NEVANLINNA THEOREMS IN PUSH-FORWARD VERSION

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I. Introduction

We consider a polynomial map f of $\mathbb{C}^n$, i.e., a holomporphic map $f : \mathbb{C}^n_z \to \mathbb{C}^n_w, z = (z_1, \ldots, z_n) \mapsto (f_1(z), \ldots, f_n(z))$, where $\mathbb{C}^n_z = \mathbb{C}^n_w = \mathbb{C}^n, z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ are the coordinate systems for $\mathbb{C}^n_z$ and $\mathbb{C}^n_w$, respectively, and $f_1, \ldots, f_n \in \mathbb{C}[z_1, \ldots, z_n]$. For any polynomial map $f : \mathbb{C}^n_z \to \mathbb{C}^n_w$ with $\det(Df) \neq 0$, it is naturally associated a dominant rational map $F : \mathbb{P}^n_z - - \to \mathbb{P}^n_w$ defined by $[z_0 : z_1 : \ldots : z_n] \mapsto [z_0^{\deg f} : F_1(z_0, z_1, \ldots, z_n) : \ldots : F_n(z_0, \ldots, z_n)]$, where F_i is the homogeneous polynomial of degree $\deg f := \max_{1 \leq t \leq n} \deg f_t$ uniquely determined by $F_i(1, z_1, \ldots, z_n) = f_i(z_1, \ldots, z_n)$ for $i = 1, 2, \ldots, n$.

There is the well-known Jacobian problem which was raised by Keller in 1939 [**K**] and is still unknown (cf.[**V**]): If $f : \mathbb{C}^n_z \to \mathbb{C}^n_w$ is a polynomial map with the Jacobian $\det(Df) = 1$, then f has an inverse of polynomial map. In [**J**, corollary 4], we have proved: Let $f : \mathbb{C}^n_z \to \mathbb{C}^n_w$ be a polynomial map with $\det(Df) = 1$. Then f has an inverse of polynomial map if and only if $\text{supp}F_*D_{z_0} = \text{supp}D_{w_0}$, where D_{z_0} is the divisor given by $z_0 = 0$ and $F_*D_{z_0}$ is the push-forward current which is indeed a divisor.

From the above result, it leads us to take attention to the push-forward divisor $F_*D_{z_0}$. In order to investigate general push-forward divisors, in this paper, we establish the Nevanlinna main theorems in push-forward version which are analogous to the ones in the value distribution theory. We shall study any polynomial map $f : \mathbb{C}^n_z \to \mathbb{C}^n_w$ with $\det Df \neq 0$, its associated a dominant rational map $F : \mathbb{P}^n_z - - \to \mathbb{P}^n_w$ and any divisor D on $\mathbb{P}^n_z$. We shall prove the first main theorem, the second main theorem, the defect relation and some other results. For proving these theorems, besides the modified traditional method in the value distribution theory, some estimate from [**J**, theorem 2] about push-forward currents will be used.

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2. Preliminaries

Meromorphic maps Let M and N be connected complex manifolds and let S be a proper analytic subset of M. Let $f : M - S \to N$ be a holomorphic map. The closed graph G of f is the closure of the graph of f over $M - S$ in $M \times N$. Let $\pi : G \to M$ and $\widehat{f} : G \to N$ be the natural projections. The map is said to be *meromorphic* on M, denoted by $f : M - - \to N$, if G is analytic in $M \times N$ and if π is proper. The *indeterminacy* $I_f = \{x \in M \mid \#\widehat{f}(\pi^{-1}(x)) > 1\}$ is analytic, where $\#B$ means the cardinacy of a set B, and is contained in S. We know $\operatorname{codim} I_f \geq 2$. We assume $S = I_f$.

If $B \subset M$ is a subset, we define the *image of* B *by* f is the set

$$f(B) = \widehat{f}(\pi^{-1}(B)) = \{y \in N \mid (x,y) \in G, \text{ for some } x \in B\}.$$

Currents on complex spaces Let X be a reduced, pure n-dimensional complex space and $\operatorname{Reg}(X)$ be the set of all regular points of X. We can define currents on X (cf.[**D**, p.14]). Since the problem is local, we assume that there is an embedding $j : X \to \Omega$, where $\Omega \subset \mathbb{C}^N$ is an open subset (i.e., X is identified with a closed analytic subset of Ω). We define

$$\mathcal{E}^{p,q}(X) = \tilde{j}^* \mathcal{E}^{p,q}(\Omega)$$

with the quotient topology, where

$$\tilde{j}^* : \mathcal{E}^{p,q}(\Omega) \to \mathcal{E}^{p,q}(\operatorname{Reg}(X))$$

is the usual pull-back map and $\mathcal{E}^{p,q}(M)$ is the space of all (p,q)-forms on a manifold M. It is known that the definition of $\mathcal{E}^{p,q}$ is indepedent of the choice of the embedding j (see [**D**, p.14]). Then we define $\mathcal{D}^{p,q}(X) = \{\sigma \in \mathcal{E}^{p,T}(X) \mid \sigma \textit{ has compact support on } X\}$ with the inductive limit topology. The dual space $\mathcal{D}'^{p,q}(X)$ is defined as the space of (p,q) bidimensional *currents* on X.

The operators $d, \partial, \bar{\partial}$ and push-forward of currents (by proper holomorphic maps) then are defined for currents on such complex spaces as defined on manifolds.

Divisors on complex spaces For any reduced, pure n-dimensional complex space $(X, \mathcal{O})$, we denote $\mathcal{M}$ the sheaf of germs of meromorphic functions on X and denote $\mathcal{M}^*$ the sheaf (of multiplication groups) of invertible elements in $\mathcal{M}$. Similary $\mathcal{O}^*$ is the sheaf of invertible elements in $\mathcal{O}$. A *divisor* D on X is a global section of the sheaf $\mathcal{M}^*/\mathcal{O}^*$. A divisor D on X also can be described by giving an open cover U_i of X and for each i an element $f_i \in \Gamma(U_i, \mathcal{M}^*)$ such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$ for any i and j.

If $F : \mathbb{P}^n_z \to \mathbb{P}^n_w$ is a dominant rational (also meromorphic) map. Let G be the closed graph and $\pi, \widehat{F}$ are the natural projections. For any divisor D on $\mathbb{P}^n_z$, we can pull back it on G as a divisor π^*D in an obvious way.

The Carlson-Griffiths singular form Let $D_1, \ldots, D_q$ be divisors on $\mathbb{P}^n_z$ such that $\mathrm{supp}D_1, \ldots, \mathrm{supp}D_q$ are manifolds located in normal crossings and each $D_j = D_{g_j}$, where $g_j \in \mathbb{C}[z_0, \ldots, z_n]$ is a homogeneous polynomial of degree p_j. Denote $D = \sum_{j=1}^q D_j$. Each D_j is also given by the system $\{U_i, g_j/z_i^{p_j}\}_{0\le i\le n}$, where $U_i = \{[z_0 : \ldots : z_n] \in \mathbb{P}^n \mid z_i \neq 0\}$. The associated holomorphic line bundle L_{D_j} of D_j has the Hermitian metric $h_j = \{U_i, h_{ji}\}_{0\le i\le n}$, where

$$h_{ji} = \frac{|z_i|^{2p_j}}{(|z_0|^2 + \ldots + |z_n|^2)^{p_j}}.$$

Let $L = L_{D_1} \otimes \ldots \otimes L_{D_q}$. Then the Hermitian metric h of L is $h = \{U_i, h_{1i} \cdot \ldots \cdot h_{qi}\}_{0\le i\le n}$. For each D_j, as the section $\{g_j/z_i^{p_j}\}_{0\le i\le n}$ of L_{D_j}, a globally defined function $\|D_j\|^2$ on $\mathbb{P}^n_z$ is defined by

$$\|D_j\|^2 \mid U_i = \frac{h_{ji}|g_j|^2}{eC_{g_j}|z_i|^{2p_j}} = \frac{|g_j|^2}{eC_{g_j}(|z_0|^2 + \ldots + |z_n|^2)^{p_j}},$$

where $C_{g_j} > 0$ is a constant such that

$$|g_j|^2 \le C_{g_j}(|z_0|^2 + \ldots + |z_n|^2)^{p_j}$$

for all $z_0, \ldots, z_n \in \mathbb{C}$. We know the Chern form $c(K_{\mathbb{P}^n_z})$ of the canonical bundle $K_{\mathbb{P}^n_z}$

$$c(K_{\mathbb{P}^n_z}) = -(n+1)\Omega_{\mathbb{P}^n_z}, \tag{2.1}$$

which is also defined as the Ricci form of the volume form $\Omega_{\mathbb{P}^n_z}$ on $\mathbb{P}^n_z$. Let's recall the notions of Ricci form and volume form. Let M be any complex manifold. The canonical bundle K_M of M is the holomorphic line bundle whose transition functions are the Jacobian of the coordinate change mappings in the intersection of domains in a covering of M, i.e., let $\{U_\alpha, W^\alpha\}_\alpha$ be a coordinate syatem covering of M, then on $U_\alpha \cap U_\beta$, the transition functions $g_{\alpha\beta} = \det(\partial w^\alpha_j / \partial w^\beta_k)$. If $\Phi_\alpha = \prod_{v=1}^n \frac{\sqrt{-1}}{2\pi} dw^\alpha_v \wedge d\overline{w}^\alpha_v$ on U_α is the local Euclidean volume form, then a positive (n,n)-form Ω which is defined locally on U_α as $\lambda_\alpha \Phi_\alpha$ is a global form on M if and only if $\lambda_\beta = |g_{\alpha\beta}|^2 \lambda_\alpha$ in $U_\alpha \cap U_\beta$. Such (n,n)-form Ω is called a *volume form*. The *Ricci form* of Ω, denoted by $\operatorname{Ric}\Omega$, is defined by $\operatorname{Ric}\Omega \mid U_\alpha = dd^c \log \lambda_\alpha$.

The *Carlson-Griffiths Singular volume form* Ψ on $\mathbb{P}^n_z - \operatorname{supp}D$ is defined by (cf. [**SHA**, p.79])

$$\Psi = \frac{C\Omega_{\mathbb{P}^n_z}}{\prod_{j=1}^q (\log \|D_j\|^2)^2 \|D_j\|^2},$$

where the constant $C > 0$ is determined by the following properties

(2.2) $\operatorname{Ric}\Psi > 0$;

(2.3) $(\operatorname{Ric}\Psi)^n > \Psi$;

(2.4) $\int_{\mathbb{P}^n_z - \operatorname{supp}D} (\operatorname{Ric}\Psi)^n < +\infty$;

(2.5) $\operatorname{Ric}\Psi \mid (\mathbb{P}^n_z - \operatorname{supp}D) = c(L_D) + c(K_{\mathbb{P}^n_z}) - \sum_{j=1}^q dd^c \log(\|D_j\|^2)^2.$

3. Push-forward of currents by F

Let $f : \mathbb{C}^n_z \to \mathbb{C}^n_w$ be a polynomial map with the Jacobian $\det(Df) \neq 0$. Let $F : \mathbb{P}^n_z - - \to \mathbb{P}^n_w$ be its associated dominant rational map. Let G be the closed graph of F, and π and $\widehat{F}$ be the projections. G is an irreducible, pure complex n-dimensional analytic subset in $\mathbb{P}^n_z \times \mathbb{P}^n_w$, so G is regarded as an irreducible reduced complex space. Therefore π and $\widehat{F}$ are proper holomorphic maps from complex space onto complex manifolds.

For any divisor D on $\mathbb{P}^n_z$, we pull back it on G as a divisor π^*D. Then we obtain a pushforward current $\widehat{F}_*(\pi^*D)$ on $\mathbb{P}^n_w$. We want to show that this push-forward current is indeed a divisor. Before doing that, we need the following lemma. The proof below is due to Shiffman.

Lemma 3.1 *Let M and N be n-dimensional complex manifolds and let $f : M \to N$ be a surjective proper holomorphic map. If D is a divisor on M, then the current f_*D is a divisor on N.*

Proof Let $A = \mathrm{supp} D$, and $\bar{f} = f \mid A : A \to N$. We assume that $\operatorname{codim} \bar{f}(A) = 1$. Let $S = \{x \in A \mid \dim \bar{f}^{-1}(\bar{f}(x)) \geq 1\}$. Then because of $\operatorname{codim} \bar{f}(A) = 1$,

$$\operatorname{codim} \bar{f}(S) \geq 2.$$

We first show that $f_*D \mid N - \bar{f}(S) \in \mathcal{D}'^{1,1}(n - \bar{f}(S))$ is a divisor. In fact, for any point $w \in N - \bar{f}(S)$, $\bar{f}^{-1}(w)$ is a finite set. Then there is an open neighborhood $W(w)$ of w in $N - \bar{f}(S)$, and finite disjoint open subsets $U_1(w), \ldots, U_r(w)$ in M such that for each $U_i(w)$, there is a holomorphic function $g_i \in \mathcal{O}(U_i(w))$,

$$f^{-1}(W(w)) \cap A = \cup_{i=1}^r U_i(w) \cap A, \text{ and}$$
$$D \mid U_i(w) = dd^c \log |g_i|^2, \text{ for } i = 1, 2, \ldots, r,$$

where the Poincaré-Lelong formula is used. Let $J_f = \{z \in M \mid z \textit{ is a critical point of } f\}$. Then $f(J_f) \subset N$ is an analytic subset. Since $W(w) - f(J_f)$ is connected, there are integers $\lambda_1, \ldots, \lambda_r$ such that for any $u \in W(w) - f(J_f)$, there is an open neighborhood $W(u)$ of u in $W(w) - f(J_f)$, and disjoint open subsets $U_{1,1}(u), \ldots, U_{1,\lambda_1}(u) \subset U_1(w); \ldots; U_{r,1}(u), \ldots, U_{r,\lambda_r}(u) \subset U_r(w)$, so that $f \mid U_{i,j}(u) : U_{i,j}(u) \to W(u)$ is biholomorphic for all $i = 1, 2, \ldots;\ 1 \leq j \leq \lambda_i$. Then

$$
\begin{aligned}
f_*D \mid W(u) &= \sum_{i=1}^{r}\sum^{\lambda_i} dd^c \log \left| g_i \circ (f \mid U_{i,j}(u))^{-1} \right|^2 \\
&= dd^c \log \left| \prod_{i=1}^{r}\prod_{j=1}^{\lambda_i} g_i \circ (f \mid U_{i,j}(u))^{-1} \right|^2 \\
&= dd^c \log |g|^2 ,
\end{aligned}
$$

where $g = \prod_{i=j}^{r}\prod_{j=1}^{\lambda_i} g_i \circ (f|U_{i,j}(u))^{-1}$ on $W(u)$. g is a well-defined holomorphic function on $W(w) - f(J_f)$, which can be extended on $W(w)$ holomorphically. Therefore we have proved that f_*D is a divisor on $N - \bar{f}(S)$.

Let $V = \mathrm{supp} f_*D \mid N - \bar{f}(S)$. V has a decomposition $V = \cup_j V_j$, where V_j are irreducible hypersurfaces on $N - f(S)$. Since we have proved $f_*D \mid N - f(S) = \sum_j n_j V_j$, V_j has an extension $\widetilde{V_j}$ in N for all j.

It suffices to show the current $T = \sum_j n_j \widetilde{V_j} - f_*D \in \mathcal{D}'^{1,1}(N)$ must be zero. Since $T \mid N - \bar{f}(S) = 0$, and $\dim_{\mathbb{R}} \bar{f}(S) \leq 2n - 4$. Then the current $T = 0$ follows from the following lemma. **QED**

Lemma 3.2 *Let $0 < p < 2n$, and $\Omega \subset \mathbb{C}^n$ be and open subset and $E \subset \Omega$ be a closed subset with $h^p(E) = 0$, where h^p is the Hausdorff measure of order p. If $\sigma \in \mathcal{D}'^p(\Omega)$ is d-closed and of order 0, then $\|\sigma\|(E) = 0$.*

Proof This is a special case of Federer [**F**, 4.1.20], or cf. [**Sh**, lemma A.2]. **QED**

Now we can prove that the current $\widehat{F}_*(\pi^*D)$ is a divisor. In fact, by Hironaka's theorem of resolution of singularities [**H**], there is a modification $\sigma : G' \to G$, where G' is a compact complex manifold. It follows that $\widehat{F}_*D = (\widehat{F} \circ \sigma)_*(\sigma^*D)$. Thus $\widehat{F}_*D$ is a divisor on $\mathbb{P}^n_w$ by the lemma above.

In [**J**], we proved that let $f : M - - \to N$ be a surjective meromorphic map, where M and N are compact connected complex manifolds of complex n-dimension. Let $\mathcal{L}$ be a semi-positive holomorphic line bundle over M with a nozero holomorphic section s. The locus of s on M is denoted by V as an analytic hypersurface. Then the image $f(V)$ is also an analytic hypersurface on N.

We take an open covering $\{U_\alpha\}$ of M and a Hermitian metric $h = \{h_\alpha\}$ of $\mathcal{L}$ such that the curvature of $(\mathcal{L}, h)$ is semi-positive. Let the given holomorphic section $s = \{s_\alpha\}$. Then we have a globally defined function on M : $\|s\|^2 = h_\alpha |s_\alpha|^2$ on U_α. Put $\varphi = -\log \|s\|^2$. By [J], $f_*\varphi$ is the plurisubharmonic exhaustion function of $N - f(V)$. By the lemma 3.1, if we denote D_s to be the divisor determined by s, $f_*D_s = \tilde{f}_*(\pi^* D_s)$ is also a divisor. Then for any point $w \in N \cap F(V)$, there exists an open neighborhood U_1 of w in N and a holomorphic function $g \in \mathcal{O}(U_1)$ such that $f_*D_s = dd^c \log |g|^2$. We notice that $f_*\varphi \in C^\infty(N - f(V \cup J_f \cup I_f))$. Then we can present

lemma 3.3 (See [J, theorem 2]) *Let $f, M, N, \mathcal{L}, s$ and φ be as above. Let w be any given point in $N \cap f(V)$ with a neighborhood U_1 as above. Then there exists an open neighborhood U of w with $U \subset U_1$ and a positive constant number $C = C(w, f, g)$ such that*

$$0 \leq f_*\varphi(u) \leq -\log |g(u)|^2 + C$$

for all $u \in U - f(V \cup J_f \cap I_f)$.

We can apply this theorem to any dominant rational map F : $\mathbb{P}^n_z - - \to \mathbb{P}^n_w$ and any holomorphic section s because of the fact that any hypersurface V on $\mathbb{P}^n_z$ should be a locus of some holomorphic section of some positive holomorphic line bundle over $\mathbb{P}^n_z$. For the section s, it is associated a globally defined function $\varphi = \|s\|$ on $\mathbb{P}^n_z$. It was proved that $F_*\varphi$ is an exhaustion plurisubharmonic function for $\mathbb{P}^n_w - F(V)$. Furthermore, the lemma 3.3 said that

$$F_*\varphi \in \mathcal{L}^1_{loc}(\mathbb{P}^n_w) \quad . \tag{3.4}$$

4. Notations in the value distribution theory

Let f and F be as before. Assume $\deg f > 1$. Consider the inclusion map $i : \mathbb{C}^n_w \hookrightarrow \mathbb{P}^n_w$, $(w_1, \dots, w_n) \mapsto [1 : w_1 : \dots : w_n]$, which identifies $i(\mathbb{C}^n_w) \cong \mathbb{C}^n_w$. We use $(w_1, \dots, w_n)$ as coordinates system on $i(\mathbb{C}^n_w)$.

On $i(\mathbb{C}^n_w)$, we let

$$\varphi = dd^c(|w_1|^2 + \ldots + |w_n|^2), \quad \omega = dd^c \log(|w_1|^2 + \ldots + |w_n|^2),$$
$$\sigma_P = d^c \log(|w_1|^2 + \ldots + |w_n|^2) \wedge \omega^P; \quad \omega = \omega_{n-1},$$
$$B(r) = \left\{[1 : w_1 : \ldots : w_n] \in i(\mathbb{C}^n_w) \mid |w_1|^2 + \ldots + |w_n|^2 < r^2\right\},$$
$$S(r) = \left\{[1 : w_1 : \ldots : w_n] \in i(\mathbb{C}^n_w) \mid |w_1|^2 + \ldots + |w_n|^2 = r^2\right\}.$$

Lemma 4.1 (Jenson-Lelong formula) *Let T be a real valued function and $T \in \mathcal{L}^1_{loc}(\mathbb{C}^n)$ such that dd^cT is of order 0. Then for $0 < r_0 < r$, one has*

$$\int_{R_0}^{r} \frac{dt}{t} \int_{B(t)} dd_cT \wedge \Omega^{n-1} = \frac{1}{2} \int_{S(r)} T \wedge \sigma_{n-1} - \frac{1}{2} \int_{S(r_0)} T \wedge \sigma_{n-1} + C,$$

where the constant C is independent of r.

Proof See [**Sh**, lemma 2.3]. **QED**

We define the *characteristic function of F* by

$$T_{*F}(r, r_0) = \int_{r_0}^{r} \frac{dt}{t} \int_{B(t)} F_* \Omega_{\mathbb{P}^n_z} \wedge \Omega^{n-1},$$

where $\Omega_{\mathbb{P}^n_z}$ is the Fubini-Study metric form on $\mathbb{P}^n_z$, and $F_* = \widehat{F}_* \pi^*$.

For any positive current χ on $i(\mathbb{C}^n_w)$ of $(n-1, n-1)$ bidimension, we define the *counting function* of χ by

$$N(\chi; r, r_0) = \int_{r_0}^{r} \frac{dt}{t^{2n-1}} \int_{B(t)} \chi \wedge \Omega^{n-1}.$$

Note that if χ is d-closed, by Stokes' theorem,

$$N(\chi; r, r_0) = \int_{r_0}^{r} \frac{dt}{t} \int_{B(t)} \chi \wedge \Omega^{n-1}.$$

Abbreviately, we denote

$$N_{*F}(D_g; r, r_0) = N(F_*D_g, r, r_0),$$

where D_g is a divisor on $\mathbb{P}_z^n$ given by a homogeneous polynomial g.

5. The first main theorem

Let $0 \neq g \in \mathbb{C}[z_0, z_1, \ldots, z_n]$ be any homogeneous polynomial. Denote D_g be its associated divisor on $\mathbb{P}_z^n$. Put

$$\varphi_g = \log \frac{eC_g(|z_0|^2 + \ldots + |z_n|^2)^{\deg g}}{|g(z_0, \ldots, z_n)|^2},$$

where C_g is a positive constant satisfying

$$|g(z_0, \ldots, z_n)|^2 \leq C_g(|z_0|^2 + \ldots + |z_n|^2)^{\deg g}$$

for all $z_0, \ldots, z_n \in \mathbb{C}$. Thus $\varphi_g \geq 0$ and $\varphi_g \in C^\infty(\mathbb{P}_z^n - \mathrm{supp} D_g) \cap \mathcal{L}_{loc}^1(\mathbb{P}_z^n)$.

Apply the Poincaré-Lelong formula, we see

$$dd^c \varphi_g = \deg g \cdot \Omega_{\mathbb{P}_z^n} - D_g, \qquad \text{on } \mathbb{P}_z^n.$$

Then

$$F_* dd^c \varphi_g = \deg g \cdot F_* \Omega_{\mathbb{P}_z^n} - F_* D_g, \qquad \text{on } \mathbb{P}_w^n.$$

Since $\widehat{F}_*, \pi^*$ commute with d, d^c, we see F_* commutes with dd^c, then we restrict the above relation $i(\mathbb{C}_w^n)$ to obtain

$$dd^c F_* \varphi_g = \deg g \cdot F_* \Omega_{\mathbb{P}_z^n} - F_* D_g, \qquad \text{on } i(\mathbb{P}_w^n).$$

By (3.4), it follows that

$$F_* \varphi_g \in \mathcal{L}_{loc}^1(\mathbb{C}_w^n).$$

Then by applying the lemma 4.1, we have proved

Theorem 5.1 (First main theorem) *Let f, F be as before. Then*

$$\deg g \cdot T_{*F}(r, r_0) = N_{*F}(D_g; r, r_0) + \frac{1}{2} \int\limits_{S(r)} F_* \varphi_g \sigma + O(1)$$

for $r \gg r_0$.

Corollary 5.2 (Nevanlinna inequality)

$$N_{*F}(D_g; r, r_0) \leq \deg g T_{*F}(r, r_0) + O(1).$$

Proof Note $\varphi_g \geq 0$, then $\int_{S(r)} F_* \varphi_g \sigma \leq 0$. **QED**

We would like to give the following proposition to close this section. The inequality here is conjectured to be equality which remains a problem.

Proposition 5.3 $\overline{\lim}_{r\to\infty} \frac{T_{*F}(r,r_0)}{\log r} \leq (\deg f)^{n-1}$.

Proof

$$\overline{\lim}_{r\to\infty} \frac{T_{*F}(r, r_0)}{\log r} \leq \overline{\lim}_{r\to\infty} \int\limits_{B(r)} F_* \omega_{\mathbb{P}^n_z} \wedge \omega^{n-1}$$

$$= \int\limits_{\mathbb{P}^n_z} F_* \omega_{\mathbb{P}^n_z} \wedge \omega^{n-1}$$

$$= \int\limits_{G} \pi^* \omega_{\mathbb{P}^n_z} \wedge \widehat{F}^* \omega^{n-1}$$

$$= \int\limits_{\mathbb{C}^n_z} dd^c \log(1 + |z_1|^2 + \ldots + |z_n|^2) \wedge \\ \wedge f^*(dd^c \log(|w_1|^2 + \ldots + |w_n|^2))^{n-1}$$

$$= \lim{}_{r\to\infty} \left[\int_{r_0}^{r} \frac{dt}{t} \int\limits_{B(t,\mathbb{C}^n_z)} dd^c \log(1 + |z_1|^2 + \ldots + |z_n|^2) \wedge \\ \wedge (dd^c \log(|f_1|^2 + \ldots + |f_n|^2))^{n-1} \right] / \log r,$$

where G is the closed graph of F, and $B(r,\mathbb{C}_z^n) = \{(z_1,\ldots,z_n) \in \mathbb{C}_z^n \mid |z_1|^2+\ldots+|z_n|^2 < r^2\}$.

Since

$$
\begin{aligned}
&\int_{r_0}^{r} \frac{dt}{t} \int_{B(t,\mathbb{C}_z^n)} dd^c \log(1+|z_1|^2+\ldots+|z_n|^2))^{n-1} \wedge \\
&\qquad\qquad (dd^c \log(|f_1|^2+\ldots+|f_n|^2))^{n-1} \\
&\leq \frac{1}{2} \int_{\partial B(r,\mathbb{C}_z^n)} \log|f|^2 (dd^c \log|f|^2)^{n-2} \wedge dd^c \log(1+|z|^2) \wedge \\
&\qquad\qquad \wedge d^c \log(1+|z|^2) \\
&\quad -\frac{1}{2} \int_{\partial B(r_0,\mathbb{C}_z^n)} \log|f|^2 (dd^c \log|f|^2)^{n-2} \wedge dd^c \log(1+|z|^2) \wedge \\
&\qquad\qquad \wedge d^c \log(1+|z|^2) + O(1) \\
&\leq (\deg f \cdot \log r + \frac{A}{2}) \int_{\partial B(r,\mathbb{C}_z^n)} (dd^c \log|f|^2)^{n-2} \wedge dd^c \log(1+|z|^2) \wedge \\
&\qquad\qquad \wedge d^c \log(1+|z|^2) + O(1) \\
&\leq (\deg f \cdot \log r + A) \int_{B(r,\mathbb{C}_z^n)} (dd^c \log|f|^2)^{n-2} \wedge (dd^c \log(1+|z|^2))^2 \\
&\qquad\qquad + O(1),
\end{aligned}
$$

where $|f|^2 = |f_1|^2+\ldots+|f_n|^2$ and the positive constant A is independent of r, we then have

$$
\begin{aligned}
&\overline{\lim}_{r\to\infty} \frac{T_{*F}(r,r_0)}{\log r} \\
&\leq \lim_{r\to\infty} \frac{(\deg f \cdot \log r + A)}{\log r} \int_{B(r,\mathbb{C}_z^n)} (dd^c \log|f|^2)^{n-2} \wedge \\
&\qquad\qquad \wedge (dd^c \log(1+|z|^2))^2 \\
&\leq \deg f \int_{\mathbb{C}_z^n} (dd^c \log|f|^2)^{n-2} \wedge (dd^c \log(1+|z|^2))^2
\end{aligned}
$$

$$\begin{aligned}
&\leq \ldots\ldots \\
&\leq (\deg f)^{n-1} \int_{\mathbb{C}^n_z} (dd^c \log(1+|z|^2))^n \\
&= (\deg f)^{n-1}.
\end{aligned}$$

QED

6. The second main theorem

In this section, we shall prove the second main theorem. Let f, F be as before. Let D_{J_F} be the *ramification divisor* of the meromorphic map F on $\mathbb{P}^n_z$, i.e., locally on $\mathbb{P}^n_z - I_F$, it is given by the Jacobian determinant of F, and then it is extended on $\mathbb{P}^n_z$ (cf.[**SH**, p.73]). Let D_{J_F} be determined by a unique (up to a constant factor) homogeneous polynomial $J_F \in \mathbb{C}[z_0, \ldots, z_n]$. We use λ_F to denote the sheets number of F.

Theorem 6.1 (Second main theorem) *Let f, F be as before. Let $D_1, \ldots, D_q$ be divisors on $\mathbb{P}^n_z$ so that* $\mathrm{supp}D_1, \ldots, \mathrm{supp}D_q$ *are manifolds located in normal crossings. Suppose each $D_j = D_{g_j}$, where $g_j \in \mathbb{C}[z_0, \ldots, z_n]$ is homogeneous polynomial of degree p_j, for $j = 1, 2, \ldots, q$. Denote $D = \sum_{j=1}^{q} D_j$. Then*

$$\begin{aligned}
&\left(\sum_{j=1}^{q} p_j - (n+1)\right) T_{*F}(r, r_0) \\
&\qquad\qquad \leq N_{*F}(D; r, r_0) + N_{*F}(D_{J_F}; r, r_0) + \log r + O(1)
\end{aligned}$$

for $r \gg r_0$.

Proof For any $w \in i(\mathbb{C}^n_w) - F(\mathrm{supp}D_{J_F})$, there is an open neighborhood $W(w)$ of w in $i(\mathbb{C}^n_w) - F(\mathrm{supp}D_{J_F})$ and λ_F disjoint open subsets $U_1(w), \ldots, U_{\lambda_F}(w)$ in $\mathbb{C}^n_z$ such that

$$F^{-1}(W(w)) = \bigcup_{j=1}^{\lambda_f} U_j(w) \text{ and}$$
the restriction $F \mid U_i(w)$ is biholomorphic.

Consider the Carlson-Griffiths singular volume form Ψ,

$$\begin{aligned}
&F_*\Psi \mid W(w) \\
&= \sum_{i=1}^{\lambda_F} (F \mid U_i(w))_*(\Psi \mid U_i(w)) \\
&= \sum_{i=1}^{\lambda_F} (F \mid U_i(w))_* \frac{dd^c \log(1+|z_1|^2+\ldots+|z_n|^2))^n}{\prod_{j=1}^q (\log\|D_j\|^2)^2\|D_j\|^2} \\
&= \sum_{i=1}^{\lambda_F} \left((F \mid U_i(w))^{-1}\right)^* \frac{n!(dd^c(|z_1|^2+\ldots+|z_n|^2))^n}{(1+|z_1|^2+\ldots+|z_n|^2)^{n+1}} \times \\
&\qquad \times \frac{1}{\prod_{j=1}^q (\log\|D_j\|^2)^2\|D_j\|^2} \\
&= \tilde{\xi}(dd^c(|w_1|^2+\ldots+|w_n|^2))^n,
\end{aligned}$$

where

$$\tilde{\xi} = \sum_{i=1}^{\lambda_F} \frac{n!}{((F \mid U_i(w))^{-1})^* X(w)},$$

and

$$X(w) = \left[|\det D(F \mid U_i(w)) \mid (1+\sum_{t=1}^n |z_t|^2)^{n+1} \times \right.$$
$$\left. \times \prod_{j=1}^q (\log\|D_j\|^2)^2\|D_j\|^2 \right].$$

Put

$$\xi = \prod_{i=1}^{\lambda_F} \frac{n!}{((F \mid U_i(w))^{-1})^* X(w)}$$

Apply the Poincaré-Lelong formula, $dd^c \log\|D_j\|^2 = -c(L_{D_j}) + D_j$, so

$$\begin{aligned}
dd^c \log \xi &= F_*c(L_{D_j}) + F_*c(K_{\mathbb{P}^n_z}) \\
&\qquad - \sum_{j=1}^q F_* dd^c \log(\log\|D_j\|^2)^2 - F_*D - F_*D_{J_F} \\
&= F_* \operatorname{Ric}\Psi - F_*D - F_*D_{J_F},
\end{aligned}$$

where the formula (2.5) was used.

By (2.1) we obtain

$$
\begin{aligned}
&N(dd^c \log \xi; r, r_0) + \sum_{j=1}^{q} N(F_* dd^c \log(\log \|D_j\|^2)^2; r, r_0) \\
&= \left(\sum_{j=1}^{q} p_j - (n+1)\right) T_{*F}(r, r_0) \\
&\qquad - N_{*F}(D; r, r_0) - N_{*F}(D_{J_F}; r, r_0) \qquad (6.2)
\end{aligned}
$$

To prove the theorem, it suffices to estimate the left hand side of the above identity.

Let's estimate $N(F_* dd^c \log(\log \|D_j\|^2)^2; r, r_0)$ first. Since

$$\log(\log \|D_j\|^2)^2 = \log\left(\log \frac{|g_j(z_0, \ldots, z_n)|^2}{eC_{g_j}(|z_0|^2 + \ldots + |z_n|^2)^{\deg g_j})^2}\right)^2,$$

then

$$\varphi_{g_j} = \log \frac{eC_{g_j}(|z_0|^2 + \ldots + |z_n|^2)^{\deg g_j}}{|g_j(z_0, \ldots, z_n)|^2} \geq 1.$$

Thus

$$0 \leq F_* \log(\log \|D_j\|^2)^2 \leq 2F^* \log \varphi_{g_j} + 2 \log 2. \qquad (6.3)$$

Take any $w \in i(\mathbb{C}_w^n)$, take $W(w)$ and $U_1(w), \ldots, U_{\lambda_F}(w)$ as before, we then have

$$
\begin{aligned}
F^* \log \varphi_{g_j} \mid W(w) &= \sum_{v=1}^{\lambda_F} (F \mid U_v(w))_* \log \varphi_{g_j} \\
&= \log \prod_{v=1}^{\lambda_F} \varphi_{g_j} \circ (F \mid U_v(w))^{-1} \qquad (6.4) \\
&= \lambda_F \log \prod_{v=1}^{\lambda_F} \left(\varphi_{g_j} \circ (F \mid U_v(w))^{-1}\right)^{1/\lambda_F}
\end{aligned}
$$

$$\begin{aligned}&\le \lambda_F \log \sum_{v=1}^{\lambda_F} \varphi_{g_j} \circ (F \mid U_v(w))^{-1} - \lambda_F \log \lambda_F. \\ &= \lambda_F \log F_* \varphi_{g_j} - \lambda_F \log \lambda_F.\end{aligned} \tag{6.4}$$

By the proof of the lemma 3.3, we also know

$$F_* \log(\log \|D_j\|^2)^2 \in \mathcal{L}^1_{loc}(i(\mathbb{C}^n)).$$

Thus apply the lemma 4.1 and by (6.3), (6.4), we obtain

$$\begin{aligned}&\sum_{j=1}^{q} N(F_* dd^c \log(\log \|D_j\|^2)^2; r, r_0) \\ &\quad \le -\frac{1}{2} \sum_{j=1}^{q} \int_{S(r)} F_* \log(\log \|D_j\|^2)^2 \sigma + O(1) \\ &\quad \le \sum_{j=1}^{q} \int_{S(r)} (F_* \log \varphi_{g_j}) \sigma + O(1) \\ &\quad \le \lambda_F \sum_{j=1}^{q} \int_{S(r)} (\log F_* \varphi_{g_j}) \sigma + O(1) \\ &\quad \le \lambda_F \sum_{j=1}^{q} \log \int_{S(r)} (F_* \varphi_{g_j}) \sigma + O(1) \\ &\quad \le \lambda_F \sum_{j=1}^{q} \log(2 \deg g_j \, T_{*F}(r, r_0)) + O(1) \\ &\quad \le O(\log T_{*F}(r, r_0))\end{aligned} \tag{6.5}$$

for $r \gg r_0$. Here the last second inequality is due to the first main theorem 5.1.

Next we estimate the term $N(dd^c \log \xi; r, r_0)$ in (6.2). By the previous argument, we see $\log \xi \in \mathcal{L}^1_{loc}(i(\mathbb{C}^n_w))$. Then

$$
\begin{aligned}
N(dd^c \log \xi; r, r_0) &= \frac{1}{2} \int\limits_{S(r)} \log \xi^{1/\lambda_F} \sigma + O(1) \\
&= \frac{\lambda_F}{2} \int\limits_{S(r)} \log \xi^{1/\lambda_F} \sigma + O(1) \\
&\leq \frac{\lambda_F}{2} \int\limits_{S(r)} \log \tilde{\xi} \sigma + O(1) \\
&\quad \textit{(By the definitions of } \xi \textit{ and } \tilde{\xi}, \\
&\quad \textit{and by } \xi^{1/\lambda_F} \leq \tilde{\xi}/\lambda_F) \\
&= \frac{\lambda_F}{2} \int\limits_{S(r)} \log c \tilde{\xi}^{\frac{1}{n}} \sigma + O(1) \\
&\leq \frac{\lambda_F}{2} \log \int\limits_{S(r)} c \tilde{\xi}^{\frac{1}{n}} \sigma + O(1),
\end{aligned}
\tag{6.6}
$$

where $c = \frac{1}{(n!)^{1/n}}$. Since

$$
\begin{gathered}
\int\limits_{B(t)} (c\tilde{\xi}^{\frac{1}{n}}) \varphi^n = 2 \int\limits_0^r \left(\int\limits_{S(t)} (c\tilde{\xi}^{\frac{1}{n}} \sigma) \right) t^{2n-1} dt, \\
\int\limits_{S(r)} (c\tilde{\xi}^{\frac{1}{n}}) \sigma = \frac{1}{2r^{2n-1}} \frac{d}{dr} \int\limits_{B(r)} (c\tilde{\xi}^{\frac{1}{n}}) \varphi^n.
\end{gathered}
\tag{6.7}
$$

Put

$$
\widehat{T}(r, r_0) = \int\limits_{r_0}^r \frac{dt}{t^{2n-1}} \int\limits_{B(t)} (c\tilde{\xi}^{\frac{1}{n}}) \varphi^n. \tag{6.8}
$$

From (6.6), (6.7) and (6.8), it follows

$$N(dd^c \log \xi; r, r_0) \\ \leq \frac{\lambda_F n}{2} \log \left(\frac{1}{2nr^{2n-1}} \frac{d}{dr} \left(r^{2n-1} \frac{d\widehat{T}}{dr} \right) \right) + O(1). \tag{6.9}$$

By the classical result in the value distribution theory (cf.[**Sha**, p.84]), (6.9) implies that for any $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, so that $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ and there is a subset $E = E(\epsilon) \subset \mathbb{R}_+$ with finite δ-measure, such that

$$N(dd^c \log \xi; r, r_0) \\ \leq \epsilon \log r + O\left(\log \widehat{T}(r, r_0)\right) + O(1), \tag{6.10}$$

for all $r \in \mathbb{R}_+ - E$.

To complete estimating (6.10), we estimate the term $\widehat{T}(r, r_0)$. Let $F_*\Psi \mid (i(\mathbb{C}^n_w) - F(\mathrm{supp}D \cup \mathrm{supp}D_{J_F})) = \sum_{j,k}^n R_{jk} dw_j \wedge d\overline{w_k}$, where the matrix $R = (R_{jk})$ is positive definite. Recall the definition of $\tilde{\xi}$ and (2.3),

$$\begin{aligned} &\tilde{\xi}(dd^c(|w_1|^2 + \ldots + |w_n|^2))^n \\ &= F_*\Psi \\ &\leq F_*(\mathrm{Ric}\,\Psi)^n \\ &= n! \det R (dd^c(|w_1|^2 + \ldots + |w_n|^2))^n. \end{aligned}$$

Thus $\tilde{\xi} \leq n! \det R$ holds on $i(\mathbb{C}^n_w) - F(\mathrm{supp}D \cup \mathrm{supp}D_{J_F})$. By Hardamard inequality: for any positive definity matrix R, $(\det R)^{1/n} \leq \frac{1}{n} \mathrm{tr}\, R$. Then we have

$$\begin{aligned} &\left(\frac{\tilde{\xi}}{n!}\right)^{1/n} \left(dd^c(|w_1|^2 + \ldots + |w_n|^2)\right)^n \\ &\leq \frac{1}{n} \mathrm{tr}\, R \left(dd^c(|w_1|^2 + \ldots + |w_n|^2)\right)^n \\ &\leq \frac{1}{n} \sum_{j=1}^n R_{jj} \left(dd^c(|w_1|^2 + \ldots + |w_n|^2)\right)^n \\ &= F_* \mathrm{Ric}\,\Psi \wedge \left(dd^c(|w_1|^2 + \ldots + |w_n|^2)\right)^{n-1}, \end{aligned} \tag{6.11}$$

where since $\left(dd^c(|w_1|^2+\ldots+|w_n|^2)\right)^{n-1}=(n-1)!\sum_{j=1}^{n}\left(\frac{\sqrt{-1}}{2\pi}\right)^{n-1}$ $dw_1\wedge d\overline{w_1}\wedge\ldots(\wedge dw_j)^{\text{omit}}\wedge(d\overline{w_j})^{\text{omit}}\wedge\ldots\wedge dw_n\wedge d\overline{w_n}$, $F_*\operatorname{Ric}\Psi$ $\wedge\ \left(dd^c(|w_1|^2+\ldots+|w_n|^2)\right)^{n-1}\ =\ \frac{1}{n}\sum_{j=1}^{n}R_{jj}\left(dd^c(|w_1|^2+\ldots+|w_n|^2)\right)^n$ by direct computation.

Recall the definition of $\widehat{T}(r,r_0)$ and (6.4), we see

$$\begin{aligned}
\widehat{T}(r,r_0)&=\int_{r_0}^{r}\frac{dt}{t^{2n-1}}\int_{B(t)}(c\tilde{\xi})^{1/n}\varphi^n\\
&\le\int_{r_0}^{r}\frac{dt}{t^{2n-1}}\int_{B(t)}F_*\operatorname{Ric}\Psi\wedge\left(dd^c(|w_1|^2+\ldots+|w_n|^2)\right)^{n-1}\\
&=N(F_*\operatorname{Ric}\Psi;r,r_0)\\
&=\Big(\sum_{j=1}^{q}p_j-(n+1)\Big)T_{*F}(r,r_0)\\
&\qquad\qquad+N\left(F_*dd^c\log(\log\|D_j\|^2)^2;r,r_0\right)\\
&\le O(T_{*F}(r,r_0)).
\end{aligned}$$

Together with (6.10), for any $\epsilon>0$, and $r\in\mathbb{R}_+-E$,

$$\begin{aligned}
N(dd^c\log\xi;r,r_0)&\le\epsilon\,\log r+O(\log T_{*F}(r,r_0))+O(1)\\
&\le\epsilon\,\log r+O(\log^+\log r)+O(1).
\end{aligned}\tag{6.13}$$

Here the proposition (5.3) is used. Also by a classical result (cf. **[Sha,** remark 1, p.88]), we obtain

$$N(dd^c\log\xi;r,r_0)\le\epsilon\,\log r+O(\log^+\log r)\tag{6.13}$$

for all $r\gg r_0$.

Combinating (6.2), (6.5) and (6.13), we proved the theorem.

QED

7. Other results

For any divisor $D=D_g$ on $\mathbb{P}^n_z$, we define the *defect of D under f* by

$$\delta_{*F} = \deg g - \overline{\lim}_{r\to\infty} \frac{N_{*F}(D; r, r_0)}{T_{*F}(r, r_0)},$$

where $\delta_{*F}(D)$ is independent of the choice of r_0. By the second main theorem, we have the following

Theorem 7.1 (Defect relation) *Let $f, F, D_1, \ldots, D_q$ be as in the theorem 6.1. Then*

$$\sum_{j=1}^{q} \delta_{*F}(D_j) \leq n + 1 + \deg(J_F).$$

Theorem 7.2 *Let f, F be as above. Let D be a divisor given by a hyperplane on $\mathbb{P}^n_w$ with* $\operatorname{codim}(\operatorname{supp} D \cap F(\operatorname{supp} D_{J_F})) \geq 2$. *Let $F^*D = D_g$, and suppose that* $\operatorname{supp} D_{g_1}$ *is smooth and g_1 divides g, where $g_1, g \in \mathbb{C}[z_0, \ldots, z_n]$ are homogeneous polynomials. Then*

$$\deg g_1 \leq \lambda_F + n + 1 + \deg J_F.$$

More precisely,

$$\deg g_1 \leq \lambda_F \underline{\lim}_{r\to\infty} \frac{\log r}{T_{*F}(r, r_0)} + n + 1 + \deg J_F.$$

Proof If we can show that

$$N_{*F}(D_{g_1}; r, r_0) \leq \lambda_F N(D : r, r_0), \tag{7.3}$$

then by the theorem 6.1, for any $\epsilon > 0$ and $0 < r_0 < r < +\infty$,

$$\begin{aligned}
&(\deg g_1 - (n+1))T_{*F}(r, r_0) \\
&\quad \leq N_{*F}(D_{g_1}; r, r_0) + N_{*F}(D_{J_F}; r, r_0) + \epsilon \log r + O(1) \\
&\quad \leq \lambda_F N(D; r, r_0) + \deg J_F \cdot T_{*F}(r, r_0) + \epsilon \log r + O(1) \\
&\quad \leq \lambda_F \log r + \deg J_F \cdot T_{*F}(r, r_0) + \epsilon \log + O(1).
\end{aligned}$$

Here the hypothesis that D is a hyperplane is used.

Now we prove (7.3). By the proof of [**Dr**, lemma 3.2], we know

$$F^*D \le [\mathrm{supp}F^*D] + D_{J_F}$$

holds on $i(\mathbb{C}^n_z)$, where $[\mathrm{supp}F^*D]$ is the current by integration on $\mathrm{supp}F^*D$. Then if $U \subset i(\mathbb{C}^n_z)$ is an open subset with $U \cap \mathrm{supp}D_{J_P} = \emptyset$,

$$[\mathrm{supp}F^*D] \mid U \le F^*D \mid U \le [\mathrm{supp}F^*D] \mid U.$$

$$\text{i.e., } F^*D \mid U = [\mathrm{supp}F^*D] \mid U. \tag{7.4}$$

For any $w \in i(\mathbb{C}^n_w) - F(\mathrm{supp}D_{J_F})$. There is an open neighborhood $W(w)$ of w in $i(\mathbb{C}^n_w) - F(\mathrm{supp}D_{J_F})$ and λ_F disjoint open subsets $U_1(w), \ldots, U_{\lambda_F}(w)$ in $i(\mathbb{C}^n_z)$, such that $F^{-1}(W(w)) = \bigcup_{i=1}^{\lambda_F} U_i(w)$, and $F \mid U_i(w) : U_i(w) \to W(w)$ is biholomorphic. Then

$$\begin{aligned} F_*D_{g_1} &\le F_*D_g \mid W(w) = F_*[\mathrm{supp}D_g] \mid W(w) \\ &= \sum_{i=1}^{\lambda_F} (F \mid U_i(w))^{-1*}[\mathrm{supp}D_g] \\ &= \sum_{i=1}^{\lambda_F} \left[\mathrm{supp}(F \mid U_i(w))^{-1*}[\mathrm{supp}D_g]\right] \\ &= \sum_{i=1}^{\lambda_F} [\mathrm{supp}D] \\ &= \lambda_F D. \end{aligned}$$

Thus $F_*D_g \mid i(\mathbb{C}^n_w) - F(\mathrm{supp}D_{J_F}) = \lambda_F D$. Since $\mathrm{codim}\,(\mathrm{supp}D \cap F(\mathrm{supp}D_{J_F})) \ge 2$, then $F_*D_g \mid i(\mathbb{C}^n_w) = \lambda_F D$, i.e.,

$$F_*D_{g_1} \le F_*D_g = \lambda_F D$$

holds on $i(\mathbb{C}^n_w)$. This proves (7.3). **QED**

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