17. Large RE Sets

Post's idea for solving his problem was to show that a sufficiently large RE set could not have degree **0'**, and then producing a non–recursive RE set which was this large. Although this idea did not solve Post's problem, it led to many interesting results, which we shall explore briefly.

We introduce our first type of large RE set. A simple set is an RE set whose complement is infinite but includes no infinite RE set. Note that a simple set is not recursive, since its complement is not RE. We shall see later that every RE degree other than **0** is the degree of a simple set; so we cannot show that a simple set cannot have degree **0'**. However, we shall prove the weaker result that a simple set cannot be RE complete. In order to do this, we first find a characterization of the RE complete sets.

A creating function for a set A is a recursive real F such that $F(e) \in A \leftrightarrow F(e) \in W_e$ for all e. A creative set is an RE set which has a creating function. For example, if A is defined by $e \in A \leftrightarrow e \in W_e$, the A is creative with creating function I_1^1. A creative set is in some sense effectively non–recursive; for the creating function F shows that A^c is not equal to any W_e and hence is not RE.

17.1. PROPOSITION. A set is RE complete iff it is creative.

Proof. Let A be RE complete. Then there is a recursive real F such that $W_e(e) \leftrightarrow F(e) \in A$. Using the RE Parameter Theorem, pick a recursive total S so that $W_{S(e)}(x) \leftrightarrow W_e(F(x))$. Then

$$F(S(e)) \in A \leftrightarrow W_{S(e)}(S(e)) \leftrightarrow F(S(e)) \in W_e.$$

Hence $G(e) = F(S(e))$ defines a creating function for A.

Suppose that A is creative with creating function F, and let B be any RE set. Pick a total recursive S so that $W_{S(x)}(y) \leftrightarrow x \in B$. Then

$$x \in B \leftrightarrow F(S(x)) \in W_{S(x)} \leftrightarrow F(S(x)) \in A;$$

so B is reducible to A. □

17.2. PROPOSITION. A simple set is not RE complete.

Proof. By 17.1, it is enough to show that a creative set A is not simple. Let F be a creating function for A and let g be an index of the empty set. Choose a recursive total S so that $W_{S(e)}(x) \leftrightarrow W_e(x) \vee x = F(e)$; then $W_{S(e)} = W_e \cup \{F(e)\}$. If $W_e \subseteq A^c$, then $F(e) \in W_e \leftrightarrow F(e) \in A$ shows that $F(e) \notin W_e$ and $W_{S(e)} \subseteq A^c$. Now define a recursive real G inductively by $G(0) = g$, $G(n+1) = S(G(n))$. By induction on n, $W_{G(n)} \subseteq A^c$ and $F(G(n)) \in W_{G(n+1)} - W_{G(n)}$. Thus if we set $x \in B \leftrightarrow \exists n(x \in W_{G(n)})$, then B is an infinite RE subset of A^c. Thus A is not simple. □

Now we turn to another type of large RE set. A <u>hypersimple</u> set is a coinfinite RE set A such that there is no recursive real F for which

$$\forall x \exists y(y \notin A \;\&\; x < y \leq F(x)).$$

17.3. PROPOSITION. Every hypersimple set is simple.

Proof. Let A be RE and suppose that there is an infinite RE subset B of A^c. Define an RE R by $R(x,y) \leftrightarrow y \in B \;\&\; x < y$. By the Selector Theorem, there is a recursive selector G for R. Since B is infinite, G is total; and for all x, $G(x) \notin A \;\&\; x < G(x)$. Hence A is not hypersimple. □

17.4. PROPOSITION. There is a simple set which is not hypersimple.

Proof. Define $R(x,y) \leftrightarrow y \in W_x \;\&\; y > 2x$. Then R is RE; so by the Selector Theorem, there is a recursive selector F for R. Let A be the range of F. Since $y \in A \leftrightarrow \exists x \mathcal{G}_F(x,y)$ and $\mathcal{G}_F$ is RE by the Graph Theorem, A is RE. If $F(x)$ is defined, then $F(x) > 2x$. Hence for each z, there are at most z numbers in A which are $\leq 2z$; so there is a $y \notin A$ such that $z \leq y \leq 2z$. It follows that A is coinfinite and not hypersimple. Finally, suppose that W_e is an infinite RE subset of A^c. Then $\exists y R(e,y)$; so $F(e)$ is defined and in $W_e \cap A$, a contradiction. □

To prove an analogue of 17.2 for hypersimple sets, we need some definitions. First observe that if we have an algorithm for computing A from an oracle for B, then we may use values given by the oracle to compute a number z

and then ask the oracle if $z \in B$. Imagine a simple algorithm which never does this. Then our algorithm computes a finite number of numbers; asks the oracle which of them are in B; and then decides if the input is in A. We might as well ask the oracle about all numbers less then some number z; i. e., we might as well ask the oracle for $\overline{\chi_B}(z)$.

We now put all of this into a definition. We say that A is truth–table reducible (abbreviated tt–reducible) to B if there is a recursive real F and a recursive set C such that

$$x \in A \leftrightarrow \overline{\chi_B}(F(x)) \in C$$

for all x. B is truth–table complete (abbreviated tt–complete) if B is RE and every RE set is tt–reducible to B. It is easy to see that

$$\begin{aligned} A \text{ is reducible to } B &\rightarrow A \text{ is tt–reducible to } B \\ &\rightarrow A \text{ is recursive in } B. \end{aligned}$$

Thus complete RE sets are tt–complete and tt–complete sets have degree $\mathbf{0}'$. (Both converses are false.)

17.5. PROPOSITION. A hypersimple set is not tt–complete.

Proof. Suppose that A is tt–complete. Choose a recursive real F and a recursive set C such that

$$W_e(e) \leftrightarrow \overline{\chi_A}(F(e)) \in C$$

for all e. Let $J_n = \{(n)_i : i < lh(n)\}$. Then every finite set is J_n for some n. Moreover, the relation $\overline{\chi_{J_n}}(F(x)) \notin C$ is recursive and hence RE; so there is a recursive real S such that $W_{S(n)}(x) \leftrightarrow \overline{\chi_{J_n}}(F(x)) \notin C$. Then

$$\overline{\chi_A}(F(S(n)) \in C \leftrightarrow W_{S(n)}(S(n)) \leftrightarrow \overline{\chi_{J_n}}(F(S(n))) \notin C.$$

It follows that $(\exists y < F(S(n)))(y \in A \leftrightarrow y \in J_n)$.

To show that A is not hypersimple, it will suffice to show how to compute from x a z such that $(\exists y \notin A)(x < y \leq z)$. First compute an m such that every subset of $\{0,1,\ldots,x\}$ is J_n for some $n \leq m$. Let z be the largest of the $F(S(n))$ for

$n \leq m$. There is an $n \leq m$ such that $J_n = \{y: y \leq x \ \& \ y \notin A\}$. Hence there is a $y < F(S(n)) \leq z$ such that $y \in A \leftrightarrow (y \leq x \ \& \ y \notin A)$. But this clearly implies that $y \notin A$ and $x < y$. □

For the results of this section to have any interest, we must know that there are sets which are hypersimple (and hence simple).

17.6. PROPOSITION (DEKKER). If **a** is RE and $\mathbf{a} \neq \mathbf{0}$, then there is a hypersimple set of degree **a**.

Proof. Let A be an RE set of degree **a**. Since $\mathbf{a} \neq \mathbf{0}$, A is non–recursive and hence infinite. By 14.7, there is a one–one recursive real F with range A. Define an RE set B by

$$x \in B \leftrightarrow \exists y(x < y \ \& \ F(y) < F(x)).$$

We will show that B is hypersimple and dg $B = \mathbf{a}$.

To show that B is coinfinite, let z be given. Choose $x > z$ so that $F(x)$ is as small as possible. Then clearly $x \notin B$. Now suppose that there is a recursive real G such that for every z there is a $y \in B^C$ such that $z < y \leq G(z)$. We obtain a contradiction by showing that A is recursive. Given a, $F(y) \geq a$ for large enough y (since F is one–one); so there is an z such that $a \leq F(y)$ for $z < y \leq G(z)$. We can find such a z by examining $z = 0$, $z = 1$, etc. We claim that $a \in A$ iff $a = F(x)$ for some $x \leq G(z)$. If not, $a = F(x)$ for some $x > G(z)$. Choose $y \notin B$ such that $z < y \leq G(z) < x$. Since $y \notin B$, $F(y) < F(x) = a$, contradicting the choice of z.

It remains to show that $B \equiv_R A$. Clearly

$$x \in B \leftrightarrow (\exists a < F(x))(a \in A \ \& \ (\forall y \leq x)(a \neq F(y)));$$

so $B \leq_R A$. Suppose that we want to use an oracle for B to compute whether or not $a \in A$. Since B is coinfinite, we can find an $x \notin B$ such that $F(x) > a$ by trial. Since $x \notin B$, $a \in A$ iff $a = F(y)$ for some $y < x$. □

17.7. COROLLARY. There is an RE set of degree $\mathbf{0}'$ which is not tt–complete.

Proof. By 17.5. □

A maximal set is a coinfinite RE set A such that for every coinfinite RE set B including A, $B-A$ is finite. Thus a maximal set is a coinfinite RE set with as few RE sets as possible including it.

It is fairly easy to show that a maximal set is hypersimple. However, it is not a simple matter to show that maximal sets exist; this was done by Friedberg. The final result of a series of investigations of this question is the following theorem of Martin: an RE degree **a** contains a maximal set iff $\mathbf{a}' = \mathbf{0}''$. Thus this notion of largeness does tell us more about the degree than our previous notions, but does not tell us that the degree cannot be $\mathbf{0}'$.

18. Function of Reals

We now extend our notion of a function to allow reals as arguments. (We could allow all total functions as arguments; but this would complicate matters without really adding anything, since a function can be replaced by its contraction.) We use lower case Greek letters, usually α, β, and γ, for reals. When the value of m is not important, we write $\vec{\alpha}$ for $\alpha_1,\ldots,\alpha_m$. We use $\mathbb{R}$ for the class of reals and $\mathbb{R}^{m,k}$ for the class of all $(m+k)$–tuples $(\alpha_1,\ldots,\alpha_m,x_1,\ldots,x_k)$. An (m,k)–ary function is a mapping of a subset of $\mathbb{R}^{m,k}$ into ω. (Thus a $(0,k)$–ary function is just a k–ary function.) From now on, a function is always an (m,k)–ary function for some m and k. Such a function is total if its domain is all of $\mathbb{R}^{m,k}$. An (m,k)–ary relation is a subset of $\mathbb{R}^{m,k}$. We define the representing function of such a relation as before.

Note that the real arguments to a function or relation must precede the number arguments. It may sometimes be convenient to write them in a different order. It is then understood that we are to move all real arguments to the left of all number arguments without otherwise changing the order of the arguments.

Now we consider how to extend the idea of computability. The new