Modular Representations of Algebraic Groups

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These lectures provide an introduction to the modular representation theory of semisimple algebraic groups. Sections 1 and 2 assume only a basic acquaintance with the theory of algebraic groups and with the standard language of representation theory. Later sections, however, employ the theory of group schemes, and so make more demands on the reader. Nevertheless, it should become clear that the study of positive characteristic phenomena is ideally suited to the approach defined by these techniques.

Many important topics as well as many proofs are omitted or only barely sketched due to lack of time. The reader may consult the papers in the bibliography for further information.

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1. ELEMENTARY THEORY

Throughout we will fix an algebraically closed field k. Unless explicitly stated to the contrary, we assume that k has positive characteristic p.

(1.1) RATIONAL MODULES. Let G be an affine algebraic group defined over k. A finite dimensional kG-module V is said to be rational if the associated homomorphism $\rho_V: G \to GL(V)$ is a morphism of algebraic groups.[1] By a rational G-module we mean a kG-module V which is a union of rational finite dimensional submodules in the above sense. We let $\underline{M}_G$ be the category whose objects are the rational G-modules and whose morphisms are the G-module homomorphisms.

(1.2) EXAMPLES/REMARKS. (a) $\underline{M}_G$ is an abelian category which is closed under the formation of tensor products, direct limits, and duals of finite dimensional modules. Further, it is easy to see that $\underline{M}_G$ possesses enough injective objects.

(b) The coordinate ring $k[G]$ of G is a rational G-module relative to the <u>left translation action</u> of G: $(g\cdot f)(x) = f(xg)$, $f \in k[G]$, $g,x \in G$. We may also view $k[G]$ as a rational right G-module by using the <u>right</u> translation action of G: $(f\cdot g)(x) = f(gx)$.

(c) $k[G]$ has a well-known commutative Hopf alge-

[1] For the most part, all modules are taken to be <u>left</u> modules.

bra structure, with comultiplication $\Delta: k[G] \to k[G] \otimes k[G]$, counit $\varepsilon: k[G] \to k$, and antipode $\eta: k[G] \to k[G]$. Then $\underline{M}_G$ is isomorphic to the category of comodules for $k[G]$ (see [14; 1.1] for more details).

(d) Let g be the Lie algebra of G, that is, the Lie algebra of all k-derivations D of $k[G]$ satisfying the identity $(1 \otimes D)\Delta = \Delta D$. Then g becomes a rational G-module, called the adjoint module, as follows. Take $g.D$, $g \in G$, $D \in g$, to be the derivation defined by $(g.D)(f) = g\cdot(D(g^{-1}\cdot f))$. It is easy to check that g.D satisfies the required identity to be an element of g.

(1.3) NOTATION. We now take up the case when G is a (connected) semisimple algebraic group. For simplicity, we will assume that G is simply connected. We list below some of the standard notation that we will use throughout:

$B = T.U$	Fixed Borel subgroup with maximal torus T, unipotent radical U
$B^- = T.U^-$	Opposite Borel subgroup
Φ	Root system of T in G
Φ^+	Positive roots defined by B
$\Pi = \{\alpha_1,\ldots,\alpha_\ell\}$	Fundamental roots in Φ^+
W	Weyl group
w_o	Long word in W
$<,>$	W-invariant, symmetric, positive definite bilinear form on $E = \mathbb{Z}\Phi \otimes \mathbb{R}$

$\alpha^\vee$	coroot $2\alpha/\langle\alpha,\alpha\rangle$, $\alpha \in \Phi$
Λ	Weight lattice in $\mathbb{E}$ spanned by the fundamental dominant weights $\omega_1,\ldots,\omega_\ell$ (where $\langle\omega_i,\alpha_j^\vee\rangle = \delta_{ij}$, $1 \leq i,j \leq \ell$)
Λ^+	Dominant weights
ρ	$\omega_1 +\ldots+ \omega_\ell$
Λ_r^+	$\lambda \in \Lambda^+$ satisfying $0 \leq \langle\lambda,\alpha_j^\vee\rangle < p^r$, $1 \leq j \leq \ell$; r a positive integer
$\leq$	Partial order on Λ given by $\lambda \leq \mu$ iff $\mu-\lambda$ is a sum of positive roots
$\lambda \to \lambda^*$	Opposition involution on $\mathbb{E}$ defined by $\lambda^* = -w_0(\lambda)$
$P_J = L_J.U_J$, $J \subseteq \Pi$	Parabolic subgroup containing B with Levi factor L_J having J as fundamental roots

If V is a rational G-module and $\lambda \in \Lambda$, let $V_\lambda = \{v \in V \mid t.v = \lambda(t)v,\ t \in T\}$ denote the λ-weight space for the action of T on V. If $V_\lambda \neq 0$, we say that λ is a <u>weight</u> of T in V. We can now state the following basic result, due to Chevalley [12]:

(1.4) THEOREM. Let G be a semisimple, simply connected algebraic group over k.

(a) Let V be an irreducible rational G-module. Among the weights of T in V there is a unique maximal weight λ (relative to $\leq$), called the <u>high weight</u> of V. Necessarily, $\lambda \in \Lambda^+$. Any two irreducible rational

G-modules with the same high weights are G-isomorphic.

(b) Conversely, given $\lambda \in \Lambda^+$, there exists an irreducible rational G-module, denoted $S(\lambda)$, of high weight λ.

Let us make several comments concerning the proof of this result. The first part (a) is relatively straightforward and parallels closely the proof of the corresponding result for complex Lie algebras. As for (b), there are several ways to construct $S(\lambda)$, all of them important. The original method of Chevalley involves getting $S(\lambda)$ from an appropriate linear system on the projective variety G/B. This is closely related to the theory of induced modules taken up in Section 2 below. Secondly, one can obtain $S(\lambda)$ from the corresponding complex irreducible module by reduction mod p (see [44; §12]). Thirdly, there is a direct argument of Borel which uses only the normality of $k[G]$ and an elementary result of Chevalley (see [27; 31.4] for details).

(1.5) THE FROBENIUS MORPHISM. The next basic result in the modular representation theory is the tensor product theorem. To explain this result, it is convenient to assume that the semisimple group is defined and split over the prime field $k_o = GF(p)$. (It is always possible to arrange this.) Thus, in particular, $k[G] = k_o[G] \otimes k$, where $k_o[G]$ is the k_o-algebra of regular functions on G defined over k_o. We consider the Frobenius morphism

$\sigma: G \to G$ defined by means of its comorphism $\sigma^*: k[G] \to k[G]$ $(f \otimes c \to f^p \otimes c,\ f \in k_o[G],\ c \in k)$.[2] Now if V is a rational G-module, we can, for a positive integer r, form a new rational G-module $V^{(r)}$ as follows. As a k-vector space $V^{(r)} = V$, but the effect of applying $g \in G$ to $v \in V^{(r)}$ is now $\sigma^r(g).v$. Thus, $\rho_V(r) = \rho_V \circ \sigma^r$. We say that $V^{(r)}$ is obtained from V by <u>twisting</u> by σ^r. We may assume that T is defined and split over k_o. Then clearly the weights of T in $V^{(r)}$ are the p^r-multiples of the weights of T in V. Thus, $S(\lambda)^{(r)} \cong S(p^r\lambda)$.

(1.6) TENSOR PRODUCT THEOREM. Let G be a semisimple, simply connected algebraic group defined and split over k_o. For $\lambda \in \Lambda^+$, let $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^r\lambda_r$ be its p-adic expansion (so that $\lambda_i \in \Lambda_1^+$, cf. (1.3)). Then $S(\lambda) \cong S(\lambda_0) \otimes S(\lambda_1)^{(1)} \otimes \ldots \otimes S(\lambda_r)^{(r)}$.

This result was originally proved in general by Steinberg. We will briefly sketch an argument from [15]. It is given in two steps:

1) Let $V(\mathfrak{g})$ be the restricted enveloping algebra of the Lie algebra $\mathfrak{g}$ of G. The adjoint action of G on $\mathfrak{g}$ induces a rational action of G on the quotient of $V(\mathfrak{g})$ by its Jacobson radical J. Furthermore, since G is connected, it must fix elementwise the finitely many central primitive idempotents of $V(\mathfrak{g})/J$. Given a

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[2]For example, if $G = SL_n(k)$, then σ is the endomorphism of G which raises each matrix entry of $g \in G$ to the <u>pth</u> power

restricted irreducible g-module S, it corresponds to a central primitive idempotent of $V(g)/J$. It follows, from the simple connectivity of G, that S extends (uniquely) to an irreducible rational G-module.

2) If S is an irreducible rational G-module, let S_1 be an irreducible g-submodule. We have a G-isomorphism $\mathrm{Hom}_g(S_1,S) \otimes S_1 \to S$ given by $\phi \otimes s \to \phi(s)$, $\phi \in \mathrm{Hom}_g(S_1,S)$, $s \in S_1$. Here $\mathrm{Hom}_g(S_1,S)$ is given its natural structure as a rational G-module, using 1) above. Applying this to $S(\lambda)$, $\lambda \in \Lambda_1^+$, we see easily that $S(\lambda)$ is g-irreducible. Now, for arbitrary $\lambda \in \Lambda^+$, write $\lambda = \lambda_0 + p\theta$ where $\lambda_0 \in \Lambda_1^+$ and $\theta \in \Lambda^+$. Clearly, $S(\lambda)$ is a G-composition factor of $S(\lambda_0) \otimes S(\theta)^{(1)}$, whence any irreducible g-submodule of $S(\lambda)$ is g-isomorphic to $S(\lambda_0)$. A bookkeeping argument involving high weights establishes that $S(\lambda) \cong S(\lambda_0) \otimes S(\theta)^{(1)}$. The proof can now be completed by induction on r.

As a bonus of the above argument, we get the result of Curtis that the $S(\lambda)$, $\lambda \in \Lambda_1^+$, are exactly the distinct irreducible restricted g-modules.

(1.7) FORMAL CHARACTERS. Given a finite dimensional rational G-module V, we can form its <u>formal</u> <u>character</u>

$$\mathrm{ch}\, V = \sum_{\lambda} \dim V_\lambda \, e^\lambda$$

in the integral group algebra $\mathbb{Z}[\Lambda]$ of Λ (so $e^\lambda . e^\mu = e^{\lambda+\mu}$).

One is then led to ask for an explicit formula for ch $S(\lambda)$ as well as for dim $S(\lambda)$. Both of these problems were completely solved in the case of $k = \mathbb{C}$ in a series of famous papers written by Weyl in the 1920's; for example, Weyl's character formula states that

$$\operatorname{ch} S(\lambda) = \sum_{w\in W} \det(w)\, e^{w(\lambda+\rho)} / \sum_{w\in W} \det(w)\, e^{w(\rho)}.$$

At present no such formula is known in positive characteristic, although a relevant conjecture has been formulated (cf. Section 5). The answer is not even known in general for $G = SL_n(k)$!!

2. INDUCED MODULES

In this section, we will study the process of obtaining rational modules for an algebraic group from those of a closed subgroup.

(2.1) BASIC DEFINITIONS. Let H be a closed subgroup of an affine algebraic group G over k. Let $\pi\colon G \to G/H$ be the quotient morphism, where G/H denotes the variety of right cosets Hg. Given a rational H-module W, we will construct a sheaf L_W on G/H. First suppose W is finite dimensional. For U an open subvariety of G/H, set $\Gamma(U,L_W) = \mathrm{Map}_H(\pi^{-1}(U),W)$, the space of all morphisms $f\colon \pi^{-1}(U) \to W$ (of varieties) such that $f(hg) = h.f(g)$, $h \in H$, $g \in \pi^{-1}(U)$. In general, set $L_W = \varinjlim_{W'} L_{W'}$, the direct limit taken over all finite dimensional submodules W' of W. We call L_W the

<u>sheaf induced from</u> W. It is clear that L_W is actually a module for the structure sheaf $\mathcal{O}_{G/H}$ of G/H, and in fact one can show it is quasi-coherent (and coherent if $\dim W < \infty$).[3]

The space $W|^G = \mathrm{Map}_H(G,W)$ of global sections of L_W carries a natural G-module structure by setting $(g.f)(g') = f(g'g)$, $f \in W|^G$, $g,g' \in G$. Since $\mathrm{Map}_H(G,W)$ is clearly a G-submodule of $\mathrm{Hom}_{k\text{-alg}}(k[W],k[G]) \cong k[G] \otimes W$, it follows that $W|^G$ is in fact rational for W finite dimensional, hence rational in general by (1.2). Also, it is clear that the map $\mathrm{Ev}(W)\colon W|^G \to W$ ($f \to f(1)$) is H-equivariant, and it follows directly that given any rational G-module V and H-module homomorphism $\phi\colon V \to W$, there exists a unique G-module homomorphism $\Phi\colon V \to W|^G$ such that $\mathrm{Ev}(W)\circ\Phi = \phi$. This fact (<u>Frobenius reciprocity</u>) merely expresses the fact that $|^G$, which is called the <u>induction functor</u>, is the right adjoint to the restriction functor $\underline{M}_G \to \underline{M}_H$ ($V \to V|_H$). At times we will denote $|^G$ by $|_H^G$ when H needs to be mentioned.

(2.2) EXAMPLES AND ELEMENTARY PROPERTIES. We give below an number of examples involving induced modules.

(2.2.1) TENSOR IDENTITY. If $V \in \mathrm{Ob}(\underline{M}_G)$, $W \in \mathrm{Ob}(\underline{M}_H)$, then $W|^G \otimes V \cong (W \otimes V|_H)|^G$. To establish this, we can

[3]The quasi-coherence is straightforward if π is locally trivial in the Zariski topology. If W is finite dimensional, one can show in general that L_W is locally free, cf. [18].

reduce to the case in which V is finite dimensional.
For $K \in Ob(\underline{M}_G)$, we have $Hom_G(K,W|^G \otimes V) \cong Hom_G(K \otimes V^*,W|^G)$ $\cong Hom_H(K \otimes V^*)|_H,W) \cong Hom_H(K|_H,W \otimes V|_H) \cong Hom_G(K,(W \otimes V|_H)|^G)$, whence the conclusion.

(2.2.2) COORDINATE RINGS. We have $k[H]|^G \cong k[G]$; $k|^G \cong k[G/H]$. Here k in $k|^G$ denotes the one-dimensional trivial H-module.

(2.2.3) PARABOLIC INDUCTION. Suppose that H is a parabolic subgroup of G. Since G/H is complete, it follows that $W|^G = \Gamma(G/H,L_W)$ is finite dimensional for any finite dimensional rational H-module W. Let us consider the case in which G is semisimple, simply connected, and $H = B$ is a Borel subgroup. For a rational character λ on B, it will be convenient to let λ also denote the one-dimensional rational B-module defined by λ. Now for H and G arbitrary, it follows immediately that $Ev(W) = 0$ iff $W|^G = 0$, for any rational H-module W. In particular, if $\lambda|^G \neq 0$, we see the image of the dual map $Ev(\lambda)^*$ is a B-stable line in $(\lambda|^G)^*$ of weight $-\lambda$. Hence, $-\lambda|^G \neq 0$ implies that $\lambda \in \Lambda^+$. Conversely, if $\lambda \in \Lambda^+$, the existence of a nonzero B-module homomorphism
$S(\lambda)^* \to S(\lambda)^*/[U ,S(\lambda)^*] \cong -\lambda$ guarantees that $-\lambda|^G \neq 0$.
Below are some basic properties of $-\lambda|^G$ for $\lambda \in \Lambda^+$.

a) $-\lambda|^G$ has the irreducible module $S(\lambda)^* \cong S(\lambda^*)$ as its socle (= maximal completely reducible submodule).

Hence, $-\lambda|^G$ is indecomposable.

This is immediate from reciprocity.

b) $-\lambda|^G$ has a unique B-stable line. It has weight λ^* and λ^* is the maximal weight in $-\lambda|^G$ (with respect to $\leq$).

If $-\lambda|^G$ has B-stable line isomorphic to μ, then the G-submodule D it generates is a cyclic U^--module. This follows easily from the Bruhat decomposition of G. From a) it follows that $\lambda^* \leq \mu$. Now (2.2.1) above and reciprocity imply that $-\mu|^G$ also has a B-stable line, isomorphic to λ. Thus, $\lambda^* = \mu$ and D is the socle of $-\lambda|^G$. The desired result follows.

c) The formal character ch $-\lambda|^G$ is given by Weyl's formula (1.7) (with λ^* in place of λ). Hence, dim $-\lambda|^G$ is given by Weyl's dimension formula.

This is more difficult, cf. (3.3.4) below.

It follows that $\operatorname{ch} -\lambda|^G = \sum_{\lambda^* \geq \mu \in \Lambda^+} m_\lambda(\mu) \operatorname{ch} S(\mu)$,

where $m_\lambda(\mu) = [-\lambda|^G : S(\mu)]$ is the multiplicity of $S(\mu)$ as a composition factor of $-\lambda|^G$. Over the years there has been intense interest in the structure of $-\lambda|^G$ (or equivalently in the duals $(-\lambda|^G)^*$, popularly known as Weyl modules). In particular, an explicit determination of the matrix $[m_\lambda(\mu)]$ (or its inverse) would essentially solve the above problem of calculating ch $S(\lambda)$, cf. (1.7).

(2.2.4) SMITH'S THEOREM [43]. Fix a parabolic subgroup $P = P_J$ and let $P^- = L_J.U_J^-$ be the opposite parabolic subgroup. Fix $S(\lambda)$ and let V be the L_J-socle of the space $S(\lambda)^{U_J^-}$ of U_J^--fixed points. We make several observations:

a) V is irreducible as an L_J-module.

This is clear since B^- has a unique fixed line in $S(\lambda)$, and, a fortiori, in V.

b) Now view V as a rational P-module by making U_J act trivially. Then $Ev(V)$ maps $(V|_P^G)^{U_J^-}$ injectively into V.

In fact, if $f \in \ker Ev(V)$, then $f(1) = 0$. Thus, for $g \in P$, $u \in U_J^-$, $f(gu) = (u.f)(g) = f(g) = g.f(1) = 0$, so $f = 0$ on the dense subset $P_J.U_J^-$ of G. Hence, $f = 0$. (Notice b) holds for an arbitrary rational P-module.)

c) Since $-\lambda^*$ is the "low" weight in V, we get a nonzero B-module homomorphism $V \to -\lambda^*$ which lifts to a nonzero P-module homomorphism $V \to -\lambda^*|_B^P$, necessarily an inclusion by a). Similarly, there is a nonzero P-module homomorphism $S(\lambda) \to -\lambda^*|_B^P$, whose image is contained in V since $S(\lambda)$ is a cyclic B-module, generated by a nonzero $-\lambda^*$-weight vector.

d) Thus, by c), $S(\lambda) \subseteq V|_P^G$. Hence, $S(\lambda)^{U_J^-} \subseteq (V|_P^G)^{U_J^-}$, an L_J-submodule of V by b).

<u>It follows therefore that</u> $S(\lambda)^{U_J^-}$ <u>is an irreducible</u> L_J-<u>module</u>.

(2.2.5) ITERATED INDUCTION. If $P_1,\ldots,P_n$ is any sequence of parabolic subgroups containing B and if V is a rational B-module, let $V|^{P_1,\ldots,P_n}$ denote $V|^{P_1}|_B|^{P_2}\ldots|_B|^{P_n}$, the result of successively restricting to B then inducing to P_i. <u>Then</u> $V|^{P_1,\ldots,P_n} \cong V|^G$ <u>as</u> P_n-<u>modules provided that</u> $P_1\ldots P_n = G$. The reader will find a proof of this fact in [16]. Let us merely point out an application. Let $w_o = s_{\beta_1}\ldots s_{\beta_N}$ be a reduced expression for the long word w_o. Setting $P_i = P_{\{\beta_i\}}$, we have $P_1\ldots P_N = G$. Now if V is a rational B-module which extends to a rational P_i-module, we have from (2.2.1) and the isomorphism $k[P_i/B] \cong k$ that $V|_B^{P_i} \cong V$. Thus, we get the following <u>extension theorem</u> [16]: <u>A rational</u> B-<u>module extends to a rational</u> G-<u>module iff it extends to a rational</u> P-<u>module for each minimal parabolic subgroup</u> $P \supseteq B$.

3. INFINITESIMAL METHODS

The theory of group schemes aims to rehabilitate in positive characteristic the classical algebraic group-Lie algebra correspondence. Below we will indicate several applications of this point of view. First, we introduce some preliminary terminology, mostly taken from [20].

Let $\underline{M}_k\underline{E}$ denote the category of k-functors: an object in $\underline{M}_k\underline{E}$ is a functor from the category $\underline{M}_k$ of commutative k-algebras to the category $\underline{E}$ of sets. The category $\underline{Sch}/k$ of k-schemes embeds naturally in $\underline{M}_k\underline{E}$: to the k-scheme X we associate the functor (still denoted X) $R \to X(R) = \mathrm{Hom}_k(\mathrm{Spec}\ R, X)$, $R \in \mathrm{Ob}(\underline{M}_k)$. Intermediate between $\underline{Sch}/k$ and $\underline{M}_k\underline{E}$ is the full subcategory $\underline{M}_k\tilde{\underline{E}}$ of sheaves in the fppf topology on $\underline{M}_k^{op}$. The inclusion morphism $\underline{M}_k\tilde{\underline{E}} \to \underline{M}_k\underline{E}$ admits a left adjoint $\sim$: $\underline{M}_k\underline{E} \to \underline{M}_k\tilde{\underline{E}}$ $(X \to \tilde{X})$ commuting with finite projective limits, called sheafification. The point behind the introduction of $\underline{M}_k\tilde{\underline{E}}$ here is roughly to enlarge $\underline{Sch}/k$ suitably in order to facilitate many natural constructions (especially of a group-theoretic nature).

Next, recall that a k-group G is a functor from $\underline{M}_k$ to the category $\underline{Gr}$ of groups. An affine k-group (or affine k-group scheme) G is a k-group which is representable (by its coordinate ring $k[G]$). An affine algebraic group G in the classical sense defines an affine k-group, still denoted G, by setting $G(R) = \mathrm{Hom}_{k\text{-alg}}(k[G], R)$, $R \in \mathrm{Ob}(\underline{M}_k)$. The Hopf algebra structure on $k[G]$ endows $G(R)$ with a group structure in a well-known way. The standard notions of rational representations, etc., all apply in the more general setting of affine k-groups.

We cannot enter into further details of the above here. The reader may wish to consult [20], especially Ch. II,§1, and Ch. II,§§1,2,3, for more details. The formalism involving $\underline{M}_k\tilde{\underline{E}}$ will enter only in a technical way below which the reader may wish to ignore at first reading.

(3.1) FROBENIUS KERNELS. Let G be an affine k-group. Assume G is defined over the prime field k_o, and let $\sigma\colon G \to G$ be the Frobenius morphism (1.5). Now let H be a closed subgroup scheme of G. For a positive integer r, we define the <u>rth infinitesimal thickening</u> of H, denoted HG_r, by means of the pull-back diagram

$$\begin{tikzcd} HG_r \arrow[r] \arrow[d] & H \arrow[d, hook, "\text{incl.}"] \\ G \arrow[r, "\sigma^r"'] & G \end{tikzcd}$$

Thus, HG_r is a closed subgroup scheme of G. When $H = \{e\}$, the trivial subgroup, we denote HG_r by just G_r and we call it the <u>rth Frobenius kernel</u> of G.

EXAMPLES. a) Let $G = SL_n$. Then G_r is given by $G_r(R) = \{[a_{ij}] \in SL_n(R) \mid a_{ij}^{p^r} = \delta_{ij}\}$.

b) Assume G is connected. For $r = 1$, the dual Hopf algebra $k[G_1]^*$ of $k[G_1]$ is isomorphic to the restricted enveloping algebra $V(\mathfrak{g})$ of the Lie algebra of G. In particular, $\underline{M}_{G_1}$ is isomorphic to the category of restricted $\mathfrak{g}$-modules.

c) Let G be a semisimple, simply connected algebraic group defined and split over k_o. It is easy to see that $\widetilde{G/G_r} \cong G$ for all r. Here $\widetilde{G/G_r}$ is the sheafification of the k-group G/G_r defined by

$(G/G_r)(R) = G(R)/G_r(R)$, $R \in \mathrm{Ob}(\underline{M}_k)$. Similarly, $G_r\tilde{/}G_s \cong G_{r-s}$ for $r > s$ (cf. [23; 3.6]). It is then easy to extend the argument of (1.6) to see that the map $\lambda \to S(\lambda)|_{G_r}$, $\lambda \in \Lambda_r^+$, is a bijection between Λ_r^+ and representatives from the distinct isomorphism classes of irreducible rational G_r-modules. (In a different form, this result is due to Humphreys [26] for $r > 1$.) More generally, for an arbitrary closed subgroup scheme H of G, one has that $HG_r\tilde{/}G_r \cong H\tilde{/}H \times_G G_r$, and the irreducible rational HG_r-modules have the form $V \otimes W$ where V is an irreducible rational $H\tilde{/}H \times_G G_r$-module and $W \cong S(\lambda)|_{HG_r}$ for some $\lambda \in \Lambda_r^+$.

(3.2) MACKEY THEORY. Let $f\colon H \to G$ be a morphism of affine k-groups. Given a rational G-module V, we can regard it as a rational H-module by means of the morphism f. Thus, we obtain a <u>restriction functor</u> $f^*\colon \underline{M}_G \to \underline{M}_H$ which, as in Section 2, admits a right adjoint, called <u>induction</u>, $f_*\colon \underline{M}_H \to \underline{M}_G$ (see [14;1.2] for more details). The functor f_* is left exact, and, as usual, we let $R^nf_*\colon \underline{M}_H \to \underline{M}_G$ denote its <u>nth</u> right derived functor. When no confusion results, we often write $V|_H$ for $f^*(V)$ and $W|^G$ or $W|_H^G$ for $f_*(W)$.

Now fix $f\colon H \to G$ and let L be a closed subgroup scheme of G. Let $i\colon L \to G$ be the inclusion morphism. Consider the morphism (*) $L \times H \to G$ (in <u>Sch</u>/k) given by the composition of $i \times f$ with the multiplication map in G. Let $j: L \times_G H \to H$ be the natural inclusion morphism of affine k-groups and let $f': L \times_G H \to L$ be the morphism obtained by restricting f to $L \times_H H$. Now we can state

(3.2.1) THEOREM [18]. With the above notations, assume in addition that the morphism (*) is an epimorphism in $\underline{\tilde{M}}_k\underline{E}$. Then we have a natural isomorphism of functors

$$R^n j_* \circ f'^* \cong f^* \circ R^n i_*: \underline{M}_L \to \underline{M}_H, \quad n \geq 0.$$

We remark that the morphism (*) is an epimorphism in case G is a reduced algebraic group and $G(k) = L(k)f(H(k))$. If H is a closed subgroup scheme of G and if we write $H \cap L$ for $H \times_G L$, we can express the $n = 0$ case of (3.2.1) as $V|^G|_H \cong V|_{H \cap L}|^H$ for $V \in \mathrm{Ob}(\underline{M}_L)$!!.

Now assume that G is reduced and of finite type over k, and that H and L are closed subgroups such that L has an open orbit Ω in $G\tilde{/}H$. Choose $f \in \Omega(k)$ and let x be a representative for f in $G(k)$. Let $L_x = H^x \times_G L$, where $H^x = x^{-1}Hx$, be the stabilizer of x in L. For $V \in \mathrm{Ob}(\underline{M}_H)$, let V^x denote the rational H-module obtained by making H^x act on V through the morphism $H^x \to H$ $(h^x \to h)$. Now we have

(3.2.2) THEOREM [18]. Assume that $G/H - \Omega$ has codimen-

sion ≥ 2. Then for a rational H-module V, we have
$V|^{G}|_{L} \cong V^{x}|_{L_x}|^{L}$.

In addition to the proofs of (3.2.1) and (3.2.2), [18] gives a version of (3.2.2) valid for the higher derived functors of induction.

(3.3) APPLICATIONS. Let G be a semisimple, simply connected algebraic group, defined and split over k_o.

a) PARABOLIC INDUCTION THEOREM. Let P_J and P_K be two standard parabolic subgroups such that $J^* \cup K = \Pi$. One verifies that the P_K-orbit Ω of w_oP_J/P_J in G/P_J is open and that $\operatorname{codim}(G/P_J - \Omega) \geq 2$. <u>Therefore</u>, <u>we obtain from</u> (3.2.2) <u>that</u> $V|^{G}|_{P_K} \cong V^{w_o}|_{P_K \cap P_J^{w_o}}|^{P_K}$ <u>for</u> $V \in \operatorname{Ob}(\underline{M}_{P_J})$ [18].

b) THE STEINBERG MODULES. For each positive integer r, define the <u>rth</u> Steinberg module $St(r)$ to be $S((p^r-1)\rho)$. Below we sketch a proof that $St(r) \cong -(p^r-1)\rho|_B^G$. The argument is given in several steps:

1) $\dim St(r) = p^{rN}$, $N = |\Phi^+|$. This follows from the $r = 1$ case (in view of (1.6)) where one argues directly, cf. [18] for details.

2) Since $St(r)$ is G_r-irreducible (3.1), reciprocity gives an inclusion $i\colon St(r)|_{G_r} \to -(p^r-1)\rho|_{B_r}^{G_r}$ of G_r-modules.

3) Applying (3.2.1) for $L = B_r$, $H = U_r^-$, we obtain,

using (2.2.2) and the fact that $B_r \times_{G_r} U_r^-$ is the trivial group, that $-(p^r-1)\rho|_{B_r}^{G_r}|_{U_r^-} \cong k[U_r^-]$, so has dimension equal to p^{rN}. Thus, i is an isomorphism by 1) above.

4) Similarly, $-(p^r-1)\rho|_B^{BG_r}|_{G_r} \cong -(p^r-1)\rho|_{B_r}^{G_r}$, so there is an isomorphism $j\colon St(r)|_{BG_r} \cong -(p^r-1)\rho|_B^{BG_r}$ which factors as $St(r)|_{BG_r} \overset{\beta}{\to} -(p^r-1)\rho|_B^G|_{BG_r} \overset{\alpha}{\to} -(p^r-1)\rho|_B^{BG_r}$. By (2.2.3b), α is an injection on its B-socle, so α is an injection. It follows that β is an isomorphism, which clearly proves our claim.

c) BUNDLE COHOMOLOGY. Consider the following commutative diagram, the square being a pull-back:

$$\begin{tikzcd}
G \arrow[r, "\sigma^r"] & G & \\
& & BG_r \arrow[ul, "b"'] \\
BG_r \arrow[uu, "b"] \arrow[r, "c"] & B \arrow[uu, "a"] \arrow[ur, "a'"'] &
\end{tikzcd}$$

Here a, a', b are the natural inclusion morphisms. By (3.2.1), we have that $R^n b_* \circ c^* \cong \sigma^{r*} \circ R^n a_* \cong \sigma^{r*} \circ R^n b_* \circ a'_*$ (since a'_* is exact [14;4.2]) for all $n \geq 0$. Let $V \in Ob(\underline{M}_B)$. If we apply the above to $V^{(r)} \otimes -(p^r-1)\rho$, using b) above and the fact (2.2.1) that the exact functor $- \otimes St(r)$ commutes with induction, we obtain immediately that

(3.3.1) $R^n a_*(V^{(r)} \otimes -(p^r-1)\rho) \cong R^n a_*(V)^{(r)} \otimes St(r),\ n \geq 0.$

(This argument was first discovered by E. Cline and reproduced in [18] in detail.) Using the easily derived fact that $R^n a_*(V) \cong H^n(G/B, L_V)$ (sheaf cohomology, cf. [19] for example) we obtain from (3.3.1) the following important result discovered independently by H. Andersen [5] and W. Haboush [25]:

(3.3.2) THEOREM. For a rational B-module V, we have an isomorphism of rational G-modules:

$$H^n(G/B, L_V)^{(r)} \otimes St(r) \cong H^n(G/B, L_{V^{(r)} \otimes -(p^r-1)\rho}),\ n \geq 0.$$

This theorem has several important consequences.

(3.3.3) COROLLARY. Let $\lambda \in \Lambda^+$. Then $H^n(G/B, L_{-\lambda}) = 0$ for all positive integers n.

This result, due to G. Kempf [36], follows easily from (3.3.2) using the ampleness of the line bundle $L_{-\rho}$, cf. [5] for more details.

(3.3.4) COROLLARY. For $\lambda \in \Lambda^+$, $ch\ {-\lambda}|^G$ is given by Weyl's formula (1.7) (with λ^* in place of λ).

This is a well-known consequence of (3.3.3), cf. [24] for the argument.

(3.3.5) COROLLARY. For $V \in Ob(\underline{M}_B)$, there is an injection $H^n(G/B,L_V)^{(r)} \to H^n(G/B,L_V(r))$ of rational G-modules for all $n \geq 0$, $r \geq 0$.

This result, which had been conjectured earlier by Cline, Scott, and the author, was proved by Andersen in [5]. It follows easily from (3.3.2).

4. RATIONAL COHOMOLOGY

The rational cohomology of affine algebraic groups was first studied by Hochschild in the early 1960's. It plays an important role in the representation theory, and in this section we will survey some of the work done to date.

(4.1) BASIC DEFINITIONS. Let G be an affine k-group. Let F_G be the functor from the category $\underline{M}_G$ to the category of k-vector spaces which assigns to each $V \in \mathrm{Ob}(\underline{M}_G)$ the space V^G of G-fixed points.[4] It is trivial to see that $\underline{M}_G$ possesses enough injectives, so we can speak of the n<u>th</u> right derived functor R^nF_G of F_G, $n \geq 0$. For $V \in \mathrm{Ob}(\underline{M}_G)$, write $H^n(G,V) = R^nF_G(V)$, the n<u>th</u> <u>rational</u> <u>cohomology</u> <u>group</u> <u>of</u> G <u>with</u> <u>coefficients</u> <u>in</u> V. Similarly, we can define the rational Ext_G-groups, $\mathrm{Ext}^n_G(V,-) = R^n\mathrm{Hom}_G(V,-)$, $V \in \mathrm{Ob}(\underline{M}_G)$.

Now let H be a closed subgroup scheme of G and let W be a fixed rational G-module. By reciprocity, we have that $\mathrm{Hom}_H(W,-) = K\circ J$, where $J = |_H^G$ is induction

[4] More precisely, if we view V as a k[G]-comodule with structure map Δ_V: $V \to k[G] \otimes V$, then $V^G = \{v \in V | \Delta_V(v) = 1 \otimes v\}$.

from $\underline{M}_H$ to $\underline{M}_G$ and $K = \mathrm{Hom}_G(W,-)$. It is immediate that J takes injective objects in $\underline{M}_H$ to injective objects in $\underline{M}_G$, so there is a Grothendieck spectral sequence

$$(4.1.1) \qquad E_2^{s,t} = \mathrm{Ext}_G^s(W, R^tJ(V)) \Rightarrow \mathrm{Ext}_H^{s+t}(W,V)$$

for all rational H-modules V.

(4.2) SOME BASIC RESULTS. In this section G will be a fixed semisimple, simply connected algebraic group, defined and split over k_o.

(4.2.1) TRANSFER THEOREM [19]. Let V be a rational G-module. Then the restriction map on cohomology induces a natural isomorphism $H^n(G,V) \to H^n(B,V)$, $n \geq 0$, of cohomology groups.

To prove this result, we will use the spectral sequence (4.1.1) with $H = B$. It is not hard to see that $R^tJ(Z) \cong H^t(G/B, L_Z)$ for every rational B-module Z. Now if $Z = V \in \mathrm{Ob}(\underline{M}_G)$, we obtain from (2.2.1) that $R^tJ(V) \cong V \otimes R^tJ(\underline{0})$, where $\underline{0}$ denotes the one-dimensional trivial B-module. It follows from (3.3.3) that $H^t(G/B, L_{\underline{0}}) = 0$ for $t > 0$. Taking W to be the trivial G-module, our spectral sequence collapses to give the desired result.

(4.2.2) REMARKS: a) The same argument shows that for $V \in \mathrm{Ob}(\underline{M}_G)$ and $\lambda \in \Lambda^+$, we have that $\mathrm{Ext}_G^n(V, -\lambda|^G) \cong \mathrm{Ext}_B^n(V,-\lambda)$, $n \geq 0$. See [19].

b) One can also show for r a positive integer and $V \in \mathrm{Ob}(\underline{M}_G)$ that there is an isomorphism $H^n(G,V) \to H^n(BG_r,V)$, $n \geq 0$.

(4.2.3) THEOREM [19]. Let $V \in \mathrm{Ob}(\underline{M}_G)$, $\lambda \in \Lambda^+$. Suppose that $-\lambda$ is not strictly greater (in the partial order $\geq$ (1.3)) than any weight η of T in V. Then $\mathrm{Ext}^n_G(V,-\lambda|^G) = 0$, $n > 0$.

For the proof see [19;3.2]. This result has the following important consequence (also taken from [19]):

(4.2.4) COROLLARY. Let $\lambda,\mu \in \Lambda^+$.

a) $-\lambda|^G \otimes -\mu|^G$ is G-acyclic, i.e., $H^n(G,-\lambda|^G \otimes -\mu|^G) = 0$ for all positive n.

b) $-\lambda|^G \otimes -\mu$ is B-acyclic.

(4.2.5) REMARK. Recent work of Wang Jain-pain [45] shows that $-\lambda|^G \otimes -\mu|^G$ $(\lambda,\mu \in \Lambda^+)$ has a G-filtration with sections isomorphic to induced modules $-\omega|^G$ $(\omega \in \Lambda^+)$, at least as long as p is sufficiently large. It follows from (4.2.4) therefore that an arbitrary tensor product $-\lambda_1|^G \otimes\ldots\otimes -\lambda_n|^G$ $(\lambda_i \in \Lambda^+)$ is G-acyclic. Using the parabolic induction theorem (3.3a), D. Vella has determined (unpublished as yet) conditions which guarantee that $-\lambda|^G$, $\lambda \in \Lambda^+$, has an L-filtration with sections isomorphic to induced modules $-\omega|^L$ (ω a dominant weight for L), where L is a Levi factor of a suitable parabolic subgroup of G. In particular, this means $-\lambda|^G$ is L-acyclic.

The above results have an interesting (though formal) application, observed first in [42] (and motivated by similar results in characteristic 0), to the representation theory. For V,W finite dimensional rational B-modules, we define

$$\chi(V,W) = \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{Ext}_B^i(V,W).$$

(It is not hard to see that the Ext-groups here are finite dimensional and vanish for i sufficiently large, cf. [19].) Now for $\lambda,\mu \in \Lambda^+$, (4.2.4) implies that $\chi(\mu^*,-\lambda|^G) = \delta_{\lambda,\mu}$ (Kronecker delta). Thus, $\mathrm{ch}\ -\lambda|^G = \sum_{\lambda\in\Lambda+} \chi(\mu^*,-\lambda|^G)\ \mathrm{ch}\ -\mu|^G$, and so, by the additivity of χ, we get that

(4.2.6) $$\mathrm{ch}\ S(\lambda) = \sum_{\mu\in\Lambda^+} \chi(\mu^*,S(\lambda))\ \mathrm{ch}\ -\mu|^G.$$

(4.3) FURTHER RESULTS. Let G be a semisimple, simply connected algebraic group, defined and split over k_o.

(4.3.1) GENERIC COHOMOLOGY. For $q = p^d$, let $G(q)$ denote the subgroup of G consisting of GF(q)-rational points. Then, given a rational G-module V, we can consider the classical discrete cohomology groups $H^n(G(q),V)$. The "generic cohomology" arises from the stability of these groups. More precisely:

(4.3.1.1) THEOREM [19]. Let V be a finite dimensional rational G-module. For a fixed n, the cohomology groups $H^n(G(q),V)$ achieve a stable value $H^n_{gen}(G,V)$ as $d \to \infty$ which is given in terms of rational cohomology by $H^n(G,V^{(r)})$ for $r >> 0$.

Besides the proof of this result, [19] contains arithmetic conditions on r and d which guarantee that $H^n_{gen}(G,V) \cong H^n(G(q),V) \cong H^n(G,V^{(r)})$. The above result has also been extended to include the twisted groups [9].

We next introduce a variation on the Kostant partition function P. Namely, for $n = 0,1,\ldots,$ and $\lambda \in \Lambda$, let $P_n(\lambda)$ denote the number of ways λ can be written as a sum of n positive roots. Thus, we have that $P = P_0 + P_1 + \ldots$. In the following result, we assume G is of simple type.

(4.3.1.2) THEOREM [22], [23]. Let $\lambda \in \Lambda^+$ satisfy the condition $\langle\lambda+\rho,\alpha_0^\vee\rangle \leq p$, where α_0 is the maximal short root in Φ^+. Assume also that $p \geq 2h$, where h is the Coxeter number of Φ. Then for $0 \leq n < (p-2h+2)/4$ and all $r \geq 1$,

$$\dim H^n(G,S(\lambda)^{(r)}) = \begin{cases} 0 & n \text{ odd} \\ \sum_{w\in W} \det(w)P_m(w(\lambda+\rho)-\rho), & n=2m \text{ is even.} \end{cases}$$

In view of (4.3.1.1), the above formula calculates also the generic cohomology of $S(\lambda)$ in a range of degrees.

(4.3.2) INFINITESIMAL COHOMOLOGY. Let G be as above, and consider the relationship between the cohomology of G and that of its Frobenius kernels G_r (3.1).

(4.3.2.1) THEOREM [14]. Let V be a finite dimensional rational G-module.

a) The natural restriction map $H^n(G,V) \to \varprojlim H^n(B_r,V)$ is an isomorphism for each n.

b) The natural restriction map $H^n(G,V) \to \varprojlim H^n(G_r,V)$ is an isomorphism for $n \leq 2$, and an injection for all n.

In b) above, it turns out that $H^1(G,V) \cong H^1(G_r,V)$ for $r >> 0$, at least so long as $p \neq 2$ or G has no component of type C_ℓ. In general, stability does not hold in a) or in b) (for $n = 2$).

We also would like to mention the following result.

(4.3.2.2) THEOREM [8]. Assume that $p \neq 2$ or that G has no component of type C_ℓ. Then for $\lambda \in \Lambda_r^+$, we have that $\mathrm{Ext}^1_{G_r}(S(\lambda),S(\lambda)) = 0$.

Finally, we remark that [23] contains a calculation (in a range) of the cohomology $H^*(G_r,k)$. This in fact plays an important role in the proof of (4.3.1.2).

5. LUSZTIG'S CONJECTURE

Throughout this section, G will be a fixed simple, simply connected algebraic group over k.

(5.1) THE AFFINE WEYL GROUP. Let $\mathbb{E} = \mathbb{Z}\Phi \otimes \mathbb{R}$ be the Euclidean space associated to the root system Φ of G (1.3). Let $s_{\alpha,n}$ ($\alpha \in \Phi$, $n \in \mathbb{Z}$) be the reflection of $\mathbb{E}$ about the hyperplane $H_{\alpha,n} = \{x \in \mathbb{E} | <x,\alpha^\vee> = np\}$, and let W_a (the "affine Weyl group") be the subgroup of $\mathrm{Aut}(\mathbb{E})$ generated by these $s_{\alpha,n}$. If α_o denotes the maximal short root in Φ^+, and if $S = \{s_{\alpha_1,0}, \ldots, s_{\alpha_\ell,0}, s_{\alpha_o,1}\}$, then (W_a,S) is a Coxeter system. Clearly, W_a is a semidirect product $W.T$, where T is the normal subgroup generated by the translations of $\mathbb{E}$ by p-multiples of roots. In addition to the usual action of W_a on $\mathbb{E}$, there is the "dot" action defined by $w\cdot\lambda = w(\lambda+\rho) - \rho$.

For $\mu,\lambda \in \mathbb{E}$, write $\mu \uparrow \lambda$ provided that $\mu = s_{\alpha,n}\cdot\lambda \leq \lambda$ for some $\alpha \in \Phi^+$ and integer n. We say that μ and λ are <u>strongly</u> p-<u>linked</u> (resp. p-<u>linked</u>) provided there is a sequence $\mu = \mu_0 \uparrow \mu_1 \uparrow \ldots \uparrow \mu_n = \lambda$ (resp. $\lambda = w\cdot\mu$, for some $w \in W_a$).

(5.1.1) THEOREM [3]. Let $\lambda \in \Lambda$ and let $S(\mu)$ be a G-composition factor of $H^n(G/B,L_{-\lambda^*})$ (cf. (3.3c)) for some non-negative integer n. Suppose that $w\cdot\lambda = \lambda' \in \Lambda^+$, $w \in W$. Then μ is strongly p-linked to λ'.

It follows easily from (5.1.1) and (2.1) that the high weights of the G-composition factors of an indecomposable rational G-module are p-linked.

(5.2) THE GENERIC HECKE ALGEBRA. Let $A = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ be the ring of integral Laurent polynomials in the indeterminate $q^{\frac{1}{2}}$. Given the Coxeter system (W_a, S) of (5.1) (or any Coxeter system for that matter) we define its <u>generic</u> <u>Hecke</u> <u>algebra</u> $\mathcal{H}$ to be the free A-algebra with basis T_w, $w \in W_a$, and multiplication defined by

$$\begin{aligned} T_w \cdot T_{w'} &= T_{ww'} && \text{if } \ell(ww') = \ell(w) + \ell(w') \\ (T_s+1)(T_s-q) &= 0, && s \in S. \end{aligned}$$

(Here ℓ denotes the usual length function on W_a.) Associated to $\mathcal{H}$, Kazhdan and Lusztig [34] have constructed for each pair $w, z \in W_a$, with $w \leq z$ in the Bruhat order on W_a, a polynomial $P_{w,z}(q)$. These can be described in terms of certain symmetric elements in $\mathcal{H}$: Namely, the involution $a \to \bar{a}$ of A defined by $\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}$ extends to an involution of the Hecke algebra by putting $\overline{\Sigma a_w T_w} = \Sigma \bar{a}_w T^{-1}_{w^{-1}}$. Then there exists for each $w \in W_a$ a unique element C_w such that

$$\bar{C}_w = C_w$$

$$C_w = \sum_{y \leq w} (-1)^{\ell(y)+\ell(w)} q^{(\ell(w)-2\ell(y))/2} \bar{P}_{y,w} T_y$$

where $P_{y,w} \in \mathbb{Z}[q]$ has degree at most $(\ell(w)-\ell(y)-1)/2$ and $P_{w,w} = 1$.

The reader will find details of the above in [34], as well as equivalent descriptions of the <u>Kazhdan-Lusztig polynomials</u> $P_{y,w}$. Part of the motivation behind these polynomials in general lies in the role they play in measuring the failure of Poincaré duality in the closures of the cells BwB/B $(w \in W)$ in G/B (the so-called <u>Schubert varieties</u>). In fact, $P_{y,w}(q)$ turns out to be (at least for W and $k = \mathbb{C}$) the Poincaré polynomial for a geometric cohomology (due to Goreski, MacPherson, and Deligne) of these Schubert varieties.

(5.3) THE CONJECTURE. Choose $\lambda \in \Lambda$ such that $-p< \langle\lambda+\rho,\alpha_i^\vee\rangle <0$ for $i = 0, 1,\cdots,\ell$. (The existence of λ requires that $p > h$, the Coxeter number of Φ.) Let $w\cdot\lambda \in \Lambda^+$, $w \in W_a$. According to (5.1.1), if $S(\mu^*)$ is a G-composition factor of $-(w\cdot\lambda)|^G$, then $\mu = y\cdot\lambda$, some $y \in W_a$. Now assume that $\langle w(\lambda+\rho),\alpha_0^\vee\rangle < p(p-h+2)$. Then the conjecture is that

$$(5.3.1)\quad \mathrm{ch}\ S((w\cdot\lambda)^*) = \Sigma(-1)^{\ell(w)+\ell(y)}P_{y,w}(1)\ \mathrm{ch}\ -(y\cdot\lambda)|^G,$$

where on the right side the summation is over all $y \leq w$ satisfying $y\cdot\lambda \in \Lambda^+$.

In view of (4.2.4) above, we could also express (5.3.1) in terms of an identity involving the $P_{y,w}(1)$'s and certain rational Ext-groups. For further results along these lines see [42], [7].

Although it is not yet clear where a proof of this conjecture might lie, it is tempting to speculate that some connection between rational cohomology and a geometric cohomology which involves the $P_{y,w}$'s might do the trick. This was, in fact, essentially the case in the Kazhdan-Lusztig conjecture [34] concerning Verma modules in characteristic 0 (proved independently by Brylinski and Kashiwara [11] and by Beilinson and Bernstein [10]), where (very roughly speaking) the connection between the algebraic cohomology (in the category $\mathcal{O}$ of BGG) and the geometric cohomology (alluded to above in (5.2)) was obtained via homological properties of the sheaf of algebras of differential operators on G/B.

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