MODULES WITH REGULAR GENERIC TYPES

Ivo Herzog and Philipp Rothmaler

Part I. The Basics

Part II. The Connected Case

Part III. The Ziegler Spectrum

Parts I and II

Philipp Rothmaler

Посвящается памяти моего
дорогого друга Дато Давлианидзе

For structures having a definable group action the notion of generic type makes sense. Poizat showed that having regular generics is a particularly useful model-theoretic property. In this series of papers we try to provide the reader with a systematic treatment of this in the case of modules, where - naturally - addition is taken as the corresponding group action.

I apologize for the long list of acknowledgements following in order reverse to chronological. But I experienced a lot of help and friendship in the last couple of years, which has to do with this paper in one way or another, and which I don't want to leave unmentioned. First, I would very much like to thank my colleagues and friends at Notre Dame and in Indiana for their hospitality and inspiration during the summer of 1987 without which this article would have never been written. Also I should like to thank them for letting me have a wonderful time when visiting Notre Dame during the Fall Semester of 1985 without which I would never have gone back there last summer. Second, I thank the organizers of the Paris Logic Colloquium '85 for giving me a grant so I could present there part of these results. Further, I wish to thank my British Colleagues together with their families, whose generosity I enjoyed in Spring 1985 (when the notion of finitizer occurred to me).

Last, but not least, I thank my colleagues at the Christian-Albrechts-Universität zu Kiel and the Deutsche Forschungsgemeinschaft for their helpful support.

Part I. Basics.

The purpose of this part is twofold. First it is meant to serve as a uniform base, introducing all that is needed in the sequel. This concerns both, notation and previous results. Second it should acquaint the reader with the class of modules which serves as the title for the entire article.

§0. Preliminaries.

Following Weglorz, the language we work in is L_R whose non-logical symbols are 0, +, and a unary function symbol r for every element r of the (once for all fixed, associative, unitary) ring R with the obvious interpretations. The class of all left R-modules can easily be axiomatized in L_R by a certain theory T_R. Throughout, "module" means "model of T_R". A positive primitive (pp) formula is one of the form $\exists \overline{y} \bigwedge_{i<n} (s_i(\overline{x}) + t_i(\overline{y}) = 0)$, where s_i and t_i are R-terms (or, more exactly, L_R-terms), i.e. linear expressions with coefficients from R. In a module M these define projections of solution sets of finite systems of linear equations over R, which are thus subgroups of $M^{\ell(\overline{x})}$ (here ℓ stands for length). By a pp subgroup I (mostly) mean such a definable set where $\ell(\overline{x}) = 1$. I will often not distinguish between formulas and the sets they define. So I allow myself to talk e.g. of a "pp subgroup $\varphi(x)$". Similarly, I will write $\psi \subseteq \varphi$ instead of $T_R \vdash \psi \to \varphi$. Writing " φ/ψ" (or "$\varphi/\psi(M)$") I mean a pair φ,ψ of 1-place pp formulas with $\psi \subseteq \varphi$ (or its factor group $\varphi(M)/\psi(M)$). I will say " φ/ψ is finite in M" (or in T) if $\varphi/\psi(M)$ is finite (for each (=some) $M \models T$). It is easily seen that $\varphi/\psi(A \oplus B) \cong \varphi/\psi(A) \oplus \varphi/\psi(B)$ (as abelian groups) and thus $|\varphi/\psi(A \oplus B)| = |\varphi/\psi(A)| \cdot |\varphi/\psi(B)|$.

For cardinals I will adopt the following convention: $\kappa = \lambda \pmod{\infty}$ iff $\kappa = \lambda$ or both κ and λ are infinite. Then a necessary and sufficient condition for given modules M and N to be elementarily equivalent is: $|\varphi/\psi(M)| = |\varphi/\psi(N)| \pmod{\infty}$ for all φ/ψ.

This is due to Baur, Garavaglia, and Monk; see footnote at the end of [Ga] for the history of this result. For more historical background I refer the reader to [Zie] and [PR] (see also [Ro 4]). The general theory of pure injective (p.i.) modules, p.i. indecomposables, p.i. hulls, indecomposable types etc. can also be found in these sources.. For easier reference I will quote a few below.

As $|\varphi/\psi(M)|$ is an invariant of T (which is always assumed to be complete) (mod ∞), I may define $\varphi/\psi(T)$ to be n if $|\varphi/\psi(M)| = n$ for some $M \models T$, and ω otherwise. The aforementioned criterion then shows that T is axiomatized by the collection of all the $\varphi/\psi(T)$'s.

If p is a type, p^+ denotes the collection of instances of pp formulas in p. Baur's quantifier elimination theorem implies that in T two complete types p and q over the same set coincide iff $p^+ = q^+$.

If p is in S(A), p^- denotes the collection of all pp formulas over A whose negation is in p.

Notice, $|T| = |R| + \aleph_0$ and $|T|^+$-saturated modules are p.i.

The following lemma due to B. H. Neumann will be refered to as BHN: If a coset of a group A is contained in finitely many cosets of the groups $B_0,\ldots,B_{n-1}$, then all of those where $A/A \cap B_j$ is infinite can be omitted. In particular, there is some j in n such that $A/A \cap B_j$ is finite.

Most of the notation is standard or borrowed from [Zie]. The p.i. hull of A is denoted by H(A). The complete type of A over B by t(A/B). $t^+(A/B)$ stands for $t(A/B)^+$.

If $\bar{a}_j$ are in N_j, $\bar{a} = \bar{a}_1 + \bar{a}_2$ is in $N_1 \oplus N_2$, then clearly $t^+(\bar{a}) = t^+(\bar{a}_1) \cap t^+(\bar{a}_2)$. From [PR, Ch.4, Sect.4] recall

Fact 1. If $H(\bar{a}) = N_1 \oplus N_2$ and $N_1, N_2 \neq 0$ then for a corresponding decomposition $\bar{a} = \bar{a}_1 + \bar{a}_2$ both $t^+(\bar{a}_1)$ and $t^+(\bar{a}_2)$ strictly contain $t^+(\bar{a})$. This was used in the course of proof of

Fact 2. [Zie, 4.4] p from S(0) is indecomposable (which means that H(p) = $H(\bar{a})$ is indecomposable, where $\bar{a}$ realizes p) iff for all $\psi_1,\psi_2 \in p^-$ there is some $\phi \in p^+$ such that $(\psi_1 \wedge \phi) + (\psi_2 \wedge \phi) \in p^-$.

Following Ziegler I write $\phi/\psi \in p$ instead of "$\phi \in p^+$ and $\psi \in p^-$".

Fact 3. [Zie, 7.10] Let ϕ/ψ be in the indecomposable types p and q from S(0). If $H(p) \neq H(q)$ then there is a pp $\chi \subseteq \phi$ containing ψ such that $\phi/\chi \in p$ and $\chi/\psi \in q$ or vice versa.

As common in model theory, when dealing with a complete theory T, I will work in a highly saturated (hence p.i.) universal domain, the so-called monster model of T, which contains every set I am working with. By a model I then mean an elementary substructure of the monster. In particular, any p.i. model is a direct summand of the monster, as is each p.i. submodule. Consult [SH] for this habit.

Set-theoretic inclusion is denoted by $\subseteq$, whereas $\subset$ will denote proper inclusion.

Algebraic notation mainly follows [ST]. In particular, a domain is a not necessarily commutative ring without zero divisors. R^o denotes $R \setminus 0$.

§1. The unlimited part.

The theory of unlimited types was developed by Prest. In this section I will briefly present it in the modified form given in [B-R], from which the first fact is taken. (I will work in this section in a fixed completion T of T_R; the formulas about direct sums stated in the preceding section will be used without mention).

Lemma 1. (1) There is a complete theory T_U, the so-called <u>unlimited part</u> of T, satisfying

$$\phi/\psi(T_U) = \begin{cases} 1, \text{ if } \phi/\psi(T) < \omega \\ \omega, \text{ if } \phi/\psi(T) = \omega \end{cases}$$

for all ϕ/ψ. In particular, $(T_U)_U = T_U$.

(2) A is a direct summand of a model of T_U
iff $M \equiv M \oplus A$ for any (some) model M of T
iff $U \equiv U \oplus A$ for any (some) model U of T_U
iff $\phi/\psi(T) < \omega$ implies $\phi/\psi(A) = 0$.

Thus the class of all <u>unlimited summands</u>, i.e. of all summands of models of T_U, is axiomatizable, too.

Proof: To prove T_U as given is consistent and complete, choose M,N,U satisfying (we called such a triple beautiful in [B-R]):
$M < N \models T$, M is $|T|^+$-saturated (hence p.i.), N realizes all types over $M \cup \bar{a}$ for any $\bar{a}$ from N, $U \simeq N/M$. Clearly we can write $N = M \oplus U$. Then $|\phi/\psi(M)| = |\phi/\psi(N)| = |\phi/\psi(M)| \cdot |\phi/\psi(U)|$, hence $|\phi/\psi(U)| = 1$ if $\phi/\psi(T)$ is finite. If not, $\{\phi(x) \wedge \neg\psi(x\text{-}a_i\text{-}c)$: $c \varepsilon M, i<n\}$ is consistent for each set of representatives $\{a_i : i<n\}$ of ϕ/ψ in U, which can therefore be extended, by choice of N. Thus $T_U = Th(U)$.
(2) Clearly, $M \equiv M \oplus A$ iff $|\phi/\psi(M)| < \omega$ implies $|\phi/\psi(A)| = 1$. The same is true for $U \models T_U$, whence (1) yields $M \equiv M \oplus A$ iff $U \equiv U \oplus A$. Thus it is enough to work with T_U. By (1), $\phi/\psi(T_U) \varepsilon \{1,\omega\}$, so $A \oplus B \models T_U$ and $\phi/\psi(T_U) < \omega$ imply $\phi/\psi(A \oplus B) = 0$, hence $\phi/\psi(A) = 0$. If, on the other hand, this condition holds, A is a direct summand of $U \oplus A \models T_U$. □

This implies that T_U is <u>closed under products</u>, i.e. $U \oplus U' \models T_U$ whenever U and U′ are models of T_U (or, equivalently, when $\phi/\psi(T_U) \varepsilon \{1,\omega\}$ for all ϕ/ψ).

As in [B-R], a pp type p over 0 is said to be <u>closed under finite index</u> if any ψ is in p^+ if it is a pp subgroup of finite index in some $\varphi \in p^+$.

Call p in S(0) <u>unlimited</u> if p^+ is closed under finite index. <u>$S_*(0)$</u> is the set of all unlimited types from S(0).

Next I will show that the unlimited types are exactly those which are realized in the unlimited part, and further that there is no loss in restricting to T_U when talking about unlimited types of T.

Following Prest set

$p_*^+ = \{\psi(\bar{x}) : \psi$ is pp and there is $\phi(\bar{x},\bar{a}) \varepsilon p^+$ with $\phi(\bar{x},\bar{0})/\psi(\bar{x})$ finite$\}$.

Clearly $p \varepsilon S(0)$ is unlimited iff $p^+ = p_*^+$.

Types over models have a similar feature:

Lemma 2. If $p \varepsilon S(M)$, where M is a model, then $p_*^+ = \{\phi(\bar{x})$: $\varphi(\bar{x})$ is $\phi(\bar{x},\bar{0})$ for some $\phi(\bar{x},\bar{m}) \varepsilon p^+\}$.

Proof: Let $\phi(\bar{x},\bar{m}) \varepsilon p^+$ and ϕ/ψ be finite. We have to show, p contains some formula defining a coset of ψ. There is a number n such that M $\models$ $\exists \bar{x}_0 \ldots \bar{x}_{n-1} \forall \bar{x}\ (\bigwedge_{i<j<n} [\phi(\bar{x}_i,\bar{m}) \wedge \neg\psi(\bar{x}_i\text{-}\bar{x}_j)] \wedge [\phi(\bar{x},\bar{m}) \rightarrow \bigvee_{i<n} \psi(\bar{x}\text{-}\bar{x}_i)])$.

This means, there are representatives in M for each coset of ψ which is contained in $\phi(M,\bar{m})$. Thus p must contain one of these. □

That every p_*^+ comes from some unlimited p_* (thus justifying the notation) will be established after the next lemma, which is also taken from [B-R].

Lemma 3. (1) Every filter of pp subgroups which is closed under finite index (in the above sense) is the pp part of a uniquely determined (unlimited) type from S(0).

(2) Every filter of pp subgroups in a theory closed under products is the pp part of a uniquely determined type from S(0).

Proof: (2) follows from (1). Let r be a filter as in (1). The only thing to prove is that $r \cup \{\neg\psi : \psi \text{ is pp and } \psi \notin r\}$ is consistent. As r is closed under finite conjunction, it suffices to verify that $\phi \wedge \bigwedge_{i<n} \neg\psi_i$ is consistent with T for any $\phi \varepsilon r$ and pp $\psi_i \notin r$. If not, $\phi \subseteq \bigvee_{i<n} \psi_i$. Then, by BHN, some $\phi/\phi\wedge\psi_i$ must be finite, whence $\phi\wedge\psi_i \varepsilon r$, contradicting the choice of ψ_i. □

Corollary 4. [Pr] For all $p \in S(A)$ there is a unique type $p_* \in S(0)$, the so-called <u>free part</u> of p, such that $(p_*)^+ = p_*^+$. Further, p_* is unlimited, and if $A = M$ is a p.i. model, then there is some $\bar{u} \in U \models T_U$ and some $\bar{m}$ in M such that $(M \oplus U > M$ and) $p = t(\bar{m}+\bar{u}/M)$ and $p_* = t(\bar{u})$.

Proof: Only the latter needs a proof. Let $N > M$ realize p. As M is p.i., there is some U as above with $N = M \oplus U$ and also $\bar{m}+\bar{u}$ realizing p as above. If $\phi(\bar{x}) \varepsilon p_*^+$ then $\phi(\bar{x}) = \phi(\bar{x},\bar{0})$ for some $\phi(\bar{x},\bar{m}') \varepsilon p^+$. Then $\models \phi(\bar{m}+\bar{u},\bar{m}')$ implies $\models \phi(\bar{u},\bar{0})$. Thus $p_*^+ \subseteq t^+(\bar{u})\ (= t_*^+(\bar{u}))$.

For the converse inclusion let $\phi(\overline{u}) \varepsilon t^+(\overline{u})$. Put $\chi(\overline{x},\overline{y}) = \phi(\overline{x}-\overline{y})$. Then $\chi(\overline{x},\overline{m}) \varepsilon p^+$, hence $\phi \varepsilon \overset{+}{p_*}$. □

Lemma 5 (Pillay and Prest). If p is in $S(A)$ and $B \supseteq A$ then there is $p \subseteq q \in S(B)$ with $p_* = q_*$.

Proof: I have to show that $p \cup \{\neg\psi(\overline{x},\overline{b}): \overline{b} \varepsilon B$ and $\psi(\overline{x},\overline{0}) \notin \overset{+}{p_*}\}$ is consistent. If not, there are finitely many $\phi(\overline{x},\overline{a}) \ \varepsilon p^+$, $\phi_i(\overline{x},\overline{a}) \varepsilon p^-$, $\psi_j(\overline{x},\overline{0}) \notin \overset{+}{p_*}$ and $\overline{b}_j \varepsilon B$ such that $\phi(\overline{x},\overline{a})$ is contained in the union of the $\phi_i(\overline{x},\overline{a})$ and the $\psi_j(\overline{x},\overline{b}_j)$. Since p is consistent, at least one of the ψ_j must occur, ψ_0 say. On the other hand, BHN allows us to omit those for which $\phi/\phi\wedge\psi_j$ is infinite. Thus $\phi/\phi\wedge\psi_0$ is finite, hence $\psi_0 \varepsilon \overset{+}{p_*}$; contradiction. □

Notice that in general $t_*(\overline{c}/A) = t_*(\overline{a}+\overline{c}/A)$ for all $\overline{a} \in A$.

Actually I am working in two theories, T and T_U, and it can therefore happen that some $p \in S^T(0)$ and $q \in S^{T}U(0)$ are different even though $p^+ = q^+$ (for $T \subseteq p$ and $T_U \subseteq q$). Also a little accuracy is needed when talking about pp types, since even being a filter depends on the theory (look at q containing $px=0 \wedge x\neq 0$ in $T = Th(\mathbb{Z}(p^\infty) \oplus Q)$; there q^+ is a filter, whereas in T_U, which is $Th(Q)$ in this case, every pp type containing $px=0$ contains also $x=0$).

Nevertheless, the next fact, taken from [B-R], justifies any confusion of T and T_U, at least when dealing with unlimited types.

Lemma 6. (1) $p \mapsto p_*$ defines a surjective map from $S(A)$ onto $S_*(0)$.

(2) There is a bijection between $S_*(0)$ and $S^{T}U(0)$ bringing every $q \in S_*(0)$ to some $p \in S^{T}U(0)$ so that $p^+ = q^+$.

(3) Let $q \in S(0)$ be realized in some $M \oplus U > M \models T$ (i.e. $U \models T_U$). Then q is realized by some tuple of U iff $q \in S_*(0)$.

Proof:

(1) The preceding lemma allows us to assume $A = 0$. Then, however, the assertion is trivial, since $p_* = p$ for all $p \in S_*(0)$.

(2) If $q \in S_*(0)$ then - according to Lemma 3(2) - choose $p \in S^{T}U(0)$ with $p^+ = q^+$. If $p \in S^{T}U(0)$ then - using (1) of the same lemma - choose q in $S(0)$ so that q^+ is the closure of p^+ under pp subgroups of finite index. Clearly, $q_* = q$, hence $q \in S_*(0)$.

(3) If $q = t(\bar{u})$ for some $\bar{u} \in U$ then $q = q_* \in S_*(0)$ by Lemma 1. If $q \in S_*(0)$, choose $q \subseteq p \in S(M)$ with q_* (= q) = p_*. Write $p = t(\bar{m}+\bar{u}/M)$, where $\bar{m} \in M$ and $\bar{u} \in U$. As in Corollary 4, $q = p_* = t(\bar{u})$. □

Let $\underline{S_*^+(0)}$ denote the set $\{p^+ : p \in S_*(0)\}$. From Lemma 3 it follows that $S_*^+(0)$ is closed under arbitrary unions, and contains the set of all pp subgroups of finite index and clearly also the set of all pp subgroups. The 1-types from $S_*(0)$ having these latter as pp part are denoted by **1** and **0**, correspondingly. (From now on all types are 1-types, similarly for sets of those!) There is another way of looking at $S_*^+(0)$. Let U be a model of T_U realizing all types over 0. Then every type $p \in S_*^+(0)$ corresponds to some (infinitely definable) subgroup p(U) of U. Then accordingly $\{p(U) : p \in S_*^+(0)\}$ forms a complete modular lattice under $\cap$ and + (cf. also [PR, §8.11]). $U = \mathbf{1}^+(U)$ is the greatest and $0 = \mathbf{0}^+(U)$ the smallest element in this lattice. Since this lattice does not depend on U (as long as U realizes all relevant types), I will denote it by $\underline{pp_\infty(T_U)}$. It is also called the <u>lattice of infinitely pp definable subgroups of T_U</u>. If T_U has dcc on pp subgroups then $pp_\infty(T_U)$ reduces merely to the lattice $pp(T_U)$ of pp subgroups of (some (=any) model of) T_U.

I will need also the following description of algebraic types, which appeared in the proof of [Ro 1, Lemma 2].

Lemma 7. A complete type is algebraic iff it contains a coset of a finite pp subgroup.

Proof: For the non-trivial direction, let $\phi \wedge \bigwedge_{i<n} \neg \psi_i$ be non-empty but finite, where $\phi \in p^+$, $\psi_i \in p^-$. Then there are finitely many elements $a_0,...,a_{n-1}$ (n>0) such that $\phi \subseteq \bigcup_{i<n} \psi_i \cup \bigcup_{j<n} \{a_j\}$ and no a_j can be omitted. By BHN, $\phi(x,\bar{0})/0$ is finite, for $\{a_j\}$ is a coset of the trivial group 0. □

Corollary 8. A complete type is algebraic iff $p_* = \mathbf{0}$. □

The finitizer $fin_R M$ of an R-module M is, by definition, the set of ring elements r such that rM is finite ([Ro 3]). Let us sum up some useful properties of the finitizer (cf. ibid.).

Lemma 9. Let $I = fin_R M$.

(1) I is an ideal containing the annihilator $ann_R M$.

(2) I does not depend on $M \models T$ (also neither does $ann_R M$). So the notations fin T and ann T make sense.

(3) $I = \{r \in R: M/M[r]$ is finite$\}$, where $M \models T$ and, as usual, M[r] is the pp subgroup of M defined by rx=0.

(4) fin T = ann T_U.

(5) $IM \subseteq$ acl 0, whence the factor module M/acl 0 is an R/I-module (here, as common in model theory, acl denotes algebraic closure; it is easily seen that acl 0, which is the same as acl $\varnothing$ and which is the same in each model of T, is a submodule of every model).

Proof:

(1) Let rM and r'M be finite. Then (r+r')M = rM + r'M and s(rM) and r(sM) must be finite, too. (2): "|rM| = n" is a first-order statement. (3): For every $r \in R$ consider the endomorphism h_r of the additive group of the model M of T. Then Ker h_r = M[r] and im h_r = rM. Thus $M/M[r] \simeq rM$. (4) follows from (3) and Lemma 1. (5) is clear. □

§2. Forking.

The aim of this section is to introduce, mostly without proof, module-theoretic equivalents to stability-theoretic concepts as forking, Lascar rank, and regular type.

Again I will work in a fixed completion T of T_R in this section. Let p and q be complete types. q is a nonforking (or nf-) extension of p, in terms $p \sqsubset q$ or $q \sqsupset p$, if $p \subseteq q$ and $p_* = q_*$. If $p \in S(A)$ then we also say, q does not fork over A (or q dnf/A). q is a forking extension of p, in terms $p \not\sqsubset q$ or $q \not\sqsupset p$, if $q \supseteq p$, but q is not an nf-extension of p. That this coincides with the usual notion due to Shelah was shown in [Ga] for theories closed under products (see also [Zie]) and in full generality in [P-P 1].

Lascar's U-rank is an ordinal rank on complete types, which is defined as follows (see e.g. [SH], where it is called L):

(a) $RU(q) \geq 0$ for all complete q;

(b) $RU(q) \geq \delta$, where δ is a limit ordinal, if $RU(q) \geq \alpha$ for all $\alpha < \delta$;

(c) $RU(q) \geq \alpha+1$ if there is a complete $r \not\sqsupset q$ with $RU(r) \geq \alpha$;

(d) $RU(q) = \alpha$ if $RU(q) \geq \alpha$ and $RU(q) \not\geq \alpha+1$;

(e) $RU(q) = \infty$ if $RU(q) \geq \alpha$ for any ordinal α.

As noticed in [B-R], the above definition of forking together with Lemma 1.6 shows that, for a complete type p in T, RU(p) is just the foundation rank of p_*^+ in $pp_\infty(T_U)$ (with respect to the order given by inclusion), i.e. $RU(p) = RU(p_*)$, and on $S_*(0)$ we have

(a) $RU(q) \geq 0$ for all $q \in S_*(0)$;

(b), (d), and (e) as above;

(c) $RU(q) \geq \alpha+1$ if there is an $r \in S_*(0)$ such that $RU(r) \geq \alpha$ and $q^+ \subset r^+$ (or, equivalently, $q^+(U) \supset r^+(U)$ in $pp_\infty(T_U)$). Clearly, $RU(\mathbf{O}) = 0$ and, moreover, $RU(q) = 0$ iff $q = \mathbf{O}$ (for $q \in S_*(0)$). Thus, by Corollary 8, $RU(p) = 0$ iff p is algebraic (for arbitrary complete p; this is a general fact, cf. [SH]). Further, if $p_* = \mathbf{1}$ then $RU(p) \geq RU(q)$ for all complete types q.

Forking can be defined in any stable theory ([SH]; modules are stable as shown by Baur and Fisher). Using forking, in turn, one can define a notion of independence as follows: The sets B and C are independent over a set A, in terms $B \downarrow_A C$, if t(B/A∪C) dnf/A. This kind of independence relation has a number of nice properties, however, the corresponding relation of dependence lacks one of van der Waerden's axioms: it is not transitive. Therefore Shelah introduced regular types and proved that forking dependence is in fact transitive on elements realizing regular types. Based on that he developed an elaborate dimension theory for arbitrary stable structures.

Now, a type $p \in S(A)$ is called regular if for all $B \supseteq A$ and a,b realizing p, if $a \downarrow_A B$ and $b \not\downarrow_A B$ then $a \downarrow_B b$.

I conclude this section with a module-theoretic description of this, which was obtained in [Zie] for theories closed under products and in full generality in [Pr].

Fact 1 For a complete type p in T the following are equivalent:

(1) p is regular.

(2) p_* is regular.

(3) p_* is critical in the sense that p_*^+ defines a minimal nonzero infinitely definable pp subgroup in $H(p_*)$ (in other words, p_*^+ is a maximal nonzero pp type in $Th(H(p_*))$.

Further, if p is regular, p_* is indecomposable.

§3. Stability.

As mentioned in the preceding section, all complete theories of modules are stable (it is this what we mean when we say that all modules are stable). Two important subclasses of that of stable theories appear in stability theory. I will define them only in the context of modules. That those definitions coincide with the original ones (given in terms of the power of certain Stone spaces) is due to Garavaglia (and depends on Baur's pp quantifier elimination for modules, see e.g. [Zie] for proofs and any further details concerning this section). Fix a completion T of T_R again. Remember, "types" are "1-types" now.

T is totally transcendental (t.t.) if it - or rather each (= some) of its models - satisfies the descending chain condition (dcc) on pp subgroups. T is superstable (s.s) if it satisfies the weak dcc on pp subgroups, i.e. no model contains an infinite chain of pp subgroups in which every factor is infinite.

Lemma 1 [B-R, 3.2]. T is s.s. iff T_U is t.t.
Proof: In any chain of pp subgroups of some $M \models T$, by Lemma 1.1, the infinite factors do not collapse in T_U. Hence the dcc for T_U implies the weak dcc for T.

Conversely, let $\phi_0(U) \supset \phi_1(U) \supset \phi_2(U) \supset \ldots$ be an infinite chain of pp subgroups in some $U \models T_U$. Consider $\psi_0 = \phi_0$, $\psi_{i+1} = \psi_i \wedge \phi_{i+1}$. In U the ψ_i define the same chain, and they also define a chain in every other module. But in a model M of T, in addition, the factors $\psi_i(M)/\psi_{i+1}(M)$ are infinite, by Lemma 1.1, as $M \oplus U \models T$. □

Corollary 2. A theory closed under products is s.s. iff it is t.t. □

Lascar proved the next result for arbitrary theories. In case of modules, though, it is particularly easy to prove.

Corollary 3. T is s.s. iff every complete type has an ordinal U-rank.
Proof: T is s.s. iff T_U is t.t. iff $pp_\infty(T_U) = pp(T_U)$ iff $pp_\infty(T_U)$ is well-founded iff each $p \in S_*(0)$ has an ordinal foundation rank in $pp_\infty(T_U)$. □

Let us return to t.t. theories again. If T is t.t. then, by the dcc on pp subgroups (which immediately yields the dcc on their cosets), every pp type (even with parameters) is equivalent to a single pp formula. We call such types finitely generated (f.g.). (In general, following Prest we call a complete type finitely generated if its pp part is.) From this it is not hard to derive that every model M of T is p.i., even Σ-p.i. - which means that $M^{(\omega)}$ is also p.i., for M satisfies the dcc iff $M^{(\omega)}$ does (since $\phi(M^{(\omega)}) = [\phi(M)]^{(\omega)}$ for any pp formula $\phi(x)$).

This makes the classification of models of t.t. theories of modules particularly transparent. For instance, Garavaglia used this to show that a countable t.t. theory of modules satisfies Vaught's conjecture, i.e. such a theory has either countably or continuum many isomorphism types of countable models (see also the cited literature).

That every model be p.i. can be violated even in s.s. theories (as a matter of fact, no countable module with an infinite descending chain of pp subgroups can be p.i., as there would be too many pp types to be realized). Therefore it is often very important to know whether a certain theory is t.t. or not. A much more powerful criterion for total transcendence than that of all types being finitely generated is the following, which I state without proof.

Fact 4 [P-P 2, 6.8]. T is t.t. iff every regular unlimited type over 0 is finitely generated.
Note, this implies that for total transcendence it is enough to check indecomposable unlimited types (Fact 2.1) - in T, though, for a type p in $S_*(0)$ can, in general, have finitely generated pp part when considered in T_U without being finitely generated in T.

§4. U-rank 1 modules.

From a stability-theoretic point of view the easiest theories to consider are those in which every complete type has U-rank at most 1. Theories having this property (and even their models) are also said to have U-rank 1. These are also characterized by the fact that no non-algebraic 1-type forks over the empty set. In modules we have

Lemma 1 [B-R, Corollary 2], [P-P 2, Proposition 7.1].
The following are equivalent for any completion T of T_R:

(1) T has U-rank 1.

(2) (Every (=some) model of) T_U is pp-simple, i.e. there are no proper pp subgroups in (models of) T_U.

(3) Every pp subgroup in T is either finite or has finite index in (a model of) T.

Proof: From §2 we know that T has U-rank 1 iff $pp_\infty(T_U) = \{1^+,0^+\}$. Thus the lemma follows from Lemma 1.1. □

Corollary 2 [P-P 2]. If T has U-rank 1 then there is exactly one nonzero type in $S_*(0)$. This type is regular and hence indecomposable.

Proof: The first statement is immediate using Lemma 1.6(2). Further, a pp-simple module is p.i., as there are only trivial pp types, which are realized anyway. Thus the p.i. hull of a realization $u \in U \models T_U$ of the unique unlimited type of T is contained (as a direct summand) in U, consequently pp-simple itself. Then that type is critical. It remains to apply Fact 2.1. □

Corollary 3. If $F = R/\text{fin}_R M$ is a skew-field for some (=any) $M \models T$, then T has U-rank 1.

Proof: By Lemma 1.9(4), every $U \models T_U$ is an F-module, hence a vector space over F. Vector spaces are pp-simple, since every pp formula defines a right ideal in $_RF$ (easily checked). □

§5. Regular generics.

Proceeding from the U-rank 1 case to more complicated cases by successively allowing the U-rank grow, one quickly gets into trouble caused by the fact that pp subgroups need not be submodules if the ring is not commutative. Poizat, however, introduced a class of theories containing properly that of U-rank 1 theories, which retains some of the latter's nice properties. These are the theories which have regular generic types, where, following Poizat, a <u>generic type</u> over some model M is a type t(c/M) such that t(c+m/M) dnf/0 for all $m \in M$. An arbitrary complete type is called <u>generic</u> if its non-forking extensions over some model are generic. Notice, generic types do not fork over 0.

Again, the context is a fixed completion T of T_R. Most of what is contained in this section was announced in [Ro 2].

Lemma 1. A complete type p is generic iff its free part p_* is **1**.

Proof: Using Corollary 1.4 write $p_* = t(u)$ for some $u \in U$ such that

$c = m' + u \in M \oplus U > M$. As mentioned after Lemma 1.5, $t_*(c+m/M) = t(u)$ for all m in M. As $p = t(c/M)$ is generic iff all $t(c+m/M)$ dnf/0, we have that p is generic iff all $t_*(u+m) = t(u)$. This is equivalent to $\overset{+}{t_*}(u+m) = t^+(u)$ for all $m \in M$.

I claim, $\overset{+}{t_*}(u+m) = t^+(u) \cap \overset{+}{t_*}(m)$.

For the inclusion from left to right, let $\phi \in t^+(u+m)$ and ϕ/ψ be finite. Then $\phi \in t^+(u)$ and $\phi \in t^+(m)$. Thus $\psi \in t^+(u)$ and $\psi \in \overset{+}{t_*}(m)$, since $\phi(U) = \psi(U)$. The other inclusion is left to the reader.

Consequently, p is generic iff $t^+(u) = t^+(u) \cap \overset{+}{t_*}(m)$ for all $m \in M$ iff $t^+(u) \subseteq \overset{+}{t_*}(m)$ for all m iff $t^+(u)$ contains no pp subgroup of infinite index of M iff $t(u) = \mathbf{1}$. □

Thus there is a distinguished generic type, namely **1**, and when talking of <u>the</u> generic we always mean this type.

Together with the description of regularity from §2 we get that all generics are regular iff some is (this is a general fact, see [Po]) iff **1** is.

Lemma 2. The following are equivalent:

(1) T has regular generics.

(2) **1** is regular.

(3) H(**1**) is pp-simple.

(4) There is an element a in some $U \models T_U$ such that H(a) is pp-simple and a lies in no proper pp subgroup of U (i.e. every such intersects H(a) trivially).

Proof: The equivalence of (1) - (3) has been mentioned already. For the last one notice that such an element a has a generic type over 0 iff t(a) is **1** (in T). □

Part of the corollary is contained also in [Po, Lemme 7], (4) is a special case of [Po, Lemme 2].

Corollary 3. Let T have regular generics and put $\bar{R} = R/\text{fin } T$.

(1) H(**1**) is indecomposable.

(2) $\phi(H(\mathbf{1})) \neq 0$ iff $\phi(H(\mathbf{1})) = H(\mathbf{1})$ iff $U = \phi(U)$ for all $U \models T_U$ iff $M/\phi(M)$ is finite for all (some) $M \models T$ (ϕ pp).

(3) $ann_R H(\mathbf{1}) = ann\ T_U = fin\ T$.

(4) rM is either finite or of finite index in M, for each $M \models T, r \in R$.

(5) H(**1**) is a torsion-free divisible $\bar{R}$-module.

(6) Every $U \models T_U$ is a divisible $\bar{R}$-module.

(7) fin T is a prime ideal, i.e. $\bar{R}$ a domain.

Proof:

(1) follows from Fact 2.1.

(2) is immediate from Lemma 2 and Lemma 1.1.

(3) Applying (2) to the formula rx=0 we get $ann\ T_U = ann_R H(\mathbf{1})$. The other equality was proved in §1.

(4) Applying (2) to the formula $\exists y(x=ry)$ we get, M/rM is finite iff $rH(\mathbf{1}) \neq 0$. This is the case iff $r \notin$ fin T iff rM is infinite.

(5) and (6) As in (4) one easily gets $rH(\mathbf{1}) = H(\mathbf{1})$ - and by (2), rU = U - whenever $r \notin$ fin T. Hence H(**1**) and U are divisible (over $\bar{R}$).

A module is torsion-free as an $\bar{R}$-module iff rx=0 defines in it the trivial group 0 for all $r \notin$ fin T.

If rx=0 defines a non-trivial group in H(**1**) then it defines the whole group, by pp-simplicity. But then $r \in$ fin T.

(7) easily follows from (3) and Lemma 2 (3). □

PART II. THE CONNECTED CASE.

Modules with regular generics will be investigated here in the case when they do not contain proper definable subgroups of finite index. These turn out to be divisible and, essentially, over rings without zero divisors. Most of this part is devoted to the search for a sufficiently large class of domains over which every divisible module has regular generics.

§6. Connected modules.

A module is called connected if it does not contain any proper definable subgroup of finite index.

A module is connected iff the pp part of any generic over 0 is just $\{x=x\}$:

Lemma 1. A module is connected iff it does not contain any proper pp subgroup of finite index.

Proof: For the non-trivial direction, suppose all proper pp subgroups of a module M have infinite index and let A be a definable subgroup of finite index. Then there are finitely many pp formulas ϕ_i, ψ_{ij} so that $\bigcup_i (\phi_i \wedge \bigcup_j \neg\psi_{ij})$ defines A in M. Let there be no redundant disjunct. As A has finite index, M is a finite union of cosets of the ϕ_j. By BHN there is one, ϕ_o say, which has finite index in M. By hypothesis, $\phi_o(M)$ is just M. By irredundancy, the first disjunct is not empty in M, hence all the ψ_{oj} are proper and thus of infinite index. On the other hand, $M = (M \setminus \bigcup_j \psi_{oj}) \cup \bigcup_j \psi_{oj}$. By BHN, we can omit the ψ_{oj}, whence $M \subseteq A$. □

Lemma 2. If M is connected, $fin_R M = ann_R M$.

Proof: By Lemma 1.9(3), $fin_R M = \{r \in R : M/M[r] \text{ is finite}\} = \{r \in R : M = M[r]\} = ann_R M$. □

Notice that also connectedness is an invariant of a complete theory, as is divisibility. Thus it makes sense to talk about divisible or connected complete theories. I might admit this sort of confusion also in other cases of first-order invariant properties.

Throughout, T is a completion of T_R.

Corollary 3. If T has regular generics and is connected, then $rM = M$ for each $M \models T$ and $r \in R \setminus ann_R M$.

In particular, the ring $\bar{R} = R/ann_R M$ is a domain and M a divisible $\bar{R}$-module.

Proof: Follows from the preceding lemma and Corollary 5.3. □

Corollary 4. Let $\bar{R}, T, M$ be as above. Suppose $\bar{R}$ is finite. T has regular generics and is connected iff $\bar{R}$ is a field. M is an $\bar{R}$-vector space then, hence pp-simple, and T is t.t.

Proof: Finite rings without zero divisors are skew-fields, hence fields. If, on the other hand, $\bar{R}$ is a field, then, being a vector space, M is pp-simple, hence connected and t.t., and also p.i. Thus $H(\mathbf{1})$ is $\bar{R}$, hence pp-simple. □

I am particularly interested in whether a given module with regular generics is t.t. Even in the connected case there is no hope of showing all of them are:

Example: Let $R = k[X,Y]$, the ring of polynomials over a field k in two commuting indeterminates X and Y. Let Q be its field of fractions $k(X,Y)$. Consider the R-module $M = Q/R$. That M is in fact a connected module with regular generics follows from the next section (any divisible module over a commutative domain is). As in every divisible torsion module, there are plenty of ascending chains of the form $M[r] \subset M[r^2] \subset \ldots$ However, I am going to find also a descending chain of pp subgroups (this making M non-t.t.). Clearly, $M[X]$, $YM[X]$, $Y^2M[X]$,... are pp subgroups, and since X and Y commute, they form a descending chain. It remains to show that this is proper. For this it is enough to verify that $Y^n/X + R$ is not in $Y^{n+1}M[X]$, for all n. Otherwise write $Y^n/X + R = Y^{n+1}(r/s + R)$, where $X(r/s + R) = 0$, i.e. $Xr =$

su for some $u \in R$. From the former we get some $v \in R$ with $Y^n/X = (Y^{n+1}r)/s + v$, hence $Y^n = Y^{n+1}u + Xv \in XR + Y^{n+1}R$, contradiction. □

This shows that we have to study divisible modules in further detail. I will be interested in which of these have regular generics and which of those, in turn, are t.t. Unfortunately I do not have a full answer to either of these questions. Eventually I will show though that every divisible module has regular generics and is connected if we restrict to rings which are e.g. two-sided Ore domains. This is not too much a restriction, for it includes the commutative case and the two-sided noetherian case (see below).

§7. Divisible modules.

Before turning to the results mentioned at the end of the last section I will derive some preliminary facts about pp subgroups in divisible modules which do not depend on further restrictions on the ring. Nevertheless I confine myself to domains, since the main interest is in modules with regular generics anyway.

A large subclass of that of divisible modules is that of injectives, for divisibility is equivalent to Baer's criterion restricted to principal ideals. These can differ quite a bit: e.g. over left-noetherian rings any injective module is t.t. (see cited literature), whereas the above example provides a divisible one which is not.

Recall from Robinson style model theory that a formula over a module N (i.e. with parameters from N) is <u>model consistent</u> with N if it is satisfiable in some supermodule of N (i.e. iff it is consistent, in the usual sense, with the atomic diagram of N). I will consider this notion for finite systems of equations only. For this purpose I fix some

<u>Notation</u>: For a pp formula $\phi(x)$ of the form $\exists\bar{y}\ \psi(x,\bar{y})$, where $\psi(x,\bar{y})$ is $\bigwedge_{i<n} s_i x = \sum_{j<m} s_{ij} y_j$ (and clearly $\ell(\bar{y}) = m$), put $\bar{s}_i = (s_{i0},\ldots,s_{i(m-1)}) \in R^m$ and let I_ϕ be the submodule of ${}_RR^m$ generated by

$\bar{s}_0,\ldots,\bar{s}_{n-1}$. Furthermore, let $h_\phi : {}_RR^n \to {}_RR^m$ be the homomorphism given by $(r_0,\ldots,r_{n-1}) \to (\sum_{i<n} r_i s_{ij} : j < m)$ $(= \sum_{i<n} r_i\bar{s}_i)$. □

Next I state Lemma 3.2 from [E-S] specified to this notation (notice, the restriction to one free variable in [E-S] is not essential).

Fact 1. For $\phi(x)$ as above, a module N and some $a \in N$ the following are equivalent (in the above notation):

(1) $\psi(a,\bar{y})$ is model consistent with N;

(2) $\sum_{i<n} r_i s_i a = 0$ for all $r_i \in R$ such that $\sum_{i<n} r_i\bar{s}_i = 0$ (i.e. for all $(r_0,\ldots,r_{n-1}) \in \text{Ker } h_\phi$);

(3) there is a homomorphism $g : I_\phi \to N$ with $g(\bar{s}_i) = s_i a$ for all $i < n$.

(The proof of (3) → (1) uses some amalgamated sum of N^m and ${}_RR^m$ over I_ϕ, the other implications are trivial). □

Call a submodule $M \subseteq N$ <u>n-pure</u> in N if for each n-place pp formula $\phi(\bar{x})$ and each $\bar{a} \in M^n$, $M \models \phi(\bar{a})$ iff $N \models \phi(\bar{a})$.

M is called <u>absolutely n-pure</u> if it is n-pure in every extension. The usual notion of purity is that where n is allowed to run over all natural numbers. Similarly for absolute purity. Actually I am interested only in the case $n = 1$. Absolute 1-purity is a notion between divisibility and injectivity (for a direct summand is clearly n-pure) as is the $\aleph_0$-injectivity of [E-S], which is somehow oblique to absolute 1-purity, though. It is easily seen that N is absolutely 1-pure iff every system of equations of the form $\{s_i a = \sum_{j<n} s_{ij} y_j : i < n\}$, where $a \in N$, has a solution in N if it is model consistent with N.

Lemma 2. For an arbitrary pp formula ϕ the following are equivalent (in the above notation):

(1) $\phi(N) = N$ for all absolutely 1-pure (left R-) modules N;

(2) $\phi(M) = M$ for all injective modules M;

(3) $\psi(a,\bar{y})$ is model consistent with N for every module N and each $a \in N$;

(4) $\psi(1,\bar{y})$ is model consistent with ${}_RR$;

(5) $(r_0,\ldots,r_{n-1}) \in \text{Ker } h_\phi \Rightarrow \sum_{i<n} r_i s_i = 0.$

If these conditions are violated then there is an $r \in R^o$ such that $\phi(M) \subseteq M[r]$, in particular, $\phi(M) \subseteq T(M)$, where T(M), the torsion part of M, is the set $\{a \in M : ra=0 \text{ for some } r \in R^o\}$, for every module M.

If there is some torsion-free injective $N \neq 0$ with $\phi(N) = N$ then this is true for every injective module N.

Proof:

(1) → (2): Each injective is absolutely pure.

(2) → (3): Consider an injective $M \supseteq N$ (e.g. its injective hull). Then $M \models \phi(a)$, hence $\psi(a,\bar{y})$ is realized in M.

(3) → (4): is trivial, (4) → (5) → (3) is the above fact.

(3) → (1): Let N be absolutely 1-pure and $a \in N$. Choose $M \supseteq N$ such that $M \models \exists \bar{y}\, \psi(a,\bar{y})$. As N is 1-pure in M, $N \models \phi(a)$.

Thus (1) through (5) are equivalent.

Next assume, (5) is violated. Then there are $r_i \in R$ $(i < n)$ with $r = \sum_{i<n} r_i s_i \neq 0$, but $\sum_{i<n} r_i s_{ij} = 0$ for all $j < m$, hence $\sum_{i<n} r_i (s_j a) = 0$ if $a \in \phi(M)$ (M arbitrary). Then ra=0, whereby the second assertion follows.

Thus, if there is a torsion-free N as above, the conditions (1) - (5) must hold. □

Corollary 3. Over domains, torsion-free absolutely 1-pure modules are pp-simple, and thus t.t.; torsion-free absolutely pure modules are injective.

Proof: A proper pp subgroup of an absolutely 1-pure module is in the torsion part. An absolutely pure p.i. module is injective. □

Lemma 4. If R is a domain, then (-1) → (0) → (1) for any pp formula $\phi(x)$, where

(-1) $\phi({}_R R) \neq 0$;

(0) $\phi(M) = M$ for all divisible (left R-) modules M;

(1) $\phi(N) = N$ for all absolutely 1-pure modules N (as in Lemma 2).

Proof:

As absolutely 1-pure modules are divisible, only (-1) $\to$ (0) needs a proof. Let $\phi(x)$ be $\exists \overline{y} \bigwedge_{i<n} (s_i x = \sum_{j<m} s_{ij} y_j)$. If $0 \neq r \in \phi({}_R R)$, there are $r_j \in R$ such that $s_i r = \sum_{j<m} s_{ij} r_j$ $(i < n)$. If M is divisible and $a \in M$ arbitrary, pick $b \in M$ with $a = rb$ (here we need $r \neq 0$!). Then $s_i a = s_i(rb) = \sum_{j<m} s_{ij}(r_j b)$, whence $a \in \phi(M)$. □

For the rest of this part I will be dealing with rings for which some of the converses of these implications are true. Namely, domains which are characterized by (1) $\to$ (0) I will call good in the next section, and in the appendix I will show that right Ore domains are exactly the domains satisfying (0) $\to$ (-1).

§8. Modules over Ore domains.

I am going to single out a large enough class of domains, over which every divisible module has regular generics (and is connected). This provides us with a stock of examples automatically including those which are not injective (and, as announced in [Ro 2], not even absolutely pure).

To realize this task I impose two further restrictions on the domain, the classical left Ore condition (which is (a) in Lemma 2 below) and the following less classical one.

A ring is good if the following holds for all 1-place pp formulas $\phi(x)$: If $\phi(M) = M$ for all injective M then $\phi(M) = M$ for all divisible M. (Notice also the equivalent formulations of this given by Lemma 7.2). Good domains are less exotic than their definition might let them seem to be. As a matter of fact, every right and left Ore domain is good, as I will show in the appendix. In particular, commutative domains and also right and left noetherian domains are good (cf. [ST, Ch.II] for the fact that noetherian rings are Ore). However, I do not know whether every good left Ore domain is right Ore.
Goodness guarantees plenty of generic elements:

Lemma 1. Every torsion-free element (i.e. everyone not in T(M)) in a divisible module M over a good ring realizes a generic type.

Proof: Let M be divisible and consider an arbitrary pp subgroup $A \neq M$. Since the ring is good, Lemma 7.2 implies $A \subseteq T(M)$. Hence no element outside T(M) lies in a proper pp subgroup. □

The other property I impose on the ring will make sure that the generics are regular. It is introduced - among others - in the next lemma, which is basically folklore. For completeness I enclose a proof.

Lemma 2. Let R be a domain.

(1) The following are equivalent:

(a) $Rs \cap Rr \neq 0$ for all r and s from R^o;

(b) T(M) is a submodule in every (left R-) module M;

(c) T(M) is closed under scalar multiplication in every module M.

A domain having these properties is called <u>left Ore</u>.

(2) T(M) is <u>divisor closed</u>, i.e. $Ra \cap T(M) = 0$ for all $a \in M \setminus T(M)$. In particular, T(M) is divisible if it is a module.

(3) If R is left Ore then M/T(M) is a torsion-free R-module for every (left R-) module M.

(4) If R is left Ore then torsion-free divisible (left R-) modules are injective (hence pp-simple and t.t. by Corollary 7.3).

Proof: (1) (a) →(b): For $a_i \in T(M)$ choose $s_i \in R^o$ with $s_i a_i = 0$ $(i < 2)$. Also choose $r = r_0 s_0 = r_1 s_1 \neq 0$ using (a). Then $r(a_0+a_1) = 0$, whence T(M) is a subgroup. If a_0, s_0 are as before and $s_1 \in R^o$ is arbitrary, pick r_0 and r_1 as before. Then $r_1 s_1 a_0 = 0$, hence $s_1 a_0 \in T(M)$. (b) follows. (b) → (c) is trivial. (c) → (a): Fix $s_1 \in R^o$. I will show that for all s_0 in R^o there is an $r \in R^o$ with $rs_0 \in Rs_1$. This, however, is nothing else than T(N) = N for the module $N = R/Rs_1$. As $s_1.1 \in Rs_1$, $1+Rs_1 \in T(N)$. Then, by (c), T(N) contains every $r+Rs_1$. This completes the proof of (1). (2) just uses that R is a domain. (3) follows from (1) and (2). (4): I am going to verify Baer's criterion for injectivity, which requires finding b in M for every non-zero left ideal J and every homomorphism h from J into M such that $h(r) = rb$ for all r in J.

Pick any non-zero t in J and divide h(t) by t, i.e. find a b in M with $h(t) = tb$. Given an arbitrary non-zero r in J, choose $sr = s't \neq 0$ in R using Ore's condition. Then $sh(r) = h(s't) = s'(tb) = s(rb)$. Torsion-freeness yields $h(r) = rb$. □

The result I am heading for is now

Theorem: Let M be a left module over a good left Ore domain R. M is divisible iff it has regular generics and is faithful and connected.

One half of the theorem is a special case of Corollary 6.3. I am going to prove the other in a number of steps. Remember that because of Corollary 6.4 I need not consider finite rings.

Let me point out that assuming $ann_R M$ to be 0 in the theorem is of no substantial loss, for we could just work with $\bar{R}$ instead (the annihilator being an invariant of T).

Lemma 3. A faithful module over a left Ore domain has an elementary extension which is not torsion.

Proof: If the torsion-free type $\{rx \neq 0 : r \in R^o\}$ is inconsistent with M, then $M \models \forall x \bigvee_{i<n} r_i x = 0$ for some $r_o, \ldots, r_{n-1}$ in R^o, by compactness. As R is left Ore, there is $0 \neq s \in \bigcap_{i<n} Rr_i$. Then $sM = 0$, contradicting faithfulness. □

Recall that a divisible module over a domain is certainly faithful. I call a module M <u>non-torsion</u> if it is not a torsion module i.e. iff $T(M) \neq M$.

Lemma 4. Every divisible non-torsion module over an infinite good domain is connected.

Proof: As in Lemma 1, a proper pp subgroup $\varphi(M)$ is contained in T(M). We will show that $M/\varphi(M)$ is infinite if it is non-zero. Pick an element a outside T(M). By Lemma 2(2), $Ra \cap \varphi(M) = 0$. Thus all the ra, where r runs over R, are in distinct cosets of $\varphi(M)$. The lemma now follows from the infiniteness of R. □

The next lemma completes the proof of the theorem.

Lemma 5. If M is a divisible module over an infinite good left Ore domain then $M \equiv M \oplus (M/T(M))$.

If, in addition, $M \neq T(M)$ then the generics are regular.

Proof: It suffices to show that $\phi/\psi(M/T(M)) = 0$ whenever $\phi/\psi(M)$ is finite. Assume, the former is non-zero. Then, using the pp-simplicity of M/T(M) established in Lemma 2(4), we get $\phi(M/T(M) = M/T(M)$ and $\psi(M/T(M)) = 0$. Now we apply Lemma 7.2: "Since ψ is 0 in the injective M/T(M), it is inside the torsion part in every module. In particular, $\psi(M) \subseteq T(M)$. Further, since φ defines the whole module in the torsion-free injective M/T(M), it does so in all divisible modules (R being good!). In particular, $\phi(M) = M$. Consequently, $\phi/\psi(M)$ is infinite by the bracketed statement of the preceding lemma. This completes the proof of the first assertion and also shows that a and a+T(M) have the same type if $a \in M \setminus T(M)$.

The second assertion can be derived from the fact that every element a outside T/(M) is generic and has, moreover, pp-simple p.i. hull: Namely, M/T(M) is p.i. (even injective), hence it contains the hull of each of its elements, which have therefore pp-simple hulls, too. □

The theorem is proved.

Let me finally return to the example in §7.

The non-t.t. module M considered there is divisible. The ring R is a commutative domain, hence left and right Ore. By what will be shown in the appendix those are good. Consequently, M has regular generics.

Appendix. Two-sided Ore domains are good.

First recall that the following properties are equivalent for any domain R (cf. [FA, Ch. 9, Lemma 9.3.2, and the section between 7.16 and 7.17] and [ST, Ch. II]).

(a) R is right Ore, i.e. $sR \cap rR \neq 0$ for all r and s from R^o;

(b) there is a skew-field $Q \supseteq R$ - the so-called right skew-field of fractions of R - such that $Q = \{rs^{-1}: r,s \in R; s \neq 0\}$;

(c) there is a skew-field $Q \supseteq R$ such that for all $q_0,\ldots,q_{n-1} \in Q$ there is a $t \in R^o$ with $q_i t \in R$ for all $i < n$ (n any natural number).

It turns out that these conditions are equivalent to (0) $\rightarrow$ (-1), where the latter are the corresponding conditions from Lemma 7.4:

Lemma 1. A domain R is right Ore iff the conditions below are equivalent for all pp formulas ϕ of L_R.

(-1) $\phi({}_RR) \neq 0$;

(0) $\phi(M) = M$ for all divisible left R-modules M.

Proof: Let first R be right Ore and Q its right skew-field of fractions. By Lemma 7.4 it suffices to show (0) $\rightarrow$ (-1). Let ϕ be the formula $\exists\bar{y} \bigwedge_{i<n} s_i x = \sum_{j<m} s_{ij} y_j$ and suppose (0) holds for ϕ. Being a skew-field, Q is divisible as a left R-module, too. Thus $\phi({}_RQ) = Q$. Then ${}_RQ \models \phi(1)$, hence there are $q_j \in Q$ with $s_i = \sum_{j<m} s_{ij} q_j$ $(i < n)$. Applying (c) above choose $r_j \in R$ $(j < m)$ and $t \in R^o$ such that $s_i t = \sum_{j<m} s_{ij} r_j$ $(i < n)$.

Then $0 \neq t \in \phi({}_RR)$, i.e. ϕ satisfies (-1), too.

Clearly, condition (a) above holds iff $\phi({}_RR) \neq 0$ for all ϕ of the form $\exists y_0 y_1 (x = s y_0 \wedge x = r y_1)$, where $r,s \in R^o$.

So, for the converse it is enough to show that $\phi(M) = M$ for all divisible left R-modules M and all ϕ as above. However, this is trivial. □

Lemma 2. If R is a right and left Ore domain, then condition (-1) is equivalent to the condition

(2) $\phi(N) = N$ for all injective left R-modules N.

Proof: As the first half of the above proof, noticing that Q, the right skew-field of fractions of R, is injective as a left R-module if R is also left Ore (cf. the literature cited above), for then (2) is enough to derive (-1). □

Consequently, for a right and left Ore domain conditions (-1) through (5) from Lemmata 7.2 and 7.4 are equivalent. In particular, these are good.

PART III. THE ZIEGLER SPECTRUM

Ivo Herzog

In this part of the paper we investigate the closed subset I(T) of the Ziegler Spectrum when the theory T has a regular generic.

§9. Pure-injectives

I mention a few basic facts about pure-injective modules and pure-injective hulls, all of which may be found in [PR] or [Zie].

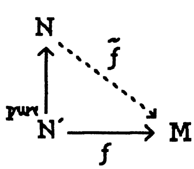
Definition 1a. A partial map f from M to N for which

$M \models \varphi(\bar{a}) \Rightarrow N \models \varphi(f\bar{a})$ for all $\bar{a} \in \operatorname{dom} f$ and ppf $\varphi(\bar{x})$ is called
$(\Leftrightarrow)$

a **partial homomorphism (isomorphism)** from M to N.

b. An embedding f of M into N is called **pure** if it is a partial isomorphism.

Definition 2. A module M is **pure-injective** if every homomorphism f from a pure submodule N′ of N to M can be lifted to a homomorphism $\tilde{f}$ from N to M:



making the above diagram commute.

Definition 3. Let M be a pure-injective module and $M \supseteq A$. Then $H_M(A)$ is a **pure-injective hull of A in M** if:

a) $H_M(A)$ is a pure-injective pure submodule of M containing A and
b) If B is a pure-injective pure submodule of M and $H_M(A) \supseteq B \supseteq A$, then $B = H_M(A)$.

The most basic properties of pure-injective modules are given by the following facts:

Fact 1. For M a pure-injective module and $M \supseteq A$, a (pure-injective) hull $H_M(A)$ of A in M exists.

Fact 2. a. M is pure-injective iff every partial homomorphism f from N to M lifts to a homomorphism $\tilde{f}$ from N to M.

b. If f is a partial isomorphism from N to M and N is pure-injective, then $\tilde{f}$ restricted to $H_N(\text{dom } f)$ is a pure imbedding.

A pp-complete type p(x) is one of the form $tp^{\pm}(A) = tp^{+}(A) \cup tp^{-}(A)$. If the theory T (a complete extension of T_R) is understood, we will denote by $S^{\pm}_I(C)$ the set of T-consistent pp-complete types over C in the set of variables indexed by I. Note that by elimination of quantifiers $S^{\pm}_I(C)$ is naturally homeomorphic to $S_I(C)$.

From Facts 1 and 2 we see that for a pure-injective M and $M \supseteq A$, $H_M(A)$ is determined by $tp^{\pm}(A/\emptyset)$ up to isomorphism. Thus it makes sense to speak about H(p) when p(x) is a pp-complete type. It also allows us to sometimes forget the subscript in " $H_M(A)$."

From Fact 2.b we can derive

Fact 3. If $c \in H_M(A)$, then $tp^{+}(c/A) \vdash_M tp(c/A)$.

Example: Let $R = Z$, the group of integers. Then the pure-injective indecomposables are the Prüfer groups $Z(p^\infty)$ for each prime p, the p-adic completion of Z for each prime p, every indecomposable finite (abelian) group and $\mathbb{Q}$, the rationals.

§ 10. The Topology

If U is an indecomposable pure-injective module then for every $a \in U$, $a \neq 0$, we know that $U = H(a) = H(tp^{\pm}(a))$. Thus there are at most $2^{|R|}$ pure-injective indecomposables up to isomorphism. Denote by T^*_R the "largest" complete theory of R-modules, i.e. $Th(\oplus U^{(\aleph_0)})$ where the U's range over all isomorphism types of indecomposable pure-injective modules over R. For more on this see [PR, §2.6]. In general, for every T, a complete extension of T_R, we let I(T) be the set of pure-injective indecomposables which occur as direct summands of models of T. $I(T^*_R)$ is just the set of all pure-injective indecomposable R-modules up to isomorphism.

In [Zie], Ziegler defines a topology on $I(T^*_R)$, a basis of which is the sets of the form $(\varphi/\psi) = \{ U \in I(T^*_R) : |\varphi(x)/\psi(x)| > 1 \}$ and the closed sets of which are exactly those of the form I(T), T a complete theory of modules. That this is indeed a topology will follow from the proposition below.

We work in T^*_R. Let $I = \{ p \in S^{\pm}_1(\emptyset) :$ p is indecomposable, i.e. H(p) is } (§0, Fact 2) endowed with the relative subspace topology which it inherits from the Stone space of all 1-types (in this space a clopen basis is given by $\{ [\sigma(x)] = \{p: \sigma(x) \in p\}: \sigma(x)$ a boolean combination of ppfs.}).
From Fact 2 of §0, one can deduce the following

Lemma [Zie,Cor.4.5]. If Δ is a finite subset of an indecomposable pp-complete type p, then there are $\varphi/\psi \in p$, i.e. $\varphi, \neg\psi \in p$, such that

$$T^*_R \vdash (\varphi \wedge \neg\psi) \to \bigwedge \Delta.$$

In the following proposition we paraphrase [Zie, Thm. 4.6].

Proposition. The sets (φ/ψ) form a basis for the quotient topology on $I(T^*_R)$ induced by the map $H: I \to I(T^*_R)$ which takes an indecomposable pp-complete type p into the isomorphism type of its hull H(p).

Proof: First we show that (φ/ψ) is open, i.e. that $H^{-1}(\varphi/\psi)$ is. Let $U \in (\varphi/\psi)$ and take p(x) with $H(p) \cong U$. We need to find an open subset $\mathcal{O}$ of I such that $p \in \mathcal{O}$ and $(\varphi/\psi) \supset H(\mathcal{O})$. Let $a \models p(x)$ and $c \in H(p)$ so that $H(p) \models \varphi(c) \wedge \neg\psi(c)$. By Fact 3 of § 9 we can take $\sigma(x,y) \in tp^+(c,a)$ such that $\sigma(x,a) \vdash_{H(p)} \varphi(x) \wedge \neg\psi(x)$.

Thus $\dfrac{\exists x\,(\sigma(x,y) \wedge \varphi(x))}{\exists x\,(\sigma(x,y) \wedge \psi(x))} \in tp^{\pm}(a)$. But if $U \in \left(\dfrac{\exists x\,(\sigma(x,y) \wedge \varphi(x))}{\exists x\,(\sigma(x,y) \wedge \psi(x))}\right)$, then clearly $U \in (\varphi/\psi)$; so take

$\mathcal{O} = [\,\exists x\,(\sigma(x,y) \wedge \varphi(x)) \wedge \neg\,(\exists x\,(\sigma(x,y) \wedge \psi(x)))\,]$.

On the other hand, let $\mathcal{O}$ be an open subset of $I(T^*_R)$ in the quotient topology induced by H. So $H^{-1}(\mathcal{O})$ is open and if $H(p) \in \mathcal{O}$, there is a formula $\varphi(x) \wedge \bigwedge_{i<n} \neg\psi_i(x)$ such that $p \in [\,\varphi(x) \wedge \bigwedge_{i<n} \neg\psi_i(x)\,]$ and $H^{-1}(\mathcal{O}) \supseteq [\,\varphi(x) \wedge \bigwedge_{i<n} \neg\psi_i(x)\,]$. By the Lemma there are $\sigma(x)$ and $\chi(x)$ such that $p \in [\,\sigma(x) \wedge \neg\chi(x)]$ and $[\,\varphi(x) \wedge \bigwedge_{i<n} \neg\psi_i(x)\,] \supseteq [\,\sigma(x) \wedge \neg\chi(x)]$. Therefore $H(p) \in H([\,\sigma(x) \wedge \neg\chi(x)]) = (\sigma/\chi)$ and $\mathcal{O} \supseteq (\sigma/\chi)$. Thus we have shown that each open subset in the quotient topology is a union of sets of the form (φ/ψ). □

The closed set I(T) turns out to be a rather important invariant of T. One can certainly perform a Cantor-Bendixson analysis on it. Given any topological space X its CB derivative X' is just $X \setminus \{$ isolated points of X $\}$ and the higher CB derivatives $X^{(\alpha)}$ are defined by recursion on the ordinals as follows:

i. $X^{(0)} = X$

ii. $X^{(\beta+1)} = (X^{(\beta)})'$ and

iii. $X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}$ if λ is a limit ordinal.

CB(X), the CB rank of X, is then defined as the greatest α for which $X^{(\alpha)} \neq \emptyset$. If no such α exists then we say CB(X) = ∞. We also talk about rk(a), the CB rank of a point $a \in X^{(\alpha)}$ as the greatest α for which $a \in X^{(\alpha)}$.

Another measure of the complexity, which we call **m-dimension**, can be [cf. PR, Chap. 10] defined on pairs of ppfs φ and ψ when $\varphi \supseteq \psi$ as follows:

i. **m-dim$_T$** $(\varphi/\psi) \geq 0$ if $T \models \exists x\ (\varphi(x) \wedge \neg\, \psi(x))$

ii. **m-dim$_T$** $(\varphi/\psi) \geq \alpha + 1$ if there is a sequence $\{\varphi_n(x)\}_{n<\omega}$ such that $\varphi = \varphi_0 \supseteq \varphi_1 \supseteq \varphi_2 \supseteq \dots \supseteq \varphi_n \supseteq \dots \supseteq \psi$ and $\dim_T(\varphi_n/\varphi_{n+1}) \geq \alpha$ for all $n < \omega$ or if there is a sequence $\{\psi_n(x)\}_{n<\omega}$ such that $\varphi \supseteq \dots \supseteq \psi_n \supseteq \dots \supseteq \psi_2 \supseteq \psi_1 \supseteq \psi_0 = \psi$ and $\dim_T(\psi_{n+1}/\psi_n) \geq \alpha$ for all $n < \omega$.

iii. **m-dim$_T$** $(\varphi/\psi) \geq \lambda$ if $\dim_T(\varphi/\psi) \geq \alpha$ for all $\alpha < \lambda$ if λ is a limit ordinal.

Then m-dim$_T(\varphi/\psi) = \alpha$ if m-dim$_T(\varphi/\psi) \geq \alpha$ and m-dim$_T(\varphi/\psi) \ngeq \alpha + 1$. If m-dim $(\varphi/\psi) \geq \alpha$ for all ordinals α then m-dim$_T(\varphi/\psi) = \infty$. By the dimension of a module M we mean m-dim$_{Th(M)}(x = x / x = 0)$ if this is not ∞; otherwise we say M does not have m-dimension.

m-dimension and CB rank are then related by the next

Theorem [Zie, Thm.8.6]. If R is a countable ring then m-dim$_T(\varphi/\psi) = \max\{ rk_{I(T)}(U) : U \in I(T) \cap (\varphi/\psi) \}$ and hence $rk_{I(T)}(U) = \min\{$ m-dim$_T(\varphi/\psi) : U \in (\varphi/\psi) \}$.

§ 11. The Hull of the Generic

In this section we consider a theory T with a regular generic 1. It is shown that if R is countable and T has m-dimension, i.e. that $CB(I(T)) < \infty$, then H(1) is the unique unlimited point in I(T) of maximal CB rank.

Suppose that R is a commutative noetherian domain. There is a bijection between Spec(R) and the set of indecomposable injective R-modules given by $\mathfrak{p} \to E(R/\mathfrak{p})$ - E denotes here the injective hull of an R-module. Let $T_{inj} = Th(\oplus E(R/\mathfrak{p})^{(\aleph_0)} : \mathfrak{p} \in Spec(R))$, the largest theory of injective R-modules. By [E-S], injectivity is an elementary property so every model of T_{inj} is injective as are all of its indecomposable direct summands. $I(T_{inj})$ is thus exactly the image of Spec(R) in $I(T^*_R)$ under the above imbedding. Prest has noted [PR,§4.7, Example 2 and §6.I] that this imbedding takes closed subsets of Spec(R) into open subsets of $I(T_{inj})$. Specifically if $\mathfrak{p}$ is a prime ideal then the image of $V(\mathfrak{p}) = \{ \mathfrak{q} \in Spec(R) : \mathfrak{q} \supseteq \mathfrak{p}\}$ is the open (in $I(T_{inj})$) subset $(\mathfrak{p}x = 0 / x = 0)$. So while the maximal ideal $\mathfrak{m}$ is a closed point in Spec(R) ($\{\mathfrak{m}\} = V(\mathfrak{m})$), in $I(T_{inj})$ it is isolated by the neighborhood $(\mathfrak{m}x = 0 / x = 0)$.

Since R is a domain, Spec (R) is an irreducible space whose generic point is the prime ideal 0. The point in $I(T_{inj})$ to which 0 corresponds is E(R), the field of quotients of R. E(R), being a field, is a pp-simple indecomposable injective (and hence pure-injective) module. So any two non-zero elements of E(R) have the same type 1 and E(R) = H(1).

Let $c \in E(R/\mathfrak{q})$. Since $E(R/\mathfrak{q})$ is indecomposable and $E(R/\mathfrak{q}) \supseteq Rc$ it follows that $E(R/\mathfrak{q}) \cong E(Rc)$. But the homomorphism $f: R \to Rc$ for which $f(1) = c$ lifts to a homorphism $\tilde{f}: E(R) \to E(Rc) \cong E(R/\mathfrak{q})$. The existence of such a map implies that $tp^+(c) \supseteq 1^+$ and that indeed for every $a \in M \models T_{inj}$, $tp^+(a) \supseteq 1^+$.

The pp-type 1^+ carries in it the least amount of positive information. In fact, $\varphi(x) \in 1^+$ iff $\varphi(M) = M$ for every $M \models T_{inj}$. Also note that each indecomposable in $I(T_{inj})$ is unlimited in T_{inj} so that T_{inj} has no pp-definable subgroups of finite index. p is thus the generic type. By Fact 1 of §2, $\mathbb{1}$ is regular and we have naturally associated the generic point of Spec(R) with the hull of generic type (a regular type, in this case) of T_{inj}.

Theorem. Suppose that R is countable and T has a regular generic p. Then $H(\mathbb{1})$ is the unique unlimited $U \in I(T)$ which is a closed point. In particular, if $0 < CB(I(T)) = \alpha < \infty$, then $H(\mathbb{1})$ is the only $U \in I(T)$ for which $rk(U) = \alpha$.

Proof: Let $U = H(\mathbb{1})$. By Lemma 2(3) of §5, U is pp-simple so $End_R U \cong \Delta$, a division ring and U is a one-dimensional (right) vector space over Δ. Now it is well known that $End_\Delta U \cong \Delta$, if we let it act on the left. From now on, we shall think of Δ as acting on the left; if we want to about the action of Δ on the right, we will denote Δ by Δ_r. We also see that $\bar{R} = R/fin_R T$ imbeds into Δ for every element $r \in R$ commutes with the action of $End_R U \cong \Delta_r$ and if $s \in fin_R T$ then $sU = 0$.

Let D be the division ring in Δ generated by $\bar{R}$. Let $\delta_1, \ldots, \delta_\alpha$ be a basis for Δ over D i.e. $\Delta = \bigoplus_{i<\alpha} D\delta_i$ as a (left) vector space over D. Then for $a \in U$, $U = \Delta a = \bigoplus_{i<\alpha} D\delta_i a$ as an $\bar{R}$-module since $D \supseteq \bar{R}$. But U is indecomposable so $\alpha = 1$ and $\Delta = D$, the division ring (in Δ) generated by $\bar{R}$.

Let $V \in I(T)$ be unlimited. Then if $\varphi(x) \in \mathbb{1}^+$, it means that $\varphi(V) = V$. Suppose moreover that $I(Th(V)) = \{V\}$ i.e. that V is a closed point in I(T). By the Theorem in §10, V has dimension zero. V has neither an ascending nor descending chain of pp-definable subgroups so we get a composition series of pp-definable subgroups of V. Using a Jordan-Hölder argument one can show that the length of a composition series is an invariant of V which we call $\mu(V)$, the multiplicity of V.

Claim: For each $\delta \in \Delta$ there is a ppf. $\sigma_\delta(x,y)$ such that $y = \delta x$ iff $U \models \sigma_\delta(x,y)$ and $\sigma_\delta(x,y)$ defines in V the graph of a bijective $\mathbb{Z}$-homomorphism (also denoted by δ).

Proof: We prove the claim for all $r \in \bar{R}$ and then show that the set of δ for which the claim holds is a division ring.

i. If $r \in \mathrm{fin}_R T$ then $rx = 0 \in 1^+$ and so $V \models rx = 0$ making V an $\bar{R}$-module. If $r \in \bar{R}$, $y = rx$ is a ppf. defining the graph of r. If $r \neq 0$ then $rU = U$ since U is pp-simple and so $r \mid x \in 1^+$. But then $V \models r \mid x$ and $rV = V$. Since $rV \cong V/\ker r$, $\mu(V/\ker r) = \mu(rV) = \mu(V)$ forcing $\ker r = 0$.

ii. If $\delta_1, \delta_2 \in \Delta$ satisfy the claim then:

$\sigma_{\delta_1+\delta_2}(x,y)$ is defined by the ppf. $\exists z\, \sigma_{\delta_1}(x,z) \wedge \sigma_{\delta_2}(x,y-z)$

$\sigma_{\delta_1\delta_2}(x,y)$ is defined by the ppf. $\exists z\ \sigma_{\delta_2}(x,z) \wedge \sigma_{\delta_1}(z,y)$ and

$\sigma_{\delta_1^{-1}}(x,y)$ is defined by $\sigma_{\delta_1}(y,x)$.

So if δ is $\delta_1+\delta_2$, $\delta_1\delta_2$ or δ_1^{-1} then $\sigma_\delta(x,y)$ is a ppf. such that $y = \delta x$ (in U) iff $U \models \sigma_\delta(x,y)$ and by hypothesis it clearly defines a function on V. If $\delta \neq 0$, then $U \models \exists x\, \sigma_\delta(x,y)$ so $\exists x\, \sigma_\delta(x,y) \in 1^+$ and $V \models \exists x\, \sigma_\delta(x,y)$ i.e. $\delta V = V$. As in case i., $\ker \delta = 0$, δ is bijective and the claim is verified.

We need to show that V is a left vector space over Δ. Let $\alpha(x)$ be the ppf. $\exists y \exists z\, (\sigma_{\delta_1+\delta_2}(x,y) \wedge \sigma_{\delta_1}(x,z) \wedge \sigma_{\delta_2}(x,y-z))$ and let $\mu(x)$ be the ppf. $\exists y \exists z\, (\sigma_{\delta_1\delta_2}(x,y) \wedge \sigma_{\delta_2}(x,z) \wedge \sigma_{\delta_1}(z,y))$. Evidently, $V \models \alpha(x)$ means that for all x in V, $(\delta_1+\delta_2)(x) = \delta_1 x + \delta_2 x$ and $V \models \mu(x)$ means that for all $x \in V$ $(\delta_1\delta_2)x = \delta_1(\delta_2 x)$. But $U \models \alpha(x) \wedge \mu(x)$ so $\alpha(x) \wedge \mu(x) \in 1^+$ and therefore $V \models \alpha(x) \wedge \mu(x)$ and V is now a vector space over Δ. Since $\Delta \supseteq \bar{R}$ and V is indecomposable, V is one-dimensional over Δ.

Now let $a \in U$, $b \in V$, $a,b \neq 0$. $tp^+(b) \supseteq 1^+ = tp^+(a)$ so there is a partial map f from U to V such that $f(a) = b$. As V is pure-injective this lifts to a map $\tilde{f}:U \to V$ which commutes with the left action of Δ so it must be an isomorphism and $U \cong V$.

U is closed in I(T) since its closure is just I(Th(U)) = {U}. By the above it follows that if V, unlimited, is another such closed point of I(T) then $U \cong V$. If $0 < CB(I(T)) = \alpha$, $I(T)^{(\alpha)}$ is a closed discrete set so all of its points are closed and unlimited. But that means that $I(T)^{(\alpha)} = \{U\}$ and the theorem is proved. □

The above theorem does not generalize to abelian structures. For example, if we consider the theory T of a torsion-free divisible abelian group and we include in the language a unary predicate P which interprets in a model of the same theory a non-trivial divisible subgroup, then I(T) will consist of two indecomposable pure-injective abelian structures: a copy of the rationals in which P interprets the trivial group 0 and a copy of the rationals again with P interpreting the whole group. It is the former of the two which is the hull of the generic so we see that the generic is regular. But both points of I(T) are T-unlimited and closed.

References.

[B-R] Baudisch, A. and Rothmaler, Ph., The stratified order in modules, pp. 44-56 in Frege Conference, Schwerin 1984, Math. Res, Akademie-Verlag, Berlin, 1984.

[E-S] Eklof, P. and Sabbagh, G., Model-completions and modules, Ann. Math. Logic, 2(3) (1971), 251-295.

[FA] Faith, C., Algebra: Rings, Modules and Categories, Springer-Verlag, Berlin New-York, 1973.

[Ga] Garavaglia, S., Forking in modules, Notre Dame J. Formal Logic, 22 (1981), 155-162.

[Po] Poizat, B., Groupes stables, avec types génériques réguliers, J. Symbolic Logic, 48(2) (1983), 339-355.

[P-P 1] Pillay, A. and Prest, M., Forking and pushouts in modules, Proc. London Math. Soc. Ser. 3., 46 (1983), 365-384.

[P-P 2] Pillay, A. and Prest, M., Modules and stability theory, Trans. Amer. Math. Soc., 300(2) (1987), 641-662.

[Pr] Prest, M., The generalized RK-order, orthogonality and regular types for modules, J. Symbolic Logic, 50(1) (1985), 202-219.

[PR] Prest, M., Model Theory and Modules, London Math. Soc. Lecture Note Series. 130, Cambridge Univ. Press, 1988.

[Ro 1] Rothmaler, Ph., Stationary types in modules, Z. math. Logik Grundl. Math., 29 (1983), 445-464.

[Ro 2] Rothmaler, Ph., On a question of Bruno Poizat, J. Symbolic Logic, 52(1) (1987), 340-341.

[Ro 3] Rothmaler, Ph., On a certain ideal, pp. 309-312 in Contributions to General Algebra, Hölder-Pichler-Tempsky (Vienna) and Teubner (Stuttgart), 1987.

[Ro 4] Rothmaler, Ph., Review of [Zie] in Zentralblatt Math. 593.16019.

[SH] Shelah, S., Classification Theory and the Number of Non-somorphic Models, North-Holland, Amsterdam, 1978.

[ST] Stenström, B., Rings of Quotients, Springer-Verlag, Berlin New-York 1975.

[Zie] Ziegler, M., Model theory of modules, Ann. Pure Appl. Logic, 26(2) (1984), 149-213.