Preface

In [22], Ivrii and Petkov introduced the notion of fundamental matrix, which now called the Hamilton map, and proved that if the Cauchy problem is C^∞ well posed for any lower order term then the characteristics are at most double and at every double characteristic point the Hamilton map has non-zero real eigenvalues, now called *effectively hyperbolic*. If the Hamilton map has no non-zero real eigenvalue, that is *noneffectively hyperbolic* case, they also proved, under some restrictions, in order that the Cauchy problem is C^∞ well posed the subprincipal symbol must lie in some interval on the real line, which depends on the reference double characteristic point. This was a breakthrough in researches on hyperbolic operators with multiple characteristics (to feel the impact of that paper, see for example [14]). They conjectured that effectively hyperbolic operator is strongly hyperbolic, that is if the Hamilton map has non-zero real eigenvalues at every double characteristic then the Cauchy problem is C^∞ well posed for any lower order term. This conjecture has been proved affirmatively in [25], [29], [30], [42]. On the other hand, the necessary condition for the C^∞ well-posedness for noneffectively hyperbolic operator, mentioned above was completed by removing the restrictions in [18] and now called the Ivrii-Petkov-Hörmander condition.

If the Hamilton map verifies some spectral restrictions, the Cauchy problem for noneffectively hyperbolic operators is C^∞ well posed under the strict Ivrii-Petkov-Hörmander condition which was proved in [24], [18]. The main remaining question is thus whether this restriction on the Hamilton map can be removed or we need other necessary conditions for the C^∞ well-posedness. Much work has been devoted to this or related questions, see [41], [44], [47], [4], [6], [20].

It has been recognized that what is crucial to the C^∞ well-posedness is not only the Hamilton map but also the behavior of the Hamilton flow near the double characteristic manifold and the Hamilton map itself is not enough to determine completely the behavior of the Hamilton flow. Strikingly enough, if the Hamilton flow lands tangentially on the double characteristic manifold then the Cauchy problem is not C^∞ well posed even though we assume the Levi conditions, only well posed in the Gevrey class of order $1 \leq s < 5$ as proved in [7] (the arguments of the proof of non well-posedness there was incomplete and then we complete the proof here). On the other hand if the Hamilton flow does not touch the double characteristic manifold tangentially then the above mentioned result still holds; the Cauchy problem is C^∞ well posed under the strict Ivrii-Petkov-Hörmander condition. A microlocal counterpart of the C^∞ well-posedness result was proved in [6].

This monograph is organized as follows. In Chapter 1, we discuss about general necessary conditions for C^∞ well-posedness. In particular, we give a little bit refined proof of Ivrii-Petkov necessary condition for the C^∞ well-posedness. In Chapter 2 we are concerned with double characteristics and we classify the hyperbolic double characteristics into effectively hyperbolic characteristics and noneffectively hyperbolic characteristics according to the spectre of the Hamilton map F_p of the principal symbol p. In Chapter 3, we make detailed studies about the behavior of noneffectively hyperbolic symbol p near the double characteristic manifold Σ assuming $H_S^3 p = 0$ on Σ with some specified vector field H_S. These studies show, in particular, the condition that $H_S^3 p = 0$ on Σ is a necessary and sufficient condition in order that the principal symbol p admits some special decomposition, we call elementary decomposition, which was proved in [44] under some restrictions on the double characteristic manifold and in [6] in full generality removing the previous restrictions. The possibility of elementary decomposition is closely related to the behavior of null bicharacteristics near the double characteristic manifold Σ. Indeed in Chapter 7, we prove that the Hamilton flow does not land tangentially on Σ if and only if the symbol p admits an elementary decomposition which was observed in [41] in the codimension (of Σ) 3 case and completed in general case in [6], [47]. In Chapter 4, assuming that $\mathrm{Ker}F_p^2 \cap \mathrm{Im}F_p^2 = \{0\}$ on a C^∞ manifold Σ and necessarily p admits an elementary decomposition, under the Levi or the strict Ivrii-Petkov-Hörmander condition we derive energy estimates, via elementary decomposition, which proves the C^∞ well-posedness of the Cauchy problem. These are classical results proved in [24] and [18]. We also show, by giving an example, that the Ivrii-Petkov-Hörmander condition is not sufficient in general to assure the C^∞ well-posedness of the Cauchy problem. In Chapter 5 we study the case that $\mathrm{Ker}F_p^2 \cap \mathrm{Im}F_p^2 \neq \{0\}$ on Σ with no null bicharacteristic landing tangentially on Σ. We derive energy estimates assuming the strict Ivrii-Petkov-Hörmander condition and hence we conclude C^∞ well-posedness. Here detailed studies about the behavior of p near Σ made in Chapter 3 provides an elementary decomposition and then we follow the arguments in [24] to derive microlocal energy estimates and [43] to show the C^∞ well-posedness. In Chapter 6 we study the case that $\mathrm{Ker}F_p^2 \cap \mathrm{Im}F_p^2 \neq \{0\}$ on Σ and hence the Hamilton flow may land tangentially on Σ. Following [7] we derive energy estimates in the Gevrey class which proves the well-posedness of the Cauchy problem in the Gevrey class $1 \leq s < 5$ under the Levi condition. In Chapter 8, using a family of exact solutions studied in [5] we prove not only the non C^∞ well-posedness of the Cauchy problem but also non Gevrey s well-posedness for $s > 5$ for a model operator of which Hamilton flow lands tangentially on the double characteristic manifold. This proves the optimality of the index 5 of Gevrey well-posedness shown in Chapter 6. In Chapter 9 we retake the model operator studied in Chapter 8 and prove that the Cauchy problem for this operator is not C^∞ well posed for *any* choice of lower order term. In fact we show that the Cauchy problem is not well posed for any lower order term in the Gevrey class of order

$s > 6$. Finally in Chapter 10, we gathered some basic facts about symplectic basis and coordinates.

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