SOME NOTES ON STABLE GROUPS

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The best examples of ω–stable groups are algebraic groups over the complex numbers. We try in Section 1 of this exposition to clarify what some model theoretic concepts mean in some quite concrete situations and write down for reference several simple facts which several model theorists have reconstructed several times each. We raise further questions directed at extensions of the Cherlin conjecture. These questions and related remarks are aimed at clarifying the distinction between the group theoretic and the geometric properties of an algebraic group. They emphasize the oft–mentioned, in the abstract, insistence that a stable group may have further structure. The comments here arise from lectures given at Notre Dame during the 1986-87 year. Several of the results arose in a number of discussions with among others Cherlin, Loveys, MacPherson, Marker, Martin, Nesin, Pillay, Steinhorn and Tanaka.

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In Section 2 we explore the difference between the use of the term 'automorphism group' in algebraic geometry and group theory. We show in Section 3 that although earlier examples have shown that the stability spectrum is not preserved by taking a finite extension of a group, every extension of Z^n by a finite group is superstable. In other words all crystallographic groups are superstable.

1. ω–STABLE AND ALGEBRAIC GROUPS

It is well known that any matrix group which is definable over the complex numbers is ω–stable of finite rank; indeed these provide the main examples of such groups. Cherlin conjectured that a simple ω–stable group of finite rank is an algebraic group over an algebraically closed field. We will discuss the meaning of 'is' in this context and some extensions of the conjecture to groups which are not simple. We proceed primarily by studying a few examples.

Since the class of ω–stable groups is closed under product, taking the product of algebraic groups of different characteristic we obtain ω–stable groups which are not algebraic. Another example is the group Z_{p^∞}. Our main interest here is with a distinction which arises already when considering algebraic groups.

There are two natural ways for a model theorist to view an algebraic subgroup of $GL(n,\mathbb{C})$ (the n x n invertible matrices over the complex numbers): as a *pure* group $(G,\cdot)$ or as an algebraic group $G^* = (G,\cdot, R_i)$ where the R_i are the restrictions to G of all relations on $\mathbb{C}^{n^2}$ definable (without parameters) in the structure $(\mathbb{C}, +,\cdot)$. We refer to the language of the second structure as the *geometric language*. We discuss below which concepts concerning algebraic groups are defined in the group language and which in the geometric. But sometimes there is no difference. The following question was essentially posed by Poizat.

1 Question: *For which affine algebraic groups are* $(G,\cdot)$ and G^* biinterpretable?

It is natural to view this question over an arbitrary algebraically closed field. In both this question and the next we have restricted for concreteness to affine algebraic groups (where the universe of the group is a Zariski closed subgrtoup of $\mathbb{C}^{n^2}$). The questions are equally meaningful for arbitrary algebraic groups (since an abstract variety can be viewed as finite union of affine varieties modulo a definable equivalence relation). By Zil'ber, Poizat, van den Dries, and Hrushovski the structures G and G^* are biinterpretable if G is simple (Theorem 4.16 of [17]). Since the projection functions are definable, the structure G^* is always ω_1– categorical. So ω_1– categoricity is a necessary but not sufficient condition for a positive answer to Question 1. It is not sufficient since $G = (\mathbb{Z}_{p2})^{\omega}$ is ω_1– categorical and (as noted by Nesin) can be viewed as an algebraic group as follows. G is elementarily equivalent to the algebraic group over an infinite field k of characteristic 2

$$\left\{\begin{pmatrix}1 & a & b\\ 0 & 1 & a\\ 0 & 0 & 1\end{pmatrix} : a,b \in k\right\}.$$

With a little more background we will see that in this case $(G,\cdot)$ is not biinterpretable with G^*. Recall that a (necessarily ω_1–categorical) structure A is *almost strongly minimal* if, possibly expanding the language to name a finite number of elements that realize a principal type, there is a definable subset D of A which is strongly minimal such that $A \subseteq \operatorname{acl} D$. But $(\mathbb{Z}_{p2})^{\omega}$ is not almost strongly minimal and, again, the existence of coordinate functions implies that G^* is always almost strongly minimal.

2 Question: *Characterize the* ω_1*–categorical affine algebraic groups.*

It is hard to imagine a purely algebraic classification since there are examples of ω_1–categorical algebraic groups which are simple, solvable but not nilpotent, and nilpotent. Every nonabelian Morley rank 2 group is ω_1–categorical. This was shown by Cherlin [4] in the solvable but not nilpotent case, by Tanaka [18] in the nilpotent case, and finally by a general argument of Lascar [12]. Our use of almost strong minimality to show the geometry is not defined in $(\mathbb{Z}_{p^2})^{\omega}$ leads to the following refinement of the question.

3 Question: *Which ω_1–categorical groups are almost strongly minimal?*

We will give examples below of algebraic groups in which the field is definable but which are not ω_1–categorical. Thus the program for proving some ω–stable group G of finite Morley rank is an algebraic group has two distinct steps:

i) Define a field k, necessarily algebraically closed, in G.
ii) Show G is an algebraic group over k.

The second step remains the sticking point for rank 2 nilpotent groups. Tanaka [18] has extended the analysis in [4] to show every such group is ω_1–categorical. All the known examples (cf. [18] and [13]) are in fact algebraic groups. Nesin has shown ([13]) how to define the field in the group language. The following observation was made by James Loveys and myself.

1.1 Lemma. *If* G *is a Morley rank 2 nilpotent group then* G *is not almost strongly minimal.*

Proof. Let A denote $(\mathbb{Z}_p)^{\omega_1}$. Then any group in our class (up to elementary equivalence) may be represented as a central extension of A by itself [4]. That is, we can represent G as A x A with the multiplication given by $(a, b)\,(c, d) = (a + c, b + d + f(a, c))$ where $f : A \times A \rightarrow A$ is a cocycle satisfying certain equations (specified in e.g. [18]). Now let α be a group homomorphism of A into A. Then it is easy to check that $\hat{\alpha}$ defined by

$\hat{\alpha}$ (a, b) = (a, b + α(a)) is an automorphism of G which fixes the center of G pointwise. To see G is not almost strongly minimal note that for any X = $\{(a_1,b_1),\ldots,(a_n,b_n)\}$ and any (a, b) with a not in the (finite) subgroup generated by the a_i, it is possible to choose infinitely many homomorphisms α of A which map $a_1,\ldots, a_n$ to 0 and differ on a. Thus the associated $\hat{\alpha}$ demonstrate that (a, b) $\notin$ acl (X $\cup$ Z (G)). If G is almost strongly minimal, for any infinite definable set W there is a finite set X with G $\subseteq$ acl (X $\cup$ W). With this contradiction we finish. $\square$

We would like to classify Morley rank 2 nilpotent nonabelian groups. All known examples are definable in algebraically closed fields. As noted before, Nesin has shown the field is definable in basic example. Is a field definable in every such example? The preceding result shows that even for the examples which are algebraic groups it is impossible to define the full geometric structure in the group language. The question which remains is whether any cocycle f which gives a rank 2 nilpotent group must be definable in the field structure.

We will explore the relations between these questions by considering some subgroups of the 2 x 2 matrices over the complex numbers. In describing these examples we not only give concrete examples but report some algebraic folklore which is helpful in the model theoretic context. We call a matrix *diagonal* if all entries off the main diagonal are zero and *scalar* if it is a diagonal matrix with all its nonzero entries equal. We fix the following notation.

$$G = GL(2,\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0\right\} \quad \text{general linear}$$

$$S = SL(2,\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1\right\} \quad \text{special linear}$$

$$Z = Z(2,\mathbb{C}) = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \neq 0\right\} \quad \text{center of } G$$

$$Z_0 = Z_0(2,\mathbb{C}) = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a^2 = 1\right\} \quad \text{center of } S$$

$PSL(2,\mathbb{C}) = S/Z_0$ projective special linear

$PGL(2,\mathbb{C}) = G/Z$ projective general linear

Note that $PGL(2,\mathbb{C})$ and $PSL(2,\mathbb{C})$ are isomorphic groups. For, PSL is defined to be $S/Z \cap S \approx Z \cdot S/Z$. But since $\mathbb{C}$ is algebraically closed each matrix A can be factored as a product of a scalar matrix and one of determinant one. So $Z \cdot S = G$. Of course these two groups are quite different over finite fields. The word 'projective' arises because the action of G on affine space becomes a faithful action on projective space when the center is factored out. The special linear group is definable in the general linear group; it is the commutator subgroup. In general the commutator subgroup is not definable. However, it is in this case. It is well known to algebraists that there is a bound on the number of multiplications needed to generate the commutator subgroup of an algebraic group over $\mathbb{C}$; a consequence of the Zil'ber indecomposability theorem extends this to any connected ω–stable group of finite Morley rank.

In the model theoretic context we say a (definable) subgroup is connected if it has no definable subgroup of finite index (equivalently has Morley degree 1). For any subset H of an ω-stable group let $\tilde{H}$ be the minimal definable subgroup containing $\tilde{H}$. We occasionally use this notation below in the context of stable groups. In such cases we implicitly assume that the group is saturated and $\tilde{H}$ is the minimal type–definable subgroup containg H (see[12]).

1.2 Definition. A *Borel* subgroup of an algebraic group G is a maximal connected solvable subgroup of G.

1.3 Fact. Every Borel subgroup of an algebraic group (over an algebraically closed field) is definable in the pure group language.

Proof. Recall that it was shown by Zil'ber [19] for ω–stable groups of finite rank and by [2] for stable groups that if H is solvable then so is $\tilde{H}$. So a subgroup of a stable group which is maximal among the solvable subgroups definable in the geometric language is definable in the group language. Now if G is an algebraic group and B is Borel, then B is maximal among all solvable closed subgroups not just the connected ones (See [11] Corollary A of Section 23) so B is definable as required. □

There may exist other maximal closed solvable groups. They will be definable but have lower rank than a Borel. There are examples of finite maximal solvable subgroups of semisimple algebraic groups (cf. [16]).

Here is a natural representation for a Borel subgroup of GL(2,$\mathbb{C}$).

$$B = B(2,\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad \neq 0\right\} \quad \text{Borel subgroup}$$

Such a representation can be explained as follows. For any n, it is easy to see that the group of upper triangular matrices is solvable. (The first commutator subgroup has ones on the diagonal, the second has zeros on the superdiagonal, the third has zeros on the superdiagonal and on the second superdiagonal, etc.) The Lie–Kolchin–Malcev theorem asserts that every solvable subgroup is conjugate to a subgroup of upper triangular matrices. Thus the group of upper triangular matrices is a Borel subgroup.

The argument for the solvability of the upper triangular matrices shows that the commutator subgroup of the upper triangular matrices is nilpotent. Nesin [15] has extended this result by showing that the commutator subgroup of a connected solvable group of finite Morley rank is nilpotent.

In our further discussion we will need two more subgroups.

$$U = U(2,\mathbb{C}) = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{C}\right\} \quad \text{unipotent}$$

$$T = T(2,\mathbb{C}) = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : ad \neq 0\right\} \quad \text{torus}$$

The properties defining these groups are expressible in the *geometric* language or more precisely in the language of matrix rings. An element a of a matrix group is called ***unipotent*** if for some n, $(a-1)^n = 0$, a closed subgroup is unipotent if all its elements are. Note that an upper triangular matrix with ones on the diagonal is unipotent. The subgroup U is a minimal definable infinite unipotent group. Note that $T(n,\mathbb{C})$ is abstractly isomorphic to a direct product of n copies of the multiplicative group of the field. That is, as a subgroup of $GL(n,\mathbb{C})$, its elements are simultaneously diagonalizable. As a maximal Abelian subgroup of a stable group, T is definable with parameters in G ($T = Z(C_G(T_0))$ for a finite subset T_0 of T).

1.4 Lemma. *Consider the pure group* G. *Suppose* $G = G_0 \times G_1$ *where both* G_0 and G_1 *are infinite and* G_0 is *∅–definable in G. Then* G *is not* ω_1*–categorical.*

Proof. We can apply the Feferman–Vaught theorem to obtain a two cardinal model of Th(G). □

For any group H and finite subgroup F, if H is ω_1–categorical then so is H/F. The converse is false; an example of an unstable group G with a finite center Z such that G/Z is ω_1–categorical is given in [14]. But this does not stop us from proving the next result.

1.5 Fact. $GL(2,\mathbb{C})$ is not ω_1–categorical.

Proof. Note that

$$GL(2,\mathbb{C}) \approx (SL(2,\mathbb{C}) \times Z)/\hat{Z}_0$$

where $\hat{Z}_0$ is the set of pairs (a,a) contained in $SL(2,\mathbb{C}) \times Z$ with $a \in Z_0$. As noted in the last lemma $SL(2,\mathbb{C}) \times Z$ has a two cardinal model and this property is preserved by factoring out a finite normal subgroup; so $GL(2,\mathbb{C})$ is not ω_1–categorical. □

So $GL(2,\mathbb{C})$ is another example of an algebraic group where the geometric language is more expressive than the group theoretic. Now the same argument shows

1.6 Fact. $B(2,\mathbb{C})$ is not ω_1–categorical.

But this fact provides an interesting anomaly. It is easy to see that $(T,\cdot) = T(2,\mathbb{C})$ is strongly minimal and so ω_1–categorical. But it is easy to verify

1.7 Proposition. $B \subseteq \operatorname{dcl}\left(T \cup \left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}\right)$.

Proof. $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & cb^{-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ if $b \neq 0$. If $b = 0$, there is nothing to prove. □

Why can't we conclude that B is ω_1–categorical? To explore this question we introduce some further terminology.

1.8 Definition. Let G_0 be a definable substructure of an L–structure G. We denote by $G|G_0$ the structure with universe G_0 and as n–ary relations all restrictions to G_0^n of Ø–definable relations on G^n. Then we say G_0 is *full* in G if the L structure G_0 is interdefinable with $G|G_0$.

Since $(B,\cdot)$ is not ω_1–categorical but $(T,\cdot)$ is, we infer that T is not full in B. Just what structure does B impose on T ? We answer this question by describing a structure on T which is definable in B and such that every automorphism α of this structure extends to an automorphism of B. Note that T is definable in the group language of B (with parameters) but is not Ø–definable in B.

To make this description note first that the scalar matrices (Z) of T, although not definable in (T,·) are definable in (B,·) as the center of B. Now the map

$$\lambda: \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} 1 & ab^{-1} \\ 0 & 1 \end{pmatrix}$$

identifies T/Z with U minus the identity element. Define an addition on T/Z by setting $A \oplus B = \gamma^{-1}(\gamma(A) \cdot \gamma(B))$. A straightforward computation verifies that $(T/Z, \oplus, \cdot)$ is a field and of course γ is an isomorphism between$(T/Z, \oplus)$ and $(U,\cdot)$. We claim the structure B imposes on T is interdefinable with the structure obtained by expanding the language to name the center and adding $\oplus$ as an operation on T/Z. To see this we show that any automorphism α of the group structure on T which induces an automorphism of the field structure on T/Z extends to an automorphism of $(B, \cdot)$. Let θ denote the action of T on U by conjugation. Then

$$B \approx U \rtimes_\theta T$$

and identifying U with T/Z via γ

$$B \approx T/Z \rtimes_\theta T.$$

Now extend α to $\hat{\alpha} : B \to B$ by $\hat{\alpha}(u,t) = (\alpha u, \alpha t)$. Direct computation verifies that $\hat{\alpha}$ is an automorphism of B.

Summing up,

1.9 Lemma. *As a subgroup of the Borel subgroup B, the torus T has a distinguished subgroup Z and T/Z is an algebraically closed field; T has no further structure.*

As a counterpoint to this example consider the Borel subgroup B' of $SL(2, \mathbb{C})$: $\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = 1\right\}$. It is easy to see B' is in the algebraic closure of the upper triangular subgroup U'. To see it is in the definable closure is harder. Define a 1–1 map from U' to the diagonal elements by sending $u \in U'$ to the unique diagonal t such that for some diagonal x, $x^2 = t$ and x conjugates $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ to u.

2 THE AUTOMORPHISM PUN

Let the multiplicative group k^x act by multiplication on the additive group k^+. It is a well known geometric fact that the 'automorphism group' of the affine line over k is isomorphic to the semidirect product of k^+ by k^x. Each element can be thought of as a pair (a,b) which acts on k by sending x to ax + b. In this section we will call this group the affine group. By 'automorphism group', the geometer does not mean the group of permutations which preserve a certain set of relations on the line. Rather he means the collection of 1 – 1 self morphisms (i.e. polynomial maps) of the line to itself whose inverses are also morphisms. The Frobenius map is the standard example that this is not the same as a bijective morphism. Note however, that a bijective definable map has a definable inverse.

4 Question: *Find a natural (in particular finite) set of relations* $\bar{R}$ *on the set of complex numbers such that the set of automorphisms of* $(\mathbb{C}, \bar{R})$ *is isomorphic to the affine group.*

Note that if the relations chosen are definable from the field structure, any field automorphism (in the usual sense) would induce an automorphism of the structure. So if we permit arbitrary abstract automorphisms the relations imposed must include some which are not definable from the field structure. Since the affine group acts strictly 2–transitively on $\mathbb{C}$, it suffices to name each orbit of three tuples to solve the question. But this language is uncountable and no countable sublanguage of it will suffice.

A more tractable problem is to find a structure with universe $\mathbb{C}$ whose group of *definable* automorphisms is isomorphic to the affine group. With Pillay, Steinhorn and Loveys we arrived at the following solution. Put on $\mathbb{C}$ the operations

$$\oplus(x,y,z) = x + y - z$$

and

$$\otimes(x,y,z,w) = ((x-z)(y-z)/(w-z)) + z.$$

That is, hide both the zero and the one.

Now naming 0 and 1 allows one to recover the field and it admits elimination of quantifiers. Suppose that $\varphi(x,y)$ defines a permutation α of k. Then in characteristic zero we will show φ is equivalent to a formula of the form $f(x,y) = 0$ where f is a linear polynomial. It has been proved (e.g. [17] Chapter 4.c) that a definable function defined on a variety is rational on a Zariski open subset of the variety.

Since we are in characteristic 0 (and the field is algebraically closed) a rational function is 1 – 1 with infinite domain only if it is the ratio of linear maps. Thus φ defines the graph of a fractional linear transformation. A routine computation shows the only fractional linear transformations which preserve the relation $\oplus(x,y,z) = w$ are of the form $ax + b$. Now (k,+) is irreducible as an ω–stable group. Thus every element is a sum of generics and so since α preserves $\oplus$ its definition as a linear function on an open set extends to all of k and we finish.

The structure described above is inadequate to solve the problem in characteristic p. In fact, no set of relations on $\mathbb{C}$ which are definable from the field structure and from which multiplication is defined can give the affine group as the group of definable automorphisms. For, the Frobenius automorphism would be definable and not in the affine group.

Given any set A and group of permutations G of A, the canonical language associated with G and A has as relations the subsets of A^n which are left invariant by the action of G. Lemma 1.9 and our last remark lead to the following question.

5 Question: *When does the canonical language for A and G have a finite basis (i.e.* $G = \mathrm{Aut}\,(A,R_1,\ldots,R_n)$?

Hrushovski [10] has shown that assuming the structure on A is ω–stable and ω–categorical the language can be taken finite; can anything reasonable be said in more generality? In [9], the proof of Representation Lemma 2 on page 63 shows that if the action of G is definable then there is a bound on the arity of relations which must be added to form a basis.

Another puzzle is to find a set of relations on $\mathbb{C} \cup \{\infty\}$ to make the definable automorphism group of the resulting structure be the fractional linear transformations. That is, to solve for the projective line the problem we have solved here for the affine line. Hrushovski has indicated informally a solution to this problem and its generalization to an arbitrary curve.

3 CRYSTALLOGRAPHIC GROUPS

In this section we show that a class of groups which have been studied intensively by both mathematicians and chemists are superstable of finite U–rank. Along the way we recall some examples of Cherlin and Rosenstein and Thomas which show the result is a little more special than one might hope. All of these are concerned with the preservation of the stability classification by finite extensions of groups.

For our purposes we take as a definition the conclusion to a theorem of Zassenhaus giving the following algebraic equivalent of the standard definition of a crystallographic group as a group of symmetries of real n–space [8].

3.1 Definition. G is a crystallographic group if G has a maximal abelian subgroup A such that $A \triangleleft G$, G/A is finite and A is isomorphic to $\mathbb{Z}^n$ for some n.

The first step in the argument that every crystallographic group is superstable is to recall from [3] that every abelian by finite group is stable. But taking finite extensions can disturb the stability spectrum. The following example was pointed out to us by Simon Thomas. Such an example was requested in [14].

3.2 Example. Let $G_1 = GL(2,\mathbb{C})$. Embed $\mathbb{Z}_2 = (\sigma)$ into the group of automorphisms of G_1 by setting

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$$

where $\bar{a}$ denotes the complex conjugate of a. Now if G_2 is the semidirect product of G_1 by Z_2 under this action, G_2 is not stable because the centralizer of σ in G_2 is isomorphic to $GL(2,\mathbb{R})$ and the reals are interpretable in $GL(2,\mathbb{R})$. Thomas had showed this interpretability result and it is contained in the discussion before Lemma 1.9 (if we replace $\mathbb{C}$ by $\mathbb{R}$ and note that the Borel group is definable in GL_2).

It is easy to see [3] that the stability class of a group G with an abelian subgroup A of finite index is determined by the $\mathbb{Z}[G/A]$ (integral group ring) module structure of A and a cocycle map from $G/A \times G/A \to A$. Since G/A is finite the cocycle can be given by naming the finitely many elements in the range. However, Example 3 of [7] shows that the module structure of A may fail to be ω–stable even if A is ω–stable and G/A is finite (indeed $G/A \approx \mathbb{Z}_2 \times \mathbb{Z}_2$). Slight variants will make the module structure not even superstable. Thus to show that every finite extension of $\mathbb{Z}^n$ (for any n) is ω–stable we must rely on some further properties of $\mathbb{Z}^n$.

3.3 Definition. An R module A is *hereditarily κ–stable* if for every ring S such that A admits an S module structure compatible with its R–module structure the S–module A is κ–stable.

The point of the definition is to demand that no S can make an appropriate descending chain of R submodules become definable. This definition leads in the natural way to the notion of structure being hereditarily ω–stable, hereditarily superstable or hereditarily stable. Since [6] showed that the stability class of module is determined by the relevant definable chain conditions (cf. [1] or [17]) the following result is obvious. (S cannot cause subgroups which do not exist to become definable!)

3.4 Lemma. *i) If there is no descending chain of* R–*submodules of* A *then* A *is hereditarily* ω–*stable.*
ii) If there is no descending chain of R–*submodules of* A *such that succesive elements of the chain are of infinite index then* A *is hereditarily superstable.*

Finally we conclude

3.5 Lemma. *For every n, the group* $\mathbb{Z}^n$ *is hereditarily superstable.*

Proof. We will show that $\mathbb{Z}^n$ contains no infinite decreasing chain of subgroups $\langle A_i : i < \omega \rangle$ such that A_{i+1} has infinite index in A_i. Since every subgroup of $\mathbb{Z}^n$ is freely generated by at most n elements (has rank at most n) it suffices to show: If $A \subseteq B \subseteq \mathbb{Z}^n$ and [B:A] is infinite then the rank of A is strictly less than the rank of B. By Lemma 15.4 of [5] we observe the following. For any free abelian group B and subgroup A it is possible to choose bases $b_1,...,b_m$ for B, $a_1,...,a_k$ for A with $k \leq m$ and nonnegative integers n_i such that $a_i = n_i b_i$ and $B/A \approx \oplus\langle b_i \rangle / \langle n_i b_i \rangle$. Thus B/A is infinite just if $k < m$ (i.e. some n_i is zero).

The U–rank of $\mathbb{Z}^n$ is bounded by n since the U–rank can be infinite only if there are arbitrarily long finite chains of subgroups with infinite index at each step. Thus we can deduce

3.6 Conclusion. Every crystallographic group is superstable with finite U–rank.

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