Abstract Homomorphisms of Big Subgroups of Algebraic Groups

BORIS WEISFEILER

This is a survey of results centered around the problem of describing automorphisms, isomorphisms, or homomorphisms between subgroups of algebraic groups. An attempt was made to follow the lectures rather closely. However some material was added to give a broader view and other material deleted in an attempt to stay within allotted space. I apologize to all the people whose results are bypassed or not fully represented.

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NOTATION

Most of our notation and terminology is standard and is explained at appropriate places; we record it here for the benefit of a reader who does not want to search for these explanations. k always denotes a field; it will be infinite unless otherwise specified; its characteristic is "char k"; $\bar{k}$ is its algebraic closure; $\mathbb{Q}_p$ is the field of p-adic numbers and $\mathbb{Z}_p$ is its ring of integers. $R_{K/k}$ denotes Weil's restriction of scalars from a ring $K \supseteq k$ to k; for a functor V from rings to sets one has $(R_{K/k}V)(L) = V(L \otimes_k K)$. If G is a group then $N_G(A)$ (resp. $Z_G(A)$) is the normalizer (resp. centralizer) in G of a subset A; if G is an algebraic group then $N_G(A)$ and $Z_G(A)$ can be considered as algebraic groups. $[A,B]$ is the commutator of subgroups A and B of G. For an algebraic or Lie group G, Lie G is its Lie algebra. For a Lie group G, μ_G is its Haar measure. For $SL_n(k)$, $SO(n,1)$ etc., $PSL_n(k)$, $PSO(n,1)$ etc., denote the quotients of the former groups by their centers. A subset X of a topological space is locally closed if it is open in its closure and it is relatively compact if its closure is compact.

1. MOTIVATION

The problem we are concerned with belongs to the following broad class: given an object having several structures, how much information is lost if some of the structures are forgotten? and what is the minimum information required to recover the complete information? Ours is the case of an algebraic group G which we consider as a functor

G: (fields) $\to$ (groups). Take the value $G(k)$ of G at a fixed field k. Does it determine the functor? More precisely: If $\alpha: G(k) \to G'(k')$ is an abstract group homomorphism, does it come from a homomorphism of functors? If it does one goes still further and asks: what are subgroups H of $G(k)$ and group homomorphisms $H \to G'(k')$ that come from a map of functors?

(1.1) A classical example of a problem of the same broad class is the Fundamental Theorem of Projective Geometry (FTPG for short). Consider the projective space $\mathbb{P}^n$ $n \geq 1$, as an algebraic variety over k. It has a distinguished family C of subvarieties: curves of degree 1 (called lines). Now consider an algebraic variety V as a functor $V: k \to V(k)$ from fields to sets. Then we have a functor $\mathbb{P}^n$: (fields) $\to$ (sets) and $\mathbb{P}^n(k)$ has, for any field k, a distinguished family of subsets: sets $L(k)$ for $L \in C$, L defined over k. Classically, $\mathbb{P}^n(k)$ is the set of 1-dimensional vector subspaces of k^{n+1} and distinguished subsets, still called lines, are the sets of one-dimensional subspaces of k^{n+1} contained in a 2-dimensional subspace of k^{n+1}. Let $\alpha: \mathbb{P}^n(k) \to \mathbb{P}^m(k')$, $n,m \geq 2$, be a bijection which, together with its inverse, carries lines into lines. Then the FTPG says that if $m,n \geq 2$, then $m = n$ and there exist a field isomorphism $\varphi: k \to k$ and a bijective map $\beta: k^{n+1} \to k'^{n+1}$ with $\beta(ax+by) = \varphi(a)\beta(x) + \varphi(b)\beta(y)$ for

$a,b \in k$, $x,y \in k^{n+1}$, such that $\alpha(kx) = k'\beta(x)$ for $x \in k^{n+1}$, $x \neq 0$. (Recall that the above properties of β are called φ-semilinearity.) We can reformulate this result in a more invariant form.

(1.2) Assign (see [BT,1.7]) to an affine algebraic variety V defined over k and to a field homomorphism $\varphi\colon k \to k'$ the algebraic variety ${}^{\varphi}V$ defined over k' by taking for the ring $k'[{}^{\varphi}V]$ of regular functions the algebra $k[V] \otimes_k k'$ (φ is used to identify k as a subfield of k'). For general V we cover V by open affine pieces $V = \cup V_i$, each defined over k, and set ${}^{\varphi}V = \cup {}^{\varphi}V_i$.

The natural injection $V(k) \to {}^{\varphi}V(k')$ is denoted φ^{o} (the notation $V(\varphi)$, the value of the functor V on the map φ, is also used in the literature).

(1.3) Now the FTPG takes the form: Let $\alpha\colon \mathbb{P}^n(k) \to \mathbb{P}^m(k')$, $n,m \geq 2$, be a bijection such that both α and α^{-1} carry lines into lines. Then there exist a field isomorphism $\varphi\colon k \to k'$ and a k'-isomorphism $\tilde{\beta}\colon {}^{\varphi}\mathbb{P}^n \to \mathbb{P}^m$ of algebraic varieties such that $\alpha = \tilde{\beta} \circ \varphi^{o}$.

(1.4) Generalizations and interpretations of the FTPG are closely related to different advances in and approaches to the problem of abstract homomorphisms of subgroups of algebraic groups. The reasons for this are many. But the fact itself is not surprising at all if one remembers that the first result in the area, a description by O. Schreier and B.L. van der Waerden (Abh. Math. Sem. Univ. Hamburg 6(1928), 303-322) of the automorphisms of the projective special linear groups $PSL_n(k)$ was based on the FTPG: for any automorphism α of $PSL_n(k)$ with $n \geq 3$, there exist a field automorphism $\varphi: k \to k$ and a matrix $A \in GL_n(k)$ such that $\alpha(\bar{S})$ is either $\overline{A\varphi(S)A^{-1}}$ or $\overline{A\varphi(S^t)^{-1}A^{-1}}$; here $\bar{S}$ is the image of $S \in SL_n(k)$ in $PSL_n(k)$, S^t is the transpose of S, and $\varphi(S) = (\varphi(s_{ij}))$ if $S = (s_{ij})$. In the more abstract language (1.2) the claim of this theorem is: There exist a field automorphism $\varphi: k \to k$ and an isomorphism $\tilde{\beta}: {}^{\varphi}PSL_n \to PSL_n$ of algebraic groups over k such that $\alpha(h) = \tilde{\beta} \circ \varphi^{\circ}(h)$ for $h \in PSL_n(k)$

Since 1928 when this result was discovered the area has been developed by many a mathematician of renown: E. Cartan, H. Freudenthal, J. Dieudonné, Hua Lo-keng, Wan Zhe-xian, I. Reiner, C. Rickart, O.T. O'Meara, A. Hahn, D. James, B. McDonald, G. Mostow, A. Borel, J. Tits, G. Prasad, G. Margulis, M. Raghunathan and many others. We will not attempt here to give a historical survey. Instead we will outline the major achievements in the subject and their interrelation. The ideal goal of the theory would be to obtain a theorem which includes all known results. This goal is, of course, unrealistic. But it is good to keep it in mind for orientation and proper perspective. For this same purpose we discuss occasionally results from adjacent areas.

2. EVIDENT RESTRICTIONS

Let G and G' be connected algebraic groups defined over fields k and k'. Let H be a subgroup of $G(k)$ and $\alpha: H \to G'(k')$ a group homomorphism.

(2.1) Suppose that $G = G''$ is the group $\mathbb{G}_a$ (so that $\mathbb{G}_a(k)$ is the additive group of k). Let B be a basis of k over its prime field. Then <u>any map</u> $B \to k$ gives rise to a homomorphism $\mathbb{G}_a(k) \to \mathbb{G}_a(k)$. Clearly, such homomorphisms are too general to permit but superficial description. Since similar constructions can be carried out for other commutative groups, we do not want our group G to be commutative. Neither do we want it to have commutative quotients. This restricts us to the case when G <u>coincides with its own commutator subgroup</u>. In particular, G is a <u>linear algebraic group</u>. Actually we often restrict ourselves to <u>semi-simple groups</u>.

(2.2) Next, if we want to recover G from a homomorphism $\alpha: H \to G'(k')$, where H is a subgroup of $G(k)$, then H should be Zariski-<u>dense</u> in G. This condition can sometimes be replaced by additional assumptions on α, H, and k' (say, k and k' are finite, $H = G(k)$, and α is an isomorphism). But in general, it is a very reasonable assumption.

(2.3) By a theorem of J. Tits (J. Algebra 20(1972), 250-270, Theorems 3,4(vi)) $G(k)$ contains Zariski-dense _free_ subgroups H if either char $k = 0$ or k is not algebraic over its prime field. For a free H no _good_ map of G into G' can of course be recovered from a general homomorphism $H \to G'(k')$. This tells us that H must be _well embedded_ in G. To specify the meaning of this "well embedded" is one of the problems of the theory. Examples of relatively small but well-embedded subgroups are (1) _full_ (of transvections or of rotations) subgroups of O.T. O'Meara and (2) _lattices_ in semi-simple groups over local fields (see §6 below). In a number of cases it has been shown that groups of type (1) and (2) are actually arithmetic (see (6.7) below and also [Va], [Se], and [VW]).

3. ISOTROPIC SEMI-SIMPLE GROUPS OVER FIELDS.

(3.1) This case can be considered as a paradigm because the situation here is well-understood and can be completely described. Let G be a connected semi-simple algebraic group defined over k (or, shorter, k-group). G is called _absolutely almost simple_ if all normal subgroups of $G(\bar{k})$ are finite central; G is _almost_ k-_simple_ if only finite central algebraic normal subgroups of G are defined over k; G is k-_isotropic_ if G contains a proper parabolic k-subgroup; $G^+(k)$ is the _normal subgroup_ of G(k) generated by U(k) where U is the unipotent radical of a minimal parabolic k-subgroup. For example, $SL_n(k)$ is a simply connected k-group. Its subgroup U is the group of upper triangular matrices with all diagonal entries equal to 1. Since $SL_n(k)$ is generated by transvections, it is generated by the conjugates of $U(k)$. Thus $(SL_n)^+(k) = SL_n(k)$.

Another example is $SO_n(f)$, the special orthogonal group of a non-degenerate (and of defect at most 1 if char $k = 2$) quadratic form $f = \sum_{1 \le i \le j \le n} a_{ij}x_i x_j$. This group is defined over k if $a_{ij} \in k$ for $1 \le i, j \le n$. It is k-isotropic iff f represents zero non-trivially over k. In this latter case $(SO_n(f))^+(k) = SO_n'(f,k)$, the spinorial kernel group.

The following theorem is a generalization of a result previously known for "most" groups.

(3.2) THEOREM. (J. Tits, Ann. Math. 80(1964), 313-329) Let G be an almost k-simple and isotropic over k algebraic k-group.

(i) If $H \supseteq G^+(k)$ is a subgroup of $G(k)$ then any normal subgroup of H is either central or contains $G^+(k)$.

(ii) One has $G(k) = Z(k) \cdot G^+(k)$ where Z is the centralizer of a maximal split k-torus of G.

(3.3) To see the relevance of this Theorem to our problem, take a homomorphism $\alpha: H \to Q$ of H into a group Q. Then by (3.2(i)) either Ker α is finite and central or Ker α contains $G^+(k)$. In other words, α is an <u>almost monomorphism or</u> an <u>almost trivial</u> map. Let us look at the latter case first. The problem is to understand $G^-(k) = G(k)/G^+(k)$. This consists of two different contributions.

(3.4) Let $\pi: \tilde{G} \to G$ be the universal cover of G. It is easily seen that $\pi(\tilde{G}(k)) \supseteq G^+(k)$. Therefore $G^-(k)$ has $G(k)/\pi(G^+(k))$ as a factor-group. This latter can be easily bounded using the exact sequence of <u>Galois cohomology</u> associated with the short exact sequence $1 \to \operatorname{Ker} \pi \to \tilde{G} \to G \to$ $\to 1$. Namely, we get the exact sequence $1 \to H^0(k,\operatorname{Ker}\pi) \to$ $\to H^0(k,\tilde{G}) \to H^0(k,G) \xrightarrow{\delta} H^1(k,\operatorname{Ker}\pi)$ or $1 \to (\operatorname{Ker}\pi)(k) \to$ $\to \tilde{G}(k) \to G(k) \xrightarrow{\delta} H^1(k,\operatorname{Ker}\pi)$. Thus δ identifies $G(k)/\pi(\tilde{G}(k))$ with a subgroup of $H^1(k,\operatorname{Ker}\pi)$. This latter is a commutative periodic group of exponent equal to the exponent of (the finite algebraic group) $\operatorname{Ker}\pi$. For example if $G = SO_n(f)$, $n \geq 3$, then $\tilde{G} = \operatorname{Spin}_n(f)$ and if $\operatorname{char} k \neq 2$ then δ is the spinor norm homomorphism. See J.-P. Serre, Cohomologie Galoisienne, (Lecture Notes in Math., vol. 5, Springer, Berlin; 1964) for details and additional examples.

(3.5) Another contribution, $\tilde{G}^-(k) = \tilde{G}(k)/\tilde{G}^+(k)$ is much more interesting. It comes from $F(k)/(F(k) \cap \tilde{G}^+(k))$ where $F = [Z_{\tilde{G}}(T), Z_{\tilde{G}}(T)]$ (see (3.2)(ii)). It is a recent and unexpected result that $\tilde{G}^-(k)$ can be nontrivial. To describe an example of such a $\tilde{G}$ consider a central division algebra D over k with $\dim_k D = N^2 < \infty$. Then there exists an (automatically simply connected) algebraic k-group G such that $G(k) \simeq SL_n(D)$, $n \geq 2$, and $G(\bar{k}) \simeq SL_{nN}(\bar{k})$. Set $SK_I(D) = D^1/[D^*,D^*]$ where D^1 is the group of elements of reduced norm 1 in D. The Dieudonné determinant establishes the isomorphism $G^-(k) \simeq SK_1(D)$. Examples of D such that $SK_1(D) \neq \{1\}$ were constructed by V. Platonov and V. Yanchevsky, and somewhat later by P. Draxl. The simplest example I know of a D with $SK_1(D) \neq 1$ is described in P. Draxl's lecture notes [Dr , §24]. Moreover there are fields k such that $SK_1(D)$ runs over all commutative finite groups when D runs over all division algebras over k, and for every countable commutative torsion group C of bounded period there is a field k and a central division algebra D over k such that $SK_1(D) = C$. Nevertheless $SK_1(D)$ is always trivial if k is a local (T. Nakayama and Y. Matsushima, 1943) or a global (S.S. Wang, 1949) field. It was because of these results that the equality $SK_1(D) = 1$ was expected to hold always. We refer the reader to [Dr], [DK], [T5] and references therein for more information. We conclude by mentioning that V. Voskresensky [Vo] gave an interesting algebro-geometric interpretation of SK_1.

(3.6) To proceed we need certain notions. An algebraic homomorphism $\beta: G \to G'$ is an <u>isogeny</u> if β is an epimorphism with finite (schematic) kernel; an isogeny β is <u>special</u> if its differential is not identically zero.

If char k = p the Frobenius map Fr: $SL_n \to SL_n$ given by $Fr(s_{ij}) = (s_{ij}^p)$ is a non-special isogeny. The standard (see e.g. [D]) map $SO_{2n+1} \to Sp_{2n}$ in characteristic 2 is a special isogeny; for pairs (G,G') a non-trivial (i.e., not an isomorphism) special isogeny exists only when char $k \leq 3$: if char k = 2 then (G,G') must have type $(B_n,C_n),(C_n,B_n),(F_4,F_4)$ and if char k = 3 then only the pair (G_2,G_2) can be involved. See [BT, §3] for more details. Using these notions the situation when α is an "almost monomorphism" (see (3.3) above) is completely described by

(3.7) THEOREM (A. Borel and J. Tits [BT]). Let G be an absolutely almost simple algebraic group defined over an (infinite) field k. Let k′ be another field and let G′ be an absolutely almost simple algebraic group defined over k′. Let H be a subgroup of G(k) containing $G^+(k)$ and let α: $H \to G'(k')$ be a homomorphism of abstract groups such that $\alpha(G^+(k))$ is Zariski dense. Assume that either G is simply connected or G′ is adjoint. Then there exist a field homomorphism φ: $k \to k'$, a special k′-isogeny β: ${}^{\varphi}G \to G'$ and a homomorphism γ: $H \to \text{center}(G'(k'))$ such that $\alpha(h) = \gamma(h) \cdot \beta(\varphi^{\circ}(h))$ for all $h \in H$.

Note that γ here (classically called a radial homomorphism) is an almost trivial map. Therefore (3.4) and (3.5) can be used to study γ. The ideas, as well as important technical steps, of the proof of (3.7) are outlined on pp. 502,503 of the Introduction to [BT] and by R. Steinberg in his Bourbaki seminar report [St2] on the paper [BT].

Theorem (3.7) was, even in the case of SL_n, a vast technical and conceptual generalization of previously known results on homomorphisms of isotropic groups over fields. It opened new avenues of research and set new standards. Among its most important achievements was a uniform treatment of all (isotropic semi-simple) algebraic groups (including exceptional ones) and all homomorphisms between them (including isomorphisms $PSL_4 \simeq PSO_6$, $SO_5 \simeq PSp_4$, triality for D_4, and the non-trivial special isogeny $SO_{2n+1} \to Sp_{2n}$ in characteristic 2). But still more remarkable was the step from isomorphisms and automorphisms to homomorphisms with dense image.

(3.8) Among crucial technical tools the proof of (3.7) uses are the existence of Bruhat decomposition in H and the algebraic geometry of the target group G′. These tools are still available if H is replaced by a (non-algebraic) group with a BN-pair (say by an isotropic classical group over an infinite-dimensional division algebra or by the infinite twisted groups ${}^2B_2$, ${}^2G_2$, and ${}^2F_4$ of M. Suzuki and R. Ree, see [St1, §11]). But it is difficult to see how the assumptions on G′ can be weakened unless methods compensating for the absence of algebraic geometry are devised.

(3.9) To conclude this discussion of isotropic groups we remark that (3.7) implies (see [T1, no. 10]) a generalization of the FTPG to irreducible Tits buildings of spherical type (the particular case of (3.7) when α is an isomorphism, $G = SL_n$, $G' = SL_m$, $m,n \geq 3$, is equivalent to the FTPG as stated in (1.1)). Conversely, an analogue of the FTPG for Tits buildings would, it seems, imply (3.7) in relative rank ≥ 2.

4. SHORT REMARKS ON O'MEARA'S METHOD

The lectures did not cover work of O.T. O'Meara and his school. There are excellent books [O'M1] and [O'M2] by O'Meara himself which have made his theory accessible to a wider audience than these notes can hope for. Therefore we discuss O'Meara's approach only briefly to indicate the features which make it conceptually different. O'Meara's method <u>associates</u> with a subgroup of $G(k)$ a <u>certain geometry</u>. This is, roughly, a Tits geometry if G is isotropic But it is some other yet <u>mysterious</u> geometry if G is anisotropic. In this latter case the geometry is constructed from a group having no parabolic k-subgroups. This geometry (and the rule used to construct it from the group) are therefore very <u>fascinating</u>. On the other hand O'Meara's method permits one to <u>probe</u> into which groups are <u>well embedded</u> (in the sense of (2.3) above). This is another conceptually important feature of the O'Meara method.

One class of such well-embedded subgroups are the <u>full</u> subgroups of $GL_n(D)$, D a division ring. We recall that a subgroup H of $GL_n(D)$ is <u>full</u> (<u>of</u> <u>transvections</u>) if for every hyperplane h in D^n and every vector $v \neq 0$, $v \in h$, there exists $A \in H$ such that $\mathrm{Im}(A - \mathrm{Id}_n) = Dv$ and $\mathrm{Ker}(A - \mathrm{Id}_n) = h$. A sample result of O.T. O'Meara [O'M3], as extended in [So], is

(4.1) THEOREM. Let H and H' be full subgroups in $PGL_n(D)$ and $PGL_{n'}(D')$ respectively, where D and D' are division rings. Let $\alpha : H \to H'$ be an isomorphism. If $n,n' \geq 3$ then $n = n'$ and there exist a ring isomorphism $\varphi : D \to D'$ (or $\varphi : D^{\circ} \to D'$) and a φ-semilinear map $\beta : D^n \to D'^n$ (or $\beta : D^{\circ n} \to D'^n$) such that $\alpha(h) = \overline{\beta} \circ h \circ \overline{\beta}^{-1}$ (or $\alpha(h) = \overline{\beta} \circ \check{h} \circ \overline{\beta}^{-1}$) for all $h \in H$. (Here $\overline{\beta}$ is the map induced by β on projective spaces and $\check{h}$ is the projective contragredient of h.)

(4.2) It has been proven in [Va] and [Se] that a full (of transvections) subgroup H of $GL_n(D)$, $n \geq 3$, contains a congruence subgroup of $GL_n(\mathcal{O})$ where $\mathcal{O}$ is a subring of D whose field of quotients is D. This is a kind of "<u>arithmeticity</u>" result. It has been extended to appropriately interpreted full subgroups of Chevalley groups, see [VW].

(4.3) We remark that O'Meara's method can be (as a paper [Ha1] of A. Hahn shows) applied to groups which are not quite full. Finally, we want to point out a similarity between O'Meara's and Mostow's (6.5) results —both impose the same conditions on both H and $\alpha(H)$ — whereas Margulis' superrigidity (6.6) is more akin to Borel-Tits' Theorem (3.7): only the condition of Zariski density is imposed on $\alpha(H)$.

5. TITS' RESULTS ON HOMOMORPHISMS OF LIE GROUPS

Again, as with O'Meara's method, the author's exposition in [T3] is beyond imitation, but, on the other hand, the results and ideas are so substantial that they may not be bypassed.

(5.1) THEOREM. Let G be an algebraic simply connected group defined over $\mathbb{R}$, the reals. Suppose $G = [G,G]$ (see (2.1)). Let H be a Lie group countable at infinity (e.g., $H = G'(\mathbb{R})$ where G' is an algebraic $\mathbb{R}$-group) and let $\alpha: G(\mathbb{R})^{o} \to H$ be a group homomorphism. Then there exist a finite-dimensional $\mathbb{R}$-algebra K, a homomorphism of rings $\varphi: \mathbb{R} \to K$ with dense (in the Hausdorff topology) image, and a homomorphism of Lie groups $\beta: {}^{\varphi}G(K) \to H$ such that $\alpha = \beta \circ \varphi^{o}$. (Here ${}^{\varphi}G$ and φ^{o} are defined in the same way as they were defined before for field homomorphisms.)

(5.2) The following example outlines the meaning of (5.1). Let $K = \mathbb{R}[\varepsilon]$ where $\varepsilon^2 = 0$ and let SO_n be the special orthogonal $\mathbb{R}$-group of the form $\sum_{1\leq i\leq n} x_i^2$. Thus $SO_n(\mathbb{R})$

is the usual special orthogonal group. $SO_n(\mathbb{R}[\varepsilon])$ is also a Lie group, an extension of $SO_n(\mathbb{R})$ by a vector group $V \simeq \mathbb{R}^{\frac{1}{2}n(n-1)}$ with $SO_n(\mathbb{R})$ acting on V via the adjoint representation. (Actually, the Lie algebra Lie $G(\mathbb{R})$ of $G(\mathbb{R})$ can be defined by Lie $G(\mathbb{R}) = \mathrm{Ker}(G(\mathbb{R}[\varepsilon]) \xrightarrow{G(r)} G(\mathbb{R}))$, where $r: \mathbb{R}[\varepsilon] \to \mathbb{R}$ is given by $r(a + b\varepsilon) = a$.) Let $d \in \mathrm{Der}_{\mathbb{Q}} \mathbb{R}$ be a derivation of $\mathbb{R}$ and $\varphi: \mathbb{R} \to \mathbb{R}[\varepsilon]$ the ring homomorphism given by $\varphi(\lambda) = \lambda + d(\lambda)\varepsilon$. The image of φ is dense in $\mathbb{R}[\varepsilon]$ if $d \neq 0$. In this case $\varphi^0: SO_n(\mathbb{R}) \to SO_n(\mathbb{R}[\varepsilon])$ is a homomorphism of Lie groups with dense image. To conclude this example we remark that $\dim \mathrm{Der}_{\mathbb{Q}} \mathbb{R} = \infty$ so that "bad" homomorphisms do exist. To construct one of them one takes a transcendental element, say t, in $\mathbb{R}$ over $\mathbb{Q}$, and then extends the differentiation d/dt from $\mathbb{Q}(t)$ to $\mathbb{R}$.

(5.3) Below we reproduce Tits' outline of the proof of (5.1). We suppose for simplicity that $(G/\mathrm{Rad}\, G)(\mathbb{R})$ has no compact factors. Then there exist algebraic $\mathbb{R}$-subgroups F_i of G, $i = 1,\ldots, d = \dim G$, isomorphic to a semi-direct product $\mathbb{G}_a \rtimes \mathbb{G}_m$ of the additive and multiplicative groups such that Lie $G = \bigoplus_{1\le i\le d}$ Lie $[F_i,F_i]$. Set $G_i = [F_i,F_i]$ and $\alpha_i = \alpha \mid G_i(\mathbb{R})$. Since $G_i(\mathbb{R}) \simeq \mathbb{R}$ we have $\alpha_i:\mathbb{R} \to H$. Let A_i be the Hausdorff closure of $\alpha_i(\mathbb{R})$. It is a commutative group. Denote by K_i the (commutative) subring of the endomorphism ring of A_i generated by the automorphisms $a \mapsto \alpha(x)a\,\alpha(x^{-1})$ for $x \in F_i$, $a \in A_i$. This gives us, for all i, homomorphisms of algebras $\varphi_i: \mathbb{R} \to K_i$ with dense image, and homomorphisms of Lie groups $\tilde{\beta}_i: {}^{\varphi_i}G_i(K_i) \to H$ such that $\alpha_i = \tilde{\beta}_i \circ \varphi_i^o$. Consider $\varphi = \oplus\varphi_i: \mathbb{R} \to \oplus K_i$, and denote by K the closure of $\varphi(\mathbb{R})$ in $\oplus K_i$. Since each K_i is still a quotient of K and $\varphi_i: \mathbb{R} \to K_i$ factors through $\mathbb{R} \xrightarrow{\varphi} K \twoheadrightarrow K_i$, there still are $\beta_i: {}^{\varphi}G_i(K) \to H$ such that $\alpha_i = \beta_i \circ \varphi^o$ for all i. Since $\mathrm{Lie}\ G(\mathbb{R}) = \oplus\mathrm{Lie}\ G_i(\mathbb{R})$ we have $\mathrm{Lie}\ {}^{\varphi}G(K) = \oplus\mathrm{Lie}\,{}^{\varphi}G_i(K)$. Therefore $U = \Pi U_i$ is a neighborhood of identity e in ${}^{\varphi}G(K)$ if the U_i are neighborhoods of e in ${}^{\varphi}G_i(K)$. We can take U_i to be so small that $\Pi U_i \to U$ is a homeomorphism Since the β_i are analytic we recover an analytic map $\beta_U = \Pi\beta_i: U \to H$. Take connected open $V_i \subseteq U_i$ so that, $e \in V_i$, $V \cdot V \subseteq U$, and $V^{-1} \subseteq U$ for $V = \Pi V_i$. Then $\alpha \circ \varphi^o: V \cap \varphi^o(G(\mathbb{R})) \to H$ coincides with β_U on V. Since α is a homomorphism and $\varphi^o(G(\mathbb{R}))$ is dense in ${}^{\varphi}G(K)$ it follows that β_U is a local homomorphism. Therefore β_U extends to a homomorphism $\beta: {}^{\varphi}G(K) \to H$. Since $\beta \circ \varphi^o$ coincides with α on $U \cap \varphi^o(G(\mathbb{R}))$ and since ${}^{\varphi}G(K)$ is connected (use the simple connectedness of G) it follows that $\alpha = \beta \circ \varphi^o$.

(5.4) Going over Tits' proof (5.3) of (5.1) the reader is most impressed by the contrast between the simplicity of the ideas and power of the result. One would expect, therefore, that the methods of [T3] would have wider applicability. And indeed they do. Tits points out in [T3, 5.1] that results similar to (5.1) can be obtained for homomorphisms of simply connected Chevalley groups over fields of characteristic different from 2 into arbitrary algebraic groups. He mentions that in characteristic 2 there are counter-examples.

6. RIGIDITY, STRONG RIGIDITY, AND SUPERRIGIDITY OF LATTICES

Let G be a semi-simple algebraic group defined over $\mathbb{R}$. Then $G(\mathbb{R})$ is a Lie group. Let Γ be a <u>lattice</u> in $G(\mathbb{R})$ (i.e., a discrete subgroup such that the volume $\int_{G/\Gamma} dg$ of G/Γ is finite; here dg is induced by a Haar measure, see [Hu, (1.1)]). Then Γ is finitely generated (see [Hu, (1.3)]). Let $x_1,\ldots,x_n$ be a set of generators. Every homomorphism $\alpha\colon \Gamma \to G(\mathbb{R})$ determines a point $(\alpha(x_1),\ldots,\alpha(x_n)) \in G(\mathbb{R})^n$. Let $R(\Gamma,G)$ be the set of $(s_i) \in G(\mathbb{R})^n$ for which $x_i \mapsto s_i$ determines a homomorphism, say $\rho\colon \Gamma \to G(\mathbb{R})$, such that $\rho(\Gamma)$ is a lattice. (This is a variation of the usual definition of $R(\Gamma,G)$). The Hausdorff topology of $G(\mathbb{R})^n$ induces a topology on $R(\Gamma,G)$, and $G(\mathbb{R})$ acts on $R(\Gamma,G)$ continuously via

$(\mathrm{Ad}\, g)(s_i) = (g\, s_i g^{-1})$. Call $\alpha \in R(\Gamma,G)$ locally (or weakly) rigid if the orbit $\mathrm{Ad}G(\mathbb{R})(\alpha)$ of α is open in $R(\Gamma,G)$.

We recall that a lattice Γ in $G(\mathbb{R})^{\circ}$ is irreducible if the projection of Γ on every simple factor of $G(\mathbb{R})^{\circ}$ is dense, see [Hu, (1.5)]; uniform if G/Γ is compact; and torsion free if Γ contains no elements of finite order (this condition always holds for an appropriate subgroup of finite index in Γ by [R, Theorem 6.11]). For a lattice Γ in $G(\mathbb{R})^{\circ}$ let id_{Γ} denote the identity embedding of Γ.

(6.1) THEOREM. (A. Weil, see [R, Theorems 6.7, and 7.66]). Let Γ be a uniform irreducible lattice in $G(\mathbb{R})^{\circ}$. If G is adjoint semi-simple and $G(\mathbb{R})^{\circ}$ has no compact factors and is not isomorphic to $PSL_2(\mathbb{R})$ then id_{Γ} is locally rigid.

This theorem was first established by A. Selberg [S] for $G = PSL_n$, $n \geq 3$. There he also pointed out an application:

(6.2) THEOREM. (Weak arithmeticity, see [R, Proposition 6.6]). Let G be as in (6.1). If Γ is a locally rigid lattice in $G(\mathbb{R})^{\circ}$, then there exist a number field $k \subseteq \mathbb{R}$, a structure of an algebraic group defined over k on G, and an element $g \in G(\mathbb{R})$ such that $g\, \Gamma\, g^{-1} \subseteq G(k)$.

(6.3) REMARK. Actually, see e.g., [M2, Lemma 1], the field $\mathbb{Q}(\mathrm{tr}\, \mathrm{Ad}\, \Gamma)$ generated over $\mathbb{Q}$ by the traces of all

$\mathrm{Ad}\gamma$, $\gamma \in \Gamma$, can be taken as k.

Weil's theorem (6.1), and therefore its corollary (6.2), were extended to certain non-uniform lattices by H. Bass, A. Borel, H. Garland, and M. Raghunathan. The methods of A. Weil were cohomological — it was at the time when deformations and their relation to cohomology were intensively explored. The proof consisted of two steps. One of them was to show that $H^1(\Gamma, \mathrm{Ad}) = 0$ (cohomology of Γ with coefficients in Lie $G(\mathbb{R})$, and the second one was to establish that the existence of deformations implies $H^1(\Gamma, \mathrm{Ad}) \neq 0$.

In the same groundbreaking paper [S], A. Selberg conjectured that any uniform irreducible lattice Γ in a group $G(\mathbb{R})^o$ assumed adjoint, semi-simple, without compact factors and different from $PSL_2(\mathbb{R})$ is arithmetic.

(6.4) Recall that a lattice Γ in $G(\mathbb{R})^o$ is called arithmetic if there exist an algebraic group H defined over $\mathbb{Q}$ and an epimorphism of Lie groups $\pi: H(\mathbb{R})^o \to G(\mathbb{R})^o$ with compact kernel such that $\Gamma \cap \pi(H(\mathbb{Z}))$ is of finite index in both Γ and $\pi(H(\mathbb{Z}))$. ($H(\mathbb{Z})$ is the set of integral matrices in $\varphi(H(\mathbb{Q}))$ where φ is a faithful $\mathbb{Q}$-representation of H; this definition of arithmeticity does not depend on φ or on the choice of a $\mathbb{Q}$-basis in the representation space of φ.) An example: let k be a totally real number field, $\mathcal{O}$ the ring of integers in k,

$f(x) = \sum_{1\le i\le n} a_i x_i^2$, $a_i \in k^*$, a quadratic form on k^n,

$G = SO_n(f)$ and $\Gamma = SO_n(f,\mathcal{O})$. Let $\sigma_0 = 1, \sigma_1, \ldots, \sigma_m$ be the embeddings of k into $\mathbb{R}$. Suppose that

$a_i < 0$, $1 \le i \le r \le n$, $a_i^{\sigma_j} > 0$ for $r < i \le n$, $j \ge 0$,

and $a_i^{\sigma_j} > 0$ for all i and all $j > 0$. Then Γ is an arithmetic lattice in $G(\mathbb{R})$ (take $H = R_{k/\mathbb{Q}}G$, then our choice of the a_i implies that $H(\mathbb{R}) = G(\mathbb{R}) \times$ (compact group)) This lattice is uniform if $f(x) = 0$ has no non-zero solutions, in particular, if $[k: \mathbb{Q}] > 1$. It is not uniform if $k = \mathbb{Q}$ and $f(x) = 0$ for some $x \in \mathbb{Q}^n$, $x \ne 0$.

In the late sixties counter-examples to the conjecture in [S] were found in the groups $SO(n,1)$, $3 \le n \le 5$, see §8. New versions of the conjecture were proposed by A. Selberg and I. Piatetski-Shapiro. The latter's version was very general, it included lattices in products of groups $G_i(k_i)$, G_i semi-simple over k_i and k_i locally compact. It was essentially I. Piatetski-Shapiro's conjecture that was subsequently established by G. Margulis (see (6.7)). However, a breakthrough was achieved by G. Mostow whose results are summed up by

(6.5) THEOREM (Strong rigidity). Let G and G' be connected adjoint algebraic semi-simple groups over $\mathbb{R}$. Suppose that $G(\mathbb{R})^0$ and $G'(\mathbb{R})^0$ have no compact factors and are not isomorphic to $PSL_2(\mathbb{R})$. Let $\alpha: \Gamma \to \Gamma'$ be an isomorphism between irreducible lattices of $G(\mathbb{R})^{o}$ and $G'(\mathbb{R})^{o}$ respectively. Then there exists an $\mathbb{R}$-isomorphism $\beta: G \to G'$ such that $\alpha(g) = \beta(g)$ for $g \in \Gamma$.

[The absence of a field homomorphism φ is explained by $\mathrm{Aut}_{\mathbb{Q}}\mathbb{R} = \{1\}$ and the absence of a "radial" homomorphism γ is ensured by the assumption that G' is adjoint, compare with (3.7). Note also that strong rigidity implies rigidity.] Mostow's original proof of (6.5) worked only for the uniform lattices. But the missing pieces for an extension to all lattices were localized and were later provided by G. Margulis, G. Prasad, and M. Raghunathan. Mostow's results led to a number of spectacular developments which culminated in Margulis'

(6.6) THEOREM (Superrigidity, see [M2]). Let G be a connected semi-simple adjoint $\mathbb{R}$-group such that $\mathrm{rk}_{\mathbb{R}} G \geq 2$ and $G(\mathbb{R})^{o}$ has no compact factors. Let G' be a connected k-simple adjoint algebraic group defined over a local field k of characteristic 0. Let $\alpha: \Gamma \to G'(k)$ be a homomorphism with Zariski-dense image of an irreducible lattice $\Gamma \subset G(\mathbb{R})^{o}$ into $G'(k)$. Then either

(i) $\alpha(\Gamma)$ is relatively compact in the Hausdorff topology of $G'(k)$, or

(ii) $k = \mathbb{R}$ or $\mathbb{C}$, $G' = G'_1 \times G'_2$ (direct product of algebraic k-groups), $\mathrm{pr}_1 \circ \alpha: \Gamma \to G'_1(k)$ has relatively compact image, and there exists a homomorphism of algebraic k-groups $\beta: G \to G'_2$ such that $pr_2 \circ \alpha(g) = \beta(g)$ for $g \in \Gamma$.

This theorem implies (6.5) if $rk_{\mathbb{R}}G \geq 2$. Indeed, for α from (6.5) we have that $\Gamma' = \alpha(\Gamma)$ is Zariski-dense and therefore (6.6)is applicable with $k = \mathbb{R}$ if $rk_{\mathbb{R}}G \geq 2$. We must have by the assumption of (6.5) on G' that $G'_1 = \{1\}$. But then (6.6) reduces to (6.5). The general version of (6.6) (see [M1] and [T2]) implies (see [T2] and [T4]) a perfect analogue of (3.7) for homomorphisms (with Zariski dense image) of lattices Γ such as in (6.6) into k-simple k-groups over an arbitrary (infinite) field k. To see the relevance of the different conditions and implications of (6.6) one need only look (and we will in (6.12)) at Margulis' proof of

(6.7) THEOREM (Arithmeticity theorem) Let G and Γ be as in (6.6). Then Γ is arithmetic.

This theorem was first proven by G. Margulis in special cases; in one of these cases a similar result was obtained by M. Raghunathan. The counter-examples in SO(n,1) and SU(n,1) (see §8) show that the gap between (6.5) and (6.6) can not be closed without additional (as compared with (6.6)) assumptions on G or α or both.

(6.8) Before proceeding further we mention several related developments. G. Prasad has contributed very much, especially in the non-archimedean case, to the study of lattices, see e.g. [Pr].

There is an ongoing investigation, led by R. Zimmer (see [Z] and [F]), of generalizations of Margulis' results to ergodic actions.

On the other hand, Y.-T. Siu has generalized the geometric version of (6.5). This version considers two compact locally symmetric spaces X and X′ of non-positive sectional curvature, of dimension ≥ 2, and having no global totally geodesic factors. The claim then is that any isomorphism of fundamental groups of X and X′ extends, modulo normalizing factors, to an isometry $X \to X'$. Y.-T. Siu [Si1,Si2] drops the assumption that X is locally symmetric but assumes that both X and X′ are compact Kählerian. He also proves in [Si1] other generalizations of (6.5). An example, by G. Mostow and Y.-T. Siu [MS], shows that non-locally symmetric X exist for which the conditions of [Si1] are satisfied.

F. Farrell and W.-C. Hsiang, e.g. [FH], studied topological generalizations of the geometric version of (6.5).

A class of compact manifolds M, dim M = n, whose universal covers are contractible but not homeomorphic to $\mathbb{R}^n$ is constructed in [Da] for $n \geq 4$. The fundamental groups $\pi_1(M)$ of such M are generated by "reflections". In view of (8.7). it is improbable that these $\pi_1(M)$ are isomorphic to lattices in Lie groups for $n \geq 30$. However, such M defy the usual techniques to establish topological rigidity.

(6.9) Margulis' proof in [M2] of (6.6) splits naturally into two parts. The first part constructs a measurable map ω between algebraic varieties, and the second part shows that ω is essentially algebraic. The first part was somewhat streamlined and conceptualized by R. Zimmer, see [Z]; we follow his exposition. In the notation of (6.6), let P (resp. P′) be a minimal $\mathbb{R}$ - (resp., k-) parabolic subgroup of G (resp., G′). Then, by a theorem of C. Moore, Γ acts ergodically on $G(\mathbb{R})/P(\mathbb{R})$. Then a result of H. Furstenberg ensures the existence of a measurable Γ-map $\omega\colon G(\mathbb{R})/P(\mathbb{R}) \to M(G'(k)/P'(k))$ from $G(\mathbb{R})/P(\mathbb{R})$ into the space of probability (positive of total mass 1) measures on $G'(k)/P'(k)$. The orbits of $G'(k)$ on $M(G'(k)/P'(k))$ are locally closed. This implies that $\omega(G(\mathbb{R})/P(\mathbb{R}))$ is (up to measure 0) contained in an orbit, say $G'(k)/D$, of $G'(k)$. It was shown by C. Moore and R. Zimmer that D is either compact, or the Zariski closure H′ of D is a proper algebraic k-subgroup of G′. In the first case $\alpha(\Gamma)$ is relatively compact. In the second case we combine ω with the natural map $G'(k)/D \to G'(k)/H'(k)$ to obtain a measurable Γ-map (also denoted by ω) $\omega\colon G(\mathbb{R})/P(\mathbb{R}) \to$

$\to G'(k)/H'(k)$. Assume for the rest of this outline that $\alpha(\Gamma)$ is not relatively compact.

(6.10) Then one must recover from ω a rational map $G \to G'$ of algebraic k-varieties (or show that ω is essentially a map into a point if $k \neq \mathbb{R}$ or $\mathbb{C}$). To succeed one must find a link connecting objects of absolutely different nature: measurable maps and rational maps. Let $\psi\colon X(\mathbb{R})^{o} \to Y(k)$ be a measurable map of points of algebraic varieties X and Y defined over $\mathbb{R}$ and k respectively with $X(\mathbb{R})$ assumed to have a measure μ_X. Call ψ μ_X-rational if, up to measure 0, ψ is a map into a point when $k \neq \mathbb{R}$ or $\mathbb{C}$ and ψ is the restriction of a rational k-map $X \to Y$ when $k = \mathbb{R}$ or $\mathbb{C}$. To establish a link between "measurable" and "rational", Margulis considers a measurable (with respect to the Lebesgue measure μ_{m+n} on $\mathbb{R}^{m+n}$) map $f\colon \mathbb{R}^{m+n} \to Y(k)$. Then he proves that if for almost all $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ the restrictions of f to $x \oplus \mathbb{R}^n$ and $\mathbb{R}^m \oplus y$ are rational with respect to the Lebesgue measures on $\mathbb{R}^n$ and $\mathbb{R}^m$, then f is μ_{m+n}-rational. To make the above theorem applicable, Margulis proves that for any $\mathbb{R}$-split subtorus $T \subseteq G$ the map $\varphi_g\colon Z = Z_{G(\mathbb{R})^o}(T(\mathbb{R})) \to G'(k)/H'(k)$ given by $\varphi_g(z) = \varphi(gz)$ is μ_U-rational for any unipotent subgroup U of Z and for almost all $g \in G$ (where μ_U is the Haar measure on U). The above statement is vacuous if

$rk_{\mathbb{R}}G = 1$ — <u>hence the restriction that</u> $rk_{\mathbb{R}}G \geq 2$. Now one increases the unipotent subgroup $U_{(r)}$, the restriction of φ_g (where $\varphi_g(u) = \varphi(gu)$) to which is $\mu_{U_{(r)}}$-rational for almost all g, by adding new root subgroups U_a one after another and applying the Fubini theorem and the theorem about maps $\mathbb{R}^m \times \mathbb{R}^n \to Y(k)$ to the φ_g: $U_{(r+1)} = U_{(r)} \times U_a \to G'(k)/H'(k)$. This completes the proof in the case when $k \neq \mathbb{R}, \mathbb{C}$. In cases $k = \mathbb{R}$ or $\mathbb{C}$ one identifies a maximal unipotent subgroup $\tilde{U}$ of $G(\mathbb{R})$ such that $\tilde{U} \cdot P(\mathbb{R})$ is a big Bruhat cell with an open subset of $G(\mathbb{R})/P(\mathbb{R})$. Thus one gets a μ_G-rational map $G(\mathbb{R}) \to G'(k)/H'(k)$. It is not difficult to lift it to a k-rational homomorphism $G \to G'$.

Needless to say, the complete proof is extremely intricate both conceptually and technically. The fact that it, together with the necessary definitions, occupies just 35 pages is due to Margulis' very condensed style of writing and his ability to extract the "concrete essence" out of every notion and proof he uses. His results are also much more general than the versions we gave here, see [M1],[T2].

(6.11) Let us outline now a derivation of (6.7) from (6.6). Since superrigidity implies local rigidity, G is defined over a number field k and Γ is contained, by (6.2), in G(k). Set $H = R_{k/\mathbb{Q}}G$. Let $\psi: \Gamma \to H(\mathbb{Q}) \simeq G(k)$ be the natural embedding. If we take $k = \mathbb{Q}(\mathrm{trAd}\,\Gamma)$, see (6.3), then, modulo replacing Γ by a subgroup of finite index, $\psi(\Gamma)$ can be assumed Zariski-dense in H. On the other hand, by the definition of $R_{k/\mathbb{Q}}$ we have that $H \simeq \prod_{\sigma} {}^{\sigma}G$, a direct product of k-groups where σ runs over all embeddings of k into $\mathbb{C}$ and ${}^{\sigma}G$ is defined in (1.2). If we identify the k-group G as the factor ${}^{\mathrm{id}}G$, then we get a k-map $\kappa\colon H \to G$. Moreover $\kappa \circ \psi = \mathrm{id}$ on Γ. Let now $G' \neq G$ be another non-trivial simple $\mathbb{R}$-factor of H and let π be the projection $H \to G'$; $\pi(\Gamma)$ is Zariski-dense in G' since it is dense in H. Suppose that $\alpha = \pi \circ \psi\colon \Gamma \to G'$ extends to a rational map $\beta\colon G \to G'$. Then the diagram

$$\begin{tikzcd} H \arrow[r, "\kappa"] \arrow[dr, "\pi"'] & G \arrow[d, "\beta"] \\ & G' \end{tikzcd}$$

commutes since $\kappa \circ \psi = \mathrm{id}_{\Gamma}$. Let us restrict it to $\mathrm{Ker}\,\kappa \subseteq H$. Then $\pi(\mathrm{Ker}\,\kappa) = G'$ by our choice of G' and π. On the other hand $\beta(\kappa(\mathrm{Ker}\,\kappa)) = \beta(e_G) = e_{G'}$, a contradiction. Thus α does not extend to $\beta\colon G \to G'$ and therefore, by (6.6(ii)), $\pi(\Gamma)$ is relatively compact in G'. Hence $H(\mathbb{R}) \simeq G(\mathbb{R}) \times$ $\times$ (compact group).

(6.12) The final step is simple. Consider the natural map $H(\mathbb{Q}) \to H(\mathbb{Q}_p)$. Since Γ is finitely generated $\psi(\Gamma) \subseteq H(\mathbb{Z}_p)$ for almost all p (namely, for all p which do not enter in the denominators of entries of generators of $\psi(\Gamma)$

considered as matrices). By (6.6(i)) we know that for all primes p, the closure $\Gamma_{(p)}$ of $\psi(\Gamma)$ in the Hausdorff topology of $H(\mathbb{Q}_p)$ is compact. Therefore $|\Gamma_{(p)}/(\Gamma_{(p)} \cap H(\mathbb{Z}_p))| < \infty$ since $H(\mathbb{Z}_p)$ is open in $H(\mathbb{Q}_p)$. Thus $H(\mathbb{Z}) \cap \Gamma = \bigcap_p (\psi(\Gamma) \cap H(\mathbb{Z}_p))$ is of finite index, say m, in Γ. Since $H(\mathbb{Z})$ is a lattice we have that $H(\mathbb{Z}) \cap \Gamma$ is of finite index $m \cdot \mathrm{vol}(G(\mathbb{R})/H(\mathbb{Z}))(\mathrm{vol}\ G(\mathbb{R})/\Gamma)^{-1}$ in $H(\mathbb{Z})$. Thus we have the situation described in (6.4).

7. QUOTIENT GROUPS OF LATTICES

The next theorem, which is a particular case of a much more general theorem due to G. Margulis [M3] and [M4], establishes a dichotomy similar to that of section (3.3).

(7.1) THEOREM. Let G and Γ be as in (6.6). Then every normal subgroup of Γ is either central or of finite index.

In view of the Arithmeticity Theorem (6.7) one can deduce (7.1) from a positive solution of the Congruence Subgroup Problem (CSP) in the cases where this latter has been established (see [Hu, §§3,4] for a discussion of isotropic groups and [Kn] for the only known solution of the CSP for anisotropic groups). However, Margulis's proof is much more uniform than the other known proofs of the CSP, and it is applicable to a wider class of groups of $\mathbb{R}$-rank ≥ 2. A more general form of the above theorem was used by Margulis [M5] to solve the CSP positively for the group D^1 of elements of reduced norm 1 in a central quaternion division algebra over a number field; this extended an earlier partial result of V. Platonov and A. Rapinchuk. The corresponding local result was obtained by B. Pollak [Po] and, for general division algebras, by C. Riehm [Ri].

(7.2) The proof of (7.1) uses two representation-theoretic notions pertaining to a locally compact group H:

Property (T) (introduced and used by Kazhdan, and therefore referred to as Kazhdan's property (T)): the identity representation of H is an isolated point of the set $\hat{H}$ of all irreducible unitary representations of H, and

Amenability: H is amenable if for every action of H on a non-empty compact X there is an invariant (under H) probability measure on X.

(7.3) It can be shown that an amenable group has property (T) iff it is compact, and that solvable Lie groups are amenable. Also we have

(7.4) THEOREM (D. Kazhdan [Ka1]). If a locally compact group H has property (T), then so does any lattice Γ in H. Therefore $\Gamma/[\Gamma,\Gamma]$ is finite for such Γ.

To apply this theorem one must exhibit (or, better, characterize) a class of groups H having property (T). This is done in

(7.5) THEOREM. A semi-simple adjoint connected Lie group H has property (T) iff it does not have $PSO(n,1)^{\circ}$, $n \geq 2$, or $PSU(n,1)$, $n \geq 1$, as factors.

Here $SO(n,1) = \{A \in SL_{n+1}(\mathbb{R}) \mid AJA^t = J\}$ and $SU(n,1) = \{A \in SL_{n+1}(\mathbb{C}) \mid AJ\bar{A}^t = J\}$ where $J = \mathrm{diag}(1,\ldots,1,-1)$. (7.5) was proven [Ka1] by D. Kazhdan for H which do not have simple factors of $\mathbb{R}$-rank 1 (actually, Kazhdan's proof was for "most" simple Lie groups of $\mathbb{R}$-rank ≥ 2; it was extended to full generality by L. Vaserstein (Funct. Anal. Appl. 2(1968), 174). Then B. Kostant showed [Ko] that a simple connected Lie group $G(\mathbb{R})$ of $\mathbb{R}$-rank 1 has property (T) iff it is not locally isomorphic to $SO(n,1)$, $n \geq 2$, or $SU(n,1)$. Recall that the family of simple Lie groups $G(\mathbb{R})$, $rk_{\mathbb{R}}G = 1$, consists of $PSO(n,1)^o$, $n \geq 2$; $PSU(n,1)$, $n \geq 1$; $PSp(n,1)$, $n \geq 1$, (the unitary group of a skew-Hermitian form of Witt index 1 over the Hamilton quaternions $\mathbb{H}$), and of a group of type F_4. (There are the following low-dimensional isomorphisms: $PSO(2,1)^o \simeq PSU(1,1) \simeq PSL_2(\mathbb{R})$, $PSO(3,1)^o \simeq PSL_2(\mathbb{C})$, $PSO(4,1)^o \simeq PSp(1,1)$, $PSO(5,1)^o \simeq PSL_2(\mathbb{H})$.) Finally, note that compact groups have (T) – for them $\hat{H}$ is discrete. The following theorem follows from (7.3) and (7.4) if $G(\mathbb{R})$ satisfies the conditions of (7.4). In general it follows from (7.1).

(7.6) THEOREM. If H is an adjoint semi-simple connected Lie group which is not locally isomorphic to $PSO(n,1)$, $n \geq 2$, or $PSU(n,1)$, $n \geq 1$, and has no compact factors, then $\Gamma/[\Gamma,\Gamma]$ is finite for any irreducible lattice Γ in H.

In the case when Γ is cocompact and G has no $\mathbb{R}$-factors of rank 1, (7.6) was previously proven by S. Kaneuki and T. Nagano [KN]. For uniform lattices it was proven by J. Bernstein and D. Kazhdan [BK]. The latter's proof is extendable to non-uniform lattices.

(7.7) The proof of Kaneuki and Nagano was based on Y. Matsushima's fundamental interpretation [Ma] of $H^1(\Gamma,\mathbb{R}) \simeq (\Gamma/[\Gamma,\Gamma]) \otimes_{\mathbb{Z}} \mathbb{R}$ through certain representation-theoretic properties of $L^2(G/\Gamma)$ (the space of square-integrable functions on G/Γ). Matsushima's results are described in [BW, Ch. II].

(7.8) We will now give a short sketch of Margulis' proof of (7.1). See also [T2] and [Z]. First, if G has no factors $PSO(n,1)^o$ or $PSU(n,1)$ then every amenable quotient of Γ is finite by (7.2), (7.3), and (7.4). The remaining cases of rank ≥ 2 groups are more difficult, they are taken care of in [M4]. In [M3] Margulis shows that non-amenable quotients of Γ from (7.1) are central. To do this he shows that for any non-amenable group N there exists a metrizable compact X on which N acts without fixing any probability measure on X. (Note that a compact X exists by definition of non-amenability in (7.2); the problem is to find a metrizable compact.) Suppose that $N = \Gamma/\Gamma_1$, $\Gamma_1 \triangleleft \Gamma$, N non-amenable. Then by a theorem of Furstenberg (which was already used in (6.9) in the case $X = G'(k)/P'(k)$) there is a measurable Γ-map $\omega: G(\mathbb{R})/P(\mathbb{R}) \to M(X)$; P and M here have the same meaning as in (6.9) and Γ acts on X via N. Let μ be the natural quasi-invariant measure on $G(\mathbb{R})/P(\mathbb{R})$. To move further we need a difficult and remarkable result of Margulis which enables us to identify $(M(X), \omega_*(\mu))$ as a measurable Γ-space with $G(\mathbb{R})/\tilde{P}(\mathbb{R})$ where $\tilde{P}$ is a proper $\mathbb{R}$-parabolic subgroup of G. But then Γ_1 acts trivially on $G(\mathbb{R})/\tilde{P}(\mathbb{R})$, which implies that Γ_1 is central. Of course, the result we have used is the heart of the argument, and the assumption $\mathrm{rk}_{\mathbb{R}} G \geq 2$ is used in its proof.

8. LATTICES IN PSO(n,1) AND PSU(n,1)

In this section k is always a number field.

(8.1) Theorems (6.6), (6.7), (7.1), (7.4), (7.5), and (7.6) are not applicable to the simple Lie groups $G = PSO(n,1)^{\circ}$, $n \geq 2$, and $G = PSU(n,1)$, $n \geq 1$. Recall that

(i) $PSO(2,1)^{\circ} \simeq PSU(1,1) \simeq PSL_2(\mathbb{R})$;

(ii) $PSO(3,1)^{o} \simeq PSL_2(\mathbb{C})$;

(iii) $PSO(n,1)^{o}$, $n \geq 2$, is the connected component of the group of isometries of an n-dimensional Lobachevsky space Λ^n;

(iv) $PSU(n,1)$ is the connected component of the group of holomorphic automorphisms of an n-dimensional ball $B^n = \{(x_i) \in \mathbb{C}^n, \Sigma|x_i|^2 \leq 1\}$; it also coincides with the connected component of the group of isometries of the Bergman metric on B^n.

(8.2) The uniform torsion-free lattices Γ in $PSO(2,1)^{o}$ are the fundamental groups of the compact Riemann surfaces $\Gamma\backslash\Lambda^2$. Therefore they are 2r-generator-1-relator groups (where r is the genus of $\Gamma\backslash\Lambda^2$). Non-uniform lattices are even worse: they contain a free subgroup of finite index. We assume in the sequel that $G \not\simeq PSL_2(\mathbb{R})$.

(8.3) Counter-examples to (7.6) in $PSO(n,1)^{o}$, $n \geq 3$, were constructed by E. Vinberg for special lattices when $3 \leq n \leq 10$. For arithmetic lattices Γ of the type described in (6.3) (with $r = 1$) J. Millson [Mi] showed that a congruence subgroup Δ of Γ contradicts (7.6). The corresponding cohomological statement — $H^1(\Delta,\mathbb{R}) \neq 0$ — was generalized in [MR]; e.g. $H^i(\Delta,\mathbb{R}) \neq 0$ for $1 \leq i \leq n$. Note that if $\Delta/[\Delta,\Delta]$ is infinite, then the same holds for all subgroups of finite index in Δ.

(8.4) Millson's proof of $|\Delta/[\Delta,\Delta]| = \infty$ is based on a simple topological observation. Namely, the fact that Λ^n contains totally geodesic subspaces of codimension 1 makes it possible to construct a non-trivial homology class $c \in H^{n-1}(\Delta\backslash\Lambda^n,\mathbb{R})$. By Poincaré duality this implies that $0 \neq H^1(\Delta;\mathbb{R}) \simeq (\Delta/[\Delta,\Delta]) \otimes_{\mathbb{Z}} \mathbb{R}$. Millson's ideas do not apply to PSU(n,1) because the ball B^n (with Bergman metric) has no totally geodesic subspaces of codimension 1.

(8.5) Counter-examples to (7.6) in PSU(n,1), $n \geq 2$. Consider arithmetic subgroups in PSU(n,1) constructed as in (6.3). Namely, take an imaginary quadratic extension K of a totally real number field k and denote by $^{-}$ the complex conjugation in $\mathbb{C}$. Then $f = \Sigma\, a_i x_i \bar{x}_i$, $a_i \in k$, is a Hermitian form on K^n. If the a_i satisfy the same conditions as in (6.3) and if, moreover, $r = 1$ then $\Gamma = PSU(f,\mathcal{O})$ is an arithmetic lattice in PSU(n,1). If $k \neq \mathbb{Q}$ D. Kazhdan has shown (see [Ka 2] and [BW, Ch. VIII]) that the lattices $PSU(f,\mathcal{O})$ have congruence subgroups Δ with infinite $\Delta/[\Delta,\Delta]$; these Δ are automatically uniform. To prove that such Δ exist Kazhdan uses powerful techniques — properties of the Weil representation of Sp_{2n} and Matsushima's [Ma] interpretation of homology of a lattice. G. Shimura [Sh] removed the condition $k \neq \mathbb{Q}$ (thus Δ can be non-uniform); his construction is more explicit and gives more information.

(8.6) <u>Counter-examples to (6.6) and (6.7) in PSO(n,1)</u> were first constructed by F. Lanner in 1950, but he did not know that they were counter-examples to a conjecture stated much later. Then V. Makarov constructed infinitely many non-arithmetic lattices in $PSO(3,1)^{o} \simeq PSL_2(\mathbb{C})$. E. Vinberg [Vi 1] understood the nature of the known examples and developed general techniques to construct more. His lattices are groups generated by reflections in totally geodesic hypersurfaces in Λ^n. Such groups are easily described in terms of generators and relations. Vinberg distinguished two ways in which arithmeticity and superrigidity can break down. Namely, suppose that Γ is a lattice in $PSO(n,1)^{o}$. Let $k = \mathbb{Q}(\operatorname{tr} \operatorname{Ad} \Gamma)$. Then, as in (6.11), it can be assumed that G is defined over k, Γ is contained in $G(k)$ and is, moreover, Zariski-dense in $M = R_{k/\mathbb{Q}}G$. Call Γ <u>quasi-arithmetic</u> if $M(\mathbb{R})^{o} \simeq PSO(n,1)^{o} \times$ $\times$(compact group). By (6.4) any arithmetic lattice is quasi-arithmetic. For a quasi-arithmetic Γ the first step (6.11) of Margulis' proof of arithmeticity works. Therefore if Γ is not arithmetic, the second step (6.12) must break down This means that for some prime p the matrix entries of the $\gamma \in \Gamma$ contain arbitrary large powers of this prime in the denominators. This contradicts also (6.6(i)). On the other hand, if Γ is not quasi-arithmetic then (6.11) does not work. This shows, as in (6.11), that (6.6(ii)) fails too. Vinberg gives in [Vi 1] examples of both uniform (for $n = 3$ and 4) and non-uniform (for $n = 3$) non-arithmetic, quasi-arithmetic lattices; he also gives examples of both uniform (for $n = 3,4$ and 5) and non-uniform (for $n = 3$ and 4) non-quasi-arithmetic lattices. A recent result [Vi 2] of Vinberg, extending his own work, and also that of V. Nikulin, and proving a conjecture of his, is

(8.7) THEOREM. There are neither uniform nor arithmetic lattices generated by reflections in $PSO(n,1)^o$, $n \geq 30$.

Thus the methods used to construct non-arithmetic lattices in $PSO(n,1)^o$ do not work for $n \geq 30$. An optimist can consider this as an indication that all lattices in $PSO(n,1)^o$ are arithmetic for large n.

(8.8) <u>Counter-examples to (6.6) and (6.7) in $PSU(n,1)$, $n \geq 2$,</u> are much more difficult to construct. The walls of a fundamental domain have codimension 1 but can not be totally geodesic (compare (8.4) and (8.6)). G. Mostow developed [Mo 2] a very deep theory of certain polyhedra in B^n. The walls of his polyhedra are surfaces of points in B^n equidistant from two given points (in the Bergman metric). These polyhedra are used to construct and study uniform non-arithmetic subgroups in $SU(n,1)$, $n = 2$. Mostow's subgroups in [Mo 2] are uniform and contained in the group of integral points of an appropriate algebraic group. Thus they are not quasi-arithmetic. Mostow's polyhedra were also used to construct interesting Kähler manifolds see [MS] and (6.8).

Using completely different ideas going back to E. Picard, Mostow constructed new both uniform and non-uniform non-quasi-arithmetic lattices in PSU(2,1); these new groups are monodromy groups of certain branched covers by B^2. The lattices mentioned above are all in PSU(2,1). However, G. Mostow has an example of a non-uniform lattice in PSU(3,1) as well.

(8.9) <u>Counter-examples to a generalization of the strong rigidity theorem (6.5)</u>. By (6.5) <u>isomorphisms</u> of lattices in $PSO(n,1)^o$, $n \geq 3$, and $PSU(n,1)$, $n \geq 2$, can be extended to isomorphisms of Lie groups. Mostow, using a generators and relations description of lattices, constructed in [Mo 2] two uniform arithmetic lattices Γ_1 and Γ_2 of PSU(2,1) and an epimorphism $\alpha\colon \Gamma_1 \to \Gamma_2$ with infinite kernel. Clearly such α can not be extended to an automorphism of PSU(2,1).

Similarly, if one looks at Vinberg's examples [Vi 1], one can find many pairs of lattices Δ_1 and Δ_2 in $PSO(3,1)^o$ such that there exist epimorphisms $\alpha\colon \Delta_1 \to \Delta_2$ with infinite kernels. Moreover, there are sequences of epimorphisms $\to \Delta_i \to \Delta_{i-1} \to \cdots \to \Delta_1$ infinite to the left and all having infinite kernels. For example, in the notation of [Vi 1], one can take $\Delta_i = \Gamma'_{r^i m}$ or $\Delta_i = \Gamma''_{r^i m}$ where r and m are fixed integers, $r > 1$, $m \geq 4$. I was not able to find in [Vi 2] such examples in $PSO(n,1)^\circ$, $n \geq 4$.

(8.10) The attention given at present by topologists to 3-manifolds has led to many results and conjectures about lattices in $PSO(3,1)^\circ = PSL_2(\mathbb{C})$. One should consult work of W. Thurston (but I can not give an exact reference). On the other hand, lattices in $PSL_2(\mathbb{C})$ are Kleinian groups, and there is a wealth of material about them, see e.g. [B]. We will stop here with a few recent results.

We say that a group M is an FQ-group if it has a free non-abelian quotient. F. Grunewald and J. Schwermer showed in [GS] that arithmetic lattices in $PSL_2(\mathbb{C})$ have an FQ-subgroup of finite index. A. Lubotzky proved that a lattice Γ in $PSL_2(\mathbb{C})$ has a subgroup Δ of finite index such that the normal closure in Γ of any element $x \neq 1$, $x \in \Delta$, is a normal subgroup of infinite index. He also mentions that M. Gromov has proved similar results for lattices in $PSO(n,1)^\circ$, $n \geq 3$.

(8.11) The notion of an FQ-group can be useful, as was pointed out to me by A. Lubotzky, in explaining the examples described in (8.3), (8.5), (8.9). Indeed, if a lattice Γ is an FQ-group, it or its subgroup of finite index can be mapped onto any finitely generated group. This would mean that the information about the commensurability class of Γ (and by (6.5) not all Γ in $PSO(n,1)^o$ or $PSU(n,1)$ are commensurable) is contained in the kernel N of a homomorphism onto a free group and in the extension of N by a free group. It would be then interesting to find out what this information is and how it is carried. (We say here that two groups are commensurable if they have isomorphic subgroups of finite index.)

9. CONCLUDING REMARKS

(9.1) Recently V. Petechuk [Pe] concluded investigation of automorphisms of $SL_n(A)$, $GL_n(A)$, and $E_n(A)$, A an arbitrary commutative ring: if $n \geq 4$ they are "standard"; for n = 3 see Petechuk, Math. Notes 31(1982). The proof is based on a description of the normal subgroup structure of $E_n(A)$. In a rough outline it goes as follows: normal subgroups correspond to ideals; therefore an automorphism α induces a bijection α^* of Spec A; for $p \in$ Spec A it induces isomorphisms $E_n(A/p) \simeq E_n(A/\alpha^*(p))$ which are standard and can be glued together to give a "standard" automorphism.

(9.2) A. Hahn [Ha2] has found a new setting — Morita equivalence — for the isomorphism theory of the classical groups over rings.

(9.3) D. James [J] and myself [We] have obtained results about homomorphisms (possibly with infinite kernels) of anisotropic algebraic groups.

(9.4) Reference [JWW] contains a list of problems pertaining to the present survey.

SUMMARY TABLE FOR §6,7,8

LATTICES IN ADJOINT SEMI-SIMPLE LIE GROUPS

In this table G is an adjoint semi-simple connected Lie group and Γ is an irreducible lattice in G

Group G / Property	$PSO(n,1)^o$, $n \geq 3$; $PSU(n,1)$, $n \geq 2$ — see (7.5)	$PSp(n,1)$, $n \geq 2$; F_4 of rank 1 — see (7.5)	$rk_{\mathbb{R}} G \geq 2$
Kazhdan's property (T), see (7.2)	no, see (7.5)	yes, see (7.5)	"yes", if G does not have $PSO(n,1)^o$ or $PSU(n,1)$ as factors; "no" otherwise, see (7.5)
$\Gamma/[\Gamma,\Gamma]$	can be infinite, see (8.3), (8.5).	finite, see (7.4) and (7.6)	finite, see (7.4) and (7.6)
Quotients	There are both amenable, see (8.3), (8.5), and non-amenable, see (8.9), infinite quotients Some lattices have other lattices as non-trivial quotients, see (8.9), (8.10) see also (8.11)	every finite quotient is not amenable see (7.2) and (7.3)	every quotient is finite, see (7.1)
Strong rigidity, see (6.5)	yes, see (6.5)	yes, see (6.5)	yes, see (6.6)
Superrigidity, see (6.6)	no, see (8.6), (8.8), but..., see (8.7)	unknown	yes, see (6.6)
Quasi-arithmetic non-arithmetic lattices, see (8.6)	in PSO(n,1) uniform (n = 3,4) non-uniform (n = 3) see (8.6), (8.7)	unknown	no, see (6.7)
non-quasi-arithmetic lattices see (8.6)	in PSO(n,1) uniform (n = 3,4,5) non-uniform (n = 3,4) see (8.6), (8.7) in PSU(n,1) uniform (n = 2) non-uniform (n = 2,3) see (8.8)	unknown	no, see (6.7)

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