

# ON AHLFORS'S THEORY OF COVERING SURFACES

DAVID DRASIN

To Wilhelm Stoll, The Vincent F. Duncan and  
Annamarie Micus Duncan Professor of Mathematics

**1. Introduction.** In [1] (see [3, pp. 214-251]) Ahlfors introduced his theory of covering surfaces. His approach was combinatorial and geometric, and showed that R. Nevanlinna's theory of meromorphic functions had topological significance, and held in differentiated form. Other accounts are in [2], [5], [9], and [12] presents a very efficient proof using Ahlfors's own framework. See [10] for an independent approach, where the conclusions are slightly weaker than in [1].

Some years ago, John Lewis asked me if there was a way to derive Nevanlinna's value distribution theory directly from the argument principle. Since Nevanlinna's approach is based on Jensen's formula, itself the integrated argument principle, it is clear that the argument principle lies behind the theory, but the connection is, to say the least, highly indirect.

In this paper we show a more transparent connection. Very little is used that is not in a first course in complex analysis, but the subtleties needed to achieve (1.5) and (1.6) show the depth of Ahlfors's own insights. In retrospect our methods have considerable intersection with those of [1], although the orientation is different. I thank H. Donnelly, A. Eremenko, D. Gottlieb, L. Lempert, M. Ramachandran and A. Weitsman for helpful discussions. The idea for the latter part of Proposition (1.8) was shown to me by S. Lalley. The influence of Miles's work [8] is also apparent; see (2.23) below.

(1.1) *Preliminaries.* (See [1], [9, Ch. 13].) Let  $a_1, \dots, a_q$  be distinct (finite) complex numbers. We develop two situations in parallel: the "base surface"  $F_0$  is either the Riemann sphere  $S^2$  or  $S^2 \setminus \bigcup_{k=1}^q D_k$ , where the  $D_k$  are disjoint continua about the  $a_k$ ; we also let  $a_{q+1} = \infty$  and take  $D_{q+1}$  accordingly. Thus,  $F_0$  is either closed or bordered.

We impose a unit mass  $\lambda(w)$  on  $F_0$  with the properties specified in [1, I.1], [9, p. 325]; this allows lengths to be assigned to (Ahlfors)

regular curves and open sets. The essential property of  $\lambda$  is that an isoperimetric inequality hold locally: each point  $p_0 \in F_0$  has a neighborhood  $U = U(p)$  such that if  $\gamma$  is a simple closed curve in  $U$  which bounds the region  $\Omega \subset U$ , then

$$(1.2) \quad \lambda(\Omega) < h\lambda(\gamma),$$

where  $h = h(p)$ . By compactness (1.2) holds on  $F_0$  with a universal  $h$  so long as the  $\lambda$ -area of  $\Omega$  is strictly bounded from one. We also let  $[\cdot]$  be the chordal metric on  $S^2$ ; it clearly satisfies (1.2).

Let  $\Delta(r) = \{|z| < r\}$ ,  $B(r) = \partial\Delta(r)$ ,  $\Delta_\lambda(w_0, \eta) = \{w; \lambda(w, w_0) < \eta\}$ ,  $B_\lambda(w_0, \eta) = \partial\Delta_\lambda(w_0, \eta)$ , and  $\Gamma_r = f(B(r))$ . We consider maps  $f : \Delta(R) \rightarrow S^2$  which preserve orientation, with  $0 < R \leq \infty$ . The most important setting is that  $f$  be meromorphic, but it is natural to require only that  $f$  be a ramified covering of  $S^2$ , in the spirit of Ahlfors-Sario [4]; according to Stoilow [11] such maps become meromorphic if  $\Delta(R)$  is given an appropriate structure. For a good account of this see, for example, [6, §2].

Consider now  $f^{-1}(F_0) \subset \Delta(R)$ ; this is a union of components  $\{G\}$ . Let  $G$  be one such component. Then for  $0 < r < R$ , set  $G(r) = G \cap \Delta(r)$ ,  $\partial G(r) = G \cap B(r)$ , and define in terms of  $\lambda$  the expressions  $S = S(r)$ ,  $L = L(r)$  for the area (including multiplicity) of  $f(G(r))$  and length of  $\Gamma_r = f(\partial G(r))$ , measured by  $\lambda$ . In this sense, the image  $\mathfrak{F}$  of  $G$  by  $f$  is a *covering surface* over  $F_0$  with  $\pi : \mathfrak{F} \rightarrow F_0$  the projection. Ahlfors also considers any  $\lambda$ -measurable subset  $D \subset F_0$ , and defines

$$S(r, D) = \frac{\lambda((\pi^{-1}(D) \cap G)(r))}{\lambda(D)}.$$

For example, if  $f$  is rational of degree  $N$  and  $D$  is any open set then  $S(r) = N + o(1) = S(r, D)$  and  $L(r) = o(1)(r \rightarrow \infty)$ .

Ahlfors's theory has significance primarily when  $\mathfrak{F}$  is *regularly exhaustible*: there exists an  $r$ -set  $A$ ,  $R$  a limit point of  $A$ , such that

$$(1.3) \quad L(r) = o(S(r)) \quad (r \rightarrow R, r \in A)$$

[5, p. 338]. If  $f$  is meromorphic or quasiregular (and nonconstant) in  $\Delta(\infty)$  and  $\lambda = [\cdot]$ , it requires but a few lines and the Schwarz

inequality to see that the full image  $\mathfrak{F}$  of  $f$  over  $F_0 = S^2$  is regularly exhaustible (cf. [5, p. 352] and [1, p. 186]). In particular, in this case  $A$  consists of nearly all large  $r$ .

We state Ahlfors's conclusions in two forms:

(1.4) THEOREM. (A) Let  $f$  be meromorphic in  $\Delta(R)$ , and  $a_1, \dots, a_q$  be distinct (finite) complex numbers. Then there exists  $h = h(a_1, \dots, a_q) > 0$  such that

$$(1.5) \quad \sum_1^q n(r, a_j) > (q-2)S(r) - hL(r).$$

(B) Let  $F_0$  be  $S^2$  or  $S^2 \setminus \cup D_k$ ,  $f : \Delta(R) \rightarrow F_0$ , and let  $\lambda$  be a unit mass as described above. Then there exists  $h = h(F_0) > 0$  such that the Euler characteristic of any (finite) covering surface  $\mathfrak{F}$  over  $F_0$  satisfies

$$(1.6) \quad \rho^+ \equiv \max(\rho, 0) \geq \rho_0 S - hL.$$

*Remarks.* 1. In (1.6) we use the definitions of  $\rho$  and  $\rho_0$  from [1]; cf. [4, p. 55]:  $\chi = -F + E - V$  ( $F$  = faces,  $E$  = edges,  $V$  = vertices). In many contemporary topological texts, what we call  $\rho$  is considered the negative of the Euler characteristic.

2. Following [9], we assume that  $F_0$  is planar:  $S^2 \setminus \cup D_j$ . Ahlfors observes [1, p. 174] that the general case follows from this by an elementary combinatorial analysis.

3. Inequality (1.6) is formally stronger than (1.5). One way to see this is that when (1.6) is used to derive the differentiated Nevanlinna theory (cf. [5, p. 148]) there is an additional branching term that is not apparent in (1.5). However, the arguments used to get these refinements are somewhat intricate; here we find that a common attack can yield both.

4. In accord with standard tradition, we use  $h$  as a positive constant which can be taken to depend only on data of the surface  $F_0$ .

For example, Picard's theorem is an immediate consequence of (1.5) or (1.6) together with (1.3); we consider (1.6). Let  $f$  be nonconstant on  $\Delta(\infty)$  and omit  $a_1, a_2, a_3$ . Let  $F_0$  be  $S^2$  with small disks  $D_k$  deleted about the  $a_k$ , so that  $\rho_0 = 1$ . By assumption, no inverse image

of any  $D_k$  can be compactly contained in any  $\Delta(r)$ , so that always  $\rho \equiv -1$ . Thus (1.3) and (1.5) are incompatible. Miles [7] shows that the main part of Nevanlinna's second fundamental theorem can be recovered from the Ahlfors theory.

(1.7) *Normal values, first fundamental theorem.* Since the argument principle gives  $n(r, a_k) - n(r, \infty)$  rather than  $n(r, a_k)$  directly, we first show that  $f$  always has many "normal" values. We have

(1.8) PROPOSITION. *Let  $r < R$ ,  $w_0 \in S^2$  and  $\eta_0 > 0$  be given. Then there exist  $K < \infty$  and  $w^* \in S^2$ , with  $[w^*, w] < \eta_0$  and  $w^*$  normal in the sense*

$$|n(r, w^*) - S(r)| < KL(r);$$

further, we may find a line  $L$  through  $w^*$  such that  $\Gamma_r$  intersects  $L \cap \{[w, w^*] < \eta_0\}$  in at most  $KL(r)$  points.

*Proof.* We assume the elementary first covering theorem of Ahlfors (the analogue of Nevanlinna's first fundamental theorem; cf. [9, pp. 328-9]): if  $D$  is an open set in  $F_0$  then  $|S(r) - S(r, D)| < h(\lambda(D))^{-1}L(r)$ . [Ahlfors also has a variant of this for coverings of "regular" curves, but that is not needed here].

In this proof, we take  $\lambda$  to be chordal measure  $[\ ]$  on  $S^2$ , and let  $S(r)$ ,  $S(r, D)$  be computed with respect to  $[\ ]$ .

For a fixed (large)  $K$ , let  $D_1 = \{w \in F_0; n(r, w) > S(r) + KL(r)\}$ , and  $D_2 = \{w; n(r, w) > S(r) - KL(r)\}$ ; here  $n(r, w)$  is the usual counting function of  $w$ -values in  $\Delta(r)$  or  $G(r)$ . Since  $S(r, D_1) = (\int_{D_1} n(r, w)d\lambda(w))\{\lambda(D_1)\}^{-1}$ , the first covering theorem yields that  $KL(r)\lambda(D_1) < hL(r)$ ; thus if  $K$  is large,  $\lambda(D_1)$  is bounded away from 1. The same analysis applies to  $D_2$ , and hence if  $K$  is sufficiently large, the set  $W$  of  $w^*$  which satisfy the Proposition has chordal measure at least .9 the measure of the ball  $\{[w, w_0] < \eta_0\}$ .

To satisfy the second condition, let us assume that  $w_0 = 0$  and, since  $\eta_0$  is small, replace the chordal metric by the Euclidean metric. Write  $\Gamma = \Gamma_r$ , and assume  $\lambda(\Gamma) < \infty$ . By making a rotation, we may assume that the intersection of  $\Gamma$  with each horizontal or vertical line contains no segment. We will show that if  $i(y_0)$  is the cardinality of  $\Gamma_r \cap \{\Im z = y_0\} \cap \{|z| < 1\}$ , then there exists a set  $Y$  of  $y$ ,  $-\frac{1}{2}\eta_0 < y < \frac{1}{2}\eta_0$  with  $\int_Y dt > .9\eta_0$  and

$$(1.9) \quad i(y) < KL(r), \quad y \in Y,$$

( $K = K(\eta)$ ).

If we grant this, it follows that there exists  $y_0 \in Y$ ,  $|y_0| < \frac{1}{4}\eta_0$ , such that the set  $\{y = y_0\} \cap \{|w| < \frac{1}{2}\eta_0\}$  has nonempty intersection with the set  $W$  constructed above. We use any  $w^* = x_0 + iy_0$  in  $W$ , with  $|x_0| < \frac{1}{2}\eta$ ,  $y_0 \in Y$ , and see that it satisfies both conditions of the Proposition.

We now produce  $y_0$  so that (1.9) holds. By our normalization,  $\Gamma \cap \Delta(w_0, 1)$  may be written as an at most countable union of graphs of continuous functions, say  $y = y_j(x)$ ,  $\alpha_j \leq x \leq \beta_j$ , with  $-1 < y_j(x) < 1$ . If  $V_j$  is the total variation of  $y_j$  on  $(\alpha_j, \beta_j)$  and  $L_j$  is the length of the graph of  $y_j$ , we have that  $V_j \leq L_j$ .

Let  $i_j(y)$  be the number of points of intersection of the graph of  $y_j$  with the line  $\{\Im z = y\}$ , so that  $i(y) = \sum_j i_j(y)$ ; then Banach's formula for total variation gives that  $V_j = \sum_{-1}^1 i_j(y) dy$ . Hence,  $\int_{-1}^1 i(y) dy < L(r)$ , so that (1.9) follows at once.

(1.10) NORMALIZATION. *Given a fixed  $r$ , we in general take  $w^* = \infty$  in Proposition 1.8.*

**2. Partitioning of  $\Delta(r)$ .** Given distinct complex numbers  $a_1, \dots, a_q$ , let  $10^{10}\eta < \inf_{i \neq j} \lambda(a_i, a_j)$ . By Proposition 1.8, we may, by decreasing  $\eta$  if necessary, choose  $a_{q+1}$  so that  $\lambda(a_{q+1}, a_k) > 10^{10}\eta$  ( $1 \leq k \leq q$ ) and then, after a Möbius transformation of  $f$  assume that  $a_{q+1} = \infty$ . This choice of  $\eta$  is in force for all that follows, so that  $\eta$  depends only on  $F_0$ . Following the ideas of Ahlfors, construct (indexing mod  $q+1$ ) Jordan arcs  $\beta_k$  ( $1 \leq k \leq q+1$ ) to join  $a_k$  to  $a_{k+1}$ . The  $\beta$ 's divide  $F_0$  with two Jordan domains  $F'$  and  $F''$ , and the preimages of the  $\beta$ 's divide  $\Delta(r)$  (or  $G$ , as appropriate) into  $N$  domains  $G_\alpha$ . We let  $F_\alpha = f(G_\alpha)$ , so that  $F_\alpha$  is contained in  $F'$  or  $F''$ . We usually ignore the specific choice of  $F'$  or  $F''$ , and write that  $f(G_\alpha) \subset F$ , where  $F$  is the relevant choice of  $F'$  or  $F''$ .

Depending on the context, we may view the domain of  $f$  as all of  $\Delta(R)$ , or as in a component  $G$  of  $f^{-1}(F_0) \cap \Delta(R)$ . Thus, the setting will determine the relevant collection of  $G_\alpha$ 's. Similarly,  $n(r, \infty)$  will

be the number of poles of  $f$  in either  $\Delta(r)$  or  $G(r)$ . We will develop our method so that the reader can readily adapt it to either situation.

We make certain inessential normalizations: the  $\beta_k$  are pointwise disjoint,  $\Gamma_r$  meets each  $\beta_k$  at finitely many points, and none of the countably many branch points of  $f$  lies on any  $\beta_k$ . Finally, we assume that in each ball  $B_\lambda(a_k, 3\eta)$  there is a line segment  $L_k$  passing through  $a_k$  such that relative to this ball,  $\beta_{k-1} \cup \beta_k = L_k \setminus a_k$ . When  $k = q + 1$ , we take  $L$  to be the line constructed in Proposition 1.8. By making an arbitrarily small change in  $r$ , we may suppose that  $\Gamma_r$  does not pass through any of the  $a_k$ .

(2.1) *Principle of the proof.* The significance of length-area is seen from elementary considerations. The work that follows is to force the hypotheses of Lemma 2.2 to be satisfied.

(2.2) LEMMA. *Let the  $G_\alpha$  be as above, and suppose  $G_\alpha$  meets  $B(r)$  in  $P(\alpha)$  points  $\zeta_{j,\alpha}$  whose images on  $S^2$  are separated by some  $\eta > 0$ . Then*

$$(2.3) \quad L(r) \geq h \sum_{\alpha} (P(\alpha) - 2)^+.$$

*Proof.* Consider a fixed  $G_\alpha$ , and  $\zeta_{1,\alpha}, \dots, \zeta_{P(\alpha),\alpha}$  on  $\tilde{G}_\alpha \cap B(r)$ , such that  $\lambda(\zeta_{i,\alpha}, \zeta_{j,\alpha}) > C\eta$ ; here the  $\zeta$ 's are listed in the order encountered on circuiting  $B(r)$  in the positive direction. Since each  $G_\alpha$  is connected, the  $\zeta$ 's are endpoints of  $P(\alpha)$  disjoint arcs  $I$  of  $B(r)$  and hence give a contribution at least  $hP(\alpha)$  to  $L(r)$ . Then if  $G_{\alpha'}$  is any other region determined by the  $\{\beta_j\}$ ,  $G_{\alpha'}$  must lie in one of the complementary domains of  $\Delta(r) \setminus G_\alpha$ .

Hence, given an initial choice of  $G_{\alpha_1}$ , choose  $G_{\alpha_2}$  so that  $G_{\alpha_2}$  is closest to  $G_{\alpha_1}$  in one of these domains (there is not a unique such  $G_{\alpha_2}$ ; in fact there are usually  $P(\alpha)$  such). Then the closures of  $G_{\alpha_2}$  and  $G_{\alpha_1}$  can have at most two points in common on  $B(r)$ . Thus,  $G_{\alpha_2}$  adds a term  $P(\alpha_2) - 2$  to  $L(r)$ , since we are forced to introduce at least  $P(\alpha_2) - 2$  new arcs  $I$  due to  $G_2$ . We exhaust the  $\{G_\alpha\}$  in this manner, and (2.3) follows.

(2.4) *The argument principle.* Now for a fixed  $k$ ,  $1 \leq k \leq q$ , let  $\beta(k)$  be the curve  $\bigcup_k^q \beta_j$ , so that  $\beta(k)$  is a Jordan arc on  $S^2$  which

joins  $a_k$  to  $\infty$ . Note that  $\beta(1) \supset \beta(2) \supset \dots$ . We also set  $\beta'(k) = \beta_{q+1} \cup \left\{ \bigcup_1^{k-1} \beta_j \right\}$ . Choose a fixed  $\theta$ , say  $\theta = 0$ , such that  $f(re^{i\theta}) \notin \bigcup_j \beta_j$ . Consider stopping times  $\Theta(k) : 0 < \theta_1 < \theta_2 < \dots < \theta_n < \theta_1 + 2\pi, n = n(k)$ , such that  $f(re^{i\theta_i}) \in \beta(k)$ ; we do not indicate the dependence on  $k$  of the  $\theta$ 's. This divides  $\Gamma_r$  into a union of arcs  $\Gamma_i = \Gamma_i^k (1 \leq i \leq n)$  each of which starts and ends on  $\beta(k)$ ;  $\Gamma_i^k$  is the image of  $\theta_i \leq t \leq \theta_{i+1}$ .

We partition the  $\Gamma_i^k$  into classes  $(I_k), (II_k)$  and  $(III_k)$ :

- ( $I_k$ ) those arcs which lie completely in  $B_\lambda(a_k, 2\eta)$ ,
- ( $II_k$ ) those arcs which lie completely in  $B_\lambda(\infty, 2\eta)$ ,
- ( $III_k$ ) the others.

If  $\gamma$  is any curve (not necessarily closed) which does not pass through  $a_k$  or  $\infty$ , we set

$$(2.5) \quad \nu_k(\gamma) = \frac{1}{2\pi} \Delta_\gamma \arg(w - a_k),$$

and note that the normalization (1.10) reduces (1.5) to an estimate from below of  $\sum_k \nu_k(\Gamma_r)$ . If  $G_\alpha = f^{-1}(F_\alpha)$ , one of the subregions of  $\Delta(r)$  determined by the  $\{\beta_j\}$  as at beginning of this §, we let

$$(2.6) \quad \nu_k(\partial F_\alpha) = \frac{1}{2\pi} \sum_\gamma \Delta_\gamma \arg(w - a_k),$$

where the sum in (2.6) is over the  $f$ -images  $\gamma$  of the arcs of  $\partial G_\alpha \cap B(r)$  (i.e., the *relative boundary* of  $G_\alpha$ ).

It is obvious that for curves  $\Gamma_i^k$  in classes  $(I_k)$  and  $(II_k)$  there can be no way to bound  $\nu_k(\Gamma_i^k)$  in terms of the length  $L(\Gamma_i^k)$ . However we have

(2.7) PROPOSITION. *Suppose  $f$  is such that  $w^* = \infty$  satisfies the conditions of Proposition 1.8. Then for  $1 \leq k \leq q$*

$$(2.8) \quad \left| \nu_k(\Gamma_r) - \sum_{(I_k)} \nu_k(\Gamma_i^k) \right| < hL(r).$$

Thus, the significant contributions to  $\nu_k(\Gamma_r)$  arise from curves

whose image winds about  $a_k$  and are close to  $a_k$ . We begin the proof here, and complete it in (2.13) below.

*Proof.* The critical case is when  $i \in (III_k)$ . Choose a (maximal) chain  $X$  of length  $p$ ,  $i_{\ell+1}, \dots, i_{\ell+p}$  such that each  $\Gamma_{\ell+j}^k \in (III_k)$ . Let  $\Gamma$  be the portion of  $\Gamma_r$ , which corresponds to  $X$ ; i.e., the image of  $\theta_{\ell+1} < t < \theta_{\ell+p+1}$ .

(2.9) LEMMA. *Let  $\Gamma$  be as above. Then*

$$(2.10) \quad |\nu_k(\Gamma)| \leq hL(\Gamma).$$

*Proof of (2.10).* Let  $S(k)$  be  $S^2$  with the open disks  $\Delta_\lambda(a_k, \eta)$  and  $\Delta_\lambda(\infty, \eta)$  deleted. Then for each  $k$   $S(k)$  is compact, so there exist  $\sigma > 0$  and  $M < \infty$  such that if  $\gamma$  is a continuum which meets  $\beta(k), \beta'(k)$  and intersects  $S(k)$ , then

$$(2.11) \quad \lambda(\gamma) > \sigma$$

and, for any choice of argument on  $S(k) \cap \beta(k)$ ,

$$|\sup \arg(w - a_k) - \inf \arg(w' - a_k)| < M \quad (w, w' \in S(k) \cap \beta(k)).$$

It is clear that by increasing  $M$  by at most  $4\pi$ , we have a similar bound when  $w$  and  $w'$  are in  $S(k) \cap \beta'(k)$ .

Let  $Q$  be the number of  $i$  such that a subcurve of  $\Gamma_i^k$  meets  $\beta'(k)$ . Then it follows from the definition of  $M$  that

$$|\nu_k(\Gamma)| < 2Q + 2M :$$

we think of  $\Gamma$  having an initial and terminal portion which does not meet  $\beta'(k)$ , and then  $Q$  intermediate portions which join  $\beta(k)$  to itself, passing through  $\beta'(k)$ . Similarly,  $L(\Gamma) > Q\sigma$ , so that (2.10) holds in the weaker form

$$(2.12) \quad |\nu_k(\Gamma)| \leq hL(\Gamma) + M.$$

It is possible to delete  $M$  in (2.12). If  $L(\Gamma) > \eta$ , it is obvious that  $M$  in (2.12) may be absorbed in the term  $hL(\Gamma)$ ; if  $L(\Gamma) < \eta$  and  $\Gamma$  meets  $B_\lambda(a_k, \eta)$  or  $B_\lambda(\infty, \eta)$ , then  $\Gamma$  is a curve both of whose endpoints are on  $\beta_k \cap B_\lambda(a_k, 3\eta)$  or  $\beta_k \cap B_\lambda(\infty, 3\eta)$ . In either case,

$\beta_k$  is a ray emanating from  $a_k$  or  $\infty$  in this region and so  $v_k(\Gamma) \equiv 0$ . Finally, if  $L(\Gamma) < \eta$  but  $\Gamma \cap \{B_\lambda(a_k, \eta) \cup B_\lambda(\infty, \eta)\} = \emptyset$ , we see that in this case  $\Gamma \equiv \Gamma_r$ , a closed curve, so that  $v_k(\Gamma) = 0$ . Hence (2.10) holds in all cases.

(2.13) *Completion of Proof of Proposition 2.7.* By the normalization (1.10) with Proposition 1.8, it is clear that  $\sum_{(II_k)} |\nu_k(\Gamma_i^k)| < KL(r)$ . Thus (2.8) is a consequence of this and (2.10).

(2.14) *An extension of Proposition 2.7.* By (2.8), the significant contribution to  $\nu_k(\Gamma_r)$  arises from portions of  $\Gamma_r$  which circuit  $a_k$  in a full revolution, and are contained in  $B_\lambda(a_k, 2\eta)$ . This can be made a bit sharper.

(2.15) LEMMA. *For each  $k \in \{1, \dots, q\}$ , let  $\Lambda_i^k$  be the arcs of  $\Gamma_r$  which lie in  $B_\lambda(a_k, 2\eta)$  and join  $\beta_{k-1}$  and  $\beta_k$ . Then*

$$(2.16) \quad |\nu_k(\Gamma_r) - \sum_i \nu_k(\Lambda_i^k)| < hL(r).$$

*Proof.* This follows at once from Proposition 2.7 and the observation that each arc  $\Gamma_i^k$  of that Proposition contains two arcs  $\Lambda_i^k$  (one which is mapped into  $F'$ , one into  $F''$ ) plus, perhaps, additional subarcs which start and end on one of  $\beta_{k-1}$  or  $\beta_k$ . Since the  $\beta$ 's are radial segments in  $B_\lambda(a_k, 3\eta)$ , the latter arcs contribute nothing to  $\nu_k(\Gamma_r)$  or  $\nu_k(\Gamma_i^k)$ .

(2.17) *More on the role of  $\infty$ .* We modify (2.5) and (2.6) to

$$(2.18) \quad \nu_k^*(\gamma) = \begin{cases} \nu_k(\gamma) & \text{if } \gamma \subset B_\lambda(a_k, 2\eta) \\ 0 & \text{otherwise} \end{cases}$$

and a similar interpretation for  $\nu_k^*(\partial F_\alpha)$  (see (2.6)).

Let  $w^* = \infty$  be normal in the sense of Proposition 1.8. We show that  $\infty$  is typical in a very strong sense.

(2.19) LEMMA (I). *Let  $\infty$  be normal. For each  $\alpha$ , let  $n(\alpha)$  be the number of poles on  $\partial G_\alpha$ , so that  $\partial G_\alpha$  is partitioned into  $n(\alpha)$  components  $\Gamma(\alpha, \beta)$ . Then with the exception of a set of  $B$  poles with*

$$(2.20) \quad \sum_{(\alpha, \beta) \in B} 1 < hL(r),$$

the following is true. For each  $k, 1 \leq k \leq q$ , the number of disjoint arcs  $\gamma \subset U_\beta \Gamma(\alpha, \beta)$  with

$$(2.21) \quad \nu_k^*(\gamma) = -\frac{1}{2}$$

plus the number of solutions to the equation

$$(2.22) \quad f(z) = a_k, \quad z \in U_\beta \Gamma(\alpha, \beta)$$

equals  $n(\alpha)$ .

(II) Conversely, let  $F_\alpha = f(G_\alpha)$  be given, choose  $k$  as in (I), and suppose that  $F_\alpha \not\subset B_\lambda(a_k, 3\eta)$ . Let the  $\{\Gamma(\alpha, \beta)\}$  be as in (I). Then with the exception of a set of  $B$  of  $(\alpha, \beta)$  as in (2.20), each  $\Gamma(\alpha, \beta)$  contains a pole.

(2.23) *Remark.* This lemma complements Miles [8], which covers the situation that  $F_\alpha \subset B_\lambda(a_k, 3\eta)$  for many  $\alpha$ ; then there may be many  $a_k$ -values not compensated by poles. In this situation,  $\nu_k(\gamma) > 0$  for many arcs  $\gamma \subset \Gamma_r$ , in contradistinction to (2.21).

*Proof.* Choose  $k$  as above. If the  $f$ -image of a  $\Gamma(\alpha, \beta)$  does not pass through  $a_k$ , then  $\Gamma(\alpha, \beta)$  contains an arc of  $B(r)$  whose image  $\gamma$  separates  $\infty$  from  $a_k$  in  $F_\alpha$ . Unless  $\gamma \subset B_\lambda(a, 3\eta)$ , ( $a = a_k, \infty$ ), the argument of (2.11) shows that  $\lambda(\gamma) > \sigma_1 > 0$ , independent of  $\alpha, k$ . By our normalization (1.10), the total number of poles so separated as  $\alpha, \beta, k$  vary satisfies (2.20). This proves (I).

Conversely, let  $n_1(\alpha, k), n_2(\alpha, k)$  be the number of solutions to (2.21) and (2.22) for a given  $\alpha$ , and circuit  $\partial F_\alpha$ . The arcs and  $a_k$ -values of (2.21) and (2.22) divide  $\partial F_\alpha$  into  $n(\alpha, k) = n_1(\alpha, k) + n_2(\alpha, k)$  portions  $\Gamma(\alpha, \beta, k)$ . To each  $\Gamma(\alpha, \beta, k)$  which does not pass through  $a_{q+1} = \infty$  corresponds a crosscut  $\gamma = \gamma(\alpha, \beta, k)$  which separates  $\infty$  from  $a_k$ . Since Proposition 1.8 holds, the argument of the paragraph immediately above shows that the number of such  $(\alpha, \beta)$  can be absorbed in (2.20). This completes the proof.

(2.24) *COROLLARY.* Let  $N^*$  be the number of pairs  $(\alpha, \beta)$  which satisfy the hypotheses of Part (II) of Lemma 2.19. Then, if  $\infty$  is normal in the sense of Proposition 1.8, we have

$$(2.25) \quad |N^* - 2n(\infty)| < hL(r).$$

In particular, if  $P$  is the number of poles which are taken in these  $\Gamma(\alpha, \beta)$ , then

$$(2.26) \quad P \leq 2n(r, \infty) + hL(r).$$

*Proof.* The poles of  $f$  correspond to regions  $F_\alpha$  which have  $\infty$  in their closure. Hence Lemma 2.19 applies. The first part of the corollary now follows since each pole is on the boundary of two  $G_\alpha$ 's. Estimate (2.26) is immediate.

### 3. Proof of (1.5)

Let  $n = n(\alpha)$  be the number of poles of  $f$  on  $G_\alpha$  relative to  $\Delta(r)$ . Note that

$$(3.1) \quad \sum_{\alpha} \sum_{\beta} 1 = \sum_{\alpha} n(\alpha) = 2n(r, \infty),$$

and that the contribution of the exceptional  $(\alpha, \beta)$  satisfies (2.20). Let the  $\{\Gamma(\alpha, \beta)\}$  be as in Lemma 2.19. Choose  $k \in \{1, \dots, q\}$ . If the  $f$ -image of a  $\Gamma(\alpha, \beta)$  does not pass through  $a_k$ , then there is an arc  $\gamma \subset \Gamma(\alpha, \beta) \cap B(r)$  whose  $f$ -image separates  $\infty$  from  $a_k$  in  $F_\alpha$ . Unless the image of  $\gamma$  lies in  $B_\lambda(a_k, 3\eta) \cup B_\lambda(\infty, 3\eta)$ , the argument of (2.11) shows that  $\lambda(\gamma) > \sigma_1 > 0$ , independent of  $\alpha$  or  $k$ . We now apply Lemma 2.2 to each of these  $n(\alpha)$  sets  $\Gamma(\alpha, \beta)$ . Let  $P(\alpha, \beta)$  be the number of  $k \in \{1, \dots, q\}$  such that, as in (2.15),  $\nu_k(\Lambda_i^k) < 0$  for an arc  $\Lambda_i^k$  of  $\Gamma(\alpha, \beta)$ , and  $P(\alpha) = \sum_{\beta} P(\alpha, \beta)$ . Then  $\nu_k(\Lambda_i^k) = \nu_k^*(\Lambda_i^k) = -\frac{1}{2}$ , so by (2.16), (3.1), (2.3) and (1.10) we have

$$\begin{aligned} (3.2) \quad \sum_k \nu_k(\Gamma_r) &\geq \sum_{k, \alpha} \nu_k(\Lambda_i^k) - hL(r) \\ &= -\frac{1}{2} \sum_{\alpha} \sum_{\beta} P(\alpha, \beta) - hL(r) \\ &= -\sum_{\alpha} n(\alpha) - \frac{1}{2} \sum_{\alpha} \sum_{\beta} \{P(\alpha, \beta) - 2\} - hL(r) \\ &\geq -\sum_{\alpha} n(\alpha) - \frac{1}{2} \sum_{\alpha} \sum_{\beta} \{P(\alpha, \beta) - 2\}^+ - hL(r) \\ &= -2n(r, \infty) - hL(r) \\ &\geq -2S(r) - hL(r). \end{aligned}$$

By the argument principle, this is (1.5).

**4. Proof of (1.6).** Recall the discussion of Euler characteristic in, say, [5, pp. 135–7], [9, pp. 322–3]. We surround each  $a_k$  ( $1 \leq k \leq q$ ) by a small disk  $D_k$ , such that  $\lambda(\zeta, a_k) \sim \eta$  for  $\zeta \in \partial D_k$ , and let  $F_0 = S^2 \setminus \cup D_k$ . Thus, there are now  $q$  crosscuts  $\beta_i$ ; what is now  $\beta_q$  consists of what in § 2 had been a connected piece of  $\beta_q$  and  $\beta_{q+1}$  which passes through  $\infty$ . We estimate  $\rho(\mathfrak{F})$  by the standard combinatorial inequality [5, p. 137], [9, p. 333]

$$(4.1) \quad \rho(\mathfrak{F}) \geq n - N$$

where  $N$  is the total number of domains  $G_\alpha = f^{-1}(F_\alpha)$  and  $n$  is the number of crosscuts  $\cup_j f^{-1}(\beta_j)$ .

(4.2) *Remark.* Each crosscut  $\gamma$  bounds two domains  $\{G_\alpha\}$ ; we will use the fact, needed for (4.1), that crosscuts  $\gamma$  which disconnect  $\mathfrak{F}$  make no net change to either side of (4.1)

Consider the arcs  $\Gamma(\alpha, \beta)$  of Lemma 2.19. In the context here,  $\Gamma(\alpha, \beta) \subset \partial G_\alpha$  and we write  $F_\alpha = f(G_\alpha)$  (so that  $F_\alpha \subset F$ , with  $F = F'$  or  $F''$ , where  $F'$  and  $F''$  are now bounded by the  $\{\beta_k\}$  and portions of the  $\cup_1^q \partial D_k$ ).

If  $\Gamma \in \Gamma(\alpha, \beta)$ , let  $\Gamma^* = \Gamma \cap \Gamma_r$  be the portion of  $\Gamma$  in the relative boundary of  $\mathfrak{F}$ ; this convention of starring will be used below. As in §2, choose  $\eta > 0$  such that, if  $j \neq k$ , then  $\lambda(\beta_j, \beta_k) > 100\eta$  (distance relative to  $\mathfrak{F}$ ). The number of pairs  $(\alpha, \beta)$  with  $\lambda(\Gamma^*(\alpha, \beta)) > \eta$  is at most  $hL(r)$ . We place these exceptional pairs  $(\alpha, \beta)$  into class (I); by Lemma 1.8 and the normalization (1.10) we may also include in (I) all  $\Gamma(\alpha, \beta)$  such that an endpoint of some  $\Gamma^*(\alpha, \beta)$  lies in an  $\eta$ -neighborhood of  $\infty$ . Thus  $(\#\{E\})$  is the cardinality of  $E$

$$(4.3) \quad \#\{(\alpha, \beta) \in (I)\} < hL(r).$$

We now introduce a significant set  $\mathcal{G}$  of pairs  $\{\alpha, \beta\}$ , which are not in (I). Let  $\partial_0 \mathfrak{F}$  be the *outer boundary* of  $\mathfrak{F}$ , i.e., the component of  $\partial \mathfrak{F}$  which intersects  $\Gamma_r$ ; thus  $\partial_0 \mathfrak{F}$  is connected and consists of part of  $\Gamma_r$  and perhaps arcs which are mapped to  $\partial D_k$  for various  $k$ . If  $(\alpha, \beta) \notin (I)$ , let (cf. Lemma 2.2)  $p = p(\alpha, \beta)$  be a maximum choice of

points  $\{\zeta\}$  on  $\partial_0 \mathfrak{F} \cap \Gamma^*(\alpha, \beta)$  such that  $\lambda(\zeta, \zeta') \geq 3\eta$ . By Proposition 1.8 and the definition of  $(I)$ , we may assume that none of the  $\zeta$  are in  $B_\lambda(\infty, 3\eta)$ . The class  $\mathcal{G}$  will consist of all  $(\alpha, \beta)$  for which  $p \neq 1$  and a certain subset of the  $(\alpha, \beta)$  for which  $p = 1$ .

If  $p = 0$  then  $\Gamma^*(\alpha, \beta) = \phi$  so that we have  $q$  crosscuts of  $\mathfrak{F}$  corresponding to this  $(\alpha, \beta)$ , and none of these crosscuts disconnect  $\mathfrak{F}$ .

Now let  $(\alpha, \beta) \notin (I)$  with  $p = 1$ , corresponding to a choice  $\zeta = \zeta_1$ . By hypothesis,  $\Gamma^*(\alpha, \beta)$  contains a cross-cut  $\gamma^*$  of  $F_\alpha$  which is contained in an  $\eta$ -neighborhood  $\Omega$  of  $\zeta_1$ . There are two possibilities. Since  $F_\alpha$  is connected, it is easy to see that either  $F_\alpha \subset \Omega$  or  $F_\alpha \supset \{F \setminus \Omega\}$ . Since  $(\alpha, \beta) \notin (I)$ , there is some  $k$  such that the two endpoints of each component  $\gamma^*$  of  $\Gamma(\alpha, \beta)$  are contained in  $\beta_k, D_k$  or  $D_{k+1}$ . Hence, when  $F_\alpha \subset \Omega$ , we see from (4.2) that all crosscuts 'over'  $\cup \beta_k$  can be ignored in computing (4.1), since each disconnects  $\mathfrak{F}$ . This is important since there can be no upper bound for the number of such components  $\Gamma(\alpha, \beta)$ . The remaining pairs  $(\alpha, \beta)$  for which  $p = 1$  are assigned to  $\mathcal{G}$ . In this case,  $F_\alpha \supset F \setminus \Omega$ , and since  $p = 1$ , it follows that  $\Gamma(\alpha, \beta)$  will contain  $q$  crosscuts of  $\mathfrak{F}$  which terminate at each  $D_k$ , and so there are at least  $q - 1$  crosscuts which do not separate  $\mathfrak{F}$ , since at least  $q - 1$  cannot meet  $\Gamma^*(\alpha, \beta)$ .

Finally, if  $p(\alpha, \beta) = p \geq 2$  and  $(\alpha, \beta) \notin (I)$ , we see that  $F_\alpha \supset F \setminus \cup_1^p \{B_\lambda(\zeta_i, 3\eta)\}$ . In this situation, there are again  $q$  crosscuts from  $\Gamma(\alpha, \beta)$ , but we are assured only that  $q - p$  do not disconnect  $\mathfrak{F}$ . However, by Lemma 2.2,

$$(4.4) \quad \sum_{\{(\alpha, \beta)\} \subset \mathcal{G}} \{p(\alpha, \beta) - 2\}^+ < hL(r).$$

If  $\mathcal{G}$  is as defined above, it follows from Proposition (1.8) and (1.10) that (3.1) holds. Let  $n_{\mathcal{G}}$  and  $N_{\mathcal{G}}$  be the contribution to  $n$  and  $N$  in (4.1) which arise from  $\{(\alpha, \beta) \in \mathcal{G}\}$ . Since  $\infty$  lies on each  $\Gamma(\alpha, \beta)$  if  $(\alpha, \beta) \in \mathcal{G}$ , we deduce from our definition of  $(I)$ , (4.3), (3.1) and (1.8) that

$$\begin{aligned}
n - N &\geq n_g - N_g - hL(r) \\
&\geq \frac{1}{2}[q^\#\{\mathcal{G} \cap \{p = 0\}\} + (q - 1)^\#\{\mathcal{G} \cap \{p = 1\}\}] \\
&\quad + \sum_{m \geq 2} (q - m)^\#\{\mathcal{G} \cap \{p = m\}\} - hL(r) \\
&\geq \sum_{q \leq 2} (q - p) + \sum_{p \geq 2} (q - 2) - \sum_g (p - 2)^+ - hL(r) \\
&\geq \frac{1}{2}(q - 2)^\#\mathcal{G} - \sum_g (p - 2)^+ - hL(r) \\
&\geq (q - 2)n(r, \infty) - hL(r) \\
&= (q - 2)S(r) - hL(r).
\end{aligned}$$

Since  $\rho_0 = q - 2$ , we have proved (1.6).

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