Inequalities in Statistics and Probability
IMS Lecture Notes–Monograph Series Vol. 5 (1984), 1–3

INTRODUCTION[1]

BY Y. L. TONG
University of Nebraska

As noted by Pólya (1967), "Inequalities play a role in most branches of mathematics and have widely different applications." This is certainly true in statistics and probability. From the viewpoint of applications, inequalities have become a useful tool in estimation and hypothesis-testing problems (such as for yielding bounds on the variances of estimators and on the probability contents of confidence regions, and for establishing monotonicity properties of the power of certain tests), in multivariate analysis, in reliability theory, and so forth. Perhaps the usefulness of inequalities can be best illustrated by the following situation in reliability theory: Under certain circumstances it is desirable or necessary to determine whether or not the reliability of a system meets a given specification. The evaluation of the true reliability of a complex system is not always feasible. But if an inequality in the form of a lower bound on the system reliability can be easily obtained, and if the lower bound already meets the specification, then one knows for sure that the system meets or exceeds the specification.

On the other hand, the theory of inequalities in statistics and probability has intrinsic interest and importance and need not rely only on applications. For deriving such inequalities one usually studies a problem from several different approaches, such as monotonicity properties via concepts of stochastic ordering of random variables and distributions, positive or negative dependence and/or association properties via a mixture of distributions or monotone transformations of random variables, Schur concavity and the notion of majorization via the diversity of the components of a vector (or matrix), and so on. Thus the study of inequalities *per se* can provide a better understanding of the interrelationships among the random variables and their transformations, and may reveal new information concerning the complicated structure of probability distributions. This in turn provides new insights, ideas, and approaches for solving a variety of problems in statistics and probability.

The general study of the theory of inequalities in statistics and probability is, of course, closely related to the developments of inequalities in mathematics. As Mitrinović pointed out (1970, p. v), although "the theory of inequalities (in mathematics) began its development from (the days of) C. F. Gauss, A. L. Cauchy, and P. L. Cebysev," it is "the classical work *Inequalities* by G. H. Hardy, J. E. Littlewood and G. Pólya (1934, 1952) . . . which transformed the field of inequalities from a collection of isolated formulas into a systematic discipline." After the publication of the second editon of their book in 1952, there have been several other volumes on mathematical inequalities; such

[1] This research was partially supported by NSF Grant MCS81–00775, A01.
AMS 1980 subject classifications. 60E15, 26D15, 62H99.
Key words and phrases: Inequalities in statistics and probability.

as Beckenbach and Bellman (1965) and Mitrinović (1970). The latest addition, the book by Marshall and Olkin (1979), contains an up-to-date treatment of the theory of majorization inequalities and its applications in linear algebra, geometry, as well as statistics and probability, and is also highly influenced by Hardy, Littlewood and Pólya (1934, 1952). Among the conference proceedings, there have been three volumes edited by Shisha (1967, 1970, 1972).

Among the books and monographs related to inequalities in statistics and probability, some chapters of the two volumes by Karlin and Studden (1966) and Karlin (1968) involve such inequalities, mainly for density functions which are totally positive. The book by Barlow and Proschan (1975) contains probability inequalities and their applications in reliability theory. The monograph by Marshall and Olkin (1979) concerns mainly inequalities in statistics and probability, and partial orderings for probability distributions, via the theory of majorization. The book by Tong (1980) deals with probability inequalities in multivariate distributions via dependence, association, and mixture of random variables and distributions, via monotonicity and diversity of the parameter vectors and other related concepts, and includes statistical applications. Each of these books contains a complete bibliography which gives sources for the research papers published in this area. Among the latest review articles, the review written by Kemperman (1981) contains a detailed description of majorization and the concept of dilation of probability measures; the paper by Eaton (1982) gives an up-to-date review of certain types of probability inequalities, and contains a few references which appeared after the publications of Marshall and Olkin (1979) and Tong (1980).

The present volume represents the most recent developments in the area of inequalities in statistics and probability. It contains 30 research and expository papers which have been grouped according to the following topics:

(a) Inequalities Via Partial Orderings: D'Abadie and Proschan (pp. 4–12), Eaton (pp. 13–25), Jensen (pp. 26–34), Karlin and Rinott (pp. 35–40).
(b) Convex and Matrix-Related Inequalities: Cohen (pp. 41–53), Das Gupta and Sarkar (pp. 54–58), Freimer and Mudholkar (pp. 59–67), Rao (pp. 68–77).
(c) Probabilistic and Distribution-Free Inequalities: Cox (pp. 78–83), Kemperman (pp. 84–103), Marshall (pp. 104–108), Vitale (pp. 109–111), Vitale (pp. 112–114).
(d) Dependence-Related Inequalities: Fink and Jodeit (pp. 115–120), Joag-Dev, Shepp and Vitale (pp. 121–126), Newman (pp. 127–140), Shaked and Tong (pp. 141–149).
(e) Inequalities in Regression and Multivariate Analysis: Bohrer and Wynn (pp. 150–155), Dharmadhikari and Joag-Dev (pp. 156–164), Lai and Wei (pp. 165–172), Perlman (pp. 173–177).
(f) Inequalities in Stochastic Optimization and Reliability: Birge and Wets (pp. 178–186), Block and Souza Borges (pp. 187–192), Savits (pp. 193–198).
(g) Inequalities in Selecting and Ordering Populations: Berger and Proschan (pp. 199–205), Chen and Sobel (pp. 206–210), Gupta, Huang and Panchapakesan (pp. 211–227).
(h) Trends and Order Restrictions: Dykstra (pp. 228–235), Magel and Wright (pp. 236–243), Robertson and Wright (pp. 244–250).

REFERENCES

BARLOW, R. E. and PROSCHAN, F. (1975). *Statistical Theory of Reliability and Life Testing*. Holt, Rinehart and Winston, New York.

BECKENBACH, E. F. and BELLMAN, R. (1965). *Inequalities*. Springer-Verlag, Berlin and New York.

EATON, M. L. (1982). A review of selected topics in multivariate probability inequalities. *Ann. Statist.* *10* 11–43.

HARDY, G. H., LITTLEWOOD, J. E., and PÓLYA, G. (1934, 1952). *Inequalities,* 1st ed., 2nd ed. Cambridge University Press, London and New York.

KARLIN, S. (1968). *Total Positivity, Vol. 1*. Stanford University Press, Stanford, CA.

KARLIN, S. and STUDDEN, W. J. (1966). *Tchebycheff Systems: With Applications in Analysis and Statistics*. Interscience, New York.

KEMPERMAN, J. H. B. (1981). Book review of Inequalities: Theory of majorization and its applications by Albert W. Marshall and Ingram Olkin; Probability inequalities in multivariate distributions by Y. L. Tong. *Bull. (New Ser.) Amer. Math. Soc.* *5* 319–324.

MARSHALL, A. W. and OLKIN, I. (1979). *Inequalities: Theory of Majorization and Its Applications*. Academic Press, New York.

MITRINOVIĆ, D. S. (1970). *Analytic Inequalities*. Springer-Verlag, Berlin and New York.

PÓLYA, G. (1967). Inequalities and the principle of nonsufficient reason. in *Inequalities,* O. Shisha ed., p. 1, Academic Press, New York.

SHISHA, O. (ed.) (1967, 1970, 1972). *Inequalities, Inequalities II, III*. Academic Press, New York.

TONG, Y. L. (1980). *Probability Inequalities in Multivariate Distributions*. Academic Press, New York.