Preface

The historical roots of the theory of bounded cohomology stretch back at least as far as Poincaré [**167**] who introduced rotation numbers in his study of circle diffeomorphisms. The Milnor–Wood inequality [**154, 204**] as generalized by Sullivan [**193**], and the theorem of Hirsch–Thurston [**109**] on foliated bundles with amenable holonomy groups were also landmark developments.

But it was not until the appearance of Gromov's seminal paper [**97**] that a number of previously distinct and isolated phenomena crystallized into a coherent subject. In [**97**] and in [**98**] Gromov indicated how many important or delicate geometric and algebraic properties of groups could be encoded and (in principle) recovered from their bounded cohomology. The essence of bounded cohomology is that it is a functor from the category of groups and homomorphisms to the category of normed vector spaces and norm-decreasing linear maps. Theorems in bounded cohomology can be restated as algebraic or topological inequalities; rigidity phenomena arise when equality is achieved (see e.g. [**31, 93, 149, 45**]).

A certain amount of activity followed; for example, the papers [**6, 27, 115, 150**] contain significant new ideas and advanced the subject. But there is a sense in which the promise of the field as suggested by Gromov has not been realized. One major shortcoming is the lack of adequate tools for computing or extracting meaningful information. There are at least two serious technical problems:

(1) The failure of the standard machinery of homological algebra (e.g. spectral sequences) to carry over to the bounded cohomology context in a straightforward way
(2) The fact that in the cases of most interest (e.g. hyperbolic groups) bounded cohomology is usually so big as to be unmanageable

Monod's monograph [**157**] addresses in a very useful way some of the most serious shortcomings of the subject by largely restricting attention to *continuous* bounded cohomology in contexts where this restriction is most informative. Burger and Monod (see especially [**33**] and [**34**]) developed the theory of continuous bounded cohomology into a powerful tool, which is of most value to people working in ergodic theory or the theory of lattices (especially in higher-rank) but is less useful for people whose main concern is the bounded cohomology of discrete groups (although Theorem 2 from [**34**] is an exception).

To get an idea of the state of the subject *ca.* 2000, we quote an excerpt from Burger–Monod [**35**], p. 19:

> Although the theory of bounded cohomology has recently found many applications in various fields ... for discrete groups it remains scarcely accessible to computation. As a matter of fact,

> almost all known results assert either a complete vanishing or
> yield intractable infinite dimensional spaces.

It is therefore a firm goal of this monograph to try to present results in terms which are concrete and elementary. We pay a great deal of attention to the case of free and surface groups, and present efficient algorithms to compute numerical invariants, whenever possible.

It is always hard for an outsider (or even an insider) to get an accurate idea of the critical (internal) questions or conjectures in a given field, whose resolution would facilitate significant progress, and of how the field does or might connect to other threads in mathematics. This monograph has a number of modest aims:

(1) To restrict attention and focus to a subfield (namely stable commutator length) which already has a number of useful and well-known applications to a wide range of geometrical contexts
(2) To carefully expose a number of foundational results in a way which should be accessible to any mathematician interested in the subject, and with a minimal number of prerequisites
(3) To develop a number of "hooks" into the subject which invite contributions from mathematicians and mathematics in what might at first glance appear to be unrelated fields (representation theory, computer science, combinatorics, etc.)
(4) To highlight the importance of hyperbolic groups in general, and free groups in particular as a critical case for understanding certain basic phenomena
(5) To give an exposition of some of my own work, and that of my collaborators, especially that part devoted to the "foundations" of the subject

Recently, there has been an outburst of activity at the intersection of low-dimensional bounded cohomology, low-dimensional dynamics, and symplectic topology (e.g. [**71, 73, 86, 169, 170, 174**], and so forth). I have done my best to discuss some of the highlights of this interaction, but I am not competent to delve into it too deeply.

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