leq q\}$ *be linear forms of* $(k + 1)$-*variables (or hyperplanes in* $\boldsymbol{P}^k$*) with algebraic coefficients. Assume that for each fixed* $v \in S$*, the linear forms* $L_{v,1}, \ldots, L_{v,q}$ *are in n-subgeneral position* $(1 \leq k \leq n < q)$ *with associated Nochka weights* $\omega_{v,1}, \ldots, \omega_{v,q}$*. Then for any* $\varepsilon > 0$ *there exists a finite set* $\mathcal{J}$ *of hyperplanes of* $\boldsymbol{P}^k(\boldsymbol{k})$ *such that the inequality*

$$\sum_{1 \leq i \leq q} \sum_{v \in S} \omega_{v,i} \lambda_{v,L_{v,i}}(x) \leq (k + 1 + \varepsilon)\, h(x)$$

holds for all points $x \in \boldsymbol{P}^k(\boldsymbol{k}) - \cup_{L \in \mathcal{J}} L$.

In terms of the proximity function, Theorem 15 takes the following forms:

Corollary *Let* $\{L_1, \ldots, L_q\}$ *be linear forms in* $(k+1)$-*variables with algebraic coefficients, in n-subgeneral position (* $1 \leq k \leq n$ *and* $q > 2n - k + 1$*). Then for any* $\varepsilon > 0$ *there exists a finite set* $\mathcal{J}$ *of hyperplanes of* $\boldsymbol{P}^k(\boldsymbol{k})$ *such that*

$$\sum_{1 \leq i \leq q} \omega_i m(x, L_i) \leq (k + 1 + \varepsilon)\, h(x)$$

holds for all points $x \in \boldsymbol{P}^k(\boldsymbol{k}) - \cup_{L \in \mathcal{J}} L$ *and where* ω_i *are the Nochka weights.*

Corollary *Let* $\{L_1, \ldots, L_q\}$ *be linear forms in* $(k+1)$-*variables with algebraic coefficients, in n-subgeneral position* $(1 \leq k \leq n)$*. Given any* $\varepsilon > 0$*, there exists a finite set* $\mathcal{J}$ *of hyperplanes of* $\boldsymbol{P}^k(\boldsymbol{k})$ *such that*

$$\sum_{1 \leq i \leq q} m(x, L_i) \leq (2n - k + 1 + \varepsilon)\, h(x)$$

holds for all points $x \in \boldsymbol{P}^k(\boldsymbol{k}) - \cup_{L \in \mathcal{J}} L$.

Corollary *Let* $\{L_1, \ldots, L_q\}$ *be hyperplanes of* $\boldsymbol{P}^n(\boldsymbol{k})$*, in general position. Then for any* $\varepsilon > 0$ *and* $1 \leq k \leq n$*, the set of points of* $\boldsymbol{P}^n(\boldsymbol{K})$ *such that*

$$\sum_{1 \leq i \leq q} m(x, L_i) \geq (2n - k + 1 + \varepsilon)\, h(x)$$

is contained a finite union of linear subspaces, $\cup_{L \in \mathcal{J}} L$, of dimension $k - 1$. In particular, the set of points of $P^n(k) - \cup_{1 \leq i \leq q} L_i$ such that

$$\sum_{1 \leq i \leq q} m(x, L_i) \geq (2n + \varepsilon) h(x)$$

is a finite set of points.

For a finite subset S of M_k of valuations containing the set S_∞ of all archimedean valuations of k. Denote by $\mathcal{O}_S$ the ring of *S-integers* of k, i.e., the set of $x \in k$ such that

$$\|x\|_v \leq 1$$

for all $v \notin S$. A point $x = (x_1, \ldots, x_n) \in k^n$ is said to be a *S-integral point* if $x_i \in \mathcal{O}_S$ for all $1 \leq i \leq n$. Let D be a *very ample effective divisor* on a projective variety V and let $1 = x_0, x_1, \ldots, x_N$ be a basis of the vector space:

$$l(D) = \{f \mid f \text{ is a rational function on the variety } V$$
$$\text{such that } f = 0 \text{ or } (f) + D \geq 0\}.$$

Then $P \to (x_1(P), \ldots, x_N(P))$ defines an embedding of $V(k) - D$ into the affine space k^N. A point P of $V(k) - D$ is said to be a *D-integral point* if $x_i(P) \in \mathcal{O}_S$ for all $1 \leq i \leq N$.

The following theorem of Ru-Wong extends the classical theorem of Thue-Siegel that $P^1 - \{3 \text{ distinct points}\}$ has finitely many integral points:

Theorem 16 *Let k be a number field and $H_1, \ldots, H_q$ be a finite set of hyperplanes of $P^n(k)$, assumed to be in general position. Let $D = \sum_{1 \leq j \leq q} H_i$, then for any integer $1 \leq k \leq n$, the set of D-integral points of $P^n(k) - D$ is contained in a finite union of linear subspaces of $P^n(k)$ of dimension $k - 1$ provided that $q > 2n - k + 1$. In particular, the set of D-integral points of $P^n(k) - \{2n + 1 \text{ hyperplanes in general position}\}$ is finite.*

More generally, let V be a projective variety, D a very ample divisor on V and let $\{\varphi_0, \ldots, \varphi_N\}$ be a basis of $l(D)$, such that

$$\Phi = [\varphi_0, \ldots, \varphi_N] : V \to \mathbf{P}^N$$

is an embedding of V into $\mathbf{P}^N$ with $V - D$ embedded in k^N. We identify V with its image $\Phi(V)$. As an immediate consequence of the main theorem we also have:

Corollary *Let V be a projective variety, D a very ample divisor on V. Let $D_1, \ldots, D_q$ be divisors in the linear system $|D|$ such that $E = D_1 + \ldots + D_q$ has, at worst, simple normal crossing singularities. If $q > 2N - k + 1$ where $N = \dim \mathrm{I}(D) - 1$ and $1 \leq k \leq n$, then the set of E-integral points of $V - E$ is contained in the intersection of a finite number of linear subspaces, of dimension $k - 1$, of $\mathbf{P}^N$ with V. In particular, the set of E-integral points of $V - E$ is finite if $q \geq 2N + 1$.*

(II) Theory of Holomorphic Curves

The Theory of holomorphic curves is the study of holomorphic maps from the complex plane into complex manifolds. More generally, one studies holomorphic maps between complex manifolds with the case of curves being the most difficult. This is due to the fact that the image of a holomorphic curve is usually of high codimension. Typically results in the theory of maps assert that, under appropriate conditions, every holomorphic map in a complex manifold M degenerates. The types of degeneracy range from the weakest form: "the image does not contain an open set" to the strongest form: "the image consists of one point". In between we have degeneration at a certain dimension. Namely, the image is contained in a complex subvariety of dimension p with $0 \leq p < n = \dim_C M$.

Manifolds with the property that every holomorphic curve $f : C \to M$ is constant is said to be *Brody-hyperbolic* [B]. If M is compact, then the concept of Brody-hyperbolic is equivalent to the concept of Kobayashi-hyperbolic [K1]. The following differential geometric description of Kobayashi-hyperbolicity is due to Royden [**Roy**]. Given a non-zero tangent vector $\xi \in T_x M$, the *infinitesimal Kobayashi-Royden (pseudo) metric* is defined by

$$0 \leq k_M(\xi) = \inf \frac{1}{r}$$

where the inf is taken over all positive real numbers r such that there exists a holomorphic map $f : \Delta_r \to M$ such that $f(0) = x$ and $f'(0) = \xi$. Here Δ_r is the disk of radius r in C. Alternatively,

$$k_M(\xi) = \sup |t| \geq 0$$

where the sup is taken over all $t \in C^*$ such that there exists a holomorphic map $f : \Delta \to M$ with $f(0)$ and $f'(0) = t\,\xi$. A complex manifold M is said to be *hyperbolic at a point* x if there exists an open neighborhood U of x and a hermitian metric $ds_U{}^2$ on TU such that $k_M \geq ds_U$ on TU. A complex manifold M is said to be *hyperbolic* if it is hyperbolic at every point. The Kobayashi pseudo-distance associate to k_M is defined by

$$d(x, y) = inf \sqrt{k_M(\gamma'(t))}$$

where the *inf* is taken over all piecewise smooth curves joining x and y. The condition that M is Kobayashi-hyperbolic is equivalent to the condition that the Kobayashi pseudo-distance is a distance; i.e., $d(x, y) > 0$ if $x \neq y$. With this distance function M is a metric space and M is said to be complete if M is a complete metric space.

The infinitesimal Kobayashi metric satisfies $k_M(t\,\xi) = |t|k_M(\xi)$, hence it is a *Finsler metric*. It has the nice property that every holomorphic map is metric decreasing. Namely, if $f : M \to N$ is holomorphic then $k_N(f_*\xi) \leq k_M(\xi)$. In particular, every biholomorphic self map of M is an *isometry* of the Kobayashi metric.

The infinitesimal Kobayashi metric does not have very good regularity in general. In this direction we have the fundamental result of Royden [R1] that the infinitesimal Kobayashi metric is always upper semi-continuous. If it is complete hyperbolic then the metric is continuous. It is well-known that the poly-discs are complete-hyperbolic but the infinitesimal Kobayashi metric is not differentiable.

As mentioned above, for compact manifolds Kobayashi-hyperbolic is equivalent to Brody-hyperbolic. In general, Kobayashi-hyperbolic implies Brody-hyperbolic but the converse may not be true if M is non-compact. The example below is very well-known.

Example (Eisenman and Taylor) Let M be the domain in C^2 given by

$$M = \left\{ (z,w) \in C^2 \mid |z| < 1, |zw| < 1 \text{ and } |w| < 1 \text{ if } z = 0 \right\}$$

Then M contains no complex lines; for if $f : C \to M$ is holomorphic then $\pi_1 \circ f$ is bounded (where π_1 is the projection onto the first coordinate), hence constant; now if $\pi_1 \circ f$ is constant then $\pi_2 \circ f$ (where π_2 is the projection onto the second coordinate) is bounded and so must be constant as well. However M is not Kobayashi-hyperbolic, because the Kobayashi distance of any point of the form $(0, w)$ from the origin is zero. This is evident by considering the connecting paths: for any positive integer n, let $f_{j,n} : \Delta \to M, j = 0, 1, 2$, be holomorphic maps defined by $f_{0,n}(z) = (z, 0), f_{1,n}(z) = (1/n, nz)$ and $f_{2,n}(z) = (1/n + z/2, w)$. Then $f_{0,n}(0) = (0,0) = p_0, f_{0,n}(1/n) = (1/n, 0) = p_1, f_{1,n}(0) = (1/n, 0) = p_1, f_{1,n}(w/n) = (1/n, w) = p_2, f_{2,n}(0) = (1/n, w) = p_2, f_{2,n}(-2/n) = (0, w)$. The Kobayashi distances between the points $0, 1/n$ and $-2/n$ on the unit disc approaches zero as n approaches ∞.

Intuitively speaking, for non-compact manifolds, the points at "infinity" plays a very important role. In fact Green [**Gn4**] showed that

Theorem 17 (Green) *Let D be a union of (possibly singular) hypersurfaces $D_1, \ldots, D_m$ hypersurfaces in a complex manifold M. Then $M - D$ is Kobayashi-hyperbolic if*

   (i) There is no non-constant holomorphic curve $f : C \to M - D$;
   (ii) There is no non-constant holomorphic curve

$$f : C \to D_{i_1} \cap \ldots \cap D_{i_k} - (D_{j_1} \cup \ldots \cup D_{j_l})$$

for all possible choices of distinct indices so that $\{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_l\} = \{1, \ldots, m\}$.

An important special case of this is the theorem:

Corollary (Green) *The complement of q hyperplanes in general position in CP^n is Kobayashi-hyperbolic for $q \geq 2n + 1$. The number $2n + 1$ is sharp.*

Let $\{P_j(z_0,\ldots,z_n) \mid 1 \le j \le k\}$ be a set of homogeneous polynomials with coefficients in an algebraic number field k. Let V be the common zeros of $P_j(i \le j \le k)$ in CP^n. We assume that V is irreducible and smooth. Lang conjectured that if V is hyperbolic then for any finite extension K of the field k, the set of rational points $V(K)$ in V over K (i.e., $V(K) = \{(z_0,\ldots,z_n) \mid$ if there exists i so that $z_i \ne 0$ and $z_j/z_i \in K$ for all $j\}$ is finite. If V is a hyperbolic affine algebraic manifold in k^N defined by polynomials $\{Q_j(z_1,\ldots,z_N) = 0, 1 \le j \le k\}$ then the set of integral points of V over K is finite. This conjecture is verified for the case of curves of genus $g \ge 2$ (Falting); for $V = P^n - \{2n + 1$ hyperplanes in general position$\}$ (Ru-Wong) and for $V =$ complement of an ample divisor of an abelian variety (Faltings).

Green ([Gn3]) also proved that if the hyperplanes are distinct but not in general position, one can still conclude that the complement is Brody-hyperbolic, namely it contains no non-trivial holomorphic curve from C. The corresponding statement in number theory:

"$P^n(k) - \{2n+1$ distinct hyperplanes$\}$ contains only finitely many integral points"

is still open. The proof of Ru-Wong for the case of hyperplanes in general position involves an extension of the Siegel-Roth-Schmidt type estimate for which the general position assumption is necessary.

Returning to the discussion of hyperbolic manifolds, the following Theorem of Brody ([B]) is very important in constructing examples.

Theorem 18 (Brody) *Small smooth deformations of a compact hyperbolic manifold are hyperbolic.*

Thus the set of compact hyperbolic manifolds is open. The following example of Brody and Green shows that it is not a closed set in general.

Example (Brody-Green) The hypersurface in CP^3 defined by

$$V_\varepsilon = \left\{ z_0^d + z_1^d + z_2^d + z_3^d + (\varepsilon z_0 z_1)^{d/2} + (\varepsilon z_0 z_2)^{d/2} = 0 \right\}$$

is hyperbolic for any $\varepsilon \neq 0$ and where $d \geq 50$ is an even integer. For $\varepsilon = 0$, V_0 is a Fermat variety which is clearly not hyperbolic. Note that by the Lefschetz theorem, V_ε is simply connected. Since V_0 is non-singular (it is the Fermat surface of degree d) it follows that V_ε is non-singular for small ε. A Fermat surface of any degree admits complex lines, for instance take μ and η be any d-th roots of -1, then $z_0 = \mu z_1$ and $z_2 = \eta z_3$ is a complex line in the Fermat surface of degree d.

For the non-compact case, the problem of deformation of hyperbolic manifolds is much more complicated, additional assumptions are needed. The concept of "hyperbolic embeddedness" is needed, we shall not get into this here. The readers are encouraged to look into the very interesting paper of Zaidenberg [**Z**].

Classically, hyperbolicity is studied via the behavior of curvature. The most well-known Theorem is the Schwarz-Pick-Ahlfors lemma:

Theorem 19 *Let M be a complex hermitian manifold with holomorphic sectional curvature bounded above by a (strictly) negative constant. Then M is Kobayashi-hyperbolic.*

For a Riemann surface, Milnor [**Mi**] (see also Yang [**Ya**]) observed that the classical condition on the holomorphic curvature can be relaxed to the condition that the curvature satisfies $K(r) \leq -(1 + \varepsilon)/(r^2 \log r)$ asymptotically where r is the geodesic distance from a point. The following higher dimensional analogue of Milnor's result is due to Greene and Wu ([**G-W**] p.113 Theorem G'). A point O of a Kähler manifold M is called a pole if the exponential map at O is a diffeomorphism of the tangent space at O onto M. Let r be the geodesic distance from O. Let S_r be the geodesic sphere and X be the outward normal. Then $Z = X - \sqrt{-1}JX$ is called the (complex) *radial direction*. The *radial curvature* is defined to be the sectional curvature of the plane determined by the radial direction. With these terminologies we can now state the Theorem of Greene and Wu:

Theorem 20 *Let M be a complex Kähler manifold with a pole such that (i)the radial curvature is everywhere non-positive and $\leq -1/(r^2 \log r)$ asymptotically, (ii)the holomorphic sectional curvature $\leq -1/r^2$ asymptotically. Then M is Kobayashi-hyperbolic.*

The next Theorem ([**K-W**]) gives a criterion of hyperbolicity without requiring information on the precise rate of decay of the curvature. A complex manifold M of complex dimension n is said to be *strongly q-concave* if there exists a continuous function φ on M such that (*i*)for all real numbers c the set $\{z \in M \mid \varphi(z) \leq c\}$ is compact, (*ii*)the Levi form $i\partial\bar{\partial}\varphi$ is semi-*negative* and has at least $n - q$ *negative* eigenvalues (in the sense of distributions) everywhere outside a compact set. Alternatively, M is strongly q-concave if there exists a continuous function φ on M such that (*i*)for all real numbers c the set $\{z \in M \mid \varphi(z) \geq c\}$ is compact, (*ii*)the Levi form $i\partial\bar{\partial}\varphi$ is semi-*positive* and has at least $n - q$ *positive* eigenvalues (in the sense of distributions) everywhere outside a compact set.

Theorem 21 (Kreuzman-Wong) *Let M be a complete Kähler manifold of complex dimension m such that both the holomorphic sectional curvature and the Ricci curvature are (strictly) negative. Assume that M is strongly 0-concave and that the universal cover is Stein then M is Kobayashi hyperbolic.*

A complete simply connected Riemannian manifold M of nonpositive Riemannian sectional curvature is said to be a *visibility manifold* if any two points at infinity (denoted $M(\infty)$) can be joined by a unique geodesic in M. A complete simply-connected Riemannian manifold with sectional curvature bounded above by a negative number (i.e., $K \leq -b^2$) is a visibility manifold. More generally a complete simply-connected Riemannian manifold with strictly negative sectional curvature (i.e., $K < 0$) and radial curvature $-K(r)$, from some fixed point, satisfying the condition

$$\int\limits_{1}^{\infty} rK(r)\, dr = \infty$$

is a visibility manifold.

Corollary (Kreuzman-Wong) *Let M be a complete Kähler manifold such that its universal cover satisfies the visibility axioms and that the Riemannian sectional curvature satisfies $-a^2 \leq K < 0$. Assume that M admits a finite volume quotient. Then M is Kobayashi hyperbolic.*

The proof of Theorem 21 (and the corollary) relies on the compacification theorem (again, we see that the "infinity" plays a crucial role) of Nadel and Tsuji [N-T] extending the result of Siu and Yau [S-T] on compacification of Kähler manifolds of finite volume and negative pinching (both above and below) of the Riemannian sectional curvature. The theorem of Siu and Yau gives more precise information on the compactification, in the case of Kähler manifolds, of the corresponding theorem of Gromov ([B-G-S]) in the Riemannian case. Both the theorem of Nadel-Tsuji and that of Siu-Yau have the origin in the work of Andreotti-Tomassini [A-T] on pseudoconcave manifolds. These theorems are natural generalization of the well-known compacification theorem for finite volume quotients of bounded symmetric domains.

Let D be an irreducible algebraic curve in CP^2. At a point p in D let $A_1, \ldots, A_k$ be local irreducible components of D containing p. Let L be a projective line through p and denote by $m_j = \min_L$ {intersection multiplicity of $L \cap A_j$}. Then $(m_1 - 1, \ldots, m_k - 1)$ are the orders of irreducible singularities at p. Let

$$b = \sum_{1 \le j \le q} (m_j - 1)$$

be the total order of singularities of D. Denote by D^* the dual curve of D. The curve D is birationally equivalent to its dual D^*. The normalization of D and D^* are isomorphic. Denote by b^* the total order of singularities of D^*.

Theorem 22 (Grauert-Peternell [G-P]) *Let D be an irreducible algebraic curve in CP^2 of genus $g \ge 2$. Assume that $b^* + \chi(D) < 0$ (where b^* is the total order of irreducible singularities of D) and that every tangent of D^* intersects D^* in at least two points. Then $CP^2 - D$ is hyperbolic.*

At present one of the major open problems in the theory of hyperbolic manifolds is the following conjecture.

Conjecture : *For a generic algebraic curve D of degree $d \ge 5$ in CP^2, the complement $CP^2 - D$ is Kobayashi-hyperbolic.*

The space of algebraic curves of degree in d in CP^2 is a projective manifold, denoted $\mathfrak{F}_d$. By a generic curve of degree d, we mean an element of $\mathfrak{F}_d - S_d$ where S_d is a subvariety of lower dimension. The conjecture is of course false without the "generic" condition. One of the difficulty of the conjecture is to describe the exceptional subvariety S_d. The interested readers are refer to the paper of Grauert [G] for many interesting ideas.

We now turn to the Second Main Theorem of Value Distribution Theory. It all begins from the fundamental work of Nevanlinna in one complex variable. Let $f : C \to CP^1$ be a holomorphic map. The characteristic function $T(f, r)$ is defined by

$$T(f, r) = \int_s^r \frac{dt}{t} \int_{\Delta_t} f^*\omega$$

where $0 < s \leq r$ and ω is the Fubini-Study metric on CP^1. For a point $a \in CP^1$, denote by $n(f, a, r)$ the number of preimages (counting multiplicities) of a inside the disk of radius r. The counting function $N(f, a, r)$ is defined by

$$N(f, a, r) = \int_s^r \frac{n(f, a, t)}{t} dt$$

The *proximity function* $m(f, a, r)$ is defined by

$$m(f, a, r) = \int_{\partial \Delta_r} \log \|f; a\| \frac{d\theta}{2\pi}$$

where $\|x; a\| = |<x, a>|/\|x\| \, \|a\|$ is the projective distance between x and a. Here $\|x\|^2 = |x_0|^2 + |x_1|^2$, $\|a\|^2 = |a_0|^2 + |a_1|^2$ and $<x, a> = x_0 a_0 + x_1 a_1$. The characteristic function, counting function and proximity function are related by the First Main Theorem of Nevanlinna ([Ne1,2,3]).

Theorem 23 (FMT) *Let $f : C \to CP^1$ be a non-constant holomorphic map and let a be a point of CP^1. Then*

$$m(f, a, r) + N(f, a, r) = T(f, r) + O(1).$$

The FMT plays a similar role in Nevanlinna Theory as the role played by the Artin-Whaple product formula in Number Theory. The counterpart of Roth's Theorem in Number Theory is The Second Main Theorem of Nevanlinna:

Theorem 24 (SMT) *Let $f : C \to CP^1$ be a non-constant holomorphic map and $\{a_1, \ldots, a_q\}$ be a finite subset of CP^1. Then for any $\varepsilon > 0$ there exists a set A of finite Lebesgue measure such that the following estimate holds for all $r \in [s, \infty) - A$:*

$$\sum_{1 \leq j \leq q} m(f, a_j, r) \leq (2 + \varepsilon) T(f, r).$$

Note that, by the FMT, the left hand side of the SMT can be replaced by $qT(f, r) - \sum_{1 \leq j \leq q} N(f, a_j, r)$.

For a point $a \in CP^1$, the defect $\delta_f(a)$ is defined to be

$$\delta_f(a) = \liminf_{r \to \infty} \frac{m(f, a, r)}{T(f, r)} == 1 - \limsup_{r \to \infty} \frac{N(f, a, r)}{T(f, r)}.$$

Corollary *Let $f : C \to CP^1$ be a non-constant holomorphic map. Then for a finite set $\{a_1, \ldots, a_q\}$ of CP^1, the sum of defects satisfies the estimate*

$$\sum_{1 \leq j \leq q} \delta_f(a_j) \leq 2.$$

The factor $2+\varepsilon$ in the SMT correspondes to the exponent $2+\varepsilon$ in Roth's Theorem. The fact that CP^1—{3 distinct point} is hyperbolic is a consequence of Nevanlinna's SMT. This corresponds to the fact in number theory that Thue-Siegel Theorem (the integral points of $P^1(k)$—{3 distinct point} is finite) follows from Roth's Theorem. The proofs of these two statements are completely analogous (cf. Ru-Wong [**R-W**]).

Nevanlinna's Theorem can be extended from CP^1 to arbitrary Riemann surface in the following form. Let M be a Riemann surface with hermitian metric

$$ds^2 = h\frac{\sqrt{-1}}{2\pi}dz \wedge d\bar{z}.$$

Denote by R the Gaussian curvature of h; i.e.,

$$R = h^{-1}\frac{\partial^2}{\partial z \partial \bar{z}} \log h.$$

Theorem 25 *Let M be a Riemann surface with hermitian metric ds^2 and let $f : C \to M$ be a non-constant holomorphic map. Let $\{a_1, \ldots, a_q\}$ be a finite set of M. Then for any $\varepsilon > 0$ there exists a set A of finite Lebesgue measure such that*

$$qT(f, ds^2, r) - \sum_{1 \le j \le q} N(f, a_j, r) \le$$

$$2\int_s^r \frac{dt}{t} \int_{|z| \le t} (f^*R)f^*ds^2 + \varepsilon T(f, ds^2, r)$$

holds for all $r \in [s, \infty) - A$. Consequently, the sum of defects satisfies the estimate

$$\sum_{1 \le j \le q} \delta(f, a_j) \le 2 \limsup \int_s^r \frac{dt}{t} \int_{|z| \le t} (f^*R)f^*ds^2/T(f, ds^2, r).$$

Here the characteristic function $T(f, ds^2, r)$ is given by

$$T(f, ds^2, r) = \int_s^r \frac{dt}{t} \int_{|z| \le t} f^*ds^2.$$

If the Gaussian curvature R is constant $(= c)$ then

$$\int_s^r \frac{dt}{t} \int_{|z| \le t} (f^*R)f^*ds^2 = cT(f, ds^2, r)$$

and Theorem 25 takes a simpler form:

Corollary *Same assumption as in Theorem 25 and assume that the Gaussian curvature $R = c$ of the Riemann surface M is constant. Then for any $\varepsilon > 0$ there exists a set A of finite Lebesgue measure such that*

$$qT(f, ds^2, r) - \sum_{1 \leq j \leq q} N(f, a_j, r) \leq (2c + \varepsilon)T(f, ds^2, r)$$

holds for all $r \in [s, \infty) - A$. Consequently, the sum of defects satisfies the estimate

$$\sum_{1 \leq j \leq q} \delta(f, a_j) \leq 2\,c.$$

For the Riemann sphere with the Fubini-Study metric, the curvature $R = 1$ (parabolic); for the torus (elliptic), the canonical metric is a flat metric, i.e. $R = 0$; and for surface of genus ≥ 2 (hyperbolic) with the canonical metric the curvature $R = -1$. Thus

Corollary (*i*)   *If $M = CP^1$ then $\sum_{1 \leq j \leq q} \delta(f, a_j) \leq 2$;*
  (*ii*)   *If $M = T = torus$ then $\sum_{1 \leq j \leq q} \delta(f, a_j) \leq 0$, in particular every non-constant holomorphic map from C into T is surjective;*
  (*iii*)   *If genus $M \geq 2$ then there is no non-constant holomorphic map from C into M, i.e., M is hyperbolic.*

The corresponding Theorems in number theory assert that the following spaces contain only finitely many integral points over any number field k:

  (*i*)   (Thue, Roth, Schmidt) P^1 - {3 distinct point};
  (*ii*)   (Siegel) T^1 - {one point};
  (*iii*)   (Mordell conjecture) compact Riemann surfaces of genus ≥ 1.

Nevanlinna's Theorem can also be generalized to higher dimension for holomorphic maps between equidimensional manifolds. This extension is due to Carlson-Griffiths [C-G] and Griffiths-King [G-K]. Let $f : C^n \to M^n$ be a holomorphic map into a projective manifold. Let D be an ample divisor on M represented as the zero set of a

holomorphic section s of a holomorphic line bundle $\mathfrak{l}$ over M. The proximity function is defined by

$$m(f, D, r) = \int_{\partial B_r} \log \frac{1}{\|s(f(z))\|} d\sigma$$

where B_r is the ball of radius r in $\mathbf{C}^n$ and $d\sigma$ is the rotationally invariant measure of the boundary, normalized so that the volume of the boundary ∂B_r is 1. Specifically,

$$d\sigma = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{n-1}.$$

Let $\mathfrak{l}$ be a holomorphic line bundle over M and let h be a hermitian metric on $\mathfrak{l}$ with Chern form ρ. The characteristic function of f is defined by

$$T(f, \mathfrak{l}, r) = \int_0^r \frac{dt}{t^{2n-1}} \int_{B_r} f^* \rho \wedge \omega^{n-1}$$

where $\omega = dd^c \|z\|^2$.

We state the SMT of Carlson-Griffiths-King in the sharper form of Wong [W2] (see also Goldberg-Grinshtein [G-G], Lang [L4] and Cherry [Ch]):

Theorem 26 *Let $\mathfrak{l}$ be a positive holomorphic line bundle over a projective manifold of dimension n and $D_1, \ldots, D_q \in |\mathfrak{l}|$ be divisors such that $D = D_1 + \ldots + D_q$ is of simple normal crossing. Let $\mathfrak{k}^*$ be the dual of the canonical bundle of M. Let $f : \mathbf{C}^n \to M^n$ be a non-degenerate (Jacobian not identically zero or equivalently, the image contains a non-empty open set). Then for any $\varepsilon > 0$, there exists a set A of finite Lebesgue measure such that the estimate*

$$\sum\nolimits_{1 \leq j \leq q} m(f, D_j, r) \leq T(f, \mathfrak{f}^*, r) + \log T(f, \mathfrak{l}, r)$$
$$+ n(1 + \varepsilon) \log \log T(f, \mathfrak{l}, r)$$
$$+ \frac{1}{2} n(1 + \varepsilon) \log \log \log T(f, \mathfrak{l}, r)$$
$$+ \frac{1}{2} n(1 + \varepsilon) \log \log r$$

holds for all $r \in [s, \infty) - A$.

Corollary *With the same assumptions as in Theorem 25 and assume in addition that f is transcendental. Then*

$$\sum\nolimits_{1 \leq j \leq q} \delta(f, D_j) \leq c_1(\mathfrak{f}^*)/c_1(\mathfrak{l}).$$

Theorem 26 holds if one replaces C^n by an affine algebraic manifold N of dimension $m \geq n = \dim M$ and under the same non-degeneracy assumption; namely, the image of the map f contains a non-empty open set. Stoll extended Theorem 26 to the case where the domain is a parabolic manifold. In this more general case, the right hand side of the estimate of Theorem 26 is more complicated; terms involving the Ricci curvature of the parabolic manifold also appears. We refer the readers to Stoll [Sto2] for details. *The corresponding statement in number theory of the estimate in Theorem 26 is conjectured by Lang. This sharper form of the Roth's Theorem is still open.*

Nevanlinna's Theory can also be extended to the non-equidimensional case under a much weaker non-degeneracy assumption. This case is much harder and much deeper; so far the only satisfactory result is the case of hyperplanes in CP^n even though there are some progress in the more general case. The main ideas of handling linearly non-degenerate holomorphic curves are contained already in Ahlfors [A] (also H. Weyl and J. Weyl [W-W]; for a different approach see Cartan [Ca]). Unlike the case of Nevanlinna and also the case of Carlson-Griffiths-King where the first derivative of the holomorphic

map contains all the necessary informations needed; the linearly non-degenerate condition involves, for curves in CP^n, derivatives of f of order up to n. The informations contained in the derivatives are related by the Plücker Formula. Ahlfor's Theory was extended by Stoll to the case of linearly non-degenerate meromorphic maps from C^m into CP^n. Stoll realized that the associated maps in the higher dimensional case, unlike the case of curves, are in general only meromorphic rather than holomorphic. This is so even if the original map is assumed to be holomorphic. Thus it is necessary to develop the whole theory for meromorphic maps. The Theory of Ahlfors and Stoll was extended by Murray where the domain is assumed to be Stein, and Wong where the domain is assumed to be affine algebraic or parabolic. The following sharper form of the SMT is due to Stoll-Wong [S-W].

Theorem 26 *Let M be an affine algebraic manifolds of dimension m and $f : M \to CP^n$ be a linearly non-degenerate meromorphic map. Let $a_1, \ldots, a_q$ be hyperplanes of CP^n in general position. Then for any $\varepsilon > 0$ there exists a set A of finite Lebesgue measure such that*

$$
\begin{aligned}
\sum_{1 \le j \le q} m(f, a_j, r) \le\ & (n+1)T(f,r) + d_M \frac{n(n+1)}{2} \Big\{ \log T(f,r) \\
& + (2 + \varepsilon) \log \log T(f,r) + \frac{n-1}{2} \log^+ r \\
& + \frac{1}{2}(5 + 3\varepsilon) \log^+ \log^+ r \Big\} \\
& + (\log^+ \log^+ \log^+ T(f,r)) \\
& + O(\log^+ \log^+ \log^+ r)
\end{aligned}
$$

for all $r \in [s, \infty) - A$ and where d_M is the degree of M.

    The SMT for linearly non-degenerate curves in CP^n correspondes to the subspace Theorem of Schmidt in number theory. The SMT can also be extended to the case of hyperplanes of CP^k in n-subgeneral position. The result is first conjectured by H. Cartan and is known as Cartan conjecture. The conjecture is first resolved in the affirmative by Nochka and also in the Thesis of Chen. The corresponding result was due to Ru and Wong using ideas from the works of Nochka and Chen.

Theorem 27 *Let M be an affine algebraic manifolds of dimension m and f : M → CP^k be a linearly non-degenerate meromorphic map. Let $a_1, \ldots, a_q$ be a hyperplanes of CP^k in n-subgeneral position ($k \leq n$). Then for any $\varepsilon > 0$ there exists a set A of finite Lebesgue measure such that*

$$\sum_{1 \leq j \leq q} m(f, a_j, r) \leq (n + 1 + \varepsilon)T(f, r)$$

for all $r \in [s, \infty) - A$.

Actually it is possible to obtain a more precise estimate as in Theorem 26.

For holomorphic curves from C into Abelian varieties, a SMT was obtained by Ochiai [**Oc**] and also by Noguchi [**Nog**] using jet metrics. However these results are not in the best possible form. In a forth coming paper we shall treat the case of holomorphic curves in spaces of constant sectional curvature. A sharp form of the SMT can be obtained via the use of Plücker's formula and also the technique of Siu described below. Recently R. Kobayashi, using a rather different method seems to obtain a fairly sharp SMT in the case of holomorphic curves in Abelian varieties.

The main ingredients of the proof are: Green-Jensen's Formula, Nevanlinna's lemma and Plücker's Formula.

Green-Jensen Formula *Let φ be a function of class C^2 or a plurisubharmonic function or a plurisuperharmonic function on C^n. Then*

$$\int\limits_{s}^{r} \frac{dt}{t^{2m-1}} \int\limits_{\|z\| \leq t} dd^c[\varphi] \wedge (dd^c\|z\|^2)^{n-1} =$$

$$\frac{1}{2} \int\limits_{\|z\|=r} \varphi d^c \log \|z\|^2 \wedge (dd^c\|z\|^2)^{n-1}$$

$$-\frac{1}{2} \int\limits_{\|z\|=s} \varphi d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{n-1}$$

where $dd^c[\varphi]$ denotes differentiation in the sense of distribution.

In particular the lemma applies to $\varphi = \log|f|$ where f is a meromorphic function.

To state the lemma of Nevanlinna we need a definition. A non-negative, non-decreasing function g defined on $[0,\infty)$ is called a *growth function* if for any $t_0 > 0$

$$\int_{t_0}^{\infty} \frac{1}{tg(t)} dt = c_0(g,t_0) < \infty.$$

A typical growth function is $g(t) = (\log(1+t))^{1+\varepsilon}$ where $\varepsilon > 0$.

Nevanlinna's Calculus Lemma *Let T be a non-negative, non-decreasing, absolutely continuous function defined on the interval $[s,\infty)$ where $s \geq 0$. Let g be a growth function. Then there exists a measurable subset $[s,\infty)$ with finite Lebesgue measure such that*

$$T'(r) \leq T(r)\, g(T(r))$$

holds for all $r \in [s,\infty) - A$.

This technical lemma is fundamental in all the estimates encountered in Nevanlinna theory. Another lemma which is of technical as well as theoretical importance is Plücker's Formula. Let S be a Riemann surface with hermitian metric h and (M,g) be a complex manifold of dimension n with constant sectional curvature c. Let $f : S \to M$ be a holomorphic curve and f_k be the k-th associate curves. Assume that $f_k \not\equiv 0$ for $1 \leq k \leq n$. Define differential forms

$$\Theta_k = dd^c \log|\Lambda_k|^2 + (k+1)c\frac{\sqrt{-1}}{2\pi}g - \frac{k(k+1)}{2}\frac{\sqrt{-1}}{2\pi}h,$$

$$1 \leq k \leq n-1.$$

We may now state the Plücker Formula (cf. [W3]):

Plücker's Formula for Spaces of Constant Curvature *Let (M,g) be a hermitian manifold of constant curvature c and S a Riemann surface with hermitian metric h. Let $f : S \to M$ be a holomorphic*

curve which is non-degenerate of order k; i.e., the associate curve
$f_k \not\equiv 0$. *Then*

$$\begin{cases} \operatorname{Ric} \Theta_1 = \Theta_2 - 2\Theta_1 + c\Omega_g \\ \operatorname{Ric} \Theta_k = \Theta_{k-1} + \Theta_{k+1} - 2\Theta_k; \ 2 \le k \le n-1 \end{cases}$$

on $S - \{\zeta \in S | \Lambda_k(\zeta) = 0\}$. *Note that* $\Theta_0 = \Theta_n \equiv 0$.

One of the main reasons that the case of curves in CP^n works well is that the associate (osculating) curves are holomorphic (to see this one can either follow the method of Ahlfors or Wong [**W3**]). If the metric connection of the target space is holomorphic then of course all higher derivatives of the curve are also holomorphic and á priori so are the associate curves (which are the wedge product of the derivatives). However, connections are usually not holomophic (almost never is, for details see Wong [**W3**]); for instance the connection of the Fubini-Study metric is not holomorphic. On the other hand, meromorphic connections do exist on projective varieties, hence osculating curves defined via these connections are also meromorphic. This is the main idea of Siu's SMT.

Theorem 28 *Let M be a projective surface (i.e., complex dimension 2) with a meromorphic connection D. Let t be a holomorphic section of a holomorphic line bundle $\mathfrak{F}$ over M such that $t \otimes D$ is holomorphic. Let $f : C \to M$ be a holomorphic curve which is non-degenerate in the sense that the image of f is not contained entirely in the pole set of D and that $f' \wedge Df' \not\equiv 0$. Let $\mathfrak{l}$ be a holomorphic line bundle over M with a non-trivial holomorphic section s such that $D = [s = 0]$ is non-singular. Then for any $\varepsilon > 0$ and $\lambda > 1$ there exists a set A of finite Lebesgue measure such that*

$$(1 - \varepsilon)T(f,r) + N(f,D,r) \le \lambda T(f, \mathfrak{F}^* \otimes \mathfrak{l}^*, r) + o(T(f, \mathfrak{l}, r)$$

for all $r \in [s, \infty) - A$.

Siu's Theorem provides some very interesting new examples even though this approach does not yet produce the "right" estimate in many of the important cases. The problem lies in the difficulty

of controlling the pole order of the meromorphic connection, making optimal estimate in the SMT unattainable.

Another long standing problem which is solved only in the last few years is the problem of moving target. A hyperplane in $CP^n = P(C^{n+1})$ may be identified with a point in the dual $P((C^{n+1})^*)$. But instead of considering fixed hyperplanes $a_1, \ldots, a_q \in P((C^{n+1})^*)$ one considers holomorphic curves $g_1, \ldots, g_q : C \to P((C^{n+1})^*)$. In the one dimensional case, Nevanlinna conjectured that the deficit estimate of a holomorphic curve $f : C \to CP^1$ remains valid if the growth of characteristic functions $T(g_j, r)$ of the moving hyperplanes is slower than the growth of the characteristic function $T(f, r)$. Chuang [Chu] made significant progress on this problem. The conjecture is finally solved by Steinmetz in 1986 for curves into CP^1. The case of curves in CP^n is solved by Ru-Stoll [R-S1,2] recently. Bardis [Ba] and O'Shea [OS] extended the Theory of moving targets to the case where the domain is also of higher dimension; deficit estimates are obtained under additional assumptions. We shall only state the SMT of Steinmetz and Ru-Stoll here. Given a family of holomorphic maps $\{g_1, \ldots, g_q\}$ from C into $P((C^{n+1})^*)$, *the field of meromorphic functions generated by* $\{g_1, \ldots, g_q\}$ is the smallest subfield $\mathfrak{B}$ of the field of meromorphic functions on C containing all the coordinate functions of $g_j, 1 \leq j \leq q$. A holomorphic curve $f : C \to CP^n$ is said to be linearly non-degenerate over $\mathfrak{B}$ if the coordinate functions of f does not satisfies any non-trivial linear equation with coefficients in $\mathfrak{B}$.

Theorem 29 *Let $f : C \to CP^n$ be a holomorphic curve and $g_1, \ldots, g_q : C \to P((C^{n+1})^*)$ be q holomorphic maps considered as moving hyperplanes of CP^n in general position. Assume that $T(g_j, r)/T(f, r) \to 0$ as $r \to \infty$ and that f is linearly non-degenerate over $\mathfrak{B}$. Then for any $\varepsilon > 0$ there exists a set A of finite Lebesgue measure such that the estimate*

$$\sum_{1 \leq j \leq q} m(f, g_j, r) \leq (n + 1 + \varepsilon) T(f, r)$$

holds for all $r \in [s, \infty) - A$. Consequently

$$\sum_{1 \leq j \leq q} \delta(f, g_j) \leq n + 1.$$

In fact Ru-Stoll obtained a version of the SMT for moving targets corresponding to the Cartan conjecture. For this they introduce a concept called k-flat (we refer the readers to [**R-S2**] for details.

Theorem 30 *Let $f : C \to CP^n$ be a holomorphic curve and $g_1, \ldots, g_q$: $C \to P((C^{n+1})^*)$ be q holomorphic maps considered as moving hyperplanes of CP^n in general position. Assume that $T(g_j, r)/T(f, r) \to 0$ as $r \to \infty$ and that the dimension of the map f is k-flat over $\mathfrak{B}$. Then for any $\varepsilon > 0$ there exists a set A of finite Lebesgue measure such that the estimate*

$$\sum_{1 \leq j \leq q} \omega_j m(f, g_j, r) \leq (2n - k + 1 + \varepsilon) \, T(f, r)$$

holds for all $r \in [s, \infty) - A$ and where the ω_j's are the Nochka weights associated to the g_j's. Consequently

$$\sum_{1 \leq j \leq q} \delta(f, g_j) \leq 2n - k + 1.$$

(III) Remarks

From the results listed in the two previous sections, the similarities between the two theories seem quite striking. The results in the Theory of curves are more complete due to the fact that there are more tools available. The analytic machineries are more powerful; the idea of Nevanlinna using invariants defined by integrals (e.g., characteristic functions, proximity functions) makes estimates easier to obtain (pointwise estimates are replaced by integral estimates). Furthermore, the proofs of the various Theorems in Nevanlinna Theory are quite uniform. The basic approach and the basic steps are essentially the same. The key ingredients are the Jensen formulas (corresponds to the Artin-Whaple's Product Formula), Ahlfors' Theory of associate (osculating) curves (corresponds to the successive minima in the geometry of numbers) and Nevanlinna's calculus lemma estimating the derivative of a positive convex increasing function by the function itself.

Even though Nevanlinna's lemma is elementary in nature, it has the effect of making many estimates routine. Without this technical

lemma Nevanlinna Theory would be much more complicated. Unfortunately, there is as yet no good analog of Nevanlinna's lemma in Number Theory. This perhaps is the main reason that the proofs in diophantine approximation are not as uniform; many estimates are obtained via ingenious process which are perhaps not so "natural". The search of a good analog of Nevanlinna's lemma should be one of the main technical goal in the theory of diophantine approximations.

If one compares the theory of successive minima to the theory of associate curves one notices that the later is much more well-developed. The center-piece of the theory of associate curves is the Formula of Plücker, relating the invariants of higher order osculating curves to that of the lower order osculating curves. The counterpart of Plücker's Formula in the theory of successive minima has yet to be developed. The precise relations among the successive minima seem rather complicated at this point. A better understanding of these fundamental relationships would go a long way in developing the theory of diophantine approximations.

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