HIGH POINTS IN THE HISTORY OF VALUE DISTRIBUTION THEORY OF SEVERAL COMPLEX VARIABLES

Wilhelm Stoll

Inaugural Lecture

Timothy O'Meara and Frank Castellino thank your for your kind introduction. I am deeply moved by your words and by the appointment to the chair. Foremost I thank the donors Vincent J. Duncan and Annamarie Micus Duncan for their generosity. My colleagues and I are most grateful for this recognition of our work by the donors and the administration of the University.

Ladies and gentlemen, colleagues, speakers and participants! This inaugural address opens the *Symposium on Value Distribution Theory in Several Complex Variables* sponsored by the University of Notre Dame. Welcome to all of you. An inaugural address, an Antrittsvorlesung, so late in life seems to be out of place and perhaps should be called an Abschiedsvorlesung. Yet, hopefully, this is premature and I can be around a few more years. Taking the hint, I will look backwards and recall some of the high points in the development of the theory. Time permits only a few topics.

Looking backwards, out of the mist of time there emerges not an abstract theory but the lively memory of those who taught me mathematics: Siegfried Kerridge, Wilhelm Germann, Wilhelm Schweizer and later at the University Hellmuth Kneser, Konrad Knopp, Erich Kamke, G. G. Lorentz and Max Müller. Also there appear those who inspired me but who were not directly my teachers: Heinz Hopf, Hermann Weyl, Rolf Nevanlinna and one who is right here with us: Shiing-shen Chern, we all welcome you. Thirty years ago you recruited me for Notre Dame. You supported the growth of this department in many ways. Your work on value distribution in several

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complex variables counts as one of your many marvelous contributions to mathematics. Thank you for coming.

The giants of the 19th century created the theory of entire functions. In this century, in 1925, with a stroke of genius, Rolf Nevanlinna extended this theory to a value distribution theory of meromorphic functions. His two *Main Theorems* are the foundation upon which Nevanlinna theory rests.

In 1933, Henri Cartan [8] proved Nevanlinna's Second Main Theorem for the case of holomorphic curves. If we view curves belonging to the theory of several dependent variables, then Cartan's paper provides the first theorem in the theory of value distribution in several complex variables. Thus let me outline his result. However, I shall use today's terminology and advancement.

For each $0 < r \in \mathbb{R}$ define the discs and circle

$$(1) \qquad \mathbb{C}[r] = \{z \in \mathbb{C} \mid |z| \leq r\} \quad \mathbb{C}(r) = \{z \in \mathbb{C} \mid |z| < r\}$$

$$(2) \qquad \mathbb{C}\langle r\rangle = \{z \in \mathbb{C} \mid |z| = r\} \qquad \mathbb{C}_* = \mathbb{C} - \{0\}.$$

An integral valued function $\nu : \mathbb{C} \to \mathbb{Z}$ is said to be a *divisor* if

$$(3) \qquad S = \operatorname{supp}\nu = \operatorname{clos}\{z \in \mathbb{C} \mid \nu(z) \neq 0\}$$

is a closed set of isolated points in $\mathbb{C}$. For all $r \geq 0$ the *counting function* n_ν of ν is defined by the finite sum

$$(4) \qquad n_\nu(r) = \sum_{z \in \mathbb{C}[r]} \nu(z).$$

For $0 < s < r \in \mathbb{R}$, the *valence function* N_ν of ν is defined by

$$(5) \qquad N_\nu(r, s) = \int_s^r n_\nu(t) \frac{dt}{t}.$$

If $h \not\equiv 0$ is an entire function, let $\mu_h(z)$ be the *zero-multiplicity* of h at z. Then $\mu_h : \mathbb{C} \to \mathbb{Z}$ is a non-negative divisor called the *zero divisor* of h.

The exterior derivative $d = \partial + \bar{\partial}$ on differential forms twists to

(6) $$d^c = \frac{i}{4\pi}(\bar{\partial} - \partial)$$

on complex manifolds. Define $\tau_o : \mathbb{C} \to \mathbb{R}$ by $\tau_o(z) = |z|^2$ for $z \in \mathbb{C}$. Define

(7) $$\sigma = d^c \log \tau_0,$$

If $r > 0$, then

(8) $$\int_{\mathbb{C}<r>} \sigma = 1.$$

If $h \not\equiv 0$ is an entire function and if $r > o$, the **Jensen Formula**

(9) $$N_{\mu_h}(r, s) = \int_{\mathbb{C}<r>} \log h\sigma - \int_{\mathbb{C}<s>} \log h\sigma$$

is a forerunner of Nevanlinna's First Main Theorem.

Let V be a normed, complex vector space of finite dimension $n + 1 > 1$. Put $V_* = V - \{0\}$. Then the multiplicative group $\mathbb{C}_*$ acts on V_*. The quotient space $\mathbb{P}(V) = V_*/\mathbb{C}_*$ is the associated projective space. The quotient map $\mathbb{P} : V_* \to \mathbb{P}(V)$ is open and holomorphic. If $M \subseteq V$, put $\mathbb{P}(M) = P(M \cap V_*)$. If W is a linear subspace of V with dimension $p + 1$, then $\mathbb{P}(W)$ is called a p-*plane* of $\mathbb{P}(V)$. If $p = n - 1$, then $\mathbb{P}(W)$ is called a *hyperplane*. The *dual* complex vector space V^* of V consists of all $\mathbb{C}$-linear functions $\mathfrak{a} : V \to \mathbb{C}$. Here $\|\mathfrak{a}\|$ is the smallest real number such that $|\mathfrak{a}(\mathfrak{x})| \leq \|\mathfrak{a}\| \, \|\mathfrak{x}\|$ for all $\mathfrak{x} \in V$. Then $\| \; \|$ is a norm on V^*. Also write $<\mathfrak{x}, \mathfrak{a}> = \mathfrak{a}(\mathfrak{x})$. Here $<\mathfrak{x}, \mathfrak{a}> = <\mathfrak{a}, \mathfrak{x}>$ indicates $(V^*)^* = V$. If $a = \mathbb{P}(\mathfrak{a}) \in \mathbb{P}(V^*)$, then $E[a] = \mathbb{P}(\ker \mathfrak{a})$ is a hyperplane in $\mathbb{P}(V)$. The assignment $a \to E[a]$ parameterizes the set of hyperplanes bijectively. The *distance* from $x = \mathbb{P}(\mathfrak{x}) \in \mathbb{P}(V)$ to $E[a]$ is measured by

(10) $$0 \leq \square x, a \square = \frac{|<\mathfrak{x}, \mathfrak{a}>|}{\|\mathfrak{x}\| \, \|\mathfrak{a}\|} \leq 1.$$

Let $f : \mathbb{C} \to \mathbb{P}(V)$ be a holomorphic map. A holomorphic map $\mathfrak{v} : \mathbb{C} \to V_*$ is called a *reduced representation* of f if $\mathbb{P} \circ \mathfrak{v} = f$. A reduced representation exists. Then $\mathfrak{w} : \mathbb{C} \to V_*$ is a reduced representation of f if and only if there is a holomorphic function $h : \mathbb{C} \to \mathbb{C}_*$ without zeroes such that $\mathfrak{w} = h\mathfrak{v}$. For $0 < s < r \in \mathbb{R}$ the *characteristic function* of f is defined by

$$(11) \qquad T_f(r,s) = \int_{\mathbb{C}\langle r\rangle} \log \|\mathfrak{v}\| \sigma - \int_{\mathbb{C}\langle s\rangle} \log \|\mathfrak{v}\| \sigma.$$

By (9), $T_f(r,s)$ does not depend on the choice of $\mathfrak{v}$, Since $\log \|\mathfrak{v}\|$ is subharmonic, $T_f \geq 0$. If f is constant, $\mathfrak{v}$ can be taken as a constant. Hence $T_f(r,s) \equiv 0$. If f is not constant, then $T_f(r,s) > 0$ and $T_f(r,s) \to \infty$ for $r \to \infty$. If $\|\ \|$ and $|||\ |||$ are two norms on V, there are constants $C_2 \geq C_1 > 0$ such that $C_1|||\mathfrak{x}||| \leq \|\mathfrak{x}\| \leq C_2|||\mathfrak{x}|||$ for all $\mathfrak{x} \in V$. Put $C = \log C_2/C_1 \geq 0$. If $0 < s < r$, then

$$(12) \qquad |T_f(r,s,\|\ \|) - T_f(r,s,|||\ |||)| \leq C.$$

Let $f : \mathbb{C} \to \mathbb{P}(V)$ and $g : \mathbb{C} \to \mathbb{P}(V^*)$ be holomorphic maps. They are called *free* if $f(z) \notin E[g(z)]$ for some $z \in \mathbb{C}$. Take reduced representations $\mathfrak{v}$ of f and $\mathfrak{w}$ of g, then (f,g) is free if and only if $<\mathfrak{v}, \mathfrak{w}> = h \not\equiv 0$. If so, the *intersection divisor* $\mu_{f,g} = \mu_h \geq 0$ does not depend on the choices of $\mathfrak{v}$ and $\mathfrak{w}$. Its *counting function* and its *valence function* are abbreviated by $n_{f,g}$ and $N_{f,g}$ respectively. The pair (f,g) is free if and only if $\square f, g, \square \not\equiv 0$. If so, for $r > 0$ the *compensation function* $m_{f,g}$ of (f,g) is defined by

$$(13) \qquad m_{f,g}(r) = \int_{\mathbb{C}\langle r\rangle} \log \frac{1}{\square f, g \square} \sigma \geq 0.$$

For $0 < s < r$, the identities (9), (11) and (13) imply the **First Main Theorem**

$$(14) \qquad T_f(r,s) + T_g(r,s) = N_{f,g}(r,s) + m_{f,g}(r) - m_{f,g}(s).$$

Cartan [8] considered the case of constant $g \equiv a \in \mathbb{P}(V^*)$ only which yields

$$T_f(r,s) = N_{f,a}(r,s) + m_{f,a}(r) - m_{f,a}(s) \tag{15}$$

which Cartan [8] mentions only implicitely. If $n = 1$, Rolf Nevanlinna proved (15) in [32] (1925).

If f or g or both are not constant and if (f, g) is free the *defect* is defined by

$$\begin{aligned} 0 \leq \delta(f,g) &= \varliminf_{r\to\infty} \frac{m_{f,g}(r)}{T_f(r,s) + T_g(r,s)} \\ &= 1 - \varlimsup_{r\to\infty} \frac{N_{f,g}(r,s)}{T_f(r,s) + T_g(r,s)} \leq 1 \end{aligned} \tag{16}$$

The map g is said to *grow slower* than f, if $T_g(r,s)/T_f(r,s) \to 0$ for $r \to \infty$. By (12), the defect does not depend on the choice of the norm on V. Also the defect is independent of s. Observe that $\mu_{f,g} = \mu_{g,f}, n_{f,g} = n_{g,f}, N_{f,g} = N_{g,f}, m_{f,g} = m_{g,f}$ and $\delta(f,g) = \delta(g,f)$. Since most investigators concentrate on constant g or on the case where g grows slower than f, this symmetry is little known.

Since the choice of the norm on V does not matter, we can choose a hermitian norm which comes from a positive definite hermitian form $(\cdot|\cdot) : V \times V \to \mathbb{C}$ with $\|\mathfrak{x}\|^2 = (\mathfrak{x}|\mathfrak{x})$ for $\mathfrak{x} \in V$. Define $\tau : V \to \mathbb{C}$ by $\tau(\mathfrak{x}) = \|\mathfrak{x}\|^2$ for $\mathfrak{x} \in V$. Then τ is of class C^∞. There is one and only one positive form Ω of bidegree(1,1) on $\mathbb{P}(V)$, called the *Fubini Study form* such that $dd^c \log \tau = \mathbb{P}^*(\Omega)$ on V_*. Let $\mathfrak{v} : \mathbb{C} \to V_*$ be a reduced representation of f. Then $f = \mathbb{P} \circ \mathfrak{v}$ implies

$$dd^c \log \|\mathfrak{v}\|^2 = \mathfrak{v}^*(\mathbb{P}^*(\Omega)) = f^*(\Omega). \tag{17}$$

If Stokes theorem and fiber integration are applied to (11) we obtain the *Ahlfors-Shimizu* definition of the *characteristic function* of f

$$T_f(r,s) = \int_s^r \int_{\mathbb{C}[t]} f^*(\Omega) \frac{dt}{t} \quad \text{for } 0 < s < r. \tag{18}$$

Here $A_f(t) = \int_{\mathbb{C}[t]} f^*(\Omega) \geq 0$ increases. Put $A_f(\infty) = \lim_{t\to\infty} A_f(t) \leq \infty$. Then

$$(19) \qquad \lim_{r\to\infty} \frac{T_f(r,s)}{\log r} = A_f(\infty).$$

Now f is constant if and only if $A_f(\infty) = 0$ and f is rational if and only if $A_f(\infty) < \infty$.

Let $\mathfrak{A} = \{a_j\}_{j\in Q}$ be a family of points $a_j \in \mathbb{P}(V^*)$ representating hyperplanes. If $P \subseteq Q$, define $\mathfrak{A}_P = \{a_j\}_{j\in P}$. For each $j \in Q$ pick $\mathfrak{a}_j \in V_*^*$ with $a_j = \mathbb{P}(\mathfrak{a}_j)$. Our definitions will not depend on the choice of $\mathfrak{a}_j$. Put $q = \#Q$. Then $\mathfrak{A}$ is said to be *linearly independent* if there is a bijective map $\lambda : \mathbb{N}[1,q] \to Q$ such that $\mathfrak{a}_{\lambda(1)}, \dots, \mathfrak{a}_{\lambda(q)}$ are linearly independent. If so, then $q \leq n+1$. Moreover $\mathfrak{A}$ is said to be *basic* if $\mathfrak{A}$ is linearly independent and $q = n+1$. Moreover $\mathfrak{A}$ is said to be in *general position* if $\mathfrak{A}_P$ is linearly independent for each $P \subseteq Q$ with $0 < \#P \leq n+1$. If N is an integer and if $q > N \geq n$, then $\mathfrak{A}$ is said to be in *N-subgeneral position* (Chen [9]) if for every subset S of Q with $\#S = N+1$, there is a subset P of S such that $\mathfrak{A}_P$ is basic.

Let $f : \mathbb{C} \to \mathbb{P}(V)$ be a holomorphic map. Then there is a unique linear subspace W of smallest dimension $k+1$ of V such that $f(\mathbb{C}) \subseteq \mathbb{P}(W)$. Then f is said to be *k-flat.* If $k = n$, then $W = V$ and f is said to be *linearly non-degenerated.*

Take $0 \leq s \in \mathbb{R}$. Let $G : \mathbb{R}[s, +\infty) \to \mathbb{R}$ and $H : \mathbb{R}[s, +\infty)$ be functions. Then $G \mathrel{\underset{\cdot\cdot}{\leq}} H$ means that there is a subset E of finite measure of $\mathbb{R}_+ = \mathbb{R}[0, +\infty)$ such that $G(r) \leq H(r)$ for all $r \in \mathbb{R}[s, +\infty) - E$.

Second Main Theorem (Cartan [8] 1933)

Let V be a hermitian vector space of dimension $n+1 > 1$. Let $f : \mathbb{C} \to \mathbb{P}(V)$ be a linearly non-degenerated, holomorphic map. Let $\mathfrak{A} = \{a_j\}_{j\in Q}$ be a finite family of "hyperplanes" $a_j \in \mathbb{P}(V^*)$ in general position with $n+1 < q = \#Q < \infty$. Take $s > 0$ and $\varepsilon > 0$. Then

$$(20) \quad \sum_{j\in Q} m_{f,a_j}(r) \mathrel{\underset{\cdot\cdot}{\leq}} (n+1)T_f(r,s) + \frac{1}{2}n(n+1)(1+\varepsilon)\log T_f(r,s) + \varepsilon \log r.$$

As a consequence, we obtain trivially

Defect Relation (Cartan [8] 1933)

Under the assumptions of the Second Main Theorem we have

$$\sum_{j\in Q}\delta(f,a_j)\leq n+1. \tag{21}$$

If $f:\mathbb{C}\to\mathbb{P}(V)$ is only k-flat, and if $\mathfrak{A}$ is in general position such that (f,a_j) is free for each $j\in Q$, Henri Cartan conjectured in 1933 that

$$\sum_{j\in Q}\delta(f,a_j)\leq 2n-k+1, \tag{22}$$

which was proven by Nochka [35] in 1982. Thus if $\#Q\geq 2n+1$ and $f(\mathbb{C})\cap E[a_j]=0$ for all $j\in Q$, then $2n+1\leq 2n-k+1$. Therefore $k=0$ and f is constant. Hence

$$\mathbb{P}(V)-\bigcup_{j\in Q}E[a_j] \tag{23}$$

is Brody-hyperbolic. In fact by a theorem of Chen [9] (22) can be improved:

Defect Relation of Cartan-Nochka-Chen

Let V be a hermitian vector space of dimension $n+1>1$. Let $f:\mathbb{C}\to\mathbb{P}(V)$ be a k-flat, holomorphic map. Let $\mathfrak{A}=\{a_j\}_{j\in Q}$ be a finite family of "hyperplanes" $a_j\in\mathbb{P}(V^*)$ in N-subgeneral position with $N\geq n$ and $N+1\leq\#Q=q<\infty$. Assume that (f,a_j) is free for each $j\in Q$. Then

$$\sum_{j\in Q}\delta(f,a_j)\leq 2N-k+1. \tag{24}$$

An alternative proof of the defect relation (21) was given by Ahlfors [1] in 1941. Also he proves a defect relation for associated maps. His proof is very powerful and works in more general situations.

Hermann and Joachim Weyl [90] lifted Ahlfors's proof to Riemann surfaces. It was simplified by H. Wu [92] in 1970, Cowen and Griffiths [17] in 1976 and Pit-Mann Wong [93] in 1976. I extended this Ahlfors-Weyl theory to non-compact Kaehler manifolds [65]. However first we have to inquire how value distribution was extended to functions and maps of several independent complex variables.

Hellmuth Kneser created such an extension in two fundamental papers [23] in 1936 and [24] in 1938. Although these papers are little remembered today, they still influence the present research in value distribution of several independent complex variables. Therefore let me explain his fundamental ideas. Again I will cast them in modern terminology and perspective.

Let M be a connected, complex manifold of dimension m. Let $f \not\equiv 0$ be a holomorphic function on M. Take $p \in M$. Let $\alpha : U' \to U$ be a biholomorphic map of an open ball U' in $\mathbb{C}^m$ centered at 0 onto an open subset U of M with $\alpha(0) = p$. Then for each integer $\lambda \geq 0$ there is a unique homogeneous polynomial P_λ of degree λ such that

$$f \circ \alpha = \sum_{\lambda=0}^{\infty} P_\lambda \tag{25}$$

where the convergence is uniform on every compact subset of U'. Since $f|U \not\equiv 0$, there is a unique number $\mu = \mu_f(p) \geq 0$ depending on f and p only such that $P_\mu \not\equiv 0$ and $P_\lambda \equiv 0$ for all $\lambda \in \mathbb{Z}$ with $0 \leq \lambda < \mu$. The number $\mu_f(p)$ is called the *zero-multiplicity* of f at p and the function $\mu_f : M \to \mathbb{Z}$ is called the *zero-divisor* of f.

An integral valued function $\nu : M \to \mathbb{Z}$ is said to be a *divisor* on M if and only if for every point $p \in M$ there is an open, connected neighborhood U of p with holomorphic functions $g \not\equiv 0$ and $h \not\equiv 0$ on U such that

$$\nu|U = \mu_g - \mu_h. \tag{26}$$

Let S be the support of ν. Then $S = \emptyset$ if and only if $\nu \equiv 0$. If $S \neq \emptyset$, then S is a pure (m−1)-dimensional analytic subset of M. Let $\mathfrak{R}(S)$ be the set of *regular points* of S and let $\sum(S) = S - \mathfrak{R}(S)$ be the set of *singular points* of S. Then $\nu|\mathfrak{R}(S)$ is locally constant.

Let $\tau : M \to \mathbb{R}_+$ be an unbounded, non-negative function of class C^∞ on M. If $B \subseteq M$ and $0 \le r \in \mathbb{R}$, abbreviate

$$(27)\qquad B[r] = \{x \in B | \tau(x) \le r^2\} \quad B(r) = \{x \in B | \tau(x) < r^2\}$$
$$(28)\qquad B\langle r\rangle = \{x \in B | \tau(x) = r^2\} \quad B_* = \{x \in B | \tau(x) > 0\}$$

Here τ is called an *exhaustion* of M if and only if $M[r]$ is compact for each $r > 0$. Abbreviate

$$(29)\qquad \upsilon = dd^c\tau \quad \omega = dd^c \log \tau \quad \sigma = d^c \log \tau \wedge \omega^{m-1}$$

Then $d\sigma = \omega^m$. The function τ is said to be *parabolic* if and only if

$$(30)\qquad \omega \ge 0 \quad \omega^m \equiv 0 \quad \upsilon^m \not\equiv 0.$$

If so, then $\upsilon \ge 0$. More over τ is said to be *strictly parabolic* if and only if τ is parabolic and $\upsilon > 0$ on M. If τ is an exhaustion and parabolic, then (M, τ) is said to be a *parabolic manifold*. If so, there is a constant $\varsigma > 0$ such that

$$(31)\qquad \int_{M[r]} \upsilon^m = \varsigma r^{2m}$$

for all $r \ge 0$. Then for almost all $r > 0$ we have

$$(32)\qquad \int_{M\langle r\rangle} \sigma = \varsigma.$$

In 1973 Griffiths and King [19] introduced parabolic manifolds. The concept was expanded in [75]. If τ is an exhaustion and strictly parabolic function, (M, τ) is said to be *a strictly parabolic manifold.* In [77] 1980 I showed that (M, τ) is strictly parabolic if and only if there is a hermitian vector space W of dimension m and a biholomorphic map $h : M \to W$ such that $\tau = \|h\|^2$. We assume that (M, τ) is strictly parabolic and we identify $M = W$ such that h becomes the identity. In this case $\varsigma = 1$ and M is a hermitian vector space, which was Kneser's starting point. We assume that $m > 1$. If $u : M\langle 1\rangle \to \mathbb{C}$ is a function such that $u\sigma$ is integrable over the unit sphere $M\langle 1\rangle$, the *mean value* of u is defined by

$$\mathfrak{M}(u) = \int_{M\langle 1\rangle} u\sigma \tag{33}$$

Let V be a hermitian vector space of dimension $n + 1 > 1$. Let $f : M \to \mathbb{P}(V)$ and $g : M \to \mathbb{P}(V^*)$ be meromorphic maps. Let I_f and I_g be the indeterminacies of f and g respectively. Then (f, g) is said to be *free* if there exists $z \in M - I_f \cup I_g$ such that $f(z) \notin E[g(z)]$. For each "unit" vector $\mathfrak{b} \in M\langle 1\rangle$ an isometric embedding $j_\mathfrak{b} : \mathbb{C} \to M$ is defined by $j_\mathfrak{b}(z) = z\mathfrak{b}$ for $z \in \mathbb{C}$. If $j_\mathfrak{b}(\mathbb{C}) \not\subseteq I_f \cup I_g$ the pull back holomorphic maps $f_\mathfrak{b} = j_\mathfrak{b}^*(f) : \mathbb{C} \to \mathbb{P}(V)$ and $g_\mathfrak{b} = j_\mathfrak{b}^*(g) : \mathbb{C} \to \mathbb{P}(V^*)$ exist and $(f_\mathfrak{b}, g_\mathfrak{b})$ is free for almost all $\mathfrak{b} \in M\langle 1\rangle$. If $0 < s < r$ the **First Main Theorem** holds

$$T_{f_\mathfrak{b}}(r, s) + T_{g_\mathfrak{b}}(r, s) = N_{f_\mathfrak{b}, g_\mathfrak{b}}(r, s) + m_{f_\mathfrak{b}, g_\mathfrak{b}}(r) - m_{f_\mathfrak{b}, g_\mathfrak{b}}(s) \tag{34}$$

Now Kneser [24] applied the operator $\mathfrak{M}$ termwise in (34) to obtain the respective value distribution functions and the **First Main Theorem**

$$T_f(r, s) + T_g(r, s) = N_{f,g}(r, s) + m_{f,g}(r) - m_{f,g}(s) \tag{35}$$

Of course Kneser considered the case $n = 1$ only. Then f is a meromorphic function. Also he assumed that $g \equiv a \in \mathbb{P}_1$ is constant. Had he stopped with the above derivation of (35), his result would have been worthless. He proceeded and expressed the value distribution functions in meaningful analytic and geometric terms. This made the paper successful.

Let Ω be the Fubini Study form on $\mathbb{P}(V)$. For $t > 0$ define A_f by

$$A_f(t) = \frac{1}{t^{2m-2}} \int_{M[t]} f^*(\Omega) \wedge v^{m-1} \geq 0. \tag{36}$$

He showed that A_f increases. Hence the limits

$$0 \leq \lim_{t \to 0} A_f(t) = A_f(0) < \infty \tag{37}$$
$$0 \leq \lim_{t \to \infty} A_f(t) = A_f(\infty) \leq \infty$$

exist. Kneser obtained the identity

$$A_f(t) = \int_{M[t]} f^*(\Omega) \wedge \omega^{m-1} + A_f(0) \tag{38}$$

for $t > 0$. Here f is constant if and only if $A_f(\infty) = 0$ and f is rational if and only if $A_f(\infty) < \infty$. Kneser proved

$$T_f(r, s) = \int_s^r A_f(t) \frac{dt}{t} \tag{39}$$

for $0 < s < r$. Moreover we have

$$\lim_{r \to \infty} \frac{T_f(r, s)}{\log r} = A_f(\infty). \tag{40}$$

A holomorphic map $\mathfrak{v} : M \to V$ is said to be a *reduced representation* of f if and only if $\dim \mathfrak{v}^{-1}(0) \leq m - 2$ and $f(z) = \mathbb{P}(\mathfrak{v}(z))$ for all $z \in M - I_f$ with $\mathfrak{v}(z) \neq 0$. In fact $I_f = \mathfrak{v}^{-1}(0)$. Reduced representations exist since M is a vector space. If $\mathfrak{v}$ is a reduced representation of f, any other reduced representation is given by $h\mathfrak{v}$, where $h : M \to \mathbb{C}_*$ is an entire function without zeroes. If $0 < s < r$, then

$$T_f(r, s) = \int_{M\langle r \rangle} \log \|\mathfrak{v}\| \sigma - \int_{M\langle s \rangle} \log \|\mathfrak{v}\| \sigma. \tag{41}$$

Since (f, g) is free, $\square f, g \square \not\equiv 0$, and for $r > 0$ the *compensation function* $m_{f,g}$ of f, g is defined

$$m_{f,g}(r) = \int_{M\langle r \rangle} \log \frac{1}{\square f, g \square} \sigma \geq 0. \tag{42}$$

Let $\nu : M \to \mathbb{Z}$ be a divisor with support S. Fot $t > 0$ the *counting function* n_ν of ν is defined by

(43) $$n_\nu(t) = \frac{1}{t^{2m-2}} \int\limits_{S[t]} \nu \upsilon^{m-1} = \int\limits_{S[t]} \nu \omega^{m-1} + n_\nu(0),$$

where the limit $n_\nu(0) = \lim_{t\to 0} n_\nu(t)$ exists. Actually since M is a vector space, $n_\nu(0) = \nu(0)$ (see Stoll [62]). For each $\mathfrak{b} \in M<1>$ with $j_\mathfrak{b}(\mathbb{C}) \not\subseteq S$, the pullback divisor $\nu_\mathfrak{b} = j_\mathfrak{b}^*(\nu)$ exists. If $t > 0$ then

(44) $$n_\nu(t) = \int\limits_{\mathfrak{b}\in M<1>} n_{\nu_\mathfrak{b}}(t)\sigma(\mathfrak{b}).$$

Thus for $0 < s < r$ the *valence function* n_ν of ν is given by

(45) $$N_\nu(r,s) = \int\limits_{\mathfrak{b}\in M<1>} N_{\nu_\mathfrak{b}}(r,s)\sigma(\mathfrak{b}) = \int\limits_s^r n_\nu(t)\frac{dt}{t}.$$

Take reduced resprentations $\mathfrak{v} : M \to \mathbb{P}(V)$ of f and $\mathfrak{w} : M \to \mathbb{P}(V^*)$ of g. Since (f,g) is free, $h = <\mathfrak{v},\mathfrak{w}> \not\equiv 0$. Then $\mu_{f,g} = \mu_h$ depends on f and g only. Put $S = h^{-1}(0)$. If $\mathfrak{b} \in M<1>$ with $j_\mathfrak{b}(\mathbb{C}) \not\subseteq S$, then $\mu_{f_\mathfrak{b},g_\mathfrak{b}} = j_\mathfrak{b}^*(\mu_{f,g})$. Hence

(46) $$N_{f,g}(r,s) = \int\limits_{\mathfrak{b}\in M<1>} N_{f_\mathfrak{b},g_\mathfrak{b}}(r,s)\sigma(\mathfrak{b}) = N_{\mu_{f,g}}(r,s).$$

Thus each term in (35) is explicitely expressed.

Actually, Kneser [24] provided a more general version of (42). For $t > 0$ the *counting function* of a pure p-dimensional analytic set S in M is defined by

(47) $$n_S(t) = \frac{1}{t^{2p}} \int\limits_{S[t]} \upsilon^p = \int\limits_{S[t]} \omega^p + n_S(0),$$

where

(48) $$n_S(0) = \lim_{t\to 0} n_S(t)$$

exists and is called the *Lelong Number* of S at 0. Kneser assumed that $0 \notin S$, then $n_S(0) = 0$. Pierre Lelong permitted $0 \in S$ and proved (46) in 1957 [26] by the use of currents. Paul Thie [87] (1967) moved that the Lelong number is an integer. This result constituted Paul Thie's theses at Notre Dame and by coincidence Pierre Lelong was present at the defense of the theses. Of course, if $0 \in S$ then $n_S(0) > 0$. Paul Thie's result proved to be most helpful in estimating volumes from below. Of course the Lelong number of S can be defined for every $x \in M$ and shall be denoted by $L_S(x)$. Yum-Tong Siu [56] (1974) proved that the sets $\{x \in M | L_S(x) \geq q\}$ is analytic for every $q \in \mathbb{N}$. The proof was simplified by Lelong [28]

Since n_S increases, the limit

$$n_S(\infty) = \lim_{t \to \infty} n_S(t) \leq \infty \tag{49}$$

exists. As an application of value distribution theory on complex spaces, I was able to show that S is affine algebraic if and only if $n_S(\infty) < \infty$ ([63]).

This result was localized by Errett Bishop [5] (1964) to extend analytic sets over higher dimensional analytic sets. His result was refined by Shiffman [47], [48], [49].

Hellmuth Kneser did not proceed to a Second Main Theorem and a Defect Relation. Also he did not consider the possible extension of his theory to parabolic manifolds or Kähler manifolds. However, he investigated another problem: the theory of functions of finite order. He solved the two dimensional case and provided the basic ideas in m-dimensions. Later he assigned the completion of these investigations to me as my thesis topic [62], [63].

Again let (M, τ) a strictly parabolic manifold of dimensions $m > 1$. Thus M is a hermitian vector space of dimension $m > 1$ and τ is the square of the norm. If $\mathfrak{x} \in M, \mathfrak{y} \in M$, then $(\mathfrak{x}|\mathfrak{y})$ is the *hermitian product* of $\mathfrak{x}$ and $\mathfrak{y}$. If $u : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function, its *order* is defined by

$$0 \leq \operatorname{Ord} u = \limsup_{r \to \infty} \frac{\log u(r)}{\log r} \leq \infty. \tag{50}$$

If $\nu \geq 0$ is a non-negative divisor, define $\operatorname{Ord} \nu = \operatorname{Ord} n_\nu$. Then

$\operatorname{Ord}\nu = \operatorname{Ord} N_\nu(\cdot, s)$. If $f : M \to \mathbb{P}(V)$ is a meromorphic map, define $\operatorname{Ord} f = \operatorname{Ord} T_f(\cdot, s)$.

If q is a non-negative integer, the *Weierstrass prime factor* is defined for all $z \in \mathbb{C}$ by

(51) $$E(z, q) = (1 - z)\exp(\sum_{p=1}^{q} \frac{1}{p} z^p).$$

For all $z \in \mathbb{C}(1)$ the *Kneser Kernel* is defined by

(52) $$e_m(z, q) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}(z^{m-1} \log E(z, q)),$$

where $\log E(0, q) = 0$.

Let $f : M \to \mathbb{C}$ be an entire function of finite order with $f(0) = 1$. Let S be the support of the zero divisor $\nu = \mu_f$ of f. Trivially $S = f^{-1}(0)$. Assume that $S \neq \emptyset$. Then there exists a largest real number $s > 0$ such that $S(s) = 0$. Since f has finite order, there is a smallest, non-negative integer q such that

(53) $$\int_s^\infty \frac{T_f(r, s)}{r^{q+2}} dr < \infty.$$

Then $q \leq \operatorname{Ord} f \leq q + 1$. Also there exists a holomorphic function F on $M(s)$ such that $F(0) = 0$ and $f|W(s) = e^F$. By the First Main Theorem the following integral converges uniformly on every compact subset of $M(s)$ and defines a holomorphic function H on $M(s)$ with $\mu_H(0) \geq q + 1$ by

(54) $$H(\mathfrak{z}) = \int_{\mathfrak{y} \in S} \nu(\mathfrak{y}) e_m(\frac{(\mathfrak{z}|\mathfrak{y})}{(\mathfrak{y}|\mathfrak{y})}, q) \omega^{m-1}(\mathfrak{y})$$

for $\mathfrak{z} \in M(s)$. Kneser [24] shows that there is a unique polynomial P of at most degree q with $P(0) = 0$ such that

(55) $$F = P|W(s) + H \qquad f|W(s) = e^{P+H}.$$

Hence $h = e^{-P} f$ is an entire function with $\mu_h = \nu = \mu_f$ and

$h|W(s) = e^H$. Thus h depends on ν only.

Given a divisor $\nu \geq 0$ on M of finite order with $S = \operatorname{supp}\nu \neq 0$, there is a largest real number $s > 0$ such that $S(s) = 0$ and a smallest, non-negative integer q such that

$$\int_s^\infty n_\nu(t)\frac{dt}{t^{q+2}} < \infty. \tag{56}$$

Then $q \leq \operatorname{Ord}\nu \leq q+1$. The integral (53) converges uniformly on every compact subset of $W(s)$ and defines a holomorphic function H on $M(s)$ with $H(0) = 0$ and $\mu_H(0) \geq q+1$ by (53). Does there exist an entire function h on M such that $h|W(s) = e^H$, such that $\mu_h = \nu$ and such that $\operatorname{Ord} h = \operatorname{Ord}\nu$? In his earlier paper, Kneser [23] (1936) proved the existance of such a canonical function if $m = 2$. It was my thesis problem to solve the case $m > 2$. His method required to show that a certain closed form was exact. If $m = 2$, this lead to a solvable ordinary differential equation. If $m > 2$, it took me two weeks to write out the system of partial differential equations to be solved, which I could not do. I asked him for advice. He said he had gone through the same terrible calculation and had been unable to solve the system. Then he threw away his notes. I followed his advice, but I found another proof ([62], [63]). Independently, Pierre Lelong ([25] 1953, [27] 1964) proved the existence of the canonical function h by another integral representation. Both solutions coincide by a uniquenen theorem of Rankin [42] (1968), who provided a third integral representation. In [64] 1953 I showed that the canonical function h of a 2m-periodic divisor is a theta function for this divisor and that any 2m-periodic meromorphic function is a quotient of two theta functions (Appell [2] 1891 if $m = 2$ and Poincaré [40] 1898 if $m \geq 2$). In 1975, Henri Skoda [58] and Gennadi Henkin [20] showed independently, that a non-negative divisor ν on a strictly pseudoconvex domain D in M with bounded valence N_ν is the zero divisor $\nu = \mu_h$ of a holomorphic function h on D with bounded characteristic. Later Henkin [21] (1978) showed, if $\operatorname{Ord}\nu < \infty$ then there is a holomorphic function h on D with $\nu = \mu_h$ and $\operatorname{Ord}\nu = \operatorname{Ord} h$. Recently, Polyakov [41] (1987) extended this result to the polydisc. Skoda [60] (1972) solved the problem for analytic sets of higher codimension in a complex vector space. For more details see [73].

The integral means method of Kneser fails on complex manifolds. Also he did not attempt to prove a Second Main Theorem and a Defect Relation. From the theory of holomorphic curves there are available the method of Cartan [8] and the method of Ahlfors [1] which was extended to Riemann surfaces by Hermann and Joachim Weyl [90], improved later by H. Wu [92].

In 1953/54 I extended the theory of Ahlfors-Weyl to meromorphic maps $f : M \to \mathbb{P}(V)$, where M is a m-dimensional, connected, complex manifold of dimension $m > 1$ endowed with a positive form χ of bidegree $(m-1, m+1)$ such that $d\chi = 0$. Here V is a hermitian vector space of dimension $n+1$. Again the targets are the hyperplanes in $\mathbb{P}(V)$ and f is linearly non-degenerated. Let $\mathfrak{A} = \{a_j\}_{j \in Q}$ be a family of hyperplanes $a_j \in \mathbb{P}(V^*)$ in general position. Then, under suitable assumptions a defect relation

(57) $$\sum_{j \in Q} \delta(f, a_j) \leq n+1$$

was obtained. Also a defect relation for associated maps was proved [65]. I cannot go into details here. The extension to $m > 1$ is based on two ideas:

(1) Let $\mathfrak{G}$ be a set of open, relative compact subsets G of M with C^∞-boundary such that $\bar{g} \in G$ for all $G \in \mathfrak{G}$, where g is open with a C^∞-boundary. Assume that for each compact subset K of M there is $G \in \mathfrak{G}$ with $G \supset K$. There the Dirichlet problem $dd^c\Psi \wedge \chi = 0$ is solved for $\bar{G} - g$ with $\Psi | \partial G = 0$ and $\Psi | \partial \bar{g} = 1$.

(2) The associated maps are defined by the use of a holomorphic differential form B of bidegree $(m-1, 0)$ such that

(58) $$0 \leq m i_{m-1} B \wedge \bar{B} \leq Y(G)\chi \qquad \text{on } \bar{G}$$

where $Y(G)$ is the smallest possible constant.

On parabolic manifolds the proof has been greatly simplified by Cowen-Griffiths [17] (1976), Pit-Mann Wong [93] (1976), Stoll [80] (1983), [82] (1985), [86] (1992). The definitions and identities (34)–(46) also hold on parabolic manifolds except, of course, for the slicing $j_{\mathfrak{b}}$ and the equality $n_\nu(0) = \nu(0)$ and (41) may be vacuous, since f may not have a global, reduced representation on M. The defect of (f, g) is defined as in (16). For an exact statement of the defect relation I refer to the papers mentioned before, but I will state the defect relation in a special case with a new variation:

Let M be a connected, complex manifold of dimension $m > 1$. Let W be a hermitian vector space of dimension m. Let $\pi : M \to W$ be a surjective, proper, holomorphic map. Then $\tau = \|\pi\|^2$ is a parabolic exhaustion of M and (M, τ) is called *a parabolic covering space of* W. Take any holomorphic form ζ of bidegree $(m, 0)$ on W without zeroes. Then the zero divisor $\beta \geq 0$ of $\pi^*(\zeta)$ does not depend on the choice of ζ and is called the *branching divisor* of π. Put $B = \operatorname{supp}\beta$. Then π is locally biholomorphic at $z \in M$ if and only if $z \in M - B$. Since π is proper and holomorphic, $B' = \pi(B)$ and $\hat{B} = \pi^{-1}(B')$ are analytic and $\pi : M - \hat{B} = W - B'$ is a covering space in the sense of topology. Its sheet number ς is given by (31).

Let V be a hermitian vector space of dimension $n + 1 > 1$. Let $f : M \to \mathbb{P}(V)$ be a linearly non-degenerated meromorphic map of transcendental growth (i.e. $A_f(\infty) = \infty$). Assume that the *Ricci defect*

$$R_f = \lim_{r\to\infty} \frac{N_\beta(r, s)}{T_p(r, s)} < \infty. \tag{59}$$

Let $\mathfrak{A} = \{a_j\}_{j\in Q}$ a finite family of hyperplanes $a_j \in \mathbb{P}(V^*)$ in general position. Then we have the **Defect Relation**

$$\sum_{j\in Q} \delta(f, a_j) \leq n + 1 + \frac{1}{2} n(n + 1) R_f. \tag{60}$$

A meromorphic map $h : M \to \mathbb{P}(V)$ is said to *separate the fibers of* π, if there is a point $x \in W - B'$ such that $\pi^{-1}(x) \wedge I_h = \emptyset$ and such that $h|\pi^{-1}(x)$ is injective. If so, and if $s > 0$, there is a constant $C(s) > 0$ such that

$$N_\beta(r, s) \leq 2(\varsigma - 1)T_h(r, s) + C(s) \tag{61}$$

for all $r > 0$ (Noguchi [38], Stoll [83]). Define

$$\mathfrak{h} = \bigcup_{k \in \mathbb{N}} \{h | h : M \to \mathbb{P}_k \text{ meromorphic, separates fibers of } \pi\} \tag{62}$$

Then the separation index of f is defined by

$$\gamma = \inf_{h \in \mathfrak{h}} \limsup_{r \to \infty} \frac{T_h(r, s)}{T_f(r, s)}. \tag{63}$$

If f separates the fibers of π, then $\gamma \leq 1$. We obtain the **Defect Relation**

$$\sum_{j \in Q} \delta(f, a_j) \leq n + 1 + n(n+1)(\varsigma - 1)\gamma \tag{64}$$

If $n = 1$, that is, if f is a meromorphic function with transcendental growth separating the fibers of π, then

$$\sum_{j \in Q} \delta(f, a_j) \leq 2\varsigma \tag{65}$$

which, in the case $m = 1$, was already proved by H. Cartan [8] (1933).

In 1977 Al Vitter [89] proved the Lemma of the logarithmic derivative for meromorphic functions on a hermitian vector space W and derived the defect relation for meromorphic maps $f : W \to \mathbb{P}(V)$ by Cartan's original method. For a detailed account see also Stoll [79], 1982. E. Bardis [3] (1990) extended the result to parabolic covering spaces of W.

In 1973–74, Carlson and Griffiths [16] and Griffiths and King [19] invented a new method to prove the defect relation. In keeping within [19], the result shall be stated only in the case of a parabolic covering space (M, τ) of a hermitian vector space of dimension $m > 1$. The advantage of the new method is, that it applies to holomorphic maps $f : M \to N$, where N is a connected, n-dimensional, compact, complex manifold. A positive holomorphic line bundle L spanned by its holomorphic sections is given on N. Then N is projective algebraic. The disadvantage of the new method is, that we have to assume that the map f is *dominant* which means that rank $f = n$. The vector space Y^* of all holomorphic sections of L have finite dimension $k + 1 > 1$. If $0 \neq \alpha \in Y^*$, the zero set $E_L[a] = \{x \in N | \alpha(x) = 0\}$ depends on

$a = \mathbb{P}(\mathfrak{a}) \in \mathbb{P}(Y^*)$ only. Let $Y = (Y^*)^*$ be the dual vector space of Y. If $x \in N$, the linear subspace $\Phi(x) = \{\mathfrak{a} \in V^* | \mathfrak{a}(x) = 0\}$ has dimension k. Thus one and only one $\varphi(x) \in \mathbb{P}(Y)$ exists such that $E[\varphi(x)] = \mathbb{P}(\Phi(x))$. The holomorphic map $\varphi : N \to \mathbb{P}(Y)$ is called the *dual classification map of* L. The value distribution functions of f are defined as those of $\varphi \circ f$. First Main Theorem holds but the defect relation so obtained is not optimal. As before we assume that f has transcendental growth and that there is given a finite family $\mathfrak{A} = \{a_j\}_{j \in Q}$ of points $a_j \in \mathbb{P}(Y^*)$. However we have to consider the geometry of $\{E_L[a_j]\}_{j \in Q}$ and not the geometry of $\{E[a_j]\}_{j \in Q}$. Define

(66) $$E_L[\mathfrak{A}] = \bigcup_{j \in Q} E_L[a_j].$$

For each $j \in Q$ take $\mathfrak{a}_j \in V_*^*$ with $a_j = \mathbb{P}(\mathfrak{a}_j)$. Take $x \in E_L[\mathfrak{A}]$. Then

(67) $$P = \{j \in Q | x \in E_L[a_j]\} = \{j \in Q | \mathfrak{a}_j(x) = 0\} \neq \emptyset$$

Put $p = \#P$. Take a bijective map $\lambda : \mathbb{N}[1,p] \to P$. There is an open, connected neighborhood U of x and a holomorphic section $\mathfrak{b} : U \to L$ such that $\mathfrak{b}(z) \neq 0$ for all $z \in U$. For each $j \in \mathbb{N}[1,p]$, there is one and only one holomorphic function h_j on U such that $\mathfrak{a}_{\lambda(j)} | U = h_j \mathfrak{b}$. Then $\mathfrak{A}$ is said to have *strictly normal crossings at* x if and only if

(68) $$dh_1(x) \wedge \ldots \wedge dh_p(x) \neq 0.$$

The definition is independent of the choices which were made. $\mathfrak{A}$ is said to have *strictly normal crossings* if $\mathfrak{A}$ has strictly normal crossings at every $x \in E_L[\mathfrak{A}]$, which we assume now.

Let K be the canonical bundle of N Let K^* be the dual bundle to K. Define

(69) $$\left[\frac{K^*}{L}\right] = \inf\{\frac{v}{w} | v \in \mathbb{N}, w \in \mathbb{N}, L^v \otimes K^w \text{ positive } \}.$$

Define R_f by (59) and γ by (63). With these assumptions and definitions, the

Defect Relation of Griffiths-King

$$\sum_{j\in Q}\delta(f,a_j) \leq \left[\frac{K^*}{L}\right] + R_f \tag{70}$$

$$\sum_{j\in Q}\delta(f,a_j) \leq \left[\frac{K^*}{L}\right] + 2(\varsigma - 1)\gamma \tag{71}$$

holds. In [75] (1977) the theory was refined and extended to general parabolic manifolds.

A difficult, major, unsolved problem is the question if "dominant" can be replaced by another assumption which does not imply $m \geq n$. For instance does (70) hold if $f(M)$ is not contained in any proper analytic subset of N? As Biancofiore [4] has shown the assumption $f(M) \not\subseteq E_L[a]$ for all $a \in \mathbb{P}(Y^*)$ does not suffice. Can the condition "strictly normal crossings" be relaxed?

Let V be a hermitian vector space of dimension $n+1 > 1$. Apply the previous theory to $N = \mathbb{P}(V)$. Let H be the hyperplane section bundle on $\mathbb{P}(V)$. Take $p \in \mathbb{N}$ and choose $L = H^p$. Then $K = H^{-n-1}$ and $L^v \otimes K^w = H^{pv-w(n+1)}$. Thus $[\frac{K^*}{L}] = \frac{n+1}{p}$. Thus (70) and (71) reads

$$\sum_{j\in Q}\delta(f,a_j) \leq \frac{n+1}{p} + R_f \tag{72}$$

$$\sum_{j\in Q}\delta(f,a_j) \leq \frac{n+1}{p} + 2(\varsigma - 1)\gamma. \tag{73}$$

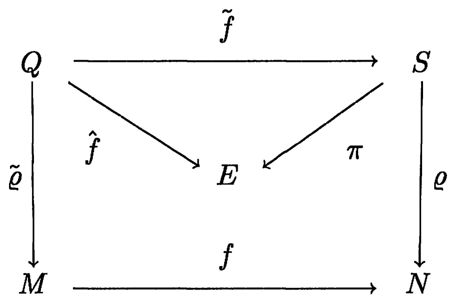
If $p = 1$, this is sharper than (60) which is due to the dominance of f.

Until now, target families of codimension 1 only where considered. Does there exist a value distribution theory for codimension $\ell > 1$. In 1958, H. Levine [30] proved an unintegrated First Main Theorem for projective planes of codimension $\ell > 1$ in $\mathbb{P}(V)$. At the 1958 Summer School at the University of Chicago, S. S. Chern asked me to find the integrated version. When I left, I told him that there is no such thing. I was much surprised when he published an integrated version [10] (1960) shortly afterwards. I failed, since I insisted on an old version to be obtained and because I had forgotten one of Max Planck's admonitions in one of his textbooks: "The energy principle is not a law of nature, but of man. Each time it fails in nature, man invents a new type of energy to restore the principle." The First Main Theorem is such a principle. In order to retain it, S. S. Chern had to admit a new, nasty term, later called the *deficit*, into the equation.

In 1965, Bott and Chern [6] extended the First Main Theorem to the equidistribution of the zeroes of holomorphic sections in hermitian vector bundles. Thus differential geometry was brought into value distribution theory. Later the theory was expanded to include all Schubert varieties associated to holomorphic vector bundles. With the work of H. Wu [91] (1968–70), F. Hirschfelder [22] (1969), L. Dektjarev [18] (1970), Michael Cowen [16] (1973), Chia-Chi Tung [88] (1973), and myself [67] (1967) [68] (1969) [69] (1970) and [76] (1978) a wide range of First Main Theorems for codimension $\ell > 1$ was established.

Mostly, they can be brought under the following scheme

(74)



Where M, N and E are connected, complex manifolds of dimensions m, n and k respectively. Here E is a compact Kähler manifold and S is an analytic subset of $N \times E$. The projections ϱ and π are surjective, open and of pure fiber dimensions q and p respectively with $n - p = \ell \geq 1$ and $m - \ell \geq 0$. The map ϱ is locally a product at every point of S. Since E is compact, ϱ is proper. Thus

(75) $$\dim S = p + k = n + q \qquad k - q = n - p = \ell.$$

The diagram is completed as a pull back by the holomorphic map f:

(76) $$Q = \{(x, z) \mid f(x) = \varrho(z)\}$$

$$(77)\qquad \tilde{\varrho}(x,z) = x \qquad \tilde{f}(x,z) = 1 \qquad \hat{f}(x,z) = \pi(z)$$
$$(78)\qquad \varrho \circ \tilde{f} = f \circ \tilde{\varrho} \qquad \hat{f} = \tilde{f} \circ \pi.$$

The map $\tilde{\varrho}$ has pure fiber dimensions q and is locally a product at every point of Q. Hence Q has pure dimension $m + q$.

For each $y \in E$, the analytic subset $S_y = \varrho(\pi^{-1}(y))$ is a pure p-dimensional analytic subset of N. The family $\mathfrak{S} = \{S_y\}_{y\in E}$ is the target family for the holomorphic map f. We assume that $E_y = f^{-1}(S_y)$ is either empty or has generically the dimension $m - \ell$. Let $\xi > 0$ be the Kähler volume for of E with

$$(79)\qquad \int_E \xi = 1$$

Let ϱ_* be the fiber integration operator. Then $\Omega = \varrho_*\pi^*(\xi)$ is a non-negative closed form of bidegree (ℓ, ℓ) and class C^∞ on N. Here Ω is the Poincaré dual of the homology class defined by $\mathfrak{S}$. Take $y \in E$, by Hodge theory or construction (H. Wu [91], Stoll [69]) there is a non-negative form $\lambda_y \geq 0$ on $E - \{y\}$ of bidegree $(k-1, k-1)$ with residue 1 at y such that

$$(80)\qquad dd^c\lambda_y = \xi \qquad \text{on } E - \{y\}$$

Then $\Lambda_y = \varrho_*\pi^*(\lambda_y) \geq 0$ is a form of bidegree $(\ell - 1, \ell - 1)$ on $N - S_y$ with

$$(81)\qquad dd^c\Lambda_y = \Omega \qquad \text{on } N - S_y$$

Let φ be a form of bidegree $(m - \ell, m - \ell)$ and of class C^∞ with compact support in M. With proper multiplicities ν_y, the Stokes Theorem, the Residue Theorem and fiber integration imply

$$(82)\qquad \int_M f^*(\Lambda_y) \wedge dd^c\varphi = -\int_M df^*(\Lambda_y) \wedge d^c\varphi$$
$$= -\int_M d\varphi \wedge d^c f^*(\Lambda_y)$$

$$= -\int_{F_y} \nu_y \varphi + \int_M \varphi \wedge dd^c f^*(\Lambda_y),$$

if E_y has pure dimension $m - \ell$. As a generalization of the Poincaré-Lelong formula we obtain the **Unintegrated First Main Theorem**

(83) $$\int_M f^*(\Omega) \wedge \varphi = \int_{F_y} \nu_y \varphi + \int_M f^*(\Lambda_y) \wedge dd^c \varphi.$$

For the integration, we assume that an exhaustion $\tau : M \to \mathbb{R}_+$ is given with

(84) $w = dd^c \log \tau \geq 0 \quad \upsilon = dd^c \tau \geq 0 \quad \sigma_\ell = d^c \log \tau \wedge w^{m-\ell}$,

Then $d\sigma_\ell = w^{m-\ell+1}$. We keep the notations (27) and (28), but do not require that τ is parabolic. For $t > 0$ the *spherical image function* is defined by

(85) $$A_f(t) = \frac{1}{t^{2m-2\ell}} \int_{M[t]} f^*(\Omega) \wedge \upsilon^{m-\ell} \geq 0.$$

For $0 < s < r$ the *characteristic function* is defined by

(89) $$T_f(r, s) = \int_s^r A_f(t) \frac{dt}{t} \geq 0.$$

Take $y \in E$ such that E_y has pure codimension ℓ or is empty. For all $t > 0$ the *counting function* is defined by

(90) $$n_{f,y}(t) = \frac{1}{t^{2m-2\ell}} \int_{E_y[t]} \nu_y \upsilon^{m-\ell} \geq 0$$

and for $0 < s < r$ the *valence function* is defined by

(91) $$N_{f,y}(r, s) = \int_s^r n_{f,y}(t) \frac{dt}{t} \geq 0.$$

For almost all $r > 0$ the *compensation function* is defined by

$$(92) \qquad m_{f,y}(r) = \frac{1}{2} \int\limits_{M<r>} f^*(\Lambda_y) \wedge \sigma_\ell \geq 0.$$

For $0 < s < r$ the *deficit* is defined by

$$(93) \qquad D_{f,y}(r,s) = \frac{1}{2} \int\limits_{M[r]-M[s]} f^*(\Lambda_y) \wedge \omega^{m-\ell+1}.$$

If $\ell = 1$ and τ is parabolic, then $\omega^m \equiv 0$ which implies $D_{f,y} \equiv 0$. However if $\ell > 1$, then this is false even if τ is parabolic. The same calculation as in (82) but respecting boundary terms yields the **First Main Theorem**

$$(94) \quad T_f(r,s) = N_{f,y}(r,s) + m_{f,y}(r) - m_{f,y}(s) - D_{f,y}(r,s).$$

A continuous form $\hat{\lambda} \geq 0$ bidegree $(k-1, k-1)$ on E exists such that $x \in E$ implies

$$(95) \qquad \hat{\lambda}(x) = \int\limits_{y \in E} \lambda_y(x) \otimes \xi(y) \geq 0.$$

The $\hat{\Lambda} = \varphi_* \pi^*(\hat{\lambda}) \geq 0$ is a continuous form of bidegree $(\ell-1, \ell-1)$ on N. For all $x \in N$, fiber integration yields

$$(96) \qquad \hat{\Lambda}(z) = \int\limits_{y \in E} \Lambda_y(z) \otimes \xi(y) \geq 0.$$

Thus we obtain

$$(97) \quad \mu_f(r) = \int\limits_{y \in E} m_{f,y}(r)\xi(y) = \frac{1}{2} \int\limits_{M<r>} f^*(\hat{\Lambda}) \wedge \sigma_\ell \geq 0$$

$$(98) \; \Delta_f(r,s) = \int\limits_{y \in E} D_{f,y}(r,s)\xi(y) = \frac{1}{2} \int\limits_{M[r]-M[s]} f^*(\hat{\Lambda}) \wedge \omega^{m-\ell-1} \geq 0$$

(99)
$$T_f(r,s) = \int_{y\in E} N_{f,y}(r,s)\xi(y) \geq 0,$$

which implies

(100)
$$\Delta_f(r,s) = \mu_f(r) - \mu_f(s).$$

For $r > 0$ define

(101)
$$B(r) = \{y \in E \mid E_y \cap M[r] \neq \emptyset\}$$
$$0 \leq b_f(r) = \int_{B(r)} \xi \leq 1$$

(102)
$$B = \{y \in E \mid E_y \neq \emptyset\}$$
$$0 \leq b_f = \int_B \xi \leq 1.$$

Then $B = \bigcup_{r>0} B(r)$ and $b_f(r) \to b_f$ for $r \to \infty$ increasingly. Now (94) implies

(103)
$$N_{f,y}(r,s) \leq T_f(r,s) + m_{f,y}(s) + D_{f,y}(r,s).$$

If $y \in E - B(r)$, then $N_{f,y}(r,s) = 0$ and (99) implies

(104)
$$\begin{aligned} T_f(r,s) &= \int_{y\in E} N_{f,y}(r,s)\xi(y) = \int_{y\in B(r)} N_{f,y}(r,s)\xi(y) \\ &\leq \int_{y\in B(r)} (T_f(r,s) + m_{f,y}(s) + D_{f,y}(r,s))\xi(y) \\ &\leq b_f(r)T_f(r,s) + \int_{y\in E} (m_{f,y}(s) + D_{f,y}(r,s))\xi(y) \\ &= b_f(r)T_f(r,s) + \mu_f(s) + \Delta_f(r,s). \end{aligned}$$

Therefore

(105)
$$0 \leq (1 - b_f(r)) \leq \frac{\mu_f(s) + \Delta_f(r,s)}{T_f(r,s)} \qquad \text{if } r > s > 0.$$

Assume that $T_f(r,s) \to \infty$ and $\Delta_f(r,s)/T_f(r,s) \to 0$ for $r \to \infty$. Then $b_f = 1$. Thus $f(M)$ intersects almost all targets S_y. Even for holomorphic curves on $\mathbb{C}$ surprising results can be obtained:

Proposition

A holomorphic map $f : \mathbb{C} \to \mathbb{P}_6$ is defined for all $z \in \mathbb{C}$ by

$$f(z) = \mathbb{P}(1, e^{\zeta z}, e^{\zeta^2 z}, \ldots, e^{\zeta^6 z}) \tag{106}$$

where $\zeta = e^{\frac{\pi i}{3}}$. If $r \geq \frac{\pi}{6}(245 + \log 7) \approx 129.3006$, then $f(\mathbb{C}[r])$ intersects at least 99% of all hyperplanes in $\mathbb{P}_6$.

Proof. A reduced representation $\mathfrak{b}$ of f is defined for all $z \in \mathbb{C}$ by

$$\mathfrak{b}(z) = (1, e^{\zeta z}, e^{\zeta^2 z}, \ldots, e^{\zeta^6 z})$$

with $\mathfrak{b}(0) = (1, \cdots, 1)$. Thus $\|\mathfrak{b}(0)\| = \sqrt{7}$. We can take $s = 0$. Thus

$$T_f(r,0) = \int_{\mathbb{C}\langle r\rangle} \log \|\mathfrak{b}\|\sigma - \frac{1}{2}\log 7$$

Observe that

$$L = \sum_{j=1}^{6} |\zeta^j - \zeta^{j-1}| = 6.$$

By Stoll [80] Proposition 15.5 page 201 we have

$$0 \leq \int_{\mathbb{C}\langle r\rangle} \log \|\mathfrak{b}\|\sigma - \frac{L}{2\pi} r \leq \frac{1}{2}\log 7.$$

Thus

$$\frac{6r - \pi \log 7}{2\pi} \leq T_f(r,0).$$

By Stoll [80] (6.66) page 140 we have $\mu_f(s) = \frac{1}{2}\sum_{p=1}^{6}\frac{1}{p} = \frac{49}{40}$ for all $s > 0$. If $r > (\pi/6)\log 7$, then

$$0 \leq 1 - b_f(r) \leq \frac{49}{40} \frac{2\pi}{6r - \pi \log 7} = \frac{49}{20} \frac{\pi}{6r - \pi \log 7}$$

Define $r_0 = \frac{\pi}{6}(245 + \log 7)$. Take $r \geq r_0$ Then

$$0 \leq 1 - b_f(r) \leq \frac{49}{20} \frac{\pi}{6r_0 - \pi \log 7} = \frac{49}{20 \times 245} = \frac{1}{100}.$$

Hence $b_f(r) \geq \frac{99}{100}$, q.e.d.

This calculation was made possible by a theorem of Shiffman-Weyl. The method can be greatly improved, see Molzon, Shiffman, and Sibony [31] (1981), and Lelong and Gruman [29] (1986).

In 1929 Rolf Nevanlinna [33] conjectured that his defect relation remains valid, if the constant target points $a_j \in \mathbb{P}_1$ are replaced by "target" functions $g_j : \mathbb{C} \to \mathbb{P}_1$ which move slower than the "hunter" function $f : \mathbb{C} \to \mathbb{P}_1$, that is, if

$$(107) \qquad T_{g_j}(r, s)/T_f(r, s) \to 0 \qquad \text{for } r \to \infty.$$

In 1964 Chi-Tai Chuang [14] proved the conjecture for entire functions $f : \mathbb{C} \to \mathbb{C}$ and created the basis for the solution of the problem. In 1986, Norbet Steinmetz [61] proved Nevanlinna's conjecture. In 1991, Ru Min and I [43] [44] [85] proved the conjecture for holomorphic curves and solved the case of the Cartan conjecture for moving targets [46]. In 1985, Charles F. Osgood [39] claimed that these theorems are a consequence of his results in diophantine approximation, but to me this implication is not self evident and still has to be established.

At the end let me state a result at Notre Dame on this subject matter, combining the work of Emmanuel Bardis [3], and Ru Min and myself [44].

At first some concepts have to be explained. Let M be a connected, complex manifold of dimension m. Let V be a hermitian vector space of finite dimension $n + 1 > 1$. Let $f : M \to \mathbb{P}(V)$ be a meromorphic map. Take $\mathfrak{a} \in V^*$ and $0 \neq \mathfrak{b} \in V^*$. Put $b = \mathbb{P}(\mathfrak{b})$. Assume that (f, b) is free. Then there exists one and only one meromorphic function $f_{\mathfrak{a},\mathfrak{b}}$ on M, called *a coordinate function*, such that for each point $p \in M$ there exists an open, connected neighborhood U of p and a reduced representation $\mathfrak{v} : U \to V$ such that

$$(108) \qquad f|U = \frac{<\mathfrak{v}, \mathfrak{a}>}{<\mathfrak{v}, \mathfrak{b}>}.$$

Here $<\mathfrak{v}, \mathfrak{b}> \not\equiv 0$ since $(f, \mathfrak{b})$ is free. Let $\mathfrak{E}_f$ be the set of all those coordinate functions of f. Trivially $\mathbb{C} \subseteq \mathfrak{E}_f$. Let $\mathfrak{M}$ be the field of meromorphic functions on M. Let $\mathfrak{K}$ be a subfield of $\mathfrak{M}$. The f is said to be *defined over* $\mathfrak{K}$ if and only if $\mathfrak{E}_f \subseteq \mathfrak{K}$. The meromorphic map f is said to be *linearly non-degenerated* over $\mathfrak{K}$ if and only if (f, g) is free for every meromorphic map $g : M \to \mathbb{P}(V^*)$ defined over $\mathfrak{K}$. Let $\mathfrak{G} = \{g_j\}_{j \in Q}$ be a finite family of meromorphic maps $g_j : M \to \mathbb{P}(V^*)$ with indeterminacy I_{g_j}. Define

$$(109) \qquad I_{\mathfrak{G}} = \bigcup_{j \in Q} I_{g_j} \qquad \mathfrak{E}_{\mathfrak{G}} = \bigcup_{j \in Q} \mathfrak{E}_{g_j}.$$

Let $\mathfrak{K}_{\mathfrak{G}} = \mathbb{C}(\mathfrak{E}_{\mathfrak{G}})$ be the extension field of $\mathfrak{E}$ in M generated by $\mathfrak{E}_{\mathfrak{G}}$. The family $\mathfrak{G}$ is said to be in *general position* if and only if there is a point $z \in M - I_{\mathfrak{G}}$ such that $\mathfrak{G}(z) = \{g_j(z)\}_{j \in Q}$ is in general position.

Theorem: Defect relation for moving target.

Let M be a connected, complex manifold of dimension M. Let W be a hermitian vector space of dimension m. Let $\pi : M \to W$ be a surjective, proper holomorphic map. Then $\tau = \|\pi\|^2$ is a parabolic exhaustion of M. Let V be a hermitian vector space of finite dimension $n + 1 > 1$. Let $\mathfrak{G} = \{g_j\}_{j \in Q}$ be a finite family of meromorphic maps $g : M \to \mathbb{P}(V^*)$ in general position. Assume at least on $k \in Q$ exists such that g_k is not constant and separates the fibers of π. Let $f : M \to \mathbb{P}(V)$ be a meromorphic map which is linearly non-degenerated over $\mathfrak{K}_{\mathfrak{G}}$. Assume that g_j grows slower than f for each $j \in Q$. Then

$$(110) \qquad \sum_{j \in Q} \delta(f, g_j) \leq n + 1.$$

During the time from 1933 to 1960 the foundation was laid. The 1960^{th} was the decade of the First Main Theorem. The 1970^{th} was the decade of the Second Main Theorem. The 1980^{th} was the decade of the moving targets. Perhaps the 1990^{th} will be a decade of refinement and of value distribution over function fields in conjunction with diophantine approximation.

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