GENERIC FORMULAS AND TYPES A LA HODGES

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§0. Introduction.

Let T be a complete theory with monster model $\mathbb{C}$ and A a subset of $\mathbb{C}$. Certain complete types $p \in S(\mathbb{C})$ have the "privilege" of being non forking over A. The smaller A is, the harder it is not to fork over it. Thus, the most "privileged" are those types that do not fork over the empty set $\varnothing$. If T is stable then, as we all know, non–forking types exist in sufficient abundance. If T happens also to have a group operation, then we are in the presence of a stable group.

The theory of stable groups has attracted much interest in recent years. There are several reasons for this interest. One is the fact that stable groups occur "in nature" more often than one might think. Another is that the theory of stable groups presents special features, due to the richer structure of the family of types. Indeed, the group itself acts on the family of types both from the left and from the right. If $p \in S(\mathbb{C})$ and g is an element of $\mathbb{C}$, then we define the left translate $gp \in S(\mathbb{C})$ of p by: $\varphi(x,\bar{a}) \in gp$ iff $\varphi(gx,\bar{a}) \in p$. In other words, gp is the type of any element of the form gc where c realizes the type p. The notion of right translate pg is defined analogously. Having these notions at hand, the following thought is quite natural: if $p \in S(\mathbb{C})$ does not fork over $\varnothing$, then it is a quite privileged type, but if it so happens that all its left translates also do not fork over $\varnothing$ then p is truly privileged. More formally, p is called a left–generic type iff gp does not fork over $\varnothing$ for all

$g \in \mathbb{C}$. The notion of right–generic type is defined similarly. These notions were introduced by Poizat in [8] (where he refers to earlier works of Zil'ber, Cherlin–Shelah [2] and his own [7], as a source of inspiration). Of course, these concepts would be uninteresting if it turned out that generic types are nonexistent or useless. Poizat proves their existence and illustrates their usefulness.

It turns out that generic types always exist in stable groups but there are very few of them. This last fact is illustrated by several results.

First, every left–generic type is also right–generic (cf. Poizat [8], Fait 5). In other words, we do not have two distinct families, of left– and right–generics, but one. Thus, we can speak simply of generic types (without "left/right" attribute).

A second result, pointed out by Hodges, is this. One might speak of a type p being generic over A, meaning that all its left– (and right–) translates do not fork over A. It turns out that if a type is generic over a small set A (i.e., $|A| < |\mathbb{C}|$) then it is generic (i.e., generic over $\varnothing$).

A third fact, due again to Poizat (Fait 6 in [8]) is that the generic types form precisely one orbit of the group action on $S(\mathbb{C})$. In other words, the group acts transitively on the family of generic types; stated in even simpler terms: if p is generic then q is generic iff $q = gp$ form some $g \in \mathbb{C}$ iff $q = ph$ for some $h \in \mathbb{C}$.

Poizat went on to "localize" the notion of genericity and defined the notion of generic formula (one that belongs to some generic type). He asked, in private conversations, for straightforward derivations of the properties of generic formulas (rather than inferring them from corresponding properties of generic types, as done in [8]). Hopefully, such proofs would be more elementary and simpler.

Wilfrid Hodges took up the challenge and found an elementary proof for a fundamental property of generic formulas (cf. 2.3 below and 1°, page 344 in [8]). From this, he gets simple proofs for the existence of generics and the equivalence of left– and right–genericity, two facts whose proofs by Poizat were heavy. We have thus an alternative approach to generic types. I lectured on Hodges' work in the Notre Dame Seminar on Stable Groups organized by A. Pillay in the Fall of 1986. This note, based on that lecture, appears with Hodges' kind approval. Also, several improvements upon a previous version are due to his useful comments (Hodges' expanded account of his work will appear in the second volume of a forthcoming book).

We should mention that another treatment dealing directly with generic formulas appears in Poizat's recently published book [9] (see esp. Section 5a). This is a new approach, more elegant than the one in [8]. It is less elementary than Hodges' method, at least in the technical sense of [3].

Still a different approach to generic types (but not formulas) is sketched in Hrushovski's [5], pp. 10–12.

I wish to thank the Mathematics Department of Notre Dame and especially J. Knight and A. Pillay for their hospitality.

§1. Preliminaries.

We use customary notations. T will be a complete stable theory with language L and monster model $\mathbb{C}$ (or G in case of groups, cf. §2 below). Models of T are always assumed to be elementary substructures of $\mathbb{C}$. a,b,... will be elements of $\mathbb{C}$, $\bar{a}, \bar{b}$,... finite sequences of such elements, A,B,... small subsets of $\mathbb{C}$ and M,N,... models of T. L(A) will be L augmented with names for the elements of A. "$\models$" will denote satisfaction in $\mathbb{C}$.

Of the various known definitions of nonforking of formulas we adopt the following one (suggested by [1], [6], [10]).

Definition 1.1. $\varphi(x,\bar{c})$ does not fork over A iff it is almost satisfied in A, i.e., every model $M \supset A$ has an element satisfying the formula $\varphi(x,\bar{c})$.

By a standard compactness argument (as, e.g., 4.3 in [4]):

Lemma 1.2. $\varphi(x,\bar{c})$ does not fork over A iff for some $\delta(\bar{x}) \in L(A)$, $\bar{x} = (x_0,\ldots,x_{k-1})$,

$$\models \exists\, \bar{x}\, \delta(\bar{x}) \wedge \forall\, \bar{x}\, (\delta(\bar{x}) \rightarrow \bigvee_{i<k} \varphi(x_i, \bar{c})).$$

(We also say, if this is the case, that $\varphi(x,\bar{c})$ is almost satisfied over A via $\delta(\bar{x})$).

§2. Generic Formulas.

From now on we assume, unless otherwise stated, that the models of T have a group operation, i.e., T is the theory of a stable group. Accordingly, we let its monster model be **G** (rather than $\mathbb{C}$). The product of a,b $\in$ **G** will be ab.

Remarks. 1. Sometimes, the name "stable group" is associated with any stable structure in which one can define a group with definable universe or, even, with a universe that is defined by a set of formulas. The results presented here can be generalized to this situation (of course, most statements must be restricted to formulas or types that are consistent with G(x), where G is the formula, or set of formulas, defining the universe of the group).

2. Another direction of generalization was pointed out by W. Hodges: for most results, it is sufficient to assume that certain formulas (rather than the whole theory) are stable. As an example, Theorem 2.4 below is true whenever $\varphi(x,\bar{y})$ is a formula such that both formulas $\varphi'(x;v,\bar{y}) = \varphi(vx,\bar{y})$ and $\varphi''(x;v,\bar{y}) = \varphi(xv,\bar{y})$ are stable (such a formula φ is called by Hodges "bistable").

The crucial concept of this note is the following:

Definition 2.1: The formula $\varphi(x,\bar{c})$ is left–generic over A if for all $g \in \mathbf{G}$, $\varphi(gx,\bar{c})$ does not fork over A.

Remark. It is obvious that the extension of "left–generic" does not change if we replace in this definition, "for all $g \in \mathbf{G}$" by "for all $g \in M$ where M is any saturated model such that $A \cup \bar{c} \subset M$". It is less obvious that we can replace the same even by "for all $g \in M$ where M is any model such that $A \cup \bar{c} \subset M$". This follows from Poizat's [8].

Lemma 2.2. $\varphi(x,\bar{c})$ is left–generic over A iff there is $\delta(\bar{x}) \in L(A)$ such that

$$(*) \qquad \models \exists \bar{x}\, \delta(\bar{x}) \wedge \forall \bar{x}\, (\delta(\bar{x}) \rightarrow \forall v \bigvee_{i<k} \varphi(vx_i, \bar{c})).$$

Moreover, δ can be chosen to be left–invariant, i.e., for all $g \in \mathbf{G}$, $\models \delta(\bar{x}) \leftrightarrow \delta(g\bar{x})$ where $g\bar{x} = (gx_0,\ldots,gx_{k-1})$.

Proof (of the "only if" direction). If φ is left–generic over A then each g has a $\delta = \delta_g \in L(A)$ "witnessing" the nonforking of $\varphi(gx,\bar{c})$ in the sense of 1.2. It follows that the type

$$q(v) = \{\exists \bar{x}\,(\delta(\bar{x}) \wedge \neg \bigvee_{i<k} \varphi(vx_i, \bar{c})) : \delta \in L(A),\ \models \exists \bar{x}\,\delta(\bar{x})\}$$

is inconsistent and hence, there are $\delta_j(\bar{x}_j)$, $j < \ell$ such that for every $g \in \mathbf{G}$, some δ_j can serve as δ_g. Taking $\delta(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{\ell-1}) = \wedge_{j<\ell}\, \delta_j(\bar{x}_j)$, we get (*). To make δ left–invariant, replace it by $\delta'(\bar{x}) = \exists v\, \delta(v\bar{x})$. □

Remark. This lemma provides an elementary characterization of left–genericity. It is elementary in a technical sense (see e.g. [3]) because it can be stated as a Σ^0_2 formula in the language of second order arithmetic.

The following lemma is the main step in Hodges' treatment of generic formulas (compare with 1° page 344 of [8]). The proof adapts the argument of 4.4 in [3] to the present situation by making a clever use of property (*) below.

Lemma 2.3 ("Main Lemma"). If $\varphi(x,\bar{c}) \vee \psi(x,\bar{c})$ is left–generic over A then so is one of $\varphi(x,\bar{c})$, $\psi(x,\bar{c})$.

Proof. We are given that for some left–invariant $\delta(\bar{x}) \in L(A)$,

$$\models \exists \bar{x}\,\delta(\bar{x}) \wedge \forall \bar{x}\,(\delta(\bar{x}) \rightarrow \forall v \bigvee_{i<k} (\varphi(vx_i, \bar{c}) \vee \psi(vx_i, \bar{c}))).$$

Denote $\varphi^*(\bar{x}, v, \bar{y}) = \bigvee_{i<k} \varphi(vx_i, \bar{y})$. Notice that:

(*) $\models \varphi^*(\bar{x}; v, \bar{y}) \leftrightarrow \varphi^*(g\bar{x}; vg^{-1}, \bar{y})$.

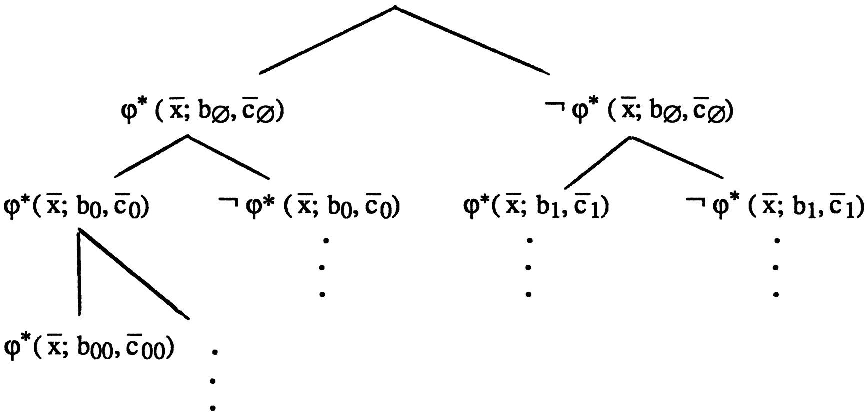
Define ψ^* in a similar way. We then have:

(**) $\models \exists \bar{x}\,\delta(\bar{x}) \wedge \forall \bar{x}\,(\delta(\bar{x}) \rightarrow \forall v\,(\varphi^*(\bar{x}; v, \bar{c}) \vee \psi^*(\bar{x}; v, \bar{c})))$.

In this proof, by the φ^*–tree of height k defined by $\langle b_t, \bar{c}_t : t \in {}^{<k}2\rangle$ we shall mean the one whose branches are

$$\{\varphi^{*s(i)}(\bar{x}; b_{s|i}, \bar{c}_{s|i}) : i < k\},$$

for $s \in {}^k2$ (here $\varphi^{*0} = \varphi^*$ and $\varphi^{*1} = \neg\varphi^*$). Pictorially:



As the theory T is stable, so is the formula φ^* and hence, there is a finite upper bound on the heights of φ^*–trees with all branches consistent. It follows that there is a largest integer k such that there is a φ^*–tree of height k all of whose branches are consistent with $\delta(\bar{x})$. (This is the only place in this proof in which any stability assumption is used). Let $\lambda(<\bar{x}_s>, <v_t, \bar{y}_t>)$, where $<\bar{x}_s> = <\bar{x}_s: s \in {}^k2>$ and $<v_t, \bar{y}_t> = <v_t, \bar{y}_t; t \in {}^{<k}2>$, be the formula stating that, for all $s \in {}^k2$, $\bar{x}_s$ satisfies $\delta(\bar{x})$ as well as the s–branch of the φ^*–tree defined by $<v_t, \bar{y}_t>$.

By the left–invariance of $\delta(\bar{x})$ and by (*), we see that $\lambda(<\bar{x}_s>,<v_t, \bar{y}_t>) \leftrightarrow \lambda(<g\,\bar{x}_s>,<v_t\, g^{-1}, \bar{y}_t>)$.

It follows that if $\lambda^*(<\bar{x}_s>) = \exists <v_t, \bar{y}_t> \lambda$ then λ^* is left–invariant. Also, by the definition of k, we know that $\models \exists <\bar{x}_s>\lambda^*$.

<u>Claim</u>. Either $\models \forall <\bar{x}_s>\ \lambda^* \rightarrow \forall v \bigvee_{s \in {}^k2} \varphi^*(\bar{x}_s; v, \bar{c}))$ or $\models \forall <\bar{x}_s>\ \lambda^* \rightarrow \forall v \bigvee_{s \in {}^k2} \psi^*(\bar{x}_s; v, \bar{c}))$, hence either $\varphi(x, \bar{c})$ or $\psi(x, \bar{c})$ is left–generic over A.

<u>Proof of the Claim</u>: If not, then there are $<\bar{a}_s'>$, h such that $\models \lambda^*(<\bar{a}_s'>) \wedge \bigwedge_{s \in {}^k2} \neg\psi^*(\bar{a}_s'; h, \bar{c})$, which implies, by (**),

(i) $\models \lambda^*(<\bar{a}_s'>) \wedge \bigwedge_{s \in {}^k 2} \varphi^*(\bar{a}_s'; h, \bar{c})$

and, also, there are $\bar{a}_s''$, g such that

(ii) $\models \lambda^*(<\bar{a}_s''> \wedge \bigwedge_{s \in {}^k 2} \neg \varphi^*(\bar{a}_s''; g, \bar{c})$.

An examination of (i) and (ii) reveals a certain gap that is easily bridged: by (*) and the left invariance of λ^*, (i) implies:

(i)′ $\models \lambda^*(<g^{-1}h\,\bar{a}_s'>) \wedge \bigwedge_{s \in {}^k 2} \varphi^*(g^{-1}h\,\bar{a}_s'; g, \bar{c})$.

By the meaning of λ^*, (i)′ and (ii) imply the existence of φ^*–trees of height k defined by sequences $<g_t', \bar{c}_t'>$ and $<g_t'', \bar{c}_t''>$ whose branches are satisfied by $<g^{-1}h\,\bar{a}_s'>$ and $<\bar{a}_s''>$ respectively. (Keep in mind that these sequences satisfy $\delta(\bar{x})$ as well). We obtain a φ^*–tree of height k+1 defined by $<g_t, \bar{c}_t; t \in {}^{<k+1}2>$ where $g_\varnothing = g$, $\bar{c}_\varnothing = \bar{c}$, $g_{0t} = g_t'$, $\bar{c}_{0t} = \bar{c}_t'$, $g_{1t} = g_t''$, $\bar{c}_{1t} = \bar{c}_t''$.

The branches of this tree are consistent with $\delta(\bar{x})$, a contradiction to the minimality of k. □

Let us remark that this proof is elementary in the sense that it can be formalized in RCA_0 (Recursive Comprehension Axiom with restricted induction, cf. [3]).

We now turn to corollaries of the Main Lemma. Notice first that 2.3 holds for right–genericity as well. This fact is immediately put to good use by Hodges:

<u>Theorem 2.4</u>. If $\varphi(x, \bar{c})$ is left–generic over A then it is right–generic over ∅. Hence, $\varphi(x, \bar{c})$ is left–generic over A iff it is right–generic over ∅ iff it is left–generic over ∅.

<u>Proof</u>. We are given that for some $\delta(\bar{x}) \in L(A)$,

$$\models \exists \bar{x}\ \delta(\bar{x}) \wedge \forall \bar{x}\,(\delta(\bar{x}) \rightarrow \bigvee_{i<k} \varphi(vx_i, \bar{c})).$$

Take $\bar{h} = <h_0,\ldots,h_{k-1}>$ such that $\models \delta(\bar{h})$. Then we have:

$\models \forall v \bigvee_{i<k} \varphi(vh_i, \bar{c})$, hence the formula $\bigvee_{i<k} \varphi(vh_i, \bar{c})$ is right–generic over ∅.

By 2.3 (applied to right–genericity) $\varphi(vh_i, \bar{c})$ is right–generic over ∅ for some i. But then, for all $g \in G$, $\varphi(vg, \bar{c}) \equiv \varphi(v(gh_i^{-1})h_i, \bar{c})$ does not fork over $\varnothing$. This means that $\varphi(v, \bar{c})$ or, if you wish, $\varphi(x, \bar{c})$ is right–generic over $\varnothing$. □

From now on, we say simply "generic" for "left– (or right)–generic over $\varnothing$".

If $X = \varphi(G, \bar{c}) = \{a \in G : \models \varphi[a, \bar{c}]\}$, then one denotes $gX = \{ga : a \in X\}$. The following easy corollary of 2.4 and its proof, shows the equivalence of 2.1 to Poizat's definition of a generic formula.

Theorem 2.5. Let $X = \varphi(G, \bar{c})$. $\varphi(x, \bar{c})$ is generic iff there are $g_0,\ldots,g_{k-1}$, such that $G = g_0X \cup g_1X \cup\ldots\cup g_{k-1}X$.

One more, quite straightforward, corollary of 2.3 is the existence of generic types (Theorem 3.1 below). The reader may go directly to the next section where we describe briefly this and a few other results on generic types; but if he selects to stay, we invite him to a discussion of the results presented so far.

Discussion. One can generalize 2.1 to formulas $\varphi(\bar{x}, \bar{c})$ with more than one free variable, by stipulating the nonforking of $\varphi(g\bar{x}, \bar{c})$ for all $g\in G$. Lemmas 2.2 and 2.3 generalize to this context (with the same proof). However, 2.4 and 2.5 are not true anymore. Indeed, let $\varphi(x,y; c)$ be the formula $x^{-1}y = c$. This formula is left generic over $\{c\}$. If c is suitably chosen then $\varphi(x,y; c)$ is not left–generic over $\varnothing$ and not right–generic over $\{c\}$. If $c=1$ then $\varphi(x,y ;c)$ is both left– and right–generic over $\varnothing$ but 2.5 does not hold for it.

This example is a particular case of a more general context. To present this context, let us return for a while to an arbitrary theory T with monster model $\mathbb{C}$. Assume nevertheless, that T has a definable group operation on a definable set G. Assume, furthermore, that we have a definable left–action of G on a definable set U. It is convenient to use a two sorted language with variables $v,u,v_1,u_1,\ldots$ ranging over G and $x,x_1,\ldots$ over U. Thus, a formula $\varphi(x, \bar{c})$ or a type $p(x)$ will always be assumed to imply $U(x)$.

In this general framework (which, by the way, has been considered also by Hrushovski in [5]) one can define left–generic formulas and prove 2.2 and 2.3. However, left–genericity over a set A does not imply left–genericity over $\emptyset$ as demonstrated by the formula $\varphi(x,y;c)$ discussed above (to fit that example into the general context, take $U = G \times G$ and define the action of G on U by $g(x,y) = (gx, gy)$). The same formula shows that 2.5 also fails.

Here is the place to ask a natural question. Returning for a moment to stable groups, is there a direct, transparent proof of the striking fact that left–genericity of a formula over a set implies left–genericity over $\emptyset$? Our discussion above shows that any such proof should use some special features that are not used in the proof of 2.3.

Back to our broader framework, we may ask ourselves where does the proof via 2.4 of the statement "left–generic over A implies left–genric over $\emptyset$" fail to generalize. The obvious obstacle is that we did not define what do we mean by right–genericity. One natural definition is the following. Say that $\varphi(x, \bar{c})$ is right–generic over A if for every $b \in U$, the formula $\varphi(vb, \bar{c})$ does not fork over A (of course, we assume that this formula implies G(v)). Lemma 2.2 holds for this notion and hence, right–genericity over a set A implies left–genericity over $\emptyset$. However, left–genericity does not imply right genericity. Still worse, the main Lemma 2.3 fails for right–generic formulas. Indeed, assume that we have an equivalence relation E on U with precisely two equivalence classes and such that G acts transitively on each class. Then, if $b,c \in U$ are representatives of the two classes then $E(x,b) \vee E(x,c)$ is right–generic over $\emptyset$ but neither E(x,b) nor E(x,c) are such (to see that such a situation can occur in a stable structure, take any stable group G, let $U = G \times \{b,c\}$ where b,c, are two distinct elements and for $(x,y), (x_1,y_1) \in U$ define $E((x,y),(x_1,y_1))$ iff $y=y_1$ and define the action of G on U by $g(x,y) = (gx,y)$). If we examine where does the proof of the main lemma fail, we are led to the conclusion that <u>if G happens to act transitively on U</u> then 2.3 does hold for right–genericity also. This much is easy to verify. Under the same assumption (that G acts transitively on U), 2.4 and hence, 2.5 are true as well.

To see that 2.4 holds indeed, we need a general fact which we state using the terminology introduced in Lemma 1.2 (and assuming that $u, v, v_1, \ldots$ range over a group G definable in a stable theory).

Theorem 2.6. If $\varphi(v, \bar{c}) \vee \psi(v, \bar{c})$ is almost satisfied over A via a left–(right)–invariant formula δ then one of φ, ψ is almost satisfied over A via a left–(right) invariant formula λ.

Sketch of the proof: the assumption implies that

$$\models \exists \bar{v}\, \delta(\bar{v}) \wedge \forall \bar{v}\, (\delta(\bar{v}) \rightarrow \forall u \bigvee_{i<k} (\varphi(uv_i, \bar{c}) \vee \psi(uv_i, \bar{c}))).$$

From this point on, proceed as in the proof of 2.3. □

Returning to the generalized 2.4, if $\varphi(x, \bar{c})$ is left–generic over A then, as in the proof of the original 2.4, we conclude that $\models \forall v \bigvee_{i<k} \varphi(vh_i, \bar{c})$ for a suitable $\bar{h}$. By 2.6, there is $i<k$ such that $\varphi(vh_i, \bar{c})$ is almost satisfied over $\varnothing$ via a right–invariant formula λ. If G acts transitively on U this implies immediately the right–genericity of $\varphi(x, \bar{c})$.

Remark. The assumption that G acts transitively on U is also needed by Hrushovski in [5] in order to get a smooth theory of generic types.

Another natural variant of right–genericity is the following. Assume that G also acts on U from the right in a definable way. In this case right–genericity has an obvious definition and 2.3 holds for both notions of one–sided genericity. However, 2.4 and 2.5 fail even if we assume the natural assumption of associativity: $(ga)h = g(ah)$ for all $g,h \in G$, $a \in U$. But again, if we assume in addition that G acts transitively on U from both the left and the right then 2.4 and 2.5 are true as well.

§3. Generic types.

We return to stable groups.

Lemma 3.1. If Γ is a (not necessarily complete) type closed under finite conjunctions all of whose formulas are generic then there is a type $p \in S(G)$ such that $p \supset \Gamma$ and all formulas of p are generic.

The proof uses 2.3 and is a standard application of Zorn's Lemma.

This result motivates the following:

Definition 3.2. A type p is generic iff all its formulas are generic.

This definition, in which p is supposed to be a complete type over any set, transforms 3.1 into Poizat's theorem on the existence of generic types. Some of the other results of [8] follow quite easily. Thus if p is generic and $q \supset p$ then q is generic iff q is a nonforking extension of p ("only if" is trivial while "if" follows using a result of Lascar stating that nonforking extensions of a given type $p \in S(A)$ can be mapped onto each other by isomorphisms over A – cf. e.g. 5.1(i) in [4]). Another, quite easily seen result is that for $p \in S(G)$, p is generic iff for all $g \in G$, gp does not fork over $\varnothing$. Also, if $p \in S(G)$ is generic then so is p^{-1}, the type of any element of the form c^{-1} where c realizes p. These and other remarkable results can be found in [8]. One result that does not appear there is due to Hodges as we mentioned in the introduction:

Theorem 3.3. If $p \in S(G)$ is left- (or right-) generic over a small set A (meaning that, for all $g \in G$, gp does not fork over A) then it is generic.

If $p \in S(G)$ is generic then so is gp for all $g \in G$. Thus g acts on the family of generic types. We close with a proof of Poizat's result, also mentioned in the introduction, that this action is transitive:

Theorem 3.4. If $p_1, p_2 \in S(G)$ are generic types then for some $g \in G$, $gp_1 = p_2$.

Proof. If $\varphi(x) \in p_2$ then $\varphi(x)$ is generic and by 2.5, there are $h_0,\ldots,h_{k-1}$ such that $\models \forall x \bigvee_{i<k} \varphi(h_i x)$.

Thus, for some h, $\varphi(hx)$ is consistent with p_1, hence, belongs to p_1. For $\varphi(x,\bar{y}) \in L$, let $\varphi'(x;v,\bar{y}) = \varphi(vx,\bar{y})$ and let $\theta_\varphi(v,\bar{y})$ be a φ'-definition of p_1. Take a small $M \langle G$. The type

$$q(v) = \{\theta_\varphi(v,\bar{c}) : \varphi(x,\bar{c}) \in p_2 \restriction M\}$$

is consistent, by the opening remark of this proof. Let g realize q. Then

$$\varphi(x) \in p_2 \restriction M \Rightarrow \varphi(gx) \in p_1 \Rightarrow \varphi(x) \in gp_1.$$

It follows that $gp_1 \restriction M = p_2 \restriction M$ and as both gp_1 and p_2 do not fork over M, we have $gp_1 = p_2$. □

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