Chapter III

QUANTIFIERS

In this chapter we consider the extension of the ideas in the preceding chapter to quantifiers. This requires that we use term extensions in the sense of Chapter I § 8. The principal difficulty consists in formulating precisely the conditions governing such term extensions. When this is taken care of, we shall find that the principal theorems of Chapter II are valid for the enlarged systems LA*, LC*, etc.; except that the number of possibilities in the decision process of Chapter II § 6 is no longer finite, so that the systems are not decidable. The system TA* and the predicate calculuses HA* and HC* are considered in § 7.

This chapter contains necessarily a lot of fussy detail. It will not be needed in the following chapters except for parts relating to variables and quantifiers.

The treatment is carried out with greater explicitness than usual.[1] Particular attention is paid to the range of variables for which a theorem is valid. In the theorem of § 8 for example, it is shown that a theorem can be proved without using any free variables not occurring in the theorem itself.

Two additional assumptions regarding $\mathfrak{S}$ are introduced at the end of § 4.

1. **Preliminary Analysis.** In an intuitive way it is clear what we want to mean by (x)A(x) and (∃x)A(x). We can get rules analogous to those in Chapter II § 2 as follows. Let $\mathfrak{S}'$ be an extension of $\mathfrak{S}$, and let x not be a term in $\mathfrak{S}'$. Let A(x) be a proposition of $\mathfrak{S}'(x)$ involving x. Then we should say (x)A(x) and (∃x)A(x) are propositions of $\mathfrak{S}'$; further

a5) (x)A(x) is true in $\mathfrak{S}'$ if A(x) is true in $\mathfrak{S}'(x)$, x being an indeterminate in $\mathfrak{S}'(x)$.[2]

a6) (∃x)A(x) is in $\mathfrak{S}'$ if, for some term t of $\mathfrak{S}'$, A(t) is in $\mathfrak{S}'$.

The parallel rules for introduction as hypothesis are:

b5) B is a consequence of (x)A(x) in $\mathfrak{S}'$ if, for some term t of $\mathfrak{S}'$, B is a consequence of A(t).

1. Gentzen did not formulate the conditions on his "Gegenstandsvariablen" with as much care as he did many other matters.

2. Note this is not the same as saying that A(t) is true in $\mathfrak{S}'$ for every term t. The latter would not be invariant under extension.

b6) B is a consequence of $(\exists x)A(x)$ in $\mathfrak{S}'$ if B is a consequence of $A(x)$ in $\mathfrak{S}'(x)$.

Although these rules are simple enough, yet their precise working out involves difficulties. Thus, suppose $A(x)$ is itself a quantified proposition, say of the form $(\exists y)\,B(x,y)$; is it or is it not in accord with our intentions to admit forming $A(t)$ when t contains y? One can easily see that it is not. In fact, the intuitive combination of a5 and b6 admits only such t's as can replace x in a valid argument of $\mathfrak{S}'(x)$. But in ordinary number theory

$$(\exists y)\ .\ x < y$$

is intuitively true for all x; on the other hand

$$(\exists y)\ .\ y + 1 < y$$

is false.

This example shows that we must exercise care in formulating rules. Indeed a rather complicated analysis is necessary.[3]

The analysis is considerably facilitated by the recognition that the role of "x" in the two statements

$$A(x) \text{ holds in } \mathfrak{S}'(x)$$
$$(x)A(x) \text{ holds in } \mathfrak{S}'$$

is quite different, just as it is quite different in the equations

$$x + x = 2x$$

$$\int_0^1 x\,dx = 1/2.$$

We can distinguish two classes of variables, and agree to keep them separate throughout. Thus we shall use "a", "b", "c" for the first usage, "x", "y", "z" for the second. Then the rule a5, for instance, can be stated: if $A(a)$ is in $\mathfrak{S}'(a)$, then $(x)\,A(x)$ is in $\mathfrak{S}'$.

Even with this help, however, we must be quite explicit in connection with phrases such as "$---_1$ occurs in $---_2$", "$---_1$ is bound in $---_2$", "$---_1$ is the result of substituting $---_2$ for $---_3$ in $---_4$", etc. The precise definitions will concern us in the next sections.

2. **Conventions of the B-language.** The complexity of the analysis of variables requires that we introduce into the U-language some technical terminology. This will be introduced, of course, as we proceed; but it will add to clearness if we first examine it as a whole, and dispose of some matters of a general

3. This complexity arises wherever bound variables occur. For careful formulations of various ideas connected with bound variables the work of Church should be consulted. See [13].

nature. The new terminology constitutes a language which will be called the B-language.

The basic nominal phrases of the B-language are summarized in Table 4. Here the symbols in Column 1 constitute the A-language of the system $\mathfrak{S}^*$ (cf. I, § 8) obtained by adjoining the term variables to $\mathfrak{S}$. The symbols in Column 2 are proper names for the categories listed at the left. Some of these can also be used as functors as in Column 3, with arguments taken from Column 5. Columns 4 and 5 give various classes of U-variables. Symbols in parentheses are not used until later chapters.

Table 4

Name of Category	U-constants Elements 1	U-constants Classes 2	U-constants Subclasses 3	U-variables Elements 4	U-variables Subclasses (or sequences) 5
Primitive constants	$e_1, e_2, \ldots$	$\mathfrak{e}$			
Term variables	$q_1, q_2, \ldots$	$\mathfrak{q}$		u, v, w	$\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$
Primitive operators	$\omega_1, \omega_2, \ldots$	Ω			
Primitive predicators	$\varphi_1, \varphi_2, \ldots$	Φ			
Real variables		$\mathfrak{r}$	$\mathfrak{h}$	a, b, c, f, g, h	$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{g}$
Apparent variables		$\mathfrak{f}$		x, y, z	$\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$
Terms		$\mathfrak{t}$	$\mathfrak{t}(\mathfrak{u})$	r, s, t	
Null class of terms		$\mathfrak{o}$			
Elementary propositions	$E_1, E_2, \ldots$	$\mathfrak{E}$	$\mathfrak{E}(\mathfrak{u})$		
Propositions	(F) (M)	$\mathfrak{P}$	$\mathfrak{P}(\mathfrak{u})$, ($\mathfrak{F}$)	A, B, C,	$\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{U}, \mathfrak{V}, \mathfrak{W}, \mathfrak{M}, \mathfrak{H}$.
Axioms		$\mathfrak{A}$	$\mathfrak{A}(\mathfrak{u})$		
Elementary theorems		$\mathfrak{S}$	$\mathfrak{S}(\mathfrak{u})$		
Theorems		$\mathfrak{T}$	$\mathfrak{T}(\mathfrak{X}, \mathfrak{u})$ $\mathfrak{T}(\mathfrak{x})$		
Null class or prosequence		O			
Null system		$\mathfrak{D}$			

Besides the notions defined in Table 4, we have already defined in Chapter II the notion of a prosequence and the following predications:

$$\text{(1a)} \qquad A \;\varepsilon\; \mathfrak{X}$$

$$\text{(1b)} \qquad \mathfrak{X} \leq \mathfrak{Y}$$

$$\text{(1c)} \qquad \mathfrak{X} \equiv \mathfrak{Y}$$

(1d) $\mathfrak{X} \subseteq \mathfrak{Y}$

(1e) $\mathfrak{X} = \mathfrak{Y}$

(1f) $A_1, \ldots, A_m \vdash B$

(1g) $\mathfrak{X} \Vdash \mathfrak{Y}$.

All but the last of these will be taken over in this chapter without change. We shall also define here the morphological predications

(2a) u occurs in t

(2b) u occurs free in A

(2c) u is bound in A,

and the morphological operations

(3a) $(\mathrm{Sb}\ {}^{s}_{u})t$

(3b) $(\mathrm{Sb}\ {}^{s}_{u})A$.

The elementary statements now are (see below)

$$\mathfrak{X} \mid \mathfrak{a} \vdash \mathfrak{Y}.$$

The predicators "$---_1 \varepsilon ---_2$", "$---_1 \subseteq ---_2$" will be used in the customary manner for indicating class membership and class inclusion respectively; also "$---_1 = ---_2$" for class identity. This usage does not conflict with (1a), (1d), (1e) but it is rather consistent with them. We shall also write the logical sum of classes $\mathfrak{a}$ and $\mathfrak{b}$ as "$\mathfrak{a} + \mathfrak{b}$" or "$\mathfrak{a}, \mathfrak{b}$". Since cardinal numbers are not involved, it is unnecessary to distinguish between a unit class and its sole element.

The predicator "$---_1 \equiv ---_2$" will be used to indicate identity in meaning; - i.e., if one will, identity of translation into the A-language. The negation of this relation will be indicated by "$---_1 \not\equiv ---_2$". Thus we have $e_1 \equiv e_1$ but $e_1 \not\equiv e_2$. This predicator will also be used in making definitions. The usage does not conflict with (1c) (in view of Remark 4 in II § 4)

The letters "i", "j", "k", "l", "m", "n" will be used for natural numbers (as subscripts, etc.). The predicators "$---_1 = ---_2$" and "$---_1 \neq ---_2$" will be used in their usual senses in that connection.

The B-language is not the same as the A-language of any of the episystems LA, LC, TA, etc. The latter is obtained simply by adding to Column 1 phrases sufficient to state particular elementary statements. If we exclude infinite classes and prosequences such an elementary statement for LC* is of the form

$$A_1, A_2, \ldots, A_m \mid a_1, a_2, \ldots, a_k \vdash B_1, B_2, \ldots, B_n.$$

The B-language is to state not only the elementary statements but the rules and morphology. Further the B-language is an interpreted language. Although we attempt to be precise as to its use, we do not attempt either to formalize it in any sense of the word, or to exhaust all the possibilities of the U-language in it.

3. **Rules for Terms and Propositions.** The formulation is given here in great detail because this appears to be the first time this has been done explicitly without assuming we are talking about symbols. Naturally this entails some prolixity. All that is necessary for the further developments is the validity of Theorems 1 and 2; and the reader may, if he prefers, take these as intuitively evident.

PRIMITIVE IDEAS FOR $\mathfrak{S}^*$. As stated in I § 8 we suppose that $\mathfrak{S}$ is a completely formalized system. We form $\mathfrak{S}^*$ by adjoining an infinite set of term variables as new primitive terms. Then the primitive ideas of $\mathfrak{S}^*$ are as follows:

Primitive terms of $\mathfrak{S}$, $(\mathfrak{e})$: $e_1, e_2, e_3, \ldots$

Term variables $(\mathfrak{q})$: $q_1, q_2, q_3, \ldots$

Primitive adjunctives: $\omega_1, \omega_2, \omega_3, \ldots$, where ω_i has m_i arguments

Primitive predicates: $\varphi_1, \varphi_2, \varphi_3, \ldots$, where φ_i has n_i arguments.

Primitive propositions: $E_1, E_2, E_3, \ldots$

FORMULATION OF $\mathfrak{S}(\mathfrak{u})$. If $\mathfrak{u} \subseteq \mathfrak{q}$, $\mathfrak{S}(\mathfrak{u})$ is the system obtained by confining the term variables to $\mathfrak{u}$. Then $\mathfrak{S}(\mathfrak{q})$ is $\mathfrak{S}^*$, while $\mathfrak{S}(\mathfrak{o})$ is $\mathfrak{S}$. The formulation of $\mathfrak{S}(\mathfrak{u})$ is then as follows:

I. <u>Terms</u> $\mathfrak{t}(\mathfrak{u})$:

(a) $\mathfrak{e} \subseteq \mathfrak{t}(\mathfrak{u})$.

(b) $\mathfrak{u} \subseteq \mathfrak{t}(\mathfrak{u})$.

(c) If $t_1, t_2, \ldots, t_{m_i} \varepsilon\ \mathfrak{t}(\mathfrak{u})$, then $\omega_i(t_1, t_2, \ldots, t_{m_i}) \varepsilon\ \mathfrak{t}(\mathfrak{u})$.[4]

II. <u>Elementary Propositions</u>, $\mathfrak{E}(\mathfrak{u})$:

(a) $E_i \ \varepsilon\ \mathfrak{E}(\mathfrak{u})$ for all $(\mathfrak{u}) \subseteq (\mathfrak{q})$.

(b) If $t_1, t_2, \ldots, t_{n_i} \varepsilon\ \mathfrak{t}(\mathfrak{u})$, then

$$\varphi_i(t_1, t_2, \ldots, t_{n_i}) \ \varepsilon\ \mathfrak{E}(\mathfrak{u}).$$[5]

III. <u>Theoretical Rules</u>. These are the same as for $\mathfrak{S}$, and are assumed to be of the form (6) in Chapter II, where $A_1, A_2, \ldots, A_m$, B are in $\mathfrak{E}(\mathfrak{u})$.

Certain further assumptions concerning $\mathfrak{S}$ are stated at the end of §4.

4. Note that an m_i is supposed associated with each ω_i and an n_i with each φ_i.

5. Cf. the preceding footnote.

DEFINITION 1. For each $u \varepsilon q$ and $t \varepsilon t(q)$ we define the predication

$$u \text{ occurs in } t^{6}$$

by recursion as follows:

(a) If $t \varepsilon e$, then u does not occur in t.

(b) If $t \varepsilon q$, then u occurs in t if and only if $u \equiv t$.

(c) If $t \equiv \omega_i(t_1, t_2, \ldots, t_{m_i})$, then u occurs in t if and only if u occurs in some t_j.

DEFINITION 2. For each $u \varepsilon q$ and any terms s and t we define the operation

$$(Sb\,{}^{s}_{u})t$$

as follows:

(a) $(Sb\,{}^{s}_{u})e_i \equiv e_i$.

(b) If $v \varepsilon q$ and $v \not\equiv u$, then $(Sb\,{}^{s}_{u})v \equiv v$.

(c) $(Sb\,{}^{s}_{u})u \equiv s$.

(d) If $t \equiv \omega_i(t_1, t_2, \ldots, t_{m_i})$,

and $\qquad (Sb\,{}^{s}_{u})t_j \equiv t_j' \quad j = 1, 2, \ldots, m_i;$

then $\qquad (Sb\,{}^{s}_{u})t \equiv \omega_i(t_1', t_2', \ldots, t_{m_i}')$.

DEFINITION 3. We define simultaneously the predications

A is in $\mathfrak{P}(u)$

u occurs free in A

x is bound in A

as follows: -

(a) If A is in $\mathfrak{E}(u)$, then A is in $\mathfrak{P}(u)$. No variable occurs, free or bound, in E_i. If

$$A \equiv \varphi_i(t_1, t_2, \ldots, t_{n_i}),$$

then u occurs free in A if and only if u occurs in some t_j in the sense of Definition 1, and no variable is bound in A.

(b) If $A \equiv B \circ C$, where $B \varepsilon \mathfrak{P}(u)$ and $C \varepsilon \mathfrak{P}(u)$, then $A \varepsilon \mathfrak{P}(u)$. The variables which occur free in A are those which occur free in either B or C or both. Likewise the variables bound in A are those bound in B or C or both.

6. At a later stage it is desirable to have also "e_i occurs in t." For this we simply change (a)(b) to "e_i occurs in e_i but not in e_j or in any $t \varepsilon q$."

(c) If $A \equiv (x)B$ or $A \equiv (\exists x)B$, where $B \,\varepsilon\, \mathfrak{P}(\mathfrak{u}, x)$, x occurs free in B[7] and x is not bound in B, then $A \,\varepsilon\, \mathfrak{P}(\mathfrak{u})$. The variables which occur free in A are those which are distinct from x and occur free in B; those bound in A are x together with those bound in B.

DEFINITION 4. A term belonging to $\mathfrak{t}(\mathfrak{r})$ will be called a *real term*; a proposition belonging to $\mathfrak{P}(\mathfrak{r})$ a *real proposition*. Likewise a term or proposition of $\mathfrak{S}(\mathfrak{o})$ will be called a *constant*.

THEOREM 1. *The class $\mathfrak{P}(\mathfrak{u})$ has the following properties*:

(a) *For each $u \,\varepsilon\, \mathfrak{q}$ and $A \,\varepsilon\, \mathfrak{P}(\mathfrak{u})$ it is definite whether u occurs in A. If it does occur, then $u \,\varepsilon\, \mathfrak{u}$.*

(b) *If $\mathfrak{u}$ is the class of all u which occur free in A, where $A \,\varepsilon\, \mathfrak{P}(\mathfrak{q})$, then $A \,\varepsilon\, \mathfrak{P}(\mathfrak{u})$.*

(c) *If $\mathfrak{u} \subseteq \mathfrak{v}$, then $\mathfrak{P}(\mathfrak{u}) \subseteq \mathfrak{P}(\mathfrak{v})$.*

The proof of Theorem 1 is by induction on the construction of A, using Definition 3.

DEFINITION 5. For each $A \,\varepsilon\, \mathfrak{P}(\mathfrak{q})$, $s \,\varepsilon\, \mathfrak{t}(\mathfrak{q})$ and $u \,\varepsilon\, \mathfrak{q}$ we define

$$(Sb\, {}^{s}_{u})A$$

as follows:

(a) If $A \equiv E_i$, $(Sb\, {}^{s}_{u})A \equiv A$.

(b) If $A \equiv \varphi_i(t_1, t_2, \ldots, t_{n_i})$,

and $\quad (Sb\, {}^{s}_{u})t_j \equiv t'_j, \qquad j = 1, 2, \ldots, n_i;$

then $\quad (Sb\, {}^{s}_{u})A \equiv \varphi_i(t'_1, t'_2, \ldots, t'_{n_i})$.

(c) If $A \equiv B \circ C$; then

$$(Sb\, {}^{s}_{v})A \equiv ((Sb\, {}^{s}_{u})B) \circ ((Sb\, {}^{s}_{u})C).$$

(d) If $A \equiv (x)B$ or $(\exists x)B$, then $(Sb\, {}^{s}_{x})A \equiv A$. If $u \not\equiv x$, and s is a real term, or if $s \equiv y$, $y \not\equiv x$, then

$$(Sb\, {}^{s}_{u})(x)B \equiv (x)(Sb\, {}^{s}_{u})B$$

$$(Sb\, {}^{s}_{u})(\exists x)B \equiv (\exists x)(Sb\, {}^{s}_{u})B.$$

Remark. It follows that $(Sb\, {}^{s}_{u})A$ may not be defined if A contains bound variables and s is a variable bound in A, or if s is a composite term containing apparent variables.

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7. This clause is optional. I shall accept it for the sake of generality, although it makes some of the later work slightly more difficult.

THEOREM 2. *The substitution operation has the following properties*:

(a) *If u does not occur free in A*,

$$(Sb\,{}^{s}_{u})A \equiv A.$$

(b) *If A $\varepsilon\,\mathfrak{P}(\mathfrak{u}, w)$ and s $\varepsilon\, t(\mathfrak{b})$, then $(Sb\,{}^{s}_{w})A$, if defined, is in $\mathfrak{P}(\mathfrak{u} + \mathfrak{b})$*.

(c) *If s $\varepsilon\, t(\mathfrak{u})$, t $\varepsilon\, t(\mathfrak{b})$, and neither u $\varepsilon\,\mathfrak{b}$ nor v $\varepsilon\,\mathfrak{u}$, then*

$$(Sb\,{}^{s}_{u})(Sb\,{}^{t}_{v})A \equiv (Sb\,{}^{t}_{v})(Sb\,{}^{s}_{u})A.$$

(d) *If s $\varepsilon\, t(\mathfrak{a})$, t $\varepsilon\, t(\mathfrak{b})$, and b is not in $\mathfrak{a}$*,

$$(Sb\,{}^{s}_{a})(Sb\,{}^{t}_{b})A \equiv (Sb\,{}^{t'}_{b})(Sb\,{}^{s}_{a})A,$$

where $t' \equiv (Sb\,{}^{s}_{a})t$.

(e) $(Sb\,{}^{u}_{u})A \equiv A.$

Remark. In the following we often represent substitution in the following more convenient manner. Let $A \equiv A(u)$, then:

$$(Sb^{s}_{u})A(u) \equiv A(s).$$

The statements of Theorem 2 are also true if we replace "A" by "r", and "$\mathfrak{P}$" by "t". The proof is by induction, using Definitions 2 and 5.[8]

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8. For instance the proof of (c) is as follows: We prove first the analogous formula for a term r, thus: If r is e_i or a variable distinct from u,v,

$$(Sb\,{}^{s}_{u})(Sb\,{}^{t}_{v})r \equiv (Sb\,{}^{s}_{u})r \equiv r;$$

$$(Sb\,{}^{t}_{v})(Sb\,{}^{s}_{u})r \equiv (Sb\,{}^{t}_{v})r \equiv r.$$

If $r \equiv u$,

$$(Sb\,{}^{s}_{u})(Sb\,{}^{t}_{v})u \equiv (Sb\,{}^{s}_{u})u \equiv s;$$

$$(Sb\,{}^{t}_{v})(Sb\,{}^{s}_{u})u \equiv (Sb\,{}^{t}_{v})s \equiv s.$$

If $r \equiv \omega_i(r_1, r_2, \ldots, r_{m_i})$ and if

$$r'_i \equiv (Sb\,{}^{s}_{u})(Sb\,{}^{t}_{v})r_i;\quad r''_i \equiv (Sb\,{}^{s}_{u})(Sb\,{}^{t}_{v})r_i;$$

then

$$(Sb\,{}^{s}_{u})(Sb\,{}^{t}_{v})r \equiv \omega_i(r'_1, r'_2, \ldots, r'_{m_i})$$

$$\equiv \omega_i(r''_1, \ldots, r''_{m_i}) \equiv (Sb\,{}^{t}_{v})(Sb\,{}^{s}_{u})r.$$

Then if A is E_i, both sides of (c) are E_i. If the above analog holds for $r_1, r_2, \ldots, r_{n_i}$, then (c) holds for $\varphi_i(r_1, \ldots, r_{n_i})$. Assuming (c) for B, C, it then follows for B o C, (x)B, and $(\exists x)B$.

PROSEQUENCES. No changes, other than the obvious ones, are required in the definition of a prosequence. We shall say that u occurs free in a prosequence $\mathfrak{X}$, if it occurs free in a constituent of $\mathfrak{X}$; it is bound in $\mathfrak{X}$ if it is bound in a constituent of $\mathfrak{X}$. We also define $(\mathrm{Sb}\,{}^{s}_{u})\mathfrak{X}$ as the prosequence formed by replacing every constituent A of $\mathfrak{X}$ by $(\mathrm{Sb}\,{}^{s}_{u})\mathrm{A}$.

4. **The Systems LA* and LC*.** We modify the formulations of the systems LA and LC to admit that the rules hold for a term extension with respect to an arbitrary class of real variables. The modified systems will be called LA* and LC*; and generally we shall use a "*" to indicate modification so as to admit quantifiers.

ELEMENTARY STATEMENTS. These are now of the form

$$\mathfrak{X} \mid \alpha \vdash \mathfrak{Y}, \tag{4}$$

where α is a class of real variables, and

$$\mathfrak{X} \subseteq \mathfrak{P}(\alpha) \qquad \mathfrak{Y} \subseteq \mathfrak{P}(\alpha). \tag{5}$$

(Thus statement (4) expresses the fact that the entailment between $\mathfrak{X}$ and $\mathfrak{Y}$ holds relative to the basic system $\mathfrak{S}(\alpha)$). The class α will be called the range.

PRIME STATEMENTS. These are the same as before except that (4) replaces II (4) and the restrictions (5) have to be satisfied.

RULES OF DERIVATION. The rules Er, C, W, $\mathcal{O}\ell$ and Or hold with the above modification for any fixed α. We have the following additional rules for the new connectives. In these it is supposed b actually occurs in A(b).

Π Universal Quantifier: If $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z} \subseteq \mathfrak{P}(\alpha)$, $\mathrm{A(b)} \in \mathfrak{P}(\alpha, \mathrm{b})$, b is not in α, x is not bound in A(b), and $\mathrm{t} \in \mathrm{t}(\alpha)$:

$$\frac{\mathfrak{X}, \mathrm{A(t)} \mid \alpha \vdash \mathfrak{Y}}{\mathfrak{X}, (\mathrm{x})\mathrm{A(x)} \mid \alpha \vdash \mathfrak{Y}} \qquad\qquad \frac{\mathfrak{X} \mid \alpha, \mathrm{b} \vdash \mathrm{A(b)}, \mathfrak{Z}}{\mathfrak{X} \mid \alpha \vdash (\mathrm{x})\mathrm{A(x)}, \mathfrak{Z}}$$

Σ Existential Quantifier: Under the same restrictions as in Π:

$$\frac{\mathfrak{X}, \mathrm{A(b)} \mid \alpha, \mathrm{b} \vdash \mathfrak{Y}}{\mathfrak{X}, (\exists \mathrm{x})\mathrm{A(x)} \mid \alpha \vdash \mathfrak{Y}} \qquad\qquad \frac{\mathfrak{X} \mid \alpha \vdash \mathrm{A(t)}, \mathfrak{Z}}{\mathfrak{X} \mid \alpha \vdash (\exists \mathrm{x})\mathrm{A(x)}, \mathfrak{Z}}$$

Remarks on these rules. 1) The remarks of II § 4 hold without change. In particular the distinctions as to parametric, principal, and component constituents all hold.

2) The only rules which make a change in α are Πr and $\Sigma\ell$. In these cases the variable b, which occurs in A(b) of the component,

but cannot occur in any other constituent, will be called the characteristic variable for that application of the rule.

3) The definitions of deduction, derivation, etc., go over without change.

ASSUMPTIONS CONCERNING $\mathfrak{S}$. In addition to the assumptions already made, we now suppose the following:

A3. The class $\mathfrak{e}$ is not void. This enters in Theorem 3 below.

A4. The rules of $\mathfrak{S}$ are invariant of a real substitution,[9] i.e., if

$$A_1(a), A_2(a), \ldots, A_m(a) \vdash B(a),$$

then for any real t,

$$A_1(t), A_2(t), \ldots, A_m(t) \vdash B(t).$$

This assumption is essential for Lemma 1 and hence for Theorem 5. This is part of the intention of the phrase "structural characterization" in I § 1. On a reasonable interpretation it follows from Definitions 5a, 5b, and 2d.

A FINITENESS RESTRICTION. We shall postulate a certain infinite class $\mathfrak{h}$ of real variables f,g,h,... with subclasses $\mathfrak{g}, \mathfrak{t}, \ldots$. We shall then impose the restriction that only a finite number of variables belonging to $\mathfrak{h}$ occur in the range of an elementary statement. It is only necessary to make this restriction for the prime statements; it will then hold automatically for any elementary theorem.

This restriction is only significant in case we wish to admit infinite prosequences and infinite classes $\mathfrak{a}$ in (4). For only a finite number of variables of any kind can occur in a term or elementary proposition.

5. **Theorems on Extensions.** The first difficulty to be overcome is that elementary statements are not immediately extensible. That is, if we have an instance of (4), and if

$$\mathfrak{x} \subseteq \mathfrak{x}', \quad \mathfrak{y} \subseteq \mathfrak{y}',$$

then we cannot conclude immediately that

$$\mathfrak{x}' \mid \mathfrak{a} \vdash \mathfrak{y}'.$$

This is because the additional parametric constituents, which it would be necessary to add to the rules to carry through a proof of Rule K, might contain the characteristic variable; and this would invalidate the inference.

9. The rules of the episystem have this character. If we are to consider a generalized approach, as in footnote[10] to II § 5, this property is required of the rules of the episystem.

LEMMA 1. Let (i) $\Delta = \Gamma_1, \Gamma_2, \ldots, \Gamma_n$, where Γ_k is

$$\mathfrak{X}_k \mid \mathfrak{a}_k \vdash \mathfrak{Y}_k,$$

be a normal derivation; (ii) $\mathfrak{b}$ be a class of real variables such that no characteristic variable of Δ is in $\mathfrak{b}$; (iii) $s \,\varepsilon\, t(\mathfrak{b})$; (iv) $a \,\varepsilon\, \mathfrak{a}_n$.

Then the sequence $\Delta' = \Gamma_1', \Gamma_2', \ldots, \Gamma_n'$, such that Γ_k' is

$$(\text{Sb}\ {}^{s}_{a})\ \mathfrak{X}_k \mid \mathfrak{b}_k \vdash (\text{Sb}\ {}^{s}_{a})\mathfrak{Y}_k,$$

where $\mathfrak{b}_k$ is obtained from $\mathfrak{a}_k$ by dropping a and then adding $\mathfrak{b}$, is a normal derivation.

Proof. We first note that our rules are such that the range of the conclusion of a rule is never larger than that in the premises. It follows from this that a characteristic variable in a normal derivation cannot occur in the final conclusion. Hence neither a nor any element of $\mathfrak{b}$ is a characteristic variable; also a occurs in every $\mathfrak{a}_k$.

Next we observe that if the restriction (5) is fulfilled for Γ_k, then it is for Γ_k'. This follows by Theorem 2b. We can therefore ignore this condition.

If Γ_k is prime so also is Γ_k'. This is clear if Γ_k is of type p1; if it is of type p2 then it follows since an axiom is unchanged by substitution (Theorem 2a).

If Γ_k follows from $\Gamma_{i_1}, \ldots, \Gamma_{i_p}$ by a rule of Chapter II, then Γ_k' follows from $\Gamma_{i_1}', \Gamma_{i_2}', \ldots, \Gamma_{i_p}'$ by the same rule. (In the case of Rule Er this requires assumption A4.)

If Γ_k follows from Γ_i by Rule Πr, then by the remark following Theorem 2 the inference is:[10]

$$\frac{\Gamma_i\ \mathfrak{X}_i \mid \mathfrak{a}_k', a, b \vdash A(b), \mathfrak{Z}_i}{\Gamma_k\ \mathfrak{X}_i \mid \mathfrak{a}_k', a \vdash (x)(\text{Sb}\ {}^{x}_{b})A(b), \mathfrak{Z}_i}.$$

The transformed inference is:

$$\frac{\Gamma_i'\ \mathfrak{X}_i' \mid \mathfrak{a}_k', b, \mathfrak{b} \vdash B(b), \mathfrak{Z}_i'}{\Gamma_k'\ \mathfrak{X}_i' \mid \mathfrak{a}_k', \mathfrak{b} \vdash (\text{Sb}\ {}^{s}_{a})(x)(\text{Sb}\ {}^{x}_{b})A(b), \mathfrak{Z}_i'},$$

where $\mathfrak{X}_i' \equiv (\text{Sb}\ {}^{s}_{a})\mathfrak{X}_i$, $\mathfrak{Z}_i' \equiv (\text{Sb}\ {}^{s}_{a})\mathfrak{Z}_i$, and $B(b) \equiv (\text{Sb}\ {}^{s}_{a})A(b)$. But by Definition 5 and Theorem 2c

$$\begin{aligned}(\text{Sb}\ {}^{s}_{a})(x)(\text{Sb}\ {}^{x}_{b})A(b) &\equiv (x)(\text{Sb}\ {}^{s}_{a})(\text{Sb}\ {}^{x}_{b})A(b)\\ &\equiv (x)(\text{Sb}\ {}^{x}_{b})(\text{Sb}\ {}^{s}_{a})A(b)\\ &\equiv (x)B(x).\end{aligned}$$

10. Here $\mathfrak{a}_k'$ is the result of deleting a from $\mathfrak{a}_k$. Note that $\mathfrak{a}_i \equiv \mathfrak{a}_k', b$.

Hence the inference from Γ_i' to Γ_k' is valid by the same rule Πr.

A similar proof holds if Γ_k follows from Γ_i by $\Sigma \ell$.

Suppose now Γ_k follows from Γ_i by a rule Σr. Then for suitably chosen b the inference is:

$$\frac{\Gamma_i \quad \mathfrak{X}_i \mid \mathfrak{a}_i', a \ \vdash (\mathrm{Sb}\,\tfrac{t}{b})A(b), \mathfrak{Z}_i}{\Gamma_k \quad \mathfrak{X}_i \mid \mathfrak{a}_i', a \ \vdash (\exists x)(\mathrm{Sb}\,\tfrac{x}{b})A(b), \mathfrak{Z}_i}.$$

Then the transformed inference is (using the same notation as in proof for Πr):

$$\frac{\Gamma_i' \quad \mathfrak{X}_i' \mid \mathfrak{a}_i', \mathfrak{b} \ \vdash (\mathrm{Sb}\,\tfrac{s}{a})(\mathrm{Sb}\,\tfrac{t}{b})A(b), \mathfrak{Z}_i'}{\Gamma_k' \quad \mathfrak{X}_i' \mid \mathfrak{a}_i', \mathfrak{b} \ \vdash (\mathrm{Sb}\,\tfrac{s}{a})(\exists x)(\mathrm{Sb}\,\tfrac{x}{b})A(b), \mathfrak{Z}_i'}.$$

But by Theorem 2d

$$(\mathrm{Sb}\,\tfrac{s}{a})(\mathrm{Sb}\,\tfrac{t}{b})A(b) \equiv (\mathrm{Sb}\,\tfrac{t'}{b})(\mathrm{Sb}\,\tfrac{s}{a})A(b)$$
$$\equiv B(t'),$$

where $t' \equiv (\mathrm{Sb}\,\tfrac{s}{a})t$. Also

$$(\mathrm{Sb}\,\tfrac{s}{a})(\exists x)(\mathrm{Sb}\,\tfrac{x}{b})A(b) \equiv (\exists x)(\mathrm{Sb}\,\tfrac{s}{a})(\mathrm{Sb}\,\tfrac{x}{b})A(b)$$
$$\equiv (\exists x)(\mathrm{Sb}\,\tfrac{x}{b})(\mathrm{Sb}\,\tfrac{s}{a})A(b)$$
$$\equiv (\exists x)B(x).$$

Thus the inference from Γ_i' to Γ_k' is also valid by Σr.

A similar proof applies if Γ_k follows from Γ_i by $\Pi \ell$.

Thus Δ' is a derivation. Since it uses the same rules in the same places as Δ does, it is a normal derivation.

LEMMA 2. <u>If $\Delta = \Gamma_1, \Gamma_2, \ldots, \Gamma_n$ is a normal derivation, and $\mathfrak{g}$ is a given infinite subclass of $\mathfrak{h}$, then there exists a normal derivation $\Delta' = \Gamma_1', \Gamma_2', \ldots, \Gamma_n'$, such that $\Gamma_n' \equiv \Gamma_n$, Γ_k' is obtained from Γ_k by changing certain variables, and the characteristic variables of Δ' are distinct from one another and belong to $\mathfrak{g}$.</u>

<u>Proof</u>. Let Δ_k be that part of Δ which constitutes a normal derivation of Γ_k. Then we show, by induction on k, that we can find a Δ_k' related to Δ_k as Δ' is to Δ in the lemma.

If Γ_k is prime, then Δ_k consists of Γ_k alone. Since no variable is characteristic, we can take $\Delta_k' \equiv \Delta_k$.

Let Γ_k be the conclusion derived from premises $\Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_p}$ by a rule R of Chapter II. Then by the hypothesis of the induction there exist normal derivations, as in the lemma, of $\Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_p}$ with characteristic variables belonging to arbitrary subclasses $\mathfrak{g}_{i_1}, \mathfrak{g}_{i_2}, \ldots, \mathfrak{g}_{i_p}$ of $\mathfrak{h}$. Take these as subclasses of $\mathfrak{g}$, no two of which have an element in common. Then if we follow $\Delta'_{i_1}, \ldots, \Delta'_{i_p}$ by an inference from $\Gamma_{i_1}, \ldots, \Gamma_{i_p}$ to Γ_{i_k} by R, we have a Δ'_k.

The same argument applies if R is one of the rules $\Pi\ell$ or Σr.

Finally let Γ_k be obtained from Γ_i by a rule R which is either Πr or $\Sigma\ell$. By the hypothesis of the induction there is a Δ'_i as stated in the lemma. Then by the argument of the first paragraph of the proof of Lemma 1 the characteristic variable, b, of R is not a characteristic variable of Δ'_i. Let $g \varepsilon \mathfrak{g}$ be also not a characteristic variable of Δ'_i. By Lemma 1 we can find a normal derivation Δ''_i of $(\mathrm{Sb}\,^{g}_{b})\Gamma_i$. This is obtained by operating on each statement of Δ_i with $(\mathrm{Sb}\,^{g}_{b})$, and has the same characteristic variables as Δ'_i. Then Δ''_i followed by R leading from Γ'_i to Γ_k is the Δ'_k sought.

THEOREM 3. <u>If</u>

$$\mathfrak{x} \mid \mathfrak{a} \vdash \mathfrak{y},$$

<u>and if $\mathfrak{b}$ is any class of variables such that</u>

$$\mathfrak{x} \subseteq \mathfrak{P}(\mathfrak{b}) \qquad \mathfrak{y} \subseteq \mathfrak{P}(\mathfrak{b});$$

<u>then</u>

$$\mathfrak{x} \mid \mathfrak{b} \vdash \mathfrak{y}.$$

<u>Proof</u>. Let $\Delta = \Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be a normal derivation of (4), such that the characteristic variables of Δ are distinct from one another and do not occur in $\mathfrak{x}$ or $\mathfrak{y}$. This is possible by Lemma 2 and the finiteness restriction at end of § 4. Let Γ_k be

$$\mathfrak{x}_k \mid \mathfrak{a}_k \vdash \mathfrak{y}_k.$$

We show, by an induction on k, that if we define Γ'_k as

$$\mathfrak{x}_k \mid \mathfrak{b}_k \vdash \mathfrak{y}_k,$$

where $\mathfrak{b}_k$ is any class satisfying the finiteness restriction such that

$$(6) \qquad \mathfrak{x}_k \subseteq \mathfrak{P}(\mathfrak{b}_k) \qquad \mathfrak{y}_k \subseteq \mathfrak{P}(\mathfrak{b}_k),$$

then Γ'_k is derivable.

If Γ_k is prime, Γ'_k is prime also.

Let Γ_k be derived from premises Γ_i (and Γ_j) by a rule R which is one of the rules O_r, $O\ell$, W of Chapter II. Then all the variables which occur in any of the premises occur also in the conclusion. Hence, if $\mathfrak{b}_k$ satisfies (6), then $\mathfrak{x}_i(\mathfrak{x}_j), \mathfrak{y}_i(\mathfrak{y}_j)$ are all in $\mathfrak{P}(\mathfrak{b}_k)$(Theorem 1, (b) and (c)). Hence, by the hypothesis of the induction, we can derive Γ_i' and Γ_j' with $\mathfrak{b}_i = \mathfrak{b}_j = \mathfrak{b}_k$. From these we can derive Γ_k' by R.

If Γ_k is derived by a Rule R which is Er, Σr, or $\Pi\ell$, then variables may occur in the premises or premise which do not occur in the conclusion. But these are not characteristic variables, and in fact we can replace all of them by e_1[11] without affecting the validity of the inference. This replacement can be made by successive applications of Lemma 1 with $s \equiv e_1$ and a one of the adventitious variables. (If necessary we can apply Lemma 2 to change the characteristic variables.) Then the same argument as in the preceding paragraph applies.

If Γ_k is derived from Γ_i by a rule Π r or $\Sigma\ell$, then the variables in the premise, other than the characteristic variable, also occur in the conclusion. Let the characteristic variable be g, and let $\mathfrak{b}_k$ satisfy (6). Then by the hypothesis of the induction we have Γ_i' with $\mathfrak{b}_i = \mathfrak{b}_k + g$. From this Γ_k' follows by the same rule as before.

THEOREM 4. <u>If</u>

$$\mathfrak{X} \mid \mathfrak{a} \vdash \mathfrak{Y},$$

<u>and if</u>

$$\mathfrak{X} \subseteq \mathfrak{X}' \qquad \mathfrak{Y} \subseteq \mathfrak{Y}',$$

$$\mathfrak{X}' \subseteq \mathfrak{P}(\mathfrak{a}') \qquad \mathfrak{Y}' \subseteq \mathfrak{P}(\mathfrak{a}');$$

<u>then</u>

$$\mathfrak{X}' \mid \mathfrak{a}' \vdash \mathfrak{Y}'.$$

<u>Proof</u>. By Theorem 3 we can replace $\mathfrak{a}$ by $\mathfrak{a}'$ in (4). Let $\mathfrak{g}$ be the class of variables in $\mathfrak{h}$ and not occurring in $\mathfrak{X}', \mathfrak{Y}'$. Then by Lemma 2 we can find a derivation Δ of

$$\mathfrak{X} \mid \mathfrak{a}' \vdash \mathfrak{Y}$$

such that all characteristic variables of Δ are in $\mathfrak{g}$. Then the proof of Chapter II, Theorem 2 applies.

THEOREM 5. <u>If</u>

$$\mathfrak{X} \mid \mathfrak{a} \vdash \mathfrak{Y},$$

<u>and if $s \varepsilon t(\mathfrak{b})$, then</u>

$$(7) \qquad (\mathrm{Sb}\ {}^{s}_{a})\mathfrak{X} \mid \mathfrak{a}', \mathfrak{b} \vdash (\mathrm{Sb}\ {}^{s}_{a})\mathfrak{Y},$$

<u>where $\mathfrak{a}'$ is obtained by removing a from $\mathfrak{a}$.</u>

11. Here we use Assumption A3. If there is a variable occurring in Γ_k we can use it instead of e_1.

Proof. If a is not in α, then (7) is

$$\mathfrak{X} \mid \alpha, \mathfrak{b} \vdash \mathfrak{Y}$$

(Theorem 2a). Hence (7) follows by Theorem 3.

We suppose then a $\varepsilon\ \alpha$. Let $\mathfrak{g}$ be a class of variables in $\mathfrak{h}$ such that none occur in $\mathfrak{b}$. Then by Lemma 2 we can find a derivation of (4) such that no characteristic variable occurs in $\mathfrak{b}$. Then we have (7) by Lemma 1.

6. **Basic Theorems of the LA* and LC* Systems.** We are now in a position to see that the principal theorems of §§ 5,6,7 in Chapter II can be carried over to the present case.

THEOREM 6. The theorems 2,3,4,5,8,9,10,11 of Chapter II hold for the enlarged LA* and LC* systems of this chapter, provided each elementary statement is assigned a range consistent with the present rules.[12]

Proof. So far as Theorem II 2 is concerned this was shown in Theorem 4. This theorem and those of § 5 show we can always have a sufficiently large class of variables, and characteristic variables can be taken so as not to bother us.

The proofs of Theorems II 3,4,5,[13]8,9,10 and the first two stages of the elimination theorem are valid without change. It is only necessary to supplement the proof of the elimination theorem with two new cases, as follows:

Case Π. $A \equiv (x)B(x)$. Then the premises are:

Γ_1 $\qquad \mathfrak{X}, B(t) \mid \alpha \vdash \mathfrak{Y},$

Γ_3 $\qquad \mathfrak{X} \mid \alpha, b \vdash B(b), \mathfrak{Z}, \qquad \mathfrak{Z} \leq \mathfrak{Y}.$

From Γ_3 and Theorem 5 (since b does not occur in $\mathfrak{X}, \mathfrak{Z}$)

$$\mathfrak{X} \mid \alpha \vdash B(t), \mathfrak{Z}.$$

From this and Γ_1 we have by the hypothesis of the induction

$$\mathfrak{X} \mid \alpha \vdash \mathfrak{Y}.$$

Case Σ. $A \equiv (\exists x)B(x)$. Then the premises are:

Γ_1 $\qquad \mathfrak{X}, B(b) \mid \alpha, b \vdash \mathfrak{Y},$

where b does not occur in $\mathfrak{X}, \mathfrak{Y}$, and

Γ_3 $\qquad \mathfrak{X} \mid \alpha \vdash B(t), \mathfrak{Z}.$

12. This range is uniquely determined in any derivation by the range of the conclusion, and the latter can be anything satisfying Theorem 4. In the case of the elimination theorem we can suppose both hypotheses have the same range.

13. In regard to II Theorem 5, the range can also be finite by Theorem 4.

From Γ_1 and Theorem 5

$$\mathfrak{X}, B(t) \mid \mathfrak{a} \vdash \mathfrak{Y}.$$

From this and Γ_3 we have by the hypothesis of the induction

$$\mathfrak{X} \mid \mathfrak{a} \vdash \mathfrak{Y}.$$

As regards II Theorem 7, it is necessary to formulate carefully the system $\mathfrak{D}$, as follows:

DEFINITION 6. The system $\mathfrak{D}_\infty$ is that specialization of $\mathfrak{S}$ in which

(a) The class $\mathfrak{e}$ contains infinitely many constituents $e_1, e_2, e_3, \ldots$.

(b) The class Ω is void.

(c) The class Φ contains infinitely many predicates of every degree of multiplicity.

(d) There are infinitely many primitive propositions $E_1, E_2, \ldots$

(e) There are no axioms and rules of procedure - in other words the relation II (6) is vacuous.

The system in which the condition (a) is relaxed to the extent of allowing there to be exactly n elements in $\mathfrak{e}$, viz., $e_1, e_2, \ldots, e_n$, will be called $\mathfrak{D}_n$.

According to this definition the terms of $\mathfrak{D}_\infty$ are the same as those of $\mathfrak{e}$; the elementary propositions are $E_1, E_2, E_3, \ldots$, together with all propositions of the form

$$\varphi_i \,(e_{k_1}, e_{k_2}, \ldots, e_{k_m}) \text{ where } m = n_i.$$

The decision process fails because there is no finite upper limit to the number of constituents in a derivation. This is because there are infinitely many possibilities for the t in Rules $\Pi\ell$ and Σ r and for the characteristic variables in Rules Π r and $\Sigma\ell$. The latter possibilities hold even in $\mathfrak{D}_n$ and $\mathfrak{r} \subsetneq \mathfrak{h}$.

Nevertheless the decision process of II § 6 can sometimes be used for discovering a derivation or for proving non-derivability. We shall illustrate this below by proving the non-derivability in LA*($\mathfrak{D}_\infty$) of

(8) $\quad \mid \mathfrak{o} \vdash (x) . A \vee B(x) :\supset\subset: A \vee . (x) . B(x).$

This will illustrate the reason for the failure of the decision process in general.

Before doing this we shall formulate the classical (truth table) evaluation because that is a necessary condition for derivability.

DEFINITION 7. A <u>valuation</u> over $\mathfrak{S}$ is any assignment of one of the values 1 or 0 to the elementary propositions of $\mathfrak{S}$ such that

(a) every axiom has the value 1,

(b) if the premises of a rule of $\mathfrak{S}$ have the value 1, so does the conclusion. (Of course the conditions (a) and (b) are vacuous if $\mathfrak{S}$ is $\mathfrak{D}_n$).

DEFINITION 8. If $\mathfrak{S}'$ is an extension of $\mathfrak{S}$ and $\mathfrak{V}$ is a valuation over $\mathfrak{S}$, then a continuation of $\mathfrak{V}$ onto $\mathfrak{S}'$ is a valuation $\mathfrak{V}'$ over $\mathfrak{S}'$ which assigns to every elementary proposition of $\mathfrak{S}$ the same value that $\mathfrak{V}$ does.

DEFINITION 9. The value of a proposition A of $\mathfrak{S}$ relative to a valuation $\mathfrak{V}$ over $\mathfrak{S}$ is defined inductively as one of the values 1 or 0 as follows:

(a) If $A \varepsilon \mathfrak{E}$, the value is that assigned in $\mathfrak{V}$.

(b) If $A \equiv B \circ C$, the value is determined from those of B and C by Table 1.

(c) If $A \equiv (x)B(x)$, let b be a term variable for $\mathfrak{S}$ and $\mathfrak{S}(b)$ the term extension of $\mathfrak{S}$ formed by adjoining b to $\mathfrak{S}$; then the value of A is 1 if that of B(b) is 1 in every continuation $\mathfrak{V}'$ of $\mathfrak{V}$ onto $\mathfrak{S}(b)$; it is 0 if there exists such a $\mathfrak{V}'$ in which B(b) has the value 0.

(d) If $A \equiv (\exists x)B(x)$, and b and $\mathfrak{V}'$ are as in c, then the value of A is 1 if that of B(b) is 1 in some $\mathfrak{V}'$; it is 0 if the value of B(b) is 0 for every $\mathfrak{V}'$.

DEFINITION 10. An elementary statement (4) is <u>valid on the classical evaluation with respect to $\mathfrak{S}$</u> if for every valuation $\mathfrak{V}$ either $\mathfrak{X}$ has a constituent with value 0 or $\mathfrak{Y}$ a constituent with value 1; it is invalid on the classical evaluation if there exists a valuation $\mathfrak{V}$ such that every constituent of $\mathfrak{X}$ has value 1 and every constituent of $\mathfrak{Y}$ has value 0.

This definition is, of course, indefinite; but it is clear that an elementary statement cannot be established as valid and invalid at the same time.

THEOREM 7. <u>A necessary condition that</u>

$$\mathfrak{X} \mid \mathfrak{a} \vdash \mathfrak{Y}$$

<u>be valid in LC*($\mathfrak{S}$) is that it be not invalid by the classical evaluation with respect to $\mathfrak{S}$.</u>

The proof of this theorem follows along the lines indicated in II §3. The details will not be given here.

We consider now the analysis of (8) in LA*. That can only be derived in LA* from

$$\Gamma_1 \qquad (x).A \vee B(x) \mid \mathfrak{o} \vdash A \vee (x)B(x).$$

This might come from $\Pi\ell$ or Vr. We consider $\Pi\ell$ first. The premise in $\Pi\ell$ would have to be the case n = 1 of Γ_2

Γ_2 $(x) . A \vee B(x), A \vee B(t_1), \ldots, A \vee B(t_n) \mid \mathfrak{o} \vdash A \vee (x)B(x)$.

The statement Γ_1 is also a special case of Γ_2, viz., for $n = 0$. Hence it is sufficient to show Γ_2 is non-derivable.

The statement Γ_2 can come from $\Pi\ell$, $V\ell$, or Vr. If it comes from $\Pi\ell$, the premise is also of form Γ_2[14]. Hence if (8) is valid some Γ_2 must be derived by one of the other rules. If Γ_2 comes from $V\ell$ one premise is the case $m = 1$ of the following:

$$\Gamma_3 \quad (x) . A \vee B(x), A \vee B(t_1), \ldots, A \vee B(t_n), B(t_1), \ldots, B(t_m) \mid \mathfrak{o} \vdash A \vee (x)B(x).$$

Since Γ_2 is the special case $m = 0$ of Γ_3, it suffices to consider Γ_3.

The statement Γ_3 can come from $\Pi\ell$, $V\ell$, or Vr. If it comes from $\Pi\ell$ the premise is again of form Γ_3. If it comes from $V\ell$ one premise is of form Γ_3 (the other contains A as constituent on the left, and is obviously valid). Hence some Γ_3 must come from Vr. Then the premises must be:

$$\Gamma_4 \quad (x) . A \vee B(x), A \vee B(t_1), \ldots, A \vee B(t_n), B(t_1), \ldots, B(t_m) \mid \mathfrak{o} \vdash A$$

$$\Gamma_5 \quad (x) . A \vee B(x), A \vee B(t_1), \ldots, A \vee B(t_n), B(t_1), \ldots, B(t_m) \mid \mathfrak{o} \vdash (x)B(x)$$

Both of these ave invalid on the classical evaluation - the first in the valuation where A is 0 and B(t) is 1 for every term t, the second in that for which A and $B(t_1), B(t_2), \ldots, B(t_m)$ are 1, but for some other t, B(t) is 0. The latter is possible since there are infinitely many terms in $\mathfrak{D}_\infty$.

THEOREM 8. <u>Under the assumption that A and B are elementary the statement</u>

$$\mid \mathfrak{o} \vdash (x) \quad A \vee B(x) \; :\supset: \; A . \vee . (x)B(x)$$

<u>is not valid in</u> $LA^*(\mathfrak{D})_\infty$.

<u>Remarks</u>. There are many variants to this treatment, and it is not possible to explore all of them. However, we may note the following:

1) If a statement is valid on the classical evaluation for $\mathfrak{S}$ it will be valid for any term extension $\mathfrak{S}'$ of $\mathfrak{S}$. For any valuation $\mathfrak{V}'$ of $\mathfrak{S}'$ will be a continuation onto $\mathfrak{S}'$ of some valuation $\mathfrak{V}$ of $\mathfrak{S}$, and it can be shown that $\mathfrak{V}'$ gives the same value to any A in $\mathfrak{S}$ that $\mathfrak{V}$ does.

2) In $\mathfrak{D}_n$ the constants $e_1, e_2, \ldots$ are indeterminates, hence there is no essential difference between them and variables - except that they cannot act as characteristic variables. A variable which is not a characteristic variable can just as well

14. Note that we are using the modified rules of Theorem II 3.

be added to $\mathfrak{e}$, and vice versa, provided appropriate changes are made in the range.

3) The combination of the two preceding remarks shows that if Γ is valid for $\mathfrak{D}_n$, it is valid for any $\mathfrak{D}_m$ with $m \geqq n$, in particular for $\mathfrak{D}_\infty$. Conversely if it is valid for $\mathfrak{D}_\infty$ it is valid for some $\mathfrak{D}_n$, viz., such that the set $e_1,\ldots,e_n$ contains all the constants which actually occur. With proper indication in the range it is valid for $\mathfrak{D}_1$, or even $\mathfrak{D}_0$.

4) These remarks show that the concept of classical validity for $\mathfrak{D}_n$ is quite different from that of k-formula in the sense of Bernays.[15] The latter is based on the interpretation of (x)A(x) as "A(t) for all t" rather than "A(b) for indeterminate b."

5) If we drop the requirement that $\mathfrak{e}$ be non-void, Theorem 3 is false as stated. This is shown by the example

$$(10) \qquad | \ \mathfrak{a} \vdash (x)A(x) \ .\supset \ (\exists x)A(x)$$

in which $\mathfrak{a}$ cannot be empty. Theorem 3 would be valid under the requirement that $\mathfrak{a} \subseteq \mathfrak{b}$, or that $\mathfrak{b}$ be non-void. Then certain statements like (10) could only be derived with a non-empty range.

Finally, in analogy with Theorem II 6, we state special results which can be derived by the decision process, as follows:

THEOREM 9. <u>The following are valid in LA*($\mathfrak{D}$), for A,B $\varepsilon\mathfrak{P}(\mathfrak{a})$, t ε t($\mathfrak{a}$):</u>

(a) $|\mathfrak{a}\vdash (x)A(x) \ . \supset . \ A(t).$

Π_0 $| \ \mathfrak{a} \vdash (y):(x)A(x) \ .\supset. \ A(y).$

Σ_0 $| \ \mathfrak{a} \vdash (y):A(y) \ .\supset. \ (\exists x).A(x).$

ΠP $| \ \mathfrak{a} \vdash (x) . \ A(x) \supset B(x):\supset: \ (x)A(x) .\supset. \ (x)B(x).$

Π_1 <u>If x does not occur in A</u>

$| \ \mathfrak{a} \vdash (x) . A \supset B(x) \ :\supset: \ A \supset (x)B(x).$

Σ_1 <u>If x does not occur in B</u>

$| \ \mathfrak{a} \vdash (x) \ . A(x) \supset B:\supset: \ (\exists x)A(x) \ .\supset. \ B.$

7. **The Systems TA*, TC*.** The additional rules for the system TA* when Π and Σ are adjoined will now be formulated. The new type of elementary statement will be:

$$(11) \qquad A \ \varepsilon \mathfrak{T}(\mathfrak{x};\mathfrak{a}),$$

15. See [3] §6, p. 56, also [47], pp. 118 ff. Cf. also footnote 2 to a5 in §1.

where

(12) $$A \varepsilon \mathfrak{P}(\alpha), \mathfrak{X} \subseteq \mathfrak{P}(\alpha).$$

The rules as given in Chapter II have to be modified by adding a range α. In t_1 this range is the class of all variables which occur in A, in t_2 the null range, and in the inferential rules it is the same in premises and conclusion. In t_3 we add the premise $\mathfrak{Y} \subseteq \mathfrak{P}(\alpha)$. We need the following new rules:

t4) If $\alpha \subseteq \mathfrak{b}$ and $A \varepsilon \mathfrak{S}(\mathfrak{X};\alpha)$, then $A \varepsilon \mathfrak{S}(\mathfrak{X};\mathfrak{b})$.

<u>Πe</u> If $t \varepsilon t(\alpha)$

$$\frac{(x).A(x)}{A(t)}$$

<u>Πi</u> If b does not occur in $\mathfrak{X}$

$$\frac{A(b)}{(x).A(x)}$$

<u>Σe</u> If b does not occur in $\mathfrak{X}$ or B,

$$\frac{(\exists x).A(x) \qquad \begin{matrix}[A(b)]\\ B\end{matrix}}{B}$$

<u>Σi</u> If $t \varepsilon t(\alpha)$

$$\frac{A(t)}{(\exists x).A(x)}$$

In these rules "$--- \varepsilon \mathfrak{S}(\mathfrak{X};\alpha)$" is understood in all cases except the premise of Πi and the right premise of Σe; in these cases the premises in full are $A(b) \varepsilon \mathfrak{S}(\mathfrak{X};\alpha,b)$ and $B \varepsilon \mathfrak{S}(\mathfrak{X},A(b);\alpha,b)$ respectively.

THEOREM 10. <u>The theorems 12-19 of Chapter II retain their validity when rules for Π and Σ are added</u>.

<u>Proof</u>. Theorem II 12 is clear.

As for Theorem II 13, it is only necessary to add the following proofs. (Note that t4 follows by Theorem 3.)

<u>Proof of Π e</u>. By the decision process of §6 (cf. Theorem 9(a))

$$\mathfrak{X},(x)A(x) \mid \alpha \vdash A(t).$$

If now

$$\mathfrak{X} \mid \alpha \vdash (x)A(x),\mathfrak{Z},$$

then by the elimination theorem

$$\mathfrak{X} \mid \alpha \vdash A(t),\mathfrak{Z}, \qquad \text{q.e.d.}$$

<u>Proof of Σ e</u>. By the second premise of Σe

$$\mathfrak{X},A(b) \mid \alpha,b \vdash B,\mathfrak{Z}.$$

We can suppose that b is not in α. Then by $\Sigma\ell$

$$\mathfrak{X},(\exists x).A(x) \mid \alpha \vdash B,\mathfrak{Z}.$$

By the first premise of Σe,

$$\mathfrak{X} \mid \alpha \vdash (\exists x)A(x), \mathfrak{B}.$$

Hence by the elimination theorem

$$\mathfrak{X} \mid \alpha \vdash B, \qquad\qquad \text{q.e.d.}$$

We need also the cases $\Pi\ell$ and $\Sigma\ell$ of Theorem II 14. The schematic proofs of these are as follows:

$$\underline{\Pi\ell}:\quad \dfrac{\dfrac{\overset{1}{(x)A(x)}}{A(t)}\,\Pi e \qquad\qquad \dfrac{\dfrac{\overset{\checkmark}{\overset{2}{A(t)}}}{B}\,Hp}{A(t)\supset B}\,\text{Pi-2}}{B}\,Pe$$

$$\underline{\Sigma\ell}:\quad \dfrac{\overset{1}{(\exists x)A(x)} \qquad\qquad \dfrac{\overset{\checkmark}{\overset{2}{A(b)}}}{B}\,Hp}{B}\,\Sigma\text{e-2}$$

The full proof of the first is as follows: Suppose $\mathfrak{X} \subseteq \mathfrak{P}(\alpha), (x)A(x) \,\varepsilon\, \mathfrak{P}(\alpha)$. Then by t1,t3,t4,

$$(x)A(x) \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, (x)A(x); \alpha).$$

Hence if $t \,\varepsilon\, t(\alpha)$, we have by Πe,

(13) $\quad A(t) \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, (x)A(x); \alpha)$.

Now suppose the premise of $\Pi\ell$, viz.,

$$B \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, A(t); \alpha).$$

Then by t3, $B \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, A(t), (x)A(x); \alpha)$.

$\therefore$ by Pi $\quad A(t) \supset B \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, (x)A(x); \alpha)$.

$\therefore$ by Pe and (13), $B \,\varepsilon\, \mathfrak{S}(\mathfrak{X}, (x)A(x); \alpha)$. q.e.d.

The proofs of Theorems II 15 and II 16 carry over without change.

For Theorem II 17 we need the following two cases:

<u>Proof of Πr</u>. By Hp. we have the rule

$$M_1 \; \dfrac{\mathfrak{X}, Z_1 \supset A(b), \ldots, Z_n \supset A(b)}{A(b)}$$

The proof scheme for Πr is then

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overset{1}{Z_1 \supset (x)A(x)} \qquad \overset{\checkmark}{\overset{2}{Z_1}}}{(x)A(x)}\,\text{Pe}}{A(b)}\,\Pi\text{e}}{Z_1 \supset A(b)}\,\text{Pi-2} \qquad \cdots \qquad Z_n \supset A(b)}{A(b)}\,\text{M}_1}{(x)A(x)}\,\Pi\text{i}$$

Similar proofs for $i = 2,3,\ldots,n$

Proof of Σr. By Hp. we have the rule

$$\text{M}_1 \ \frac{\mathfrak{X}, Z_1 \supset A(t), \ldots, Z_n \supset A(t)}{A(t)}$$

The proof scheme for Σ r is, then,

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overset{1}{Z_1 \supset . (\exists x)A(x)} \qquad \overset{\checkmark}{\overset{3}{Z_1}}}{(\exists x)A(x)}\,\text{Pe} \qquad \overset{\checkmark}{\overset{2}{(\exists x)A(x) \supset A(t)}}}{A(t)}\,\text{Pe}}{Z_1 \supset A(t)}\,\text{Pi-3} \qquad \cdots \qquad Z_n \supset A(t)}{A(t)}\,\text{M}_1}{(\exists x)A(x)}\,\Sigma\text{i}}{(\exists x)A(x)}\,\text{Pk-2}$$

similarly for $i = 2,3,\ldots,n$

Theorem II 18 and II 19 then follow without change.

8. **The Predicate Calculus.** We now consider the systems HA* and HC*. Since the introduction of apparent variables distinguishes the infinitesimal calculus from ordinary algebra, it would be in order to call each of the systems HA*, HC* a calculus when quantifiers are involved, an algebra under the circumstances of Chapter II. Thus what we have here is really a propositional calculus whereas in Chapter II we had a propositional algebra. Although this usage is not standard it has much to recommend it, and will be used here.

A propositional calculus over $\mathfrak{S}$ is, then, a system $\mathfrak{H}$ whose formulas are the propositions $\mathfrak{P}(\mathfrak{S})$, whose elementary statements are of the form

(14) $$A \ \varepsilon \ \mathfrak{H},$$

and whose theoretical rules consist of (a) a definite class of prime propositions for which (14) is asserted outright, and (b) the single rule of derivation:

$$\text{Ph} \quad \frac{A \ \varepsilon \ \mathfrak{H} \qquad (A \supset B) \ \varepsilon \ \mathfrak{H}}{B \ \varepsilon \ \mathfrak{H}}$$

The calculus HA($\mathfrak{a}$) is that calculus over $\mathfrak{D}_\infty$ in which (14) is equivalent to

$$(15) \qquad \mid \mathfrak{a} \vdash A$$

in LA*$(\mathfrak{D})_\infty$; the calculus HC*($\mathfrak{a}$) is that over $\mathfrak{D}_\infty$ in which (14) is equivalent to (15) in LC*$(\mathfrak{D})_\infty$.

THEOREM 11. <u>A set of prime propositions for the calculus HA*(HC*) consists of the propositions G obtained as follows: let G' be a prime scheme of HA (HC), or one of the schemes (a), Π_1, ΠP, Σ_0, Σ_1, of Theorem 9; let G" be obtained from G' by taking A, B, C as particular propositions of $\mathfrak{P}(\mathfrak{a}, \mathfrak{b}, \mathfrak{x})$ and $A(t) \equiv (Sb\, \frac{t}{x})A(x)$, where $\mathfrak{x}$ consists of those variables which are required by G' to be bound in A, B, C (the variables of $\mathfrak{x}$ must actually occur), and $t \, \varepsilon \, t(\mathfrak{a}, \mathfrak{b})$; then G is obtained from G" by applying Πi, with the variables of $\mathfrak{b}$ as characteristic variables, until all the variables which occur free in G belong to $\mathfrak{a}$.</u>

<u>Proof</u>. Let $\mathfrak{H}(\mathfrak{X};\mathfrak{a})$ be the system of propositions generated by Rule Ph by taking $\mathfrak{X}$ and all propositions of the above schemes as prime propositions. Then it is to be shown that

$$(16) \qquad HA^*(\mathfrak{a}) = \mathfrak{H}(0;\mathfrak{a}) \qquad (HC^*(\mathfrak{a}) = \mathfrak{H}(0;\mathfrak{a})).$$

Since all the instances of the above schemes are in HA*(HC*) by Theorem 9, II Theorem 6, and Πi, and since Ph is valid in HA*(HC*) by Theorem 10, the right side of (16) is included in the left. It suffices to show the converse. This we shall do, as in II Theorem 15, by showing that $\mathfrak{H}(\mathfrak{X};\mathfrak{a})$ satisfies the rules for $\mathfrak{T}(\mathfrak{X};\mathfrak{a})$.

So far as the rules t_1, t_2, t_3, Oe, Oi, and Pk are concerned the proof is the same as in II Theorems 20 and 21. (Note Ei is vacuous.) The validity of t_4 is obvious since every prime proposition of $\mathfrak{H}(\mathfrak{X};\mathfrak{a})$ is a fortiori a prime proposition of $\mathfrak{H}(\mathfrak{X};\mathfrak{a},\mathfrak{b})$. It remains to consider only the four new rules of §5. Of these we shall leave Πi till last, but will assume its validity in proving Σe.

<u>Πe</u>. This follows at once by (a) and Ph.

<u>Σe</u>. If the second premise is valid

$$A(b) \supset B \, \varepsilon \, \mathfrak{H}(\mathfrak{X};\mathfrak{a},b). \qquad \text{by Pi}$$

$$\therefore \qquad (x). A(x) \supset B \, \varepsilon \, \mathfrak{H}(\mathfrak{X};\mathfrak{a}) \qquad \text{by } \Pi\text{i}$$

$$(\exists x)A(x) . \supset . B \, \varepsilon \, \mathfrak{H}(\mathfrak{X};\mathfrak{a})$$

by Σ_1 and Ph. On the other hand by the first premise of Σe

$$(\exists x). A(x) \, \varepsilon \, \mathfrak{H}(\mathfrak{X},\mathfrak{a}).$$

Hence by Pe, $B \in \mathfrak{H}(\mathfrak{X}, \alpha)$.

Σi. This follows at once by Πe, Σo, and Ph.

Πi.[16] Since $A(b) \in \mathfrak{H}(\mathfrak{X}; \alpha, b)$, there exists a sequence $B_1, \ldots, B_n$ of propositions such that $B_n \equiv A(b)$ and every B_k is either 1) a member of $\mathfrak{X}$, 2) a prime proposition or 3) a consequence by Ph of some B_i and B_j preceding it. For each B_k let B_k' be $(x)(\frac{x}{b})B_k$ if b occurs in B_k, and let $B_k' \equiv B_k$ if b does not occur in B_k. Then we show by induction on k that $B_k' \in \mathfrak{H}(\mathfrak{X}, \alpha)$ for every k. It is evidently only necessary to consider the case where b occurs in B_k. In the induction we suppose $B_k \equiv A(b)$.

If $B_k \in \mathfrak{X}$, then B_k does not contain b.

If B_k is a prime proposition for $\mathfrak{H}(0; \alpha, b)$ and b occurs in B_k; then B_k' is by definition a prime proposition of $\mathfrak{H}(0; \alpha)$.

If B_k is derived by Ph from B_i and B_j, then we can suppose B_j is $B_i \supset B_k$. There are two cases according as B_i does or does not contain b. In the first case let B_i be B(b). Then B_j', which is in $\mathfrak{H}(\mathfrak{X}, \alpha)$ by the hypothesis of the induction, is

$$(x) \,.\, B(x) \supset A(x).$$

Hence by ΠP and Ph

$$(x)B(x) \,.\supset.\, (x)A(x) \in \mathfrak{H}(\mathfrak{X}; \alpha),$$

i.e.,

$$B_i' \supset (x)A(x) \in \mathfrak{H}(\mathfrak{X}, \alpha). \tag{17}$$

Since $B_i' \in \mathfrak{H}(\mathfrak{X}, \alpha)$, by the hypothesis of the induction, the conclusion follows by Pe. On the other hand if B_i does not contain b, B_j' is

$$(x) \,.\, B_i \supset A(x).$$

Since this $\in \mathfrak{H}(\mathfrak{X}; \alpha)$ we have (17) by Π_1, whence the conclusion follows as before.

Remark 1. This theorem would be easier if we allowed (x)A to be in $\mathfrak{P}$ even when x does not occur in A. We could then replace Π_1 by

$$\Vdash A \supset (x)A$$

when x does not occur in A.

Remark 2. The above proof shows that the derivation of any elementary statement of the calculus HA* or HC* can be carried out without the use of any free variables, other than those which occur in α.

16. The idea of this part of the proof is in [24].

In the formulation of the positive Heyting calculus due to the Hilbert school,[17] the calculus is generated by adjoining to the HA algebra of Chapter II the schemes (a) and Σ_0 and the rules

$$(\alpha)\quad \frac{B \supset A(b)}{B \supset (x)A(x)} \qquad\qquad (\beta)\quad \frac{A(b) \supset B}{(\exists x)A(x) \supset B}$$

where b does not occur in B. These rules are derivable in the natural system thus

$$\frac{\dfrac{\dfrac{\overset{\checkmark}{\overset{2}{B}} \qquad \overset{1}{B \supset A(b)}}{A(b)}\,\text{Pe}}{(x)A(x)}\,\Pi\text{i}}{B \supset (x)A(x)}\,\text{Pi-2} \qquad\qquad \frac{\dfrac{\overset{\checkmark}{\overset{3}{(\exists x)A(x)}} \qquad \dfrac{\overset{\checkmark}{\overset{2}{A(b)}} \qquad \overset{1}{A(b) \supset B}}{B}\,\text{Pe}}{B}\,\Sigma\text{e-2}}{(\exists x)\,.\,A(x) \supset B}\,\text{Pi-3}$$

[18]

Thus every proposition A obtained from a formula of the positive Heyting calculus by taking the formula and predicate variables to be elementary propositions of $\mathfrak{D}_\infty$ is valid in HA* for some determination of the range; and the range is determined by Theorem 3 to be the class of variables which occur in A. Conversely since the schemes of Theorem II 20 are valid in the Heyting calculus, every proposition in HA* is valid in the positive Heyting calculus. Thus we have

THEOREM 12. <u>A necessary and sufficient condition that A ε HA*(α) is that A be obtained from a formula of the positive Heyting calculus by taking its formula and predicate variables to be propositions and predicates of HA*, and that α contains the free variables occurring in the formula. Such a proposition can be shown to be in HA*(α) without using variables not already in α.</u>

The theory of quantifiers in this chapter has been grafted onto the systems LA and LC. In a similar way it can be grafted onto the systems involving negation considered in the following chapters. It is unnecessary to consider variables further.

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17. See [47] pp. 103-106. The same rules occur in [46] but of course the separation of the positive calculus is completely foreign to that work. The same rules were adopted by Heyting [45]. The characterization of the positive Heyting calculus in the text is to be understood as a definition. Strictly speaking, Σ_0 is replaced by a rule, dual to (a), which is a consequence of Σ_0 and (a).

18. Note that it is a part of the hypothesis of β that premise 1 can be derived on suppositions which do not contain b. Hence Σe is applicable.