$\mathbb{C}^n$-CAPACITY AND MULTIDIMENSIONAL MOMENT PROBLEM

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Introduction

Let K be a compact set in the n-dimensional complex space $\mathbb{C}^n$, $H(K)$ be a space of holomorphic functions on K, $H'(K)$ be the space of linear continuous functionals over $H(K)$. We will write down the value of the functional $\mu \in H'(K)$ on the function $h \in H(K)$ in the form of $<\mu, h>$. The numbers of the form $C_\nu(\mu) = <\mu, Z^\nu>$ are called the moments of the analytical functional μ, where $Z^\nu = Z_1^{\nu_1} \dots Z_n^{\nu_n}$ is a holomorphic monomial of the degree $|\nu| = \nu_1 + \dots + \nu_n$; $Z = (Z_1, \dots, Z_n) \in \mathbb{C}^n$, $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}_+^n$.

The problem arising from a number of applications (computational tomography [1], inverse problem of the potential theory [2], quadrature formulae [3], and even production functions theory [4]) is to reconstruct a functional from $H'(K)$ through its moments.

The necessary and sufficient condition of uniqueness of a functional $\mu \in H'(K)$, which has the fixed moments $\{C_\nu(\mu)\}$ is polynomial convexity of the compact set K, since polynomial convexity of K is necessary and sufficient in order that any function from $H(K)$ will be approximated by holomorphic polynomials (A. Weil, 1932).

If a functional μ is given by positive measure on the compact set $K \subset R^n \subset \mathbb{C}^n$ then the considered problem is called the classical moment problem. This classical problem is effectively and completely solved only for the case $n = 1$ (see [5]).

In connection with applications the problem of the approximate reconstruction of the functional $\mu \in H'(K)$ through the finite number of moments C_ν, $|\nu| \leq N$ is of particular interest. In the classical theory this problem is called the Markov moment problem. In order to solve this problem it is necessary to answer at least the following questions:

1. What is a guaranteed estimate of the accuracy of the possible reconstruction of the functional $\mu \in H'(K)$ if the moments $C_\nu(\mu)$, $|\nu| \leq N$ and certain norm of the functional μ are known?

2. How to find actually the functional $\mu \in H'(K)$ with a priori given moment $C_\nu(\mu)$, $|\nu| \leq N$ and with some suitable norm?

It turned out that these questions are closely connected with several modern themes from several complex variables.

Namely, for exact answer to the question 1, it is used the results of the theory of extremal plurisubharmonic functions and of the complex Monge-Ampère equation on the parabolic manifolds obtained in the papers [6]–[22] and also the theory of the Fantappie-Martineau analytical functional [23]–[27]. The modern variants of the interpolational formulae of the Jacobi type for the holomorphic functions in the hyperconvex domains [28], [29] are very useful for the answer to the question 2.

In this article we give a suitable answer to the question 1 and indicate the simplest applications. The constructive answer to the question 2 will be given in the other paper.

§1. The results.

The compact subset $K \subset \mathbb{C}^n$ is called regular (see [8], [9], [14]) if there exists (and unique) a continuous solution U_K of the following exterior Dirichlet problem for the complex Monge-Ampère equation: $U_K(Z)$ is a plurisubharmonic function in $\mathbb{C}^n \backslash K$,

$$\begin{aligned} &\det\left[\frac{\partial^2 U_K}{\partial \bar{Z}_\alpha \partial Z_\beta}(Z)\right] = 0 && \text{in } \mathbb{C}^n \backslash K \\ &U_K(Z) = \log|Z| + O(1) && \text{as } |Z| \to \infty \\ &U_K(Z) = 0 && \text{if } Z \in \partial K. \end{aligned} \tag{1.1}$$

The compact subset $K \subset \mathbb{C}^n$ is called (see [23]–[26]) linear convex if for any point $W \in \mathbb{C}^n \backslash K$ a set of complex hyperplanes passing through W and not crossing K is non-empty and contractible.

The compact K is called strictly linear convex if its boundary ∂K is smooth and for any point $W \in \partial K$ the complex tangent hyperplane $T_W^C(\partial K)$ have the unique point of contact $\{W\}$ with ∂K and this contact not higher than the first order. Any linear convex compact set K may be represented in the form of

$$K = \bigcap_{j=1}^{\infty} K_j,$$

where $K_1 \supset K_2 \supset \ldots$ is a sequence of strictly linear convex compact sets. Besides, there takes place the monotonic convergence for regular linear convex compact sets K

$$U_j(Z) \to U_K(Z) \qquad \text{for } j \to \infty, Z \in \mathbb{C}^n \backslash K, \tag{1.2}$$

where $U_j(Z) = U_{K_j}(Z)$- smooth solutions of the type (1.1) of the Monge-Ampère equation in $\mathbb{C}^n \backslash K_j$. Existence and uniqueness of such solutions for strictly linear convex compact sets is proved in [17].

We suppose without loss of generality that a linear convex compact K contains the origin of coordinates in $\mathbb{C}^n$. We define a domainK' dual to the compact set K by the formula

$$K' = \{p \in (\mathbb{C}^n)' : pZ + 1 \neq 0 \qquad \text{for } Z \in K\}.$$

For the domain K' we have such a representation

$$K' = \bigcup_{j=1}^{\infty} K'_j,$$

where $K'_1 \subset K'_2 \subset \ldots$ is a sequence of strictly linear convex domains dual to K'_j.

According to Lempert [11], [12] there exist smooth solutions $V_j = V_{K'_j}$ of the Monge-Ampère equations in the domains $K'_j \backslash \{0\}$: $V_j(p)$ is a plurisubharmonic function in $K'_j \backslash \{0\}$

$$\begin{array}{ll} \det \left[\frac{\partial^2 V_j(p)}{\partial \bar{p}_\alpha \partial p_\beta} \right] = 0 & \text{in } K'_j | \{0\} \\ V_j(p) = \log |p| + O(1) & \text{for } p \to 0 \\ V_j(p) = 0 & \text{for } p \in \partial K'_j. \end{array} \tag{1.3}$$

Besides, $O(1) = S'_j(\frac{p}{|p|}) + O_j(|p|)$, where S'_j is a smooth function on $\mathbb{C}P^{n-1}$, i.e., $S'_j(\lambda \cdot p) = S'_j(p), \forall \lambda \in \mathbb{C}$.

The following nice formula is valid ([17], p. 882)

$$V_j(p) = -U_j(Z(p)), \tag{1.4}$$

where

$$Z(p) = \frac{\partial V_j(p)}{\partial p}\left(p\frac{\partial V_j(p)}{\partial p}\right)^{-1}$$

is a diffeomorphism of the domain $K_j'\backslash\{0\}$ on $\mathbb{C}^n\backslash K_j$;

$$\frac{\partial V_j}{\partial p} = \left(\frac{\partial V_j}{\partial p_1}, \dots, \frac{\partial V_j}{\partial p_n}\right).$$

It follows from (1.2), (1.4), in particular, that there takes place a monotonic convergence

$$V_j(p) \to V_{K'}(p), \quad j \to \infty, \quad p \in K'\backslash\{0\}, \tag{1.5}$$

where $V_{K'}(p)$ is a continuous solution of the Monge-Ampère equation of the type (1.3) in the domain $K'\backslash\{0\}$.

For regular linear convex compact sets K so called [16], [22] Robin functions of the compact set K and of the domain K' are defined and continuous on $\mathbb{C}P^{n-1}$

$$\begin{aligned} S(\zeta) &= \limsup_{\lambda\to\infty}(U_K(\lambda\zeta) - \log|\lambda|) \\ S'(\zeta) &= \limsup_{\lambda\to 0}(V_{K'}(\lambda\zeta) - \log|\lambda|), \end{aligned} \tag{1.6}$$

where $\zeta \in \mathbb{C}^n : |\zeta| = 1$ is identified with a point of $\mathbb{C}P^{n-1}$.

Following Lelong [21] we shall call the functions $\gamma(\zeta) = \exp(-S(\zeta))$ and $\gamma'(\zeta) = \exp(-S'(\zeta))$, $\zeta \in \mathbb{C}P^{n-1}$ capacitative indicatrices of the compact K and of the domain K' respectively.

Due to the statement of convergence of the Robin functions from Bedford-Taylor ([22], p. 163) it follows from (1.2) and (1.5) that

$$\begin{aligned} \gamma_j(\zeta) &\to \gamma(\zeta), \quad j \to \infty, \quad \zeta \in \mathbb{C}P^{n-1} \\ \gamma_j'(\zeta) &\to \gamma'(\zeta), \quad j \to \infty, \quad \zeta \in \mathbb{C}P^{n-1}, \end{aligned} \tag{1.7}$$

where γ_j and γ_j' are capacitative indicatrices of the compact set K_j and of the domain K_j' respectively.

The following explicit relation between indicatrices γ and γ' implies from (1.3), (1.4), (1.6), (1.7)

$$\gamma(\bar{\zeta} - \frac{\partial}{\partial \zeta}\ln(\gamma'(\zeta))^2) = \frac{|\bar{\zeta} - \frac{\partial}{\partial \zeta}\ln(\gamma'(\zeta))^2|}{\gamma'(\zeta)}, \tag{1.8}$$

where $\zeta \in \mathbb{C}^n : |\zeta| = 1$.

The most important examples of linear convex and simultaneously regular compact sets are compact sets in $\mathbb{C}^n$, which are closures of the bounded linear-convex domains in $\mathbb{C}^n$ with smooth boundary or closures of the bounded convex domains in $R^n \subset \mathbb{C}^n$. In particular, for the complex ball $K = \{Z \in \mathbb{C}^n : |Z| \leq R\}$ it is well known that $U_K(Z) = \ln \frac{|Z|}{R}$. W. Stoll [10] obtained necessary and sufficient property of $U_K(Z)$ which characterizes the manifolds equivalent to the complex ball. For the real ball $K = \{Z = x + iy \in \mathbb{C}^n : |x| \leq R, y = 0\}$ M. Lundin [19] obtained the following nice formula $sh^2 U_K(Z) = \frac{1}{2}(|Z|^2 - R^2 + |Z^2 - R^2|)$.

The entire function $\hat{\mu}(\zeta)$ of the variable $\zeta \in \mathbb{C}^n$ of the form

$$\hat{\mu}(\zeta) = <\mu, \exp(i\zeta \cdot Z)>, \tag{1.9}$$

where $\zeta Z = \zeta_1 Z_1 + \ldots + \zeta_n Z_n$, is called the Fourier-Laplace transform of the analytical functional $\mu \in H'(K)$.

For the functional $\mu \in H'(K)$ where K is a regular compact set, we define semi-norms of the form

$$\begin{aligned} \|\mu\|_\delta = &\sup |<\mu, h>| \\ &h \in H(K_\delta): \quad |h(Z)| \leq 1, \quad Z \in K_\delta, \end{aligned} \tag{1.10}$$

where $K_\delta = \{Z \in \mathbb{C}^n : U_K(Z) \leq \delta\}$, $\delta > 0$, U_K satisfies (1.1).

The following result gives a sufficiently exact answer to the question 1 for functionals with support on the regular linear convex compact.

Theorem. *Let K be a regular linear convex compact in $\mathbb{C}^n$ and $\gamma'(\zeta)$ be a capacitative indicatrix of the domain K'. Then*

A) *for any $N \in \mathbb{Z}_+$ any functional $\mu \in H'(K)$ with the moments*

$C_\nu(\mu) = 0$ for $|\nu| \le N$, any $\zeta \in \mathbb{C}^n, |\zeta| = 1$, any $\lambda \in \mathbb{C}$ and for any $\delta > 0$ there takes place the following inequality

$$|\hat{\mu}(\lambda\zeta)| \le \frac{\|\mu\|\delta}{\mathrm{d}(K, K_\delta)} \left[\frac{e^{1+\delta}|\lambda| \left(1 + O_{K,\zeta}\left(\frac{|\lambda|}{N}\right)\right)}{\gamma'(\zeta) \cdot (N+1)} \right]^{N+1}, \quad (1.11)$$

where $O_{K,\zeta}(\varepsilon) \to 0$ if $\varepsilon \to 0$; $d(K_1 K_\delta) = \inf |1+p\cdot z|, z \in K, p \in K'$

B) *for any $\zeta \in \mathbb{C}^n, |\zeta| = 1$, any $N \in \mathbb{Z}_+$ there exists the functional $\mu = \mu_{N,\zeta} \in H'(K)$ with the moments $C_\nu(\mu) = 0$ for $|\nu| \le N$ and with estimates of the form*

$$|\hat{\mu}(\lambda\zeta) \ge \frac{(n-1)!}{(N+n)!} \left(\frac{|\lambda|}{|\gamma'(\zeta)|}\right)^{N+1} e^{-d_k(\zeta)\cdot|\lambda|} \left(1 - O_{K,\zeta}\left(\frac{|\lambda|^2}{\sqrt{N}}\right)\right) \quad (1.12)$$

$$\|\mu\|_\delta \le \frac{(n-1)!}{(2\pi)^n} \int\limits_{Z \in \partial K_\delta} |\omega'(\eta(Z)) \wedge \omega(Z)|, \quad (1.13)$$

where $\eta(Z)$ is any smooth $\mathbb{C}^n$-valued function of the variable $Z \in \partial K_\delta$ with the property [26]: *for all $Z \in \partial K_\delta$ and $W \in K$ we have $1 + \eta(Z) \cdot Z = 0$ and $1 + \eta(Z) \cdot W \ne 0$; $\omega(Z) = \bigwedge\limits_{j=1}^{n} dZ_j$; $\omega'(\eta) = \sum_{j=1}^n (-1)^j \eta_j \bigwedge\limits_{V \ne j} d\eta_j d_k(\zeta) > 0$.*

For the case when the compact set K is a strictly linear convex then the compact set K_δ for any $\delta > 0$ is also strictly linear convex [17]. Using in this case $\delta = 0$ and $\eta(Z) = \frac{\partial U_k(Z)}{\partial Z} \,/\, \left(Z \cdot \frac{\partial U_K(Z)}{\partial Z}\right)$ we obtain from (1.13) that the functionals $\mu_{N,\zeta}$ have a uniformly bounded norm $\|\mu\|_0$.

The theorem, roughly speaking, means that if the moments $C_\nu(\mu), |\nu| \le N$ are known for the finite measure μ with the support on the K then its Fourier-Laplace transform $\hat{\mu}(\zeta)$ is reconstructed with accuracy of the order $\|\mu\| \left(\frac{e\cdot|\zeta|}{\gamma'\left(\frac{\zeta}{|\zeta|}\right)(N+1)}\right)^{N+1}$ and not better, in general. It is important to express capacitative indicatrix $\gamma'(\zeta/|\zeta|)$ in geometric terms in order to use such an estimate. For general case it is not simple. However, the following statement is valid for the particular case when the compact set K and direction ζ are real.

Proposition. *Let K be a closure of the bounded convex domain in $R^n \subset \mathbb{C}^n$. Then the following equality is valid*

$$(\gamma'(\zeta))^{-1} = \frac{1}{4}\left[\sup_{x\in K}(\zeta\cdot x) - \inf_{x\in K}(\zeta\cdot x)\right] \tag{1.14}$$

for any real $\zeta \in R^n$, $|\zeta| = 1$.

Remark. If we drop demand of the regularity of the linear convex compact set in the theorem then the theorem is still valid if we will write in the statement that $\zeta \in \mathbb{C}P^{n-1}\backslash E$ where E is some polar subset of $\mathbb{C}P^{n-1}$. In addition, instead of the function $U_K(Z)$ of the form (1.1) it is necessary to use extreme plurisubharmonic function [8], [9] of the form

$$\begin{aligned} U_K(Z) &= \sup\{U(Z) : U \text{ is plurisubharmonic on } \mathbb{C}^n\backslash K\} \\ U(Z) &\le \log|Z| + O(1), U(Z) \le 0 \text{ on } \partial K. \end{aligned}$$

The necessary properties of the Robin function for such extremal functions are obtained by P. Lelong [21] and E. Bedford, B. Taylor [22].

This theorem supposes may be more clear interpretation in terms of the best approximations of the function $\exp(\zeta\cdot Z)$ by polynomials on the compact set K.

Let us define the numbers

$$E_N(K,f) = \inf_{P_N}\sup_{Z\in K}|f(Z) - P_N(Z)|,$$

where P_N is a polynomial of the degree N in $Z = (Z_1,\ldots,Z_n)$.

Consequence 1. *The following equality takes place*

$$\lim_{N\to\infty} N\cdot E_N^{1/N}(K, e^{i\zeta Z}) = e\cdot|\zeta|\Big/\gamma'(\zeta/|\zeta|)$$

for any regular linear convex compact set $K \subset \mathbb{C}^n$ and any $\zeta \in \mathbb{C}^n$.

Note, that the result of the consequence 1 may be considered as complement of the following general approximating result of Siciak [6], [9]. In order that $f \in H(K_\delta)$ (see (1.10) it is necessary and sufficient that

$$\overline{\lim_{N\to\infty}} E_N^{1/N}(K, f) \leq e^{-\delta}.$$

Now we will give an application of the theorem to one of computational tomography problem—to an estimate of the accuracy of the Radon transform inversion through the finite number of directions.

The transform of the type

$$R_\mu(\omega, s) = \frac{\partial}{\partial s} \int\limits_{\{x\in R^n : \omega x \leq s\}} \mu(dx),$$

where $s \in R, \omega \in S^{n-1} = \{\omega \in R^n : |\omega| = 1\}$ is called the Radon transfrom for a finite measure μ with compact support in R^n.

The finite subset Ω of the sphere S^{n-1} is called N-solvable [1] if any polynomial $P_N(x)$ of the degree N is represented in the form

$$P_N(x) = \sum_{\omega\in\Omega} P_{N,\omega}(\omega \cdot x), \tag{1.15}$$

where $P_{N,\omega}$ is a polynomial of degree N of the variable $\omega \cdot x$. For the number of elements $|\Omega|$ in Ω we have the estimate

$$|\Omega| \geq C_{N+n-1}^{n-1}. \tag{1.16}$$

Conversely, if the inequality (1.16) is held and elements in Ω are in the general position then Ω is N-solvable (see [1]).

If the Radon transform $R_\mu(\omega, s), \omega \in \Omega$ is known for the measure μ and Ω is N-solvable, then the moments $C_\nu(\mu)$ of the order $|\nu| \leq N$ are known for the measure μ due to (1.15).

Hence from the theorem we obtain the following consequence.

Consequence 2. *Let a support of the finite measure μ belong to the closure of the bounded convex domain $K \subset R^n$ and let the Radon transform $R_\mu(\omega, s)$ of the measure μ is equal to zero for directions ω belonging to N-solvable subset Ω. Then the Fourier-Laplace transform $\hat{\mu}(\zeta)$ for any $\zeta \in \mathbb{C}^n$ admits the estimate of the form (1.11).*

Note, that due to (1.14) we have $\gamma'(\zeta) = 2$ for the real unit sphere $K^1 = \{x \in R^n : |x| \leq 1\}$ and for real directions $\zeta \in S^{n-1}$. So, for this case the consequence 2 yields a preciser estimate:

$$\sup_{\{\zeta \in R^n : |\zeta| \leq \theta N\}} |\hat{\mu}(\zeta)| \leq \frac{O(1)}{\sqrt{N+1}} \cdot \left(\frac{\theta e}{2}\right)^{N+1} \cdot \|\mu\|_0$$

for any $\theta < 2/e$.

It is interesting to associate this result with the following Logan-Louis estimate (see [1]):
under the conditions of the consequence 2 we have

$$\int_{\{\zeta \in iR^n : |\zeta| \leq \theta N\}} |\hat{\mu}(\zeta)| d\zeta \leq \beta(\theta) e^{-C(\theta)(N+1)} \|\mu\|_0$$

for $K = K^1$ and for any $\theta < 1$.

§2. The proof of the theorem.

This proof essentially uses the notion of the Fantappie indicatrix of the analytical functional.

The holomorphic function of the type

$$\Phi_\mu(p) = <\mu, \frac{1}{(1+pZ)}> \tag{2.1}$$

in the domain K' is called the Fantappie indicatrix of the analytical functional $\mu \in H'(K)$. Immediately from the definition (2.1) it follows that the equality $C_\nu(\mu) = <\mu, Z^\nu> = 0$ for $|\nu| \leq N$ is equivalent to the equalities

$$\Phi_\mu^{(\nu)}(0) = \frac{d^\nu \Phi_\mu}{dp_1^{\nu_1}, \ldots, dp_n^{\nu_n}}(0) = 0, \tag{2.2}$$

for $|\nu| \leq N, \quad \nu = (\nu_1, \ldots, \nu_n)$.

The Fantappie transform $\Phi_\mu(p)$ is simply expressed through the Fourier-Laplace transform $\hat{\mu}(\zeta)$

$$\Phi_\mu(p) = i \int_0^\infty e^{-i\tau} \hat{\mu}(\tau p) d\tau, \quad p \in K'. \tag{2.3}$$

Martineau [24] obtained a general formula expressing $\hat{\mu}(\zeta)$ through $\Phi_\mu(p)$ on the basis of the Cauchy-Fantappie-Leray formula (see [27], [29]). Here we will have a need of the following elementary formula.

$$\hat{\mu}(\zeta) = \frac{1}{2\pi i} \int_{\{\lambda \in \mathbb{C}: |\lambda| = R\}} \frac{1}{\lambda} e^{-i\lambda} \Phi_\mu\left(\frac{\zeta}{\lambda}\right) d\lambda, \tag{2.4}$$

where R is such that $\zeta/\lambda \in K'$ for any $\lambda : |\lambda| = R$.

The formula (2.4) is a simple consequence of the classical Cauchy formula. In fact, substituting the Cauchy representation

$$e^{i\zeta \cdot Z} = \frac{1}{2\pi i} \int_{\lambda \in \mathbb{C}: |\lambda| = R} \frac{e^\lambda d\lambda}{\lambda - i\zeta Z}$$

in the equality (1.9) we obtain

$$\hat{\mu}(\zeta) = \frac{1}{2\pi i} \int_{|\lambda| = R} e^\lambda \frac{d\lambda}{\lambda} <\mu, \frac{1}{1 - \frac{i\zeta Z}{\lambda}}> = \frac{1}{2\pi i} \int_{|\lambda| = R} \frac{e^\lambda d\lambda}{\lambda} \Phi_\mu\left(-\frac{i\zeta}{\lambda}\right).$$

The formula (2.4) allows to obtain necessary estimate for $\hat{\mu}(\zeta)$ on the basis of suitable estimates for $\Phi_\mu(\zeta/\lambda)$. We will obtain estimates for $\Phi_\mu(\zeta/\lambda)$ from equalities (2.2) and from the following immediate estimate.

$$\|\Phi_\mu(p)\| \leq \|\mu\|_\alpha \sup_{Z \in K_\alpha} \left| \frac{1}{1 + pZ} \right|, \tag{2.5}$$

where $\|\mu\|_\alpha$ is a norm of the form (1.10), $K_\alpha = \{Z \in \mathbb{C}^n : U_K(Z) \leq \alpha\}$, $p \in K'$.

Suppose, further, $K_\delta = \{Z \in \mathbb{C}^n : U_K(Z) \leq \delta\}$, $K'_\delta = \{p \in K' : V_{K'}(p) + \delta < 0\}$, $\delta > 0$, where U_K and $V_{K'}$ are the functions satisfying (1.1), (1.5).

Consider now the plurisubharmonic function

$$\Psi_\delta(p) = \frac{1}{N+1} \ln \frac{|\Phi_\mu(p)| d(K, K_\delta)}{\|\mu\|_\delta}. \tag{2.6}$$

This function is negative in the domain $K'_\delta \subset K'$ due to (2.5). The estimate $\Psi_\delta(p) \leq \ln |p| + O(1), \quad p \in K'_\delta$ also takes place due to (2.2). Due to (1.5) the function $V_{K'}(p) + \delta$ satisfies the Monge-Ampère equation (1.3) in the domain K'_δ. As it was shown in [18], [20] such a function is extremal plurisubharmonic function in the following sense:

$$\begin{aligned} V_{K'}(p) + \delta = \sup\{V/(p) : V &\text{ is plurisubharmonic} \\ V(p) \leq 0 &\text{ and } V(p) \leq \ln |p| + O(1) \text{ in } K'_\delta\} \end{aligned} \tag{2.7}$$

So, $\Psi_\delta(p) \leq V_{K'}(p) + \delta$. From (2.6), (2.7) it follows that

$$|\Phi_\mu(p)| \leq \frac{\|\mu\|_\delta}{d(K, K_\delta)} \exp\left[(N+1)(V_{K'}(p) + \delta)\right] \tag{2.8}$$

for $p \in K'_\delta$.

Substitute now the estimate (2.8) in the formula (2.4). Taking into account (1.3), (1.5) we obtain the following inequality

$$\begin{aligned} |\hat{\mu}(\zeta)| &\leq e^R \frac{\|\mu\|_\delta}{d(K, K_\delta)} \exp\left[(N+1)\Big(\sup_{\varphi \in [O, 2\pi]} V_{K'}\left(\frac{\zeta e^{i\varphi}}{R}\right) + \delta\Big)\right] \\ &= e^R \frac{\|\mu\|_\delta}{d(K, K_\delta)} \exp\left[(N+1)\Big(\log\left|\frac{\zeta}{R}\right| - \log \gamma'\left(\frac{\zeta}{|\zeta|}\right) + \delta \right. \\ &\quad \left. + O_{K, \frac{\zeta}{|\zeta|}}\left(\left|\frac{\zeta}{R}\right|\right)\Big)\right] \end{aligned}$$

for any $\zeta \in \mathbb{C}^n$, $\delta > 0$ and for such R that $\zeta R^{-1} e^{i\varphi} \in K'_\delta$ for all $\varphi \in [O, 2\pi]$. Suppose $R = N+1$, we obtain

$$|\hat{\mu}(\zeta)| \leq \frac{\|\mu\|_\delta}{d(K, K_\delta)} \left[\frac{e^{1+\delta} |\zeta| (1 + O_{K, \zeta/|\zeta|}\left(\left|\frac{\zeta}{N+1}\right|\right)}{\gamma'\left(\frac{\zeta}{|\zeta|}\right)(N+1)}\right]^{(N+1)}.$$

The estimate (1.11), i.e. the part A) of the theorem is proved.

In order to prove the part B) of the theorem we shall have need of one more formula for the capacitative indicatrix:

$$(\gamma'(\zeta))^{-1} = \sup_{\{F\in H(K'):F(0)=0,|F(\zeta)|\leq 1,\zeta\in K'\}} \left|\zeta\frac{\partial F}{\partial \zeta}(0)\right|. \qquad (2.9)$$

The proof of (2.9) is based on the Lempert results [15]. Due to (1.3), (1.5) for the solution $V(p)$ of the Monge-Ampère equation in the domain we have an asymptotic equality

$$V(\lambda\zeta) = \log|\lambda| - \log\gamma'(\zeta) + O(|\lambda|) \text{ for } \lambda \to 0, \qquad (2.10)$$

where $p = \lambda\zeta, \zeta \in \mathbb{C}^n, |\zeta| = 1; \quad \lambda \in \mathbb{C}$.

Further, the following equality is valid (Lempert [15])

$$V_j(p) = \sup_{\{F\in H(K_j'):F(0)=0, \quad |F|\leq 1\}} \ln|F(p)|, \qquad (2.11)$$

where functions V_j satisfies (1.3).

Taking into account (1.5) from (2.11) we obtain also the equality

$$V(p) = \sup \ln|F(p)|. \qquad (2.12)$$
$$\{F \in H(K') : F(0) = O, |F| \leq 1\}$$

The equality (2.9) follows from (2.12).

Now we prove part B) of the theorem. We fix $\zeta \in \mathbb{C}^n : |\zeta| = 1$ and $N \in \mathbb{Z}_+$. Due to (2.10) there exists a function $F \in H(K')$ with the property

$$|F(p)| \leq 1, \quad p \in K' \text{ and } F(\lambda\zeta) = (\gamma'(\zeta))^{-1}\lambda + O_{K,\zeta}(\lambda^2) \text{ for } \lambda \to 0 \qquad (2.13)$$

Consider, further, a holomorphic function $\Psi(p) = F^{N+1}(p)$. We have

$$|\Psi(p)| \leq 1, \quad p \in K' \text{ and}$$
$$\frac{\partial^\nu\Psi}{\partial p_1^{\nu_1},\ldots,\partial p_n^{\nu_n}}(0) = 0 \text{ for } |\nu| \leq N. \qquad (2.14)$$

Due to the Martineau theorem [23], [24] refined in [25], [26], there exists a functional $\mu \in H'(K)$ such that its indicatrix $\Phi_\mu(p)$ satisfies the equality

$$\frac{1}{(n-1)!} D^{n-1}\Phi_\mu(p) = \Psi(p), \tag{2.15}$$

where $D\Phi = \Phi + p\frac{\partial\phi}{\partial p}$. It follows from (2.14), (2.15) that $C_\nu(\mu) = 0$ for $|\nu| \le N$. We will prove the estimate (1.13).

Let,

$$L\mu = \frac{(n-1)!}{(2\pi i)^n}\Psi(\eta(Z)) \wedge \omega'(\eta(Z)) \wedge \omega(Z),$$

where $Z \to \eta(Z)$ is any smooth mapping with the property: for any $Z \in \partial K_\delta$ and $W \in K$ we have $1 + \eta(Z) \cdot Z = 0$ and $1 + \eta(Z)W \neq 0$. Let h be any bounded holomorphic function on K_δ. Due to the Cauchy-Fantappie-Leray formula we have (see [24]–[26]):

$$<\mu, h> = \int\limits_{Z \in \partial K_\delta} L\mu \wedge h. \tag{2.16}$$

The estimate (1.13) is an immediate consequence of (2.16). We will prove now the estimate (1.12). Taking into account formulae (2.4), (2.15) we obtain the equality

$$\hat{\mu}(\lambda\zeta) = \frac{(n-1)!(-1)^{n-1}}{2\pi i} \int\limits_{\{t \in \mathbb{C} : |t| = R\}} \frac{e^{-it}}{t^n}\Psi\left(\frac{\lambda\zeta}{t}\right) dt. \tag{2.17}$$

Due to (2.13) for the function $\Psi\left(\frac{\lambda\zeta}{t}\right)$ and $|t| = N+1$ we have inequalities

$$\begin{aligned}\Psi\left(\frac{\lambda\zeta}{t}\right) &= \left[(\gamma'(\zeta))^{-1}\left(\frac{\lambda}{t} + d(\zeta)\frac{\lambda^2}{t^2}\right) + O_{K,\zeta}\left(\frac{\lambda^3}{t^3}\right)\right]^{N+1} \\ &= \left(\frac{\lambda}{\gamma'(\zeta)\cdot t}\right)^{N+1}\left[1 + d(\zeta)\frac{\lambda}{t} + O_{K,\zeta}\left(\frac{\lambda}{N+1}\right)^2\right]^{N+1}\end{aligned}$$

$$= \left(\frac{\lambda}{\gamma'(\zeta)\cdot t}\right)^{N+1}\left[\left(1+d(\zeta)\frac{\lambda}{t}\right)^{N+1}\right.$$
$$\left. + O_{K,\zeta}\left(\frac{|\lambda|^2}{N+1}\right)\right]. \tag{2.18}$$

Substituting (2.18) into (2.17) we have

$$\hat{\mu}(\lambda\zeta) = (n-1)!(-1)^{n-1}[\mathcal{J}_1 + \mathcal{J}_2], \tag{2.19}$$

where

$$\mathcal{J}_1 = \frac{1}{2\pi i}\int\limits_{|t|=R}\frac{e^{-it}}{t^n}\left(\frac{\lambda}{\gamma'(\zeta)\cdot t}\right)^{N+1}\left(1+d(\zeta)\frac{\lambda}{t}\right)^{N+1}dt$$

$$\mathcal{J}_2 = \frac{1}{2\pi i}\int\limits_{|t|=N+1}\frac{e^{-it}}{t^n}\left(\frac{\lambda}{\gamma'(\zeta)\cdot t}\right)^{N+1} O_{K,\zeta}\left(\frac{|\lambda|^2}{N+1}\right)dt$$

Computing exactly $\mathcal{J}_1$ and estimating $\mathcal{J}_2$ we find

$$\mathcal{J}_1 = \frac{((\gamma')^{-1}\lambda)^{N+1}(-i)^{N+n}}{(N+n)!}\left[\sum_{\nu=0}^{N+1}\frac{(-id\cdot\lambda)^\nu}{\nu!}\prod_{j=1}^{\nu}\left(1-\frac{n+\nu-1}{N+n+j}\right)\right]$$

$$|\mathcal{J}_2| = \frac{1}{2\pi}\frac{((\gamma')^{-1}\cdot|\lambda|\cdot e)^{N+1}}{(N+1)^{n+N}}O_{K,\zeta}\left(\frac{|\lambda|^2}{N+1}\right). \tag{2.20}$$

It follows from (2.19), (2.20) that

$$|\hat{\mu}(\lambda\zeta)| \geq \frac{(n-1)!(|\gamma'(\zeta)|^{-1}|\lambda|)^{N+1}}{(N+n)!}\times$$
$$\left[\left(1 - O_{K,\zeta}\left(\frac{|\lambda|^2}{\sqrt{N+1}}\right)\right)e^{-|d(\zeta)|\cdot|\lambda|} - \right.$$
$$\left.\frac{(N+n)!e^{N+1}}{(N+1)^{n+N}(n-1)!}O_{K,\zeta}\left(\frac{|\lambda|^2}{N+1}\right)\right]$$

$$= \frac{(n-1)!(|\gamma'(\zeta)|^{-1}\cdot|\lambda|)^{N+1}}{(N+n)!}$$
$$\left[\left(1 - O_{K,\zeta}\left(\frac{|\lambda|^2}{\sqrt{N+1}}\right)\right)e^{-|d(\zeta)||\lambda|} - O_{K,\zeta}\left(\frac{|\lambda|^2}{\sqrt{N+1}}\right)\right]$$

The estimate (1.12), and consequently, the theorem is proved.

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