Preface

Gauss hypergeometric functions and the functions in their family, such as Bessel functions, Whittaker functions, Hermite functions, Legendre polynomials and Jacobi polynomials etc. are the most fundamental and important special functions (cf. [**E–, Wa, WW**]). Many formulas related to the family have been studied and clarified together with the theory of ordinary differential equations, the theory of holomorphic functions and relations with other fields. They have been extensively used in various fields of mathematics, mathematical physics and engineering.

Euler studied the hypergeometric equation

$$x(1-x)y'' + \big(c-(a+b+1)x\big)y' - aby = 0 \tag{0.1}$$

with constant complex numbers a, b and c and he got the solution

$$F(a,b,c;x) := \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)\cdot b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)\cdot k!} x^k. \tag{0.2}$$

The series $F(a,b,c;x)$ is now called Gauss hypergeometric series or function and Gauss proved the Gauss summation formula

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{0.3}$$

when the real part of c is sufficiently large. Then in the study of this function an important concept was introduced by Riemann. That is the Riemann scheme

$$\left\{ \begin{matrix} x=0 & 1 & \infty & \\ 0 & 0 & a & ;\ x \\ 1-c & c-a-b & b & \end{matrix} \right\} \tag{0.4}$$

which describes the property of singularities of the function and Riemann proved that this property characterizes the Gauss hypergeometric function.

The equation (0.1) is a second order Fuchsian differential equation on the Riemann sphere with the three singular points $\{0,1,\infty\}$. One of the main purpose of this paper is to generalize these results to the general Fuchsian differential equation on the Riemann sphere. In fact, our study will be applied to the following three kinds of generalizations.

One of the generalizations of the Gauss hypergeometric family is the hypergeometric family containing the generalized hypergeometric function ${}_nF_{n-1}(\alpha,\beta;x)$ or the solutions of Jordan-Pochhammer equations. Some of their global structures are concretely described as in the case of the Gauss hypergeometric family.

The second generalization is a class of Fuchsian differential equations such as the Heun equation which is of order 2 and has 4 singular points in the Riemann sphere. In this case, there appear *accessory parameters*. The global structure of the generic solution is quite transcendental and the Painlevé equation which describes the deformations preserving the monodromies of solutions of the equations with an apparent singular point is interesting and has been quite deeply studied and now it becomes an important field of mathematics.

The third generalization is a class of hypergeometric functions of several variables, such as Appell's hypergeometric functions (cf. [**AK**]), Gelfand's generalized hypergeometric functions (cf. [**Ge**]) and Heckman-Opdam's hypergeometric functions (cf. [**HeO**]). The author and Shimeno [**OS**] studied the ordinary differential equations satisfied by the restrictions of Heckman-Opdam's hypergeometric function on singular lines through the origin and we found that some of the equations belong to the even family classified by Simpson [**Si**], which is now called a class of *rigid* differential equations and belongs to the first generalization in the above.

The author's original motivation related to the study in this paper is a generalization of Gauss summation formula, namely, to calculate a connection coefficient for a solution of this even family, which is solved in Chapter 12 as a direct consequence of the general formula (0.24) of certain connection coefficients described in Theorem 12.6. This paper is the author's first step to a unifying approach for these generalizations and the recent development in general Fuchsian differential equations described below with the aim of getting concrete and computable results. In this paper, we will avoid intrinsic arguments and results if possible and hence the most results can be implemented in computer programs. Moreover the arguments in this paper will be understood without referring to other papers.

Rigid differential equations are the differential equations which are uniquely determined by the data describing the local structure of their solutions at the singular points. From the point of view of the monodromy of the solutions, the rigid systems are the local systems which are uniquely determined by local monodromies around the singular points and Katz [**Kz**] studied rigid local systems by defining and using the operations called *middle convolutions* and *additions*, which enables us to construct and analyze all the rigid local systems. In fact, he proved that any irreducible rigid local system is transformed into a trivial equation $\frac{du}{dz} = 0$ by successive application of the operations. In another word, any irreducible rigid local system is obtained by successive applications of the operations to the trivial equation because the operations are invertible.

The arguments there are rather intrinsic by using perverse sheaves. Dettweiler-Reiter [**DR, DR2**] interprets Katz's operations on monodromy generators and those on the systems of Fuchsian differential equations of Schlesinger canonical form

$$\frac{du}{dx} = \sum_{j=1}^{p} \frac{A_j}{x - c_j} u \tag{0.5}$$

with constant square matrices $A_1, \dots, A_p$. These operations are useful also for non-rigid Fuchsian systems.

Here A_j are called the residue matrices of the system at the singular points $x = c_j$, which describe the local structure of the solutions. For example, the eigenvalues of the monodromy generator at $x = c_j$ are $e^{2\pi\sqrt{-1}\lambda_1}, \dots, e^{2\pi\sqrt{-1}\lambda_n}$, where $\lambda_1, \dots, \lambda_n$ are eigenvalues of A_j. The residue matrix of the system at $x = \infty$ equals $A_0 := -(A_1 + \cdots + A_p)$.

Related to the Riemann-Hilbert problem, there is a natural problem to determine the condition on matrices $B_0, B_1, \dots, B_p$ of Jordan canonical form such that there exists an irreducible system of Schlesinger canonical form with the residue matrices A_j conjugate to B_j for $j = 0, \dots, p$, respectively. An obvious necessary condition is the equality $\sum_{j=0}^{p} \operatorname{Trace} B_j = 0$. A similar problem for monodromy generators, namely its multiplicative version, is equally formulated. The latter is called a *mutiplicative* version and the former is called an *additive* version. Kostov [**Ko, Ko2, Ko3, Ko4**] called them Deligne-Simpson problems and gave an answer under a certain genericity condition. We note that the addition is a kind of a gauge

transformation

$$u(x) \mapsto (x-c)^{\lambda} u(x)$$

and the middle convolution is essentially an Euler transformation or a transformation by an Riemann-Liouville integral

$$u(x) \mapsto \frac{1}{\Gamma(\mu)} \int_c^x u(t)(x-t)^{\mu-1} dt$$

or a fractional derivation.

Crawley-Boevey [**CB**] found a relation between the Deligne-Simpson problem and representations of certain quivers and gave an explicit answer for the additive Deligne-Simpson problem in terms of a Kac-Moody root system.

Yokoyama [**Yo2**] defined operations called extensions and restrictions on the systems of Fuchsian ordinary differential equations of Okubo normal form

$$\big(x - T\big)\frac{du}{dx} = Au. \tag{0.6}$$

Here A and T are constant square matrices such that T are diagonalizable. He proved that the irreducible rigid system of Okubo normal form is transformed into a trivial equation $\frac{du}{dz} = 0$ by successive applications of his operations if the characteristic exponents are generic.

The relation between Katz's operations and Yokoyama's operations is clarified by [**O7**] and it is proved there that their algorithms of reductions of Fuchsian systems are equivalent and so are those of the constructions of the systems.

These operations are quite powerful and in fact if we fix the number of accessory parameters of the systems, they are connected into a finite number of fundamental systems (cf. [**O6**, Proposition 8.1 and Theorem 10.2] and Proposition 7.13), which is a generalization of the fact that the irreducible rigid Fuchsian system is connected to the trivial equation.

Hence it is quite useful to understand how does the property of the solutions transform under these operations. In this point of view, the system of the equations, the integral representation and the monodromy of the solutions are studied by [**DR, DR2, HY**] in the case of the Schlesinger canonical form. Moreover the equation describing the deformation preserving the monodromy of the solutions doesn't change, which is proved by [**HF**]. In the case of the Okubo normal form the corresponding transformation of the systems, that of the integral representations of the solutions and that of their connection coefficients are studied by [**Yo2**], [**Ha**] and [**Yo3**], respectively. These operation are explicit and hence it will be expected to have explicit results in general Fuchsian systems.

To avoid the specific forms of the differential equations, such as Schlesinger canonical form or Okubo normal form and moreover to make explicit calculations easier under the transformations, we introduce certain operations on differential operators with polynomial coefficients in Chapter 1. The operations in Chapter 1 enables us to equally handle equations with irregular singularities or systems of equations with several variables.

The ring of differential operators with polynomial coefficients is called a *Weyl algebra* and denoted by $W[x]$ in this paper. The endomorphisms of $W[x]$ do not give a wide class of operations and Dixmier [**Dix**] conjectured that they are the automorphisms of $W[x]$. But when we localize coordinate x, namely in the ring $W(x)$ of differential operators with coefficients in rational functions, we have a wider class of operations.

For example, the transformation of the pair $(x, \frac{d}{dx})$ into $(x, \frac{d}{dx} - h(x))$ with any rational function $h(x)$ induces an automorphism of $W(x)$. This operation is called

a *gauge transformation*. The addition in [**DR, DR2**] corresponds to this operation with $h(x) = \frac{\lambda}{x-c}$ and λ, $c \in \mathbb{C}$, which is denoted by $\mathrm{Ad}\big((x-c)^{\lambda}\big)$.

The transformation of the pair $(x, \frac{d}{dx})$ into $(-\frac{d}{dx}, x)$ defines an important automorphism L of $W[x]$, which is called a *Laplace transformation*. In some cases the Fourier transformation is introduced and it is a similar transformation. Hence we may also localize $\frac{d}{dx}$ and introduce the operators such as $\lambda(\frac{d}{dx} - c)^{-1}$ and then the transformation of the pair $(x, \frac{d}{dx})$ into $(x - \lambda(\frac{d}{dx})^{-1}, \frac{d}{dx})$ defines an endomorphism in this localized ring, which corresponds to the middle convolution or an Euler transformation or a fractional derivation and is denoted by $\mathrm{Ad}(\partial^{-\lambda})$ or mc_{λ}. But the simultaneous localizations of x and $\frac{d}{dx}$ produce the operator $(\frac{d}{dx})^{-1} \circ x^{-1} = \sum_{k=0}^{\infty} k! x^{-k-1} (\frac{d}{dx})^{-k-1}$ which is not algebraic in our sense and hence we will not introduce such a microdifferential operator in this paper and we will not allow the simultaneous localizations of the operators.

Since our equation $Pu = 0$ studied in this paper is defined on the Riemann sphere, we may replace the operator P in $W(x)$ by a suitable representative $\tilde{P} \in \mathbb{C}(x)P \cap W[x]$ with the minimal degree with respect to x and we put $\mathrm{R}\,P = \tilde{P}$. Combining these operations including this replacement gives a wider class of operations on the Weyl algebra $W[x]$. In particular, the operator corresponding to the addition is $\mathrm{RAd}\big((x-c)^{\lambda}\big)$ and that corresponding to the middle convolution is $\mathrm{RAd}(\partial^{-\mu})$ in our notation. The operations introduced in Chapter 1 correspond to certain transformations of solutions of the differential equations defined by elements of Weyl algebra and we call the calculation using these operations *fractional calculus of Weyl algebra*.

To understand our operations, we show that, in Example 1.8, our operations enables us to construct Gauss hypergeometric equations, the equations satisfied by Airy functions and Jordan-Pochhammer equations and to give integral representations of their solutions.

In this paper we mainly study ordinary differential equations and since any linear ordinary differential equation is *cyclic*, namely, it is isomorphic to a single differential operator $Pu = 0$ (cf. §1.4), we study a single ordinary differential equation $Pu = 0$ with $P \in W[x]$. In many cases, we are interested in a specific function $u(x)$ which is characterized by differential equations and if $u(x)$ is a function with the single variable x, the differential operators $P \in W(x)$ satisfying $Pu(x) = 0$ are generated by a single operator and hence it is natural to consider a single differential equation. A relation between our fractional calculus and Katz's middle convolution is briefly explained in §1.5.

In §2.1 we review fundamental results on Fuchsian ordinary differential equations. Our Weyl algebra $W[x]$ is allowed to have some parameters $\xi_1, \ldots$ and in this case the algebra is denoted by $W[x; \xi]$. The position of singular points of the equations and the characteristic exponents there are usually the parameters and the analytic continuation of the parameters naturally leads the confluence of additions (cf. §2.3).

Combining this with our construction of equations leads the confluence of the equations. In the case of Jordan-Pochhammer equations, we have versal Jordan-Pochhammer equations. In the case of Gauss hypergeometric equation, we have a unified expression of Gauss hypergeometric equation, Kummer equation and Hermite-Weber equation and get a unified integral representation of their solutions (cf. Example 2.5). After this chapter in this paper, we mainly study single Fuchsian differential equations on the Riemann sphere. Equations with irregular singularities will be discussed elsewhere (cf. [**HiO**], [**O10**]).

In Chapter 3 we examine the transformation of series expansions and contiguity relations of the solutions of Fuchsian differential equations under our operations. The results in this chapter will be used in later chapters.

The Fuchsian equation satisfied by the generalized hypergeometric series

$$
\begin{gathered}
{}_nF_{n-1}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_{n-1};x)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\dots(\alpha_n)_k}{(\beta_1)_k\dots(\beta_{n-1})_k k!}x^k \\
\text{with}\quad (\gamma)_k:=\gamma(\gamma+1)\cdots(\gamma+k-1)
\end{gathered}
\tag{0.7}
$$

is characterized by the fact that it has $(n-1)$-dimensional local holomorphic solutions at $x=1$, which is more precisely as follows. The set of characteristic exponents of the equation at $x=1$ equals $\{0,1,\dots,n-1,-\beta_n\}$ with $\alpha_1+\cdots+\alpha_n=\beta_1+\cdots+\beta_n$ and those at 0 and ∞ are $\{1-\beta_1,\dots,1-\beta_{n-1},0\}$ and $\{\alpha_1,\dots,\alpha_n\}$, respectively. Then if α_i and β_j are generic, the Fuchsian differential equation $Pu=0$ is uniquely characterized by the fact that it has the above set of characteristic exponents at each singular point 0 or 1 or ∞ and the monodromy generator around the point is *semisimple*, namely, the local solution around the singular point has no logarithmic term. We express this condition by the (generalized) Riemann scheme

$$
\begin{gathered}
\left\{\begin{matrix} x=0 & 1 & \infty & \\ 1-\beta_1 & [0]_{(n-1)} & \alpha_1 & \\ \vdots & & \vdots & ;x \\ 1-\beta_{n-1} & & \alpha_{n-1} & \\ 0 & -\beta_n & \alpha_n & \end{matrix}\right\},\quad [\lambda]_{(k)}:=\begin{pmatrix}\lambda\\ \lambda+1\\ \vdots\\ \lambda+k-1\end{pmatrix}, \\
\alpha_1+\cdots+\alpha_n=\beta_1+\cdots+\beta_n.
\end{gathered}
\tag{0.8}
$$

In particular, when $n=3$, the (generalized) Riemann scheme is

$$
\left\{\begin{matrix} x=0 & 1 & \infty & \\ 1-\beta_1 & \begin{pmatrix}0\\1\end{pmatrix} & \alpha_1 & \\ 1-\beta_2 & & \alpha_2 & ;x \\ 0 & -\beta_3 & \alpha_3 & \end{matrix}\right\}.
$$

The corresponding usual Riemann scheme is obtained from the generalized Riemann scheme by eliminating the parentheses $\Big($ and $\Big)$. Here $[0]_{(n-1)}$ in the above Riemann scheme means the characteristic exponents $0,1,\dots,n-2$ but it also indicates that the corresponding monodromy generator is semisimple in spite of integer differences of the characteristic exponents. Thus the set of (generalized) characteristic exponents $\{[0]_{(n-1)},-\beta_n\}$ at $x=1$ is defined. Here we remark that the coefficients of the Fuchsian differential operator P which is uniquely determined by the generalized Riemann scheme for generic α_i and β_j are polynomial functions of α_i and β_j and hence P is naturally defined for any α_i and β_j as is given by (13.21). Similarly the Riemann scheme of Jordan-Pochhammer equation of order p is

$$
\begin{gathered}
\left\{\begin{matrix} x=c_0 & c_1 & \cdots & c_{p-1} & \infty & \\ [0]_{(p-1)} & [0]_{(p-1)} & \cdots & [0]_{(p-1)} & [\lambda'_p]_{(p-1)} & ;x \\ \lambda_0 & \lambda_1 & \cdots & \lambda_{p-1} & \lambda_p & \end{matrix}\right\}, \\
\lambda_0+\cdots+\lambda_{p-1}+\lambda_p+(p-1)\lambda'_p=p-1.
\end{gathered}
\tag{0.9}
$$

The last equality in the above is called a *Fuchs relation*.

In Chapter 4 we define the set of generalized characteristic exponents at a regular singular point of a differential equation $Pu=0$. In fact, when the order of P is n, it is the set $\{[\lambda_1]_{(m_1)},\dots,[\lambda_k]_{(m_k)}\}$ with a partition $n=m_1+\cdots+m_k$ and complex numbers $\lambda_1,\dots,\lambda_k$. It means that the set of characteristic exponents at

the point equals

$$\{\lambda_j + \nu\,;\, \nu = 0, \dots, m_j - 1 \text{ and } j = 1, \dots, k\} \tag{0.10}$$

and the corresponding monodromy generator is semisimple if $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $1 \le i < j \le k$. In §4.1 we define the set of generalized characteristic exponents without the assumption $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $1 \le i < j \le k$. Here we only remark that when $\lambda_i = \lambda_1$ for $i = 1, \dots, k$, it is characterized by the fact that the Jordan normal form of the monodromy generator is defined by the dual partition of $n = m_1 + \cdots + m_k$ together with the usual characteristic exponents (0.10).

Thus for a single Fuchsian differential equation $Pu = 0$ on the Riemann sphere which has $p+1$ regular singular points $c_0, \dots, c_p$, we define a (generalized) Riemann scheme

$$\left\{ \begin{array}{cccc} x = c_0 & c_1 & \cdots & c_p \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{array} ; x \right\}. \tag{0.11}$$

Here $n = m_{j,1} + \cdots + m_{j,n_j}$ for $j = 0, \dots, p$, n is the order of P, $\lambda_{j,\nu} \in \mathbb{C}$ and $\{[\lambda_{j,1}]_{(m_{j,1})}, \dots, [\lambda_{j,n_j}]_{(m_{j,n_j})}\}$ is the set of generalized characteristic exponents of the equation at $x = c_j$. The $(p+1)$-tuple of partitions of n, which is denoted by $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p \\ \nu=1,\dots,n_j}}$, is called the *spectral type* of P and the Riemann scheme (0.11).

We note that the Riemann scheme (0.11) should always satisfy the Fuchs relation

$$\begin{aligned} |\{\lambda_{\mathbf{m}}\}| &:= \sum_{j=0}^{p} \sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu} - \operatorname{ord}\mathbf{m} + \tfrac{1}{2}\operatorname{idx}\mathbf{m} \\ &= 0. \end{aligned} \tag{0.12}$$

Here

$$\operatorname{idx}\mathbf{m} := \sum_{j=0}^{p} \sum_{\nu=1}^{n_j} m_{j,\nu}^2 - (p-1)\operatorname{ord}\mathbf{m} \tag{0.13}$$

and $\operatorname{idx}\mathbf{m}$ coincides with the *index of rigidity* introduced by [**Kz**].

In Chapter 4, after introducing certain representatives of conjugacy classes of matrices and some notation and concepts related to tuples of partitions, we define that the tuple $\mathbf{m}$ is *realizable* if there exists a Fuchsian differential operator P with the Riemann scheme (0.11) for generic complex numbers $\lambda_{j,\nu}$ under the condition (0.12). Furthermore, if there exists such an operator P so that $Pu = 0$ is irreducible, we define that $\mathbf{m}$ is *irreducibly realizable*.

Lastly in Chapter 4, we examine the generalized Riemann schemes of the *product* of Fuchsian differential operators and the *dual* operators.

In Chapter 5 we examine the transformations of the Riemann scheme under our operations corresponding to the additions and the middle convolutions, which define transformations within Fuchsian differential operators. The operations induce transformations of spectral types of Fuchsian differential operators, which keep the indices of rigidity invariant but change the orders in general. Looking at the spectral types, we see that the combinatorial aspect of the reduction of Fuchsian differential operators is parallel to that of systems of Schlesinger canonical form. In this chapter, we also examine the combination of these transformation and the *fractional linear transformations*.

As our interpretation of Deligne-Simpson problem introduced by Kostov, we examine the condition for the existence of a Fuchsian differential operator with a given Riemann scheme in Chapter 6. We determine the conditions on $\mathbf{m}$ such that $\mathbf{m}$ is realizable and irreducibly realizable, respectively, in Theorem 6.14. Moreover if $\mathbf{m}$ is realizable, Theorem 6.14 gives an explicit construction of the *universal Fuchsian differential operator*

$$\begin{aligned} P_{\mathbf{m}} &= \Big(\prod_{j=1}^{p}(x-c_j)^n\Big)\frac{d^n}{dx^n} + \sum_{k=0}^{n-1} a_k(x,\lambda,g)\frac{d^k}{dx^k}, \\ \lambda &= \big(\lambda_{j,\nu}\big)_{\substack{j=0,\dots,p\\ \nu=1,\dots,n_j}}, \quad g=(g_1,\dots,g_N)\in\mathbb{C}^N \end{aligned} \tag{0.14}$$

with the Riemann scheme (0.11), which has the following properties.

For fixed complex numbers $\lambda_{j,\nu}$ satisfying (0.12) the operator with the Riemann scheme (0.11) satisfying $c_0=\infty$ equals $P_{\mathbf{m}}$ for a suitable $g\in\mathbb{C}^N$ up to a left multiplication by an element of $\mathbb{C}(x)$ if $\lambda_{j,\nu}$ are "generic", namely,

$$\begin{aligned} &(\Lambda(\lambda)|\alpha)\notin\big\{-1,-2,\dots,1-(\alpha|\alpha_{\mathbf{m}})\big\} \\ &\qquad\qquad \text{for any } \alpha\in\Delta(\mathbf{m}) \text{ satisfying } (\alpha|\alpha_{\mathbf{m}})>1 \end{aligned} \tag{0.15}$$

under the notation used in (0.22). Here $g_1,\dots,g_N$ are called *accessory parameters* and if $\mathbf{m}$ is irreducibly realizable, $N=1-\frac{1}{2}\operatorname{idx}\mathbf{m}$. Example 5.6 shows the necessity of the above condition (0.15) but the condition is always satisfied if $\mathbf{m}$ is fundamental or *simply reducible* (cf. Definition 6.15 and Proposition 6.17), etc. In particular, if there is an irreducible and *locally non-degenerate* (cf. Definition 9.8) operator P with the Riemann scheme (0.11), then $\lambda_{j,\nu}$ are "generic". The simply reducible spectral type is studied in Chapter 6 §6.5, which happens to correspond to the indecomposable object studied by [**MWZ**] when the spectral type is rigid.

The coefficients $a_k(x,\lambda,g)$ of the differential operator $P_{\mathbf{m}}$ are polynomials of the variables x, λ and g. The coefficients satisfy $\frac{\partial^2 a_k}{\partial g_\nu \partial g_{\nu'}}=0$ and furthermore g_ν can be equal to suitable a_{i_ν,j_ν} under the expression $P_{\mathbf{m}}=\sum a_{i,j}(\lambda,g)x^i\frac{d^j}{dx^j}$ and the pairs (i_ν,j_ν) for $\nu=1,\dots,N$ are explicitly given in the theorem. Hence the universal operator $P_{\mathbf{m}}$ is uniquely determined from their values at generic $\lambda_{j,\nu}$ without the assumption of the irreducibility of the equation $P_{\mathbf{m}}u=0$, which is not true in the case of the systems of Schlesinger canonical form (cf. Example 9.2).

The universal operator $P_{\mathbf{m}}$ is a classically well-known operator in the case of Gauss hypergeometric equation, Jordan-Pochhammer equation or Heun's equation etc. and the theorem assures the existence of such a good operator for any realizable tuple $\mathbf{m}$. We define the tuple $\mathbf{m}$ is *rigid* if $\mathbf{m}$ is irreducibly realizable and moreover $N=0$, namely, $P_{\mathbf{m}}$ is free from accessory parameters.

In particular, the theorem gives the affirmative answer for the following question. Katz asked a question in the introduction in the book [**Kz**] whether a rigid local system is realized by a single Fuchsian differential equation $Pu=0$ without apparent singularities (cf. Corollary 10.12 iii)).

It is a natural problem to examine the Fuchsian differential equation $P_{\mathbf{m}}u=0$ with an irreducibly realizable spectral type $\mathbf{m}$ which cannot be reduced to an equation with a lower order by additions and middle convolutions. The tuple $\mathbf{m}$ with this condition is called *fundamental*.

The equation $P_{\mathbf{m}}u=0$ with an irreducibly realizable spectral type $\mathbf{m}$ can be transformed by the operation ∂_{max} (cf. Definition 5.7) into a Fuchsian equation $P_{\mathbf{m}'}v=0$ with a fundamental spectral type $\mathbf{m}'$. Namely, there exists a non-negative integer K such that $P_{\mathbf{m}'}=\partial_{\max}^K P_{\mathbf{m}}$ and we define $f\mathbf{m}:=\mathbf{m}'$. Then it turns out that a realizable tuple $\mathbf{m}$ is rigid if and only if the order of $f\mathbf{m}$, which is the order

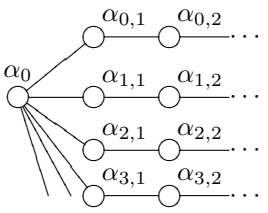
of $P_{f\mathbf{m}}$ by definition, equals 1. Note that the operator $\partial_{\max}$ is essentially a product of suitable operators $\mathrm{RAd}\big((x-c_j)^{\lambda_j}\big)$ and $\mathrm{RAd}\big(\partial^{-\mu}\big)$.

In this paper we study the transformations of several properties of the Fuchsian differential equation $P_{\mathbf{m}}u=0$ under the additions and middle convolutions. If they are understood well, the study of the properties are reduced to those of the equation $P_{f\mathbf{m}}v=0$, which are of order 1 if $\mathbf{m}$ is rigid. We note that there are many rigid spectral types $\mathbf{m}$ and for example there are 187 different rigid spectral types $\mathbf{m}$ with $\operatorname{ord}\mathbf{m}\le 8$ as are given in §13.2.

As in the case of the systems of Schlesinger canonical form studied by Crawley-Boevey [**CB**], the combinatorial aspect of transformations of the spectral type $\mathbf{m}$ of the Fuchsian differential operator P induced from our fractional operations is described in Chapter 7 by using the terminology of a Kac-Moody root system (Π, W_∞). Here Π is the fundamental system of a Kac-Moody root system with the following star-shaped Dynkin diagram and W_∞ is the Weyl group generated by the *simple reflections* s_α for $\alpha\in\Pi$. The elements of Π are called *simple roots*.

Associated to a tuple $\mathbf{m}$ of $(p+1)$ partitions of a positive integer n, we define an element $\alpha_{\mathbf{m}}$ in the positive root lattice (cf. §7.1, (7.5)):

$$
\begin{aligned}
\Pi &:= \{\alpha_0,\ \alpha_{j,\nu}\,;\ j=0,1,\dots,\ \nu=1,2,\dots\},\\
W_\infty &:= \langle s_\alpha\,;\ \alpha\in\Pi\rangle,\\
\alpha_{\mathbf{m}} &:= n\alpha_0+\sum_{j=0}^{p}\sum_{\nu=1}^{n_j-1}\Big(\sum_{i=\nu+1}^{n_j} m_{j,i}\Big)\alpha_{j,\nu},\\
(\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}}) &= \operatorname{idx}\mathbf{m},
\end{aligned}
\tag{0.16}
$$



We can define a fractional operation on $P_{\mathbf{m}}$ which is compatible with the action of $w\in W_\infty$ on the root lattice (cf. Theorem 7.5):

(0.17)

$$
\begin{array}{ccc}
\big\{P_{\mathbf{m}}:\ \text{Fuchsian differential operators with } \{\lambda_{\mathbf{m}}\}\big\} & \to & \big\{(\Lambda(\lambda),\alpha_{\mathbf{m}})\,;\ \alpha_{\mathbf{m}}\in\overline{\Delta}_+\big\}\\
\downarrow \text{fractional operations} & \circlearrowleft & \downarrow W_\infty\text{-action},\ +\tau\Lambda^0_{0,j}\\
\big\{P_{\mathbf{m}}:\ \text{Fuchsian differential operators with } \{\lambda_{\mathbf{m}}\}\big\} & \to & \big\{(\Lambda(\lambda),\alpha_{\mathbf{m}})\,;\ \alpha_{\mathbf{m}}\in\overline{\Delta}_+\big\}.
\end{array}
$$

Here $\lambda_{j,\nu}\in\mathbb{C}$, $\tau\in\mathbb{C}$, $\mathbf{m}=\big(m_{j,\nu}\big)_{\substack{j=0,\dots,p\\ \nu=1,2,\dots}}$ with $m_{j,\nu}=0$ for $\nu>n_j$,

$$
\begin{aligned}
\Lambda^0 &:= \alpha_0+\sum_{\nu=1}^{\infty}(1+\nu)\alpha_{0,\nu}+\sum_{j=1}^{p}\sum_{\nu=1}^{\infty}(1-\nu)\alpha_{j,\nu},\\
\Lambda^0_{i,j} &:= \sum_{\nu=1}^{\infty}\nu(\alpha_{i,\nu}-\alpha_{j,\nu}),\\
\Lambda_0 &:= \frac{1}{2}\alpha_0+\frac{1}{2}\sum_{j=0}^{p}\sum_{\nu=1}^{\infty}(1-\nu)\alpha_{j,\nu},\\
\Lambda(\lambda) &:= -\Lambda_0-\sum_{j=0}^{p}\sum_{\nu=1}^{\infty}\Big(\sum_{i=1}^{\nu}\lambda_{j,i}\Big)\alpha_{j,\nu}
\end{aligned}
\tag{0.18}
$$

and these linear combinations of infinite simple roots are identified with each other if their differences are in $\mathbb{C}\Lambda^0$. We note that

$$
|\{\lambda_{\mathbf{m}}\}| = (\Lambda(\lambda)+\tfrac{1}{2}\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}}).
\tag{0.19}
$$

The realizable tuples exactly correspond to the elements of the set $\overline{\Delta}_+$ of positive integer multiples of the positive roots of the Kac-Moody root system whose support contains α_0 and the rigid tuples exactly correspond to the positive real roots whose support contain α_0. For an element $w \in W_\infty$ and an element $\alpha \in \overline{\Delta}_+$ we do not consider $w\alpha$ in the commutative diagram (0.17) when $w\alpha \notin \overline{\Delta}_+$.

Hence the fact that any irreducible rigid Fuchsian equation $P_{\mathbf{m}}u = 0$ is transformed into the trivial equation $\frac{dv}{dx} = 0$ by our invertible fractional operations corresponds to the fact that there exists $w \in W_\infty$ such that $w\alpha_{\mathbf{m}} = \alpha_0$ because $\alpha_{\mathbf{m}}$ is a positive real root. The *monotone* fundamental tuples of partitions correspond to α_0 or the positive imaginary roots α in the closed negative Weyl chamber which are indivisible or satisfies $(\alpha|\alpha) < 0$. A tuple of partitions $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p \\ \nu=1,\dots,n_j}}$ is said to be monotone if $m_{j,1} \geq m_{j,2} \geq \cdots \geq m_{j,n_j}$ for $j = 0, \dots, p$. For example, we prove the exact estimate

$$\operatorname{ord} \mathbf{m} \leq 3|\operatorname{idx} \mathbf{m}| + 6 \tag{0.20}$$

for any fundamental tuple $\mathbf{m}$ in §7.2. Since we may assume

$$p \leq \tfrac{1}{2}|\operatorname{idx} \mathbf{m}| + 3 \tag{0.21}$$

for a fundamental tuple $\mathbf{m}$, there exist only finite number of monotone fundamental tuples with a fixed index of rigidity. We list the fundamental tuples of the index of rigidity 0 or -2 in Remark 6.9 or Proposition 6.10, respectively.

Our results in Chapter 3, Chapter 5 and Chapter 6 give an integral expression and a power series expression of a local solution of the universal equation $P_{\mathbf{m}}u = 0$ corresponding to the characteristic exponent whose multiplicity is free in the local monodromy. These expressions are in Chapter 8.

In §9.1 we review the monodromy of solutions of a Fuchsian differential equation from the view point of our operations. The theorems in this chapter are given by [**DR, DR2, Kz, Ko2**]. In §9.2 we review Scott's lemma [**Sc**] and related results with their proofs, which are elementary but important for the study of the irreducibility of the monodromy.

In §10.1 we examine the condition for the decomposition $P_{\mathbf{m}} = P_{\mathbf{m}'}P_{\mathbf{m}''}$ of universal operators with or without fixing the exponents $\{\lambda_{j,\nu}\}$, which implies the reducibility of the equation $P_{\mathbf{m}}u = 0$. In §10.2 we study the value of spectral parameters which makes the equation reducible and obtain Theorem 10.10. In particular we have a necessary and sufficient condition on characteristic exponents so that the monodromy of the solutions of the equation $P_{\mathbf{m}}u = 0$ with a rigid spectral type $\mathbf{m}$ is irreducible, which is given in Corollary 10.12 or Theorem 10.13. When $m_{j,1} \geq m_{j,2} \geq \cdots$ for any $j \geq 0$, the condition equals

$$(\Lambda(\lambda)|\alpha) \notin \mathbb{Z} \quad (\forall \alpha \in \Delta(\mathbf{m})). \tag{0.22}$$

Here $\Delta(\mathbf{m})$ denotes the totality of positive real roots α such that $w_{\mathbf{m}}\alpha$ are negative and $w_{\mathbf{m}}$ is the element of W_∞ with the minimal length so that $\alpha_0 = w_{\mathbf{m}}\alpha_{\mathbf{m}}$ (cf. Definition 7.8 and Proposition 7.9 v)). The number of elements of $\Delta(\mathbf{m})$ equals the length of $w_{\mathbf{m}}$, which is the minimal length of the expressions of $w_{\mathbf{m}}$ as products of simple reflections s_α with $\alpha \in \Pi$. Proposition 7.9 examines this set $\Delta(\mathbf{m})$. The set $\big\{(\alpha|\alpha_{\mathbf{m}}) \mid \alpha \in \Delta(\mathbf{m})\big\}$ gives a partition of a positive integer, which is denoted by $[\Delta(\mathbf{m})]$ and called the type of $\Delta(\mathbf{m})$ (cf. Remark 7.11 ii)). If $\mathbf{m}$ is monotone and rigid, $[\Delta(\mathbf{m})]$ is a partition of the positive integer $\operatorname{ord}\mathbf{m} + \sum_{j=0}^{p}\sum_{\nu=1}^{n_j-1}(\sum_{i=\nu+1}^{n_j} m_{j,i}) - 1$. Moreover $\mathbf{m}$ is simply reducible if and only if $[\Delta(\mathbf{m})] = 1 + \cdots + 1 = 1^{\#\Delta(\mathbf{m})}$.

In Chapter 11 we construct shift operators between rigid Fuchsian differential equations with the same spectral type such that the differences of the corresponding

characteristic exponents are integers. Theorem 11.3 gives a contiguity relation of certain solutions of the rigid Fuchsian equations, which is a generalization of the formula

$$c\big(F(a,b+1,c;x)-F(a,b,c;x)\big)=axF(a+1,b+1,c+1;x) \tag{0.23}$$

and moreover gives relations between the universal operators and the shift operators in Theorem 11.3 and Theorem 11.7. In particular, Theorem 11.7 gives a condition which assures that a universal operator is this shift operator.

The shift operators are useful for the study of Fuchsian differential equations when they are reducible because of special values of the characteristic exponents. Theorem 11.9 give a necessary condition and a sufficient condition so that the shift operator is bijective. In many cases we get a necessary and sufficient condition by this theorem. As an application of a shift operator we examine polynomial solutions of a rigid Fuchsian differential equation of Okubo type in §11.3.

In Chapter 12 we study a connection problem of the Fuchsian differential equation $P_{\mathbf{m}}u=0$. First we give Lemma 12.2 which describes the transformation of a connection coefficient under an addition and a middle convolution. In particular, for the equation $P_{\mathbf{m}}u=0$ satisfying $m_{0,n_0}=m_{1,n_1}=1$, Theorem 12.4 says that the connection coefficient $c(c_0:\lambda_{0,n_0}\rightsquigarrow c_1:\lambda_{1,n_1})$ from the local solution corresponding to the exponent λ_{0,n_0} to that corresponding to λ_{1,n_1} in the Riemann scheme (0.11) equals the connection coefficient of the reduced equation $P_{f\mathbf{m}}v=0$ up to the gamma factors which are explicitly calculated.

In particular, if the equation is rigid, Theorem 12.6 gives the connection coefficient as a quotient of products of gamma functions and an easier non-zero term. For example, when $p=2$, the easier term doesn't appear and the connection coefficient has the universal formula

$$c(c_0:\lambda_{0,n_0}\rightsquigarrow c_1:\lambda_{1,n_1})=\frac{\displaystyle\prod_{\nu=1}^{n_0-1}\Gamma\big(\lambda_{0,n_0}-\lambda_{0,\nu}+1\big)\cdot\prod_{\nu=1}^{n_1-1}\Gamma\big(\lambda_{1,\nu}-\lambda_{1,n_1}\big)}{\displaystyle\prod_{\substack{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}\\ m'_{0,n_0}=m''_{1,n_1}=1}}\Gamma\big(|\{\lambda_{\mathbf{m}'}\}|\big)}. \tag{0.24}$$

Here the notation (0.12) is used and $\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}''$ means that $\mathbf{m}=\mathbf{m}'+\mathbf{m}''$ with rigid tuples $\mathbf{m}'$ and $\mathbf{m}''$. Moreover in the right hand side of (0.24), the number of gamma factors appearing in the denominator equals to that in the numerator, the sum of the numbers $*$ in gamma factors $\Gamma(*)$ in the denominator also equals to that in the numerator and the decomposition $\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}'$ is characterized by the condition that $\alpha_{\mathbf{m}'}\in\Delta(\mathbf{m})$ or $\alpha_{\mathbf{m}''}\in\Delta(\mathbf{m})$ (cf. Corollary 12.7). The author conjectured this formula (0.24) in 2007 and proved it in 2008 (cf. [**O6**]). The proof in §12.1 based on the identity (12.8) is different from the original proof, which is explained in §12.3.

Suppose $p=2$, $\operatorname{ord}\mathbf{m}=2$, $m_{j,\nu}=1$ for $0\le j\le 2$ and $1\le\nu\le 2$, Then (0.24) equals

$$\frac{\Gamma(\lambda_{0,2}-\lambda_{0,1}+1)\cdot\Gamma(\lambda_{1,2}-\lambda_{1,1})}{\Gamma(\lambda_{0,1}+\lambda_{1,2}+\lambda_{2,1})\cdot\Gamma(\lambda_{0,1}+\lambda_{1,2}+\lambda_{2,2})}, \tag{0.25}$$

which implies (0.3) under (0.4).

The hypergeometric series $F(a,b,c;x)$ satisfies $\lim_{k\to+\infty}F(a,b,c+k;x)=1$ if $|x|\le 1$, which obviously implies $\lim_{k\to+\infty}F(a,b,c+k;1)=1$. Gauss proves the summation formula (0.3) by this limit formula and the recurrence relation $F(a,b,c;1)=\frac{(c-a)(c-b)}{c(c-a-b)}F(a,b,c+1;1)$. We have $\lim_{k\to+\infty}c(c_0:\lambda_{0,n_0}+k\rightsquigarrow c_1:\lambda_{1,n_1}-k)=1$ in the connection formula (0.24) (cf. Corollary 12.7). This suggests a

similar limit formula for a local solution of a general Fuchsian differential equation, which is given in §12.2.

In §12.3 we propose a procedure to calculate the connection coefficient (cf. Remark 12.19), which is based on the calculation of its zeros and poles. This procedure is different from the proof of Theorem 12.6 in §12.1 and useful to calculate a certain connection coefficient between local solutions with multiplicities larger than 1 in eigenvalues of local monodromies. The coefficient is defined in Definition 12.17 by using Wronskians.

In Chapter 13 we show many examples which explain our fractional calculus in this paper and also give concrete results of the calculus. In §13.1 we list all the fundamental tuples whose indices of rigidity are not smaller than -6 and in §13.2 we list all the rigid tuples whose orders are not larger than 8, most of which are calculated by a computer program `okubo` explained in §13.11. In §13.3 and §13.4 we apply our fractional calculus to Jordan-Pochhammer equations and the hypergeometric family, respectively, which helps us to understand our unifying study of rigid Fuchsian differential equations. In §13.5 we apply our fractional calculus to the even/odd family classified by [**Si**] and most of the results there have been first obtained by our calculus. In §13.6, we show some interesting identities of trigonometric functions as a consequence of the concrete value (0.24) of connection coefficients.

In §13.7, §13.8 and §13.9 we study the rigid Fuchsian differential equations of order not larger than 4 and those of order 5 or 6 and the equations belonging to 12 submaximal series classified by Roberts [**Ro**], respectively. Note that these 12 maximal series contain Yokoyama's list [**Yo**]. In §13.9.2, we explain how we read the condition of irreducibility, connection coefficients, shift operators etc. of the corresponding differential equation from the data given in §§13.7–13.9. We examine Appell's hypergeometric equations in §13.10 by our fractional calculus, which will be further discussed in another paper.

In Chapter 14 we give some problems to be studied related to the results in this paper.

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In Appendix a theorem on Coxeter groups is given, which was proved by K. Nuida through a private communication between the author and Nuida. The theorem is useful for the study of the difference of various reductions of Fuchsian differential equations (cf. Proposition 7.9 v)). The author greatly thanks Koji Nuida for allowing the author to put the theorem with its proof in this paper.

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