DEFINABILITY AND GLOBAL DEGREE THEORY[1]

S. Barry Cooper

Gödel's work [Gö34] on undecidable theories and the subsequent formalisations of the notion of a recursive function ([Tu36], [Kl36] etc.) have led to an ever deepening understanding of the nature of the non-computable universe (which as Gödel himself showed, includes sets and functions of everyday significance). The nontrivial aspect of Church's Thesis (any function not contained within one of the equivalent definitions of recursive/Turing computable, cannot be considered to be effectively computable) still provides a basis not only for classical and generalised recursion theory, but also for contemporary theoretical computer science. Recent years, in parallel with the massive increase in interest in the computable universe and the development of much subtler concepts of 'practically computable,' have seen remarkable progress with some of the most basic and challenging questions concerning the non-computable universe, results both of philosophical significance and of potentially wider technical importance.

Relativising Church's Thesis, Kleene and Post [KP54] proposed the now standard framework of the *degrees of unsolvability* $\mathcal{D}$ as the appropriate fine structure theory for ω^ω. A technical basis was found in the various equivalent notions of relative computability provided by Turing [Tu39], Kleene [Kl43], Post [Po43] and others. Within the study of $\mathcal{D}$ it has become usual to distinguish (see [Sh81]) two approaches: that of *global degree theory*, based more or less on a number of general questions concerning the structure of the degrees first stated by Rogers in his book [Ro67]; and that of *local degree theory* with its emphasis on degree structure not far removed from the degree $\mathbf{0}$ of recursive functions (in particular the *recursively enumerable*—or *r.e.*—degrees and the degrees below $\mathbf{0}'$—the degree of the coded theorems of Peano arithmetic). Of course, there is an intimate relationship between the two approaches, and the aim here is to describe some recent results showing how even the most archetypal local degree theory can be used to resolve interesting and important global questions.

§1. Notation and terminology.

We use standard notation and terminology (see for example [So87]).

For instance, corresponding to the ith Turing machine, Φ_i denotes the ith partial recursive (p.r.) functional $2^\omega \to 2^\omega$. A set A is *Turing reducible to* a set B ($A \leq_T B$) if and only if $A = \Phi_i^B$ for some $i \in \omega$, and A, B are *Turing equivalent*

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($A \equiv_T B$) if and only if $A \leq_T B$ and $B \leq_T A$. The *degree of unsolvability* or *Turing degree* of A is defined by

$$\deg(A) = \{X \in 2^\omega \mid A \equiv_T X\}.$$

We write $\leq$ for the partial ordering on $\mathcal{D}$, the set of all degrees, $\mathbf{0}$ for the least degree, consisting of all recursive sets of numbers, and $\boldsymbol{\mathcal{D}}$ for the structure $\langle \mathcal{D}, \leq \rangle$.

Kleene and Post [KP54] also defined the notion of *jump operator* on sets and degrees. Let $W_i^A = \operatorname{dom} \Phi_i^A$ denote the ith *recursively enumerable in* A (A-*r.e.*) set ($W_i = W_i^\emptyset$ being the ith r.e. set). Then the *jump* ($n+1$*th jump*) of a set A is defined by $A' = A^{(1)} = \{x \mid x \in W_x^A\}$ ($A^{(n+1)} = (A^{(n)})'$). This induces a *jump operator* on degrees defined by $\mathbf{a}' = \deg(A')$, $A \in \mathbf{a}$, with the special properties that $\mathbf{a} < \mathbf{a}'$, and $\mathbf{a}'$ is the least upper bound of the degrees of sets r.e. in $A \in \mathbf{a}$. Post's Theorem [Po48] that $X \in \Delta_{n+1}^A \Leftrightarrow X \leq_T A^{(n)}$ attaches special importance to the ascending sequence $\mathbf{a}, \mathbf{a}', \dots, \mathbf{a}^{(n)}, \dots$. We define the standard ω-*jump* of $\mathbf{a}$ by $\mathbf{a}^{(\omega)} = \deg(\oplus_{n\in\omega} A^{(n)})$, $A \in \mathbf{a}$. Kleene and Post were the first to investigate the structure $\boldsymbol{\mathcal{D}}' = \langle \mathcal{D}, \leq, ' \rangle$. They speculated ([KP54], p. 384) that the jump operator may not be capable of description within $\boldsymbol{\mathcal{D}}$ itself, a question returned to by many authors since then (for instance, in recent times, Simpson [Si77], Epstein [Ep79], Shore [Sh81] and Odifreddi [Od89]).

§2. Global degree theory.

How rich a structure is $\boldsymbol{\mathcal{D}}$ or $\boldsymbol{\mathcal{D}}'$? To what extent are degrees locally individuated? Are particular parts of the non-computable universe recognisable by their context in $\boldsymbol{\mathcal{D}}$ or $\boldsymbol{\mathcal{D}}'$? What mathematical concepts are describable within the degrees of unsolvability? Following Rogers [Ro67] global questions tend to be grouped under the following headings:

HOMOGENEITY. *For which* $\mathbf{a}, \mathbf{b} \in \mathcal{D}$ *is* $\boldsymbol{\mathcal{D}}(\geq \mathbf{a}) \equiv \boldsymbol{\mathcal{D}}(\geq \mathbf{b})$, *or* $\boldsymbol{\mathcal{D}}'(\geq \mathbf{a}) \equiv \boldsymbol{\mathcal{D}}'(\geq \mathbf{b})$?

STRONG HOMOGENEITY. *For which* $\mathbf{a}, \mathbf{b} \in \mathcal{D}$ *is* $\boldsymbol{\mathcal{D}}(\geq \mathbf{a}) \cong \boldsymbol{\mathcal{D}}(\geq \mathbf{b})$, *or* $\boldsymbol{\mathcal{D}}'(\geq \mathbf{a}) \cong \boldsymbol{\mathcal{D}}'(\geq \mathbf{b})$?

The Strong Homogeneity/Homogeneity Conjectures of Rogers [Ro67]/Yates [Ya70], respectively, refer to the special case when $\mathbf{b} = \mathbf{0}$.

AUTOMORPHISMS. *Are there any nontrivial automorphisms of* $\boldsymbol{\mathcal{D}}$ *or* $\boldsymbol{\mathcal{D}}'$?

That is, is the non-computable universe *rigid*?

DEFINABILITY. *What can we describe in terms of the structure of* $\boldsymbol{\mathcal{D}}$ *or* $\boldsymbol{\mathcal{D}}'$? *In particular:*

Is the jump $\mathbf{a}'$ *of* $\mathbf{a}$ *definable purely in terms of the structure of* $\boldsymbol{\mathcal{D}}(\geq \mathbf{a})$?

Rogers' [Ro67] chooses invariance under all automorphisms of $\boldsymbol{\mathcal{D}}$ (that is, the notion of being *order-theoretic*) as an alternative formalisation of the idea of a relation on degrees being fixed by the structure of $\boldsymbol{\mathcal{D}}$. For instance he asks (see also Question 5.12 of [Si77] and Q8. of [Ep79]):

Is the jump operator order-theoretic?

We briefly review what was previously known concerning the above questions. A more detailed discussion can be found in [Od89].

§3. Definability.

Initial segments and their relativisations provided the first source of expressive structure within the degrees of unsolvability. Lachlan [La68] used the embeddability of all countable distributive lattices as initial segments of $\mathcal{D}$ to show the undecidability and non-axiomatisability of Th$(\mathcal{D})$ (the first-order theory of $\mathcal{D}$). The best result in this direction was provided by Simpson [Si77], following Jockusch and Simpson's [JSi76] initial work on coding and definability results for $\mathcal{D}'$:

THEOREM 1 (Simpson [Si77]). *The degree of Th$(\mathcal{D})$ = the degree of Th$(\mathcal{N})$ (the theory of second-order arithmetic).*

Theorem 1 was originally proved using an ad hoc coding of Th$(\mathcal{N})$ into Th$(\mathcal{D})$, but was proved more directly by Nerode and Shore [NS80] using the countable distributive lattice initial segment embedding result.

In relation to the original question of Kleene and Post concerning the definability of the jump operator, we have:

THEOREM 2 (Shore [Sh82]). *Any relation on $\mathcal{D}(\geq \mathbf{0}^{(3)})$ which is definable in second-order arithmetic is definable in $\mathcal{D}'$.*

(The first result of this kind was proved by Simpson [Si77] with $\mathbf{0}^{(\omega)}$ in place of $\mathbf{0}^{(3)}$, the improvement to $\mathbf{0}^{(3)}$ emerging via $\mathbf{0}^{(7)}$ in Nerode and Shore [NS80], [NS80a], who also showed how to replace the jump by a parameter such as $\mathbf{0}''$ in many of these global results. There was already a natural definition of $\mathbf{0}^{(\omega)}$ in $\mathcal{D}'$, got by combining results of Enderton and Putnam [EP70] and Sacks [Sa71], as the least double-jump of an upper-bound for the arithmetical degrees.)

Without the jump much less could be said:

THEOREM 3 (Jockusch and Shore [JSh84]). *Any relation on the degrees above all the arithmetical ones is definable in $\mathcal{D}$ if and only if it is definable in second-order arithmetic. In particular, $\mathbf{0}^{(\omega)}$, and hence the ω-jump, is definable in $\mathcal{D}$.*

(The first part of Theorem 3 improves Harrington and Shore [HS81] by replacing 'hyperarithmetic' with 'arithmetical.')

There are also results concerning the existence of structural characterisations of the jump related classes of the high/low hierarchy, defined by

$$\begin{aligned}\mathrm{High}_n &= \{\mathbf{a} \leq \mathbf{0}' \mid \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}\},\\ \mathrm{Low}_n &= \{\mathbf{a} \leq \mathbf{0}' \mid \mathbf{a}^{(n)} = \mathbf{0}^{(n)}\}.\end{aligned}$$

THEOREM 4 (Shore [Sh88]). *High_n, Low_n are definable in $\mathcal{D}(\leq \mathbf{0}')$ for each $n \geq 3$.*

The above results are all proved using degree-theoretic codings, Theorems 2, 3 and 4 using developments of the Nerode/Shore coding methods. More recently, all such results have been derived using the simpler coding technique of Slaman and Woodin (see [SW86] and [OS*ta*]). The three quantifiers intrinsic to codings involving $\leq_T$ explain the best possible lower-bound $\mathbf{0}^{(3)}$ in these and other global results. Another striking definability result using these codings is:

THEOREM 5 (Slaman and Woodin [SW86]). *The set of r.e. degrees is definable in* $\mathcal{D}(\leq \mathbf{0}')$ *using a finite number of parameters.*

In another direction, there are a number of results concerning definability of particular jump ideals in $\mathcal{D}$ or $\mathcal{D}'$ (see [HS81], [NS80a] and [Sh81]). Jockusch and Shore [JSh84] obtain definability results using their theory of pseudo-jumps. For instance, with no use of codings they get:

THEOREM 6 (Jockusch and Shore [JSh84]). $\mathcal{A}$ *(= the set of arithmetical degrees) is definable in* $\mathcal{D}$.

§4. Homogeneity.

Here again we see a large gap between the situations with and without the jump. We single out the following from among the strongest previous results concerning strong homogeneity for $\mathcal{D}'$:

THEOREM 7 (Richter [Ri79]). *If* $\mathcal{D}'(\geq \mathbf{a}) \cong \mathcal{D}'(\geq \mathbf{b})$ *then* $\mathbf{a}^{(3)} = \mathbf{b}^{(3)}$.

This means, for instance, that for each $n \geq 1$, $\mathcal{D}' \not\cong \mathcal{D}'(\geq \mathbf{0}^{(n)})$. Theorem 7 essentially improves [JSol77] by a factor of one jump, and is the culmination of a sequence of results obtained by various authors, starting with Feiner's [Fe70] refutation of the Strong Homogeneity Conjecture for $\mathcal{D}'$ (see also [Ya72]). Yates [Ya70] formulated the Homogeneity Conjecture by replacing isomorphism with elementary equivalence in Rogers' original question, and Simpson [Si77] gave a negative answer. The strongest refutation is due to Shore:

THEOREM 8 (Shore [Sh81]). *If* $\mathcal{D}' \equiv \mathcal{D}'(\geq \mathbf{a})$ *then* $\mathbf{a}^{(3)} = \mathbf{0}^{(3)}$.

Without the jump information is much more difficult to obtain. Combining Theorem 4.7 from [NS80] concerning automorphisms of $\mathcal{D}$ with work of Jockusch and Soare [JSoa70] and Harrington and Kechris [HK75] on minimal covers, Shore [Sh79] disproved the Strong Homogeneity Conjecture for $\mathcal{D}$. Shore [Sh82] extended this result on isomorphisms of cones to one on elementary equivalence: *If the degree of Kleene's* $\mathcal{O} \leq \mathbf{a}^{(n)}$ *for some* n *then* $\mathcal{D} \not\equiv \mathcal{D}(\geq \mathbf{a})$, so disproving homogeneity without the jump. Improving Harrington and Shore [HS81] (by replacing 'hyperarithmetic' with 'arithmetic') we have:

THEOREM 9 (Jockusch and Shore [JSh84]). *1) If* $\mathcal{D}(\geq \mathbf{a}) \cong \mathcal{D}(\geq \mathbf{b})$ *then* $\mathbf{a}$ *and* $\mathbf{b}$ *are arithmetically equivalent.*
2) If $\mathcal{D}(\geq \mathbf{a}) \equiv \mathcal{D}$ *then* $\mathbf{a}$ *is arithmetical.*

So in particular, $\mathcal{D} \not\equiv \mathcal{D}(\geq \mathbf{0}^{(\omega)})$.

§5. Automorphisms.

Corresponding to Theorem 7 we have:

THEOREM 10 (Epstein [Ep79], Richter [Ri79]). *Let* f *be an automorphism of* $\mathcal{D}'$. *Then* $f(\mathbf{a}) = \mathbf{a}$ *for all* $\mathbf{a} \geq \mathbf{0}^{(3)}$.

Jockusch and Solovay [JSol77] were the first to prove that jump preserving automorphisms are the identity on a cone (with $\mathbf{0}^{(4)}$ instead of $\mathbf{0}^{(3)}$ as the base of the cone). Nerode and Shore [NS80] show how to replace the jump by the parameter $\mathbf{0}'$ in the above result.

Without the jump we have:

THEOREM 11 (Jockusch and Shore [JSh84]). *If ψ is an automorphism of $\mathcal{D}$ then $\psi(\mathbf{a}) = \mathbf{a}$ for all $\mathbf{a} \geq \mathbf{0}^{(\omega)}$.*

Theorem 4.7 of Nerode and Shore [NS80] was the first result of this kind for $\mathcal{D}$, improvements in the location of the base of the cone appearing in [HS81] and [Sh81].

Other restrictions on the possible automorphisms of $\mathcal{D}$ are provided by the notion of an *automorphism base* (see Jockusch and Posner [JP81]).

The situation described so far is one in which much stronger results, and simpler proofs, are possible for $\mathcal{D}'$ than for $\mathcal{D}$. The aim now is to reduce the theory for $\mathcal{D}$ to that for $\mathcal{D}'$.

§6. Pseudo-jump operators.

Jockusch and Shore [JSh83], [JSh84] observe that constructions in recursion theory relative to a set A produce a set $J(A)$ which is, from a formal point of view, very similar in definition to the jump A', or more generally αth jump $A^{(\alpha)}$ for suitable recursive ordinal α, of A. This leads them to abstract from this the notion of an *α-REA operator*, and to mimic (in a nontrivial way) completeness and cupping theorems of Friedberg [Fr57], MacIntyre [Ma77] and Posner and Robinson [PR81] for the usual αth jump to produce cones of degrees with interesting structural properties. We need below the Jockusch and Shore pseudo-jump machinery for α finite (in fact, for $\alpha = 2$).

DEFINITION. We say that J^n is an *n-REA operator* if and only if there exist $j_0, j_1, \ldots, j_{n-1} \in \omega$ such that J^n is defined by

$$J^0(A) = A,$$
$$J^{k+1}(A) = J^k(A) \oplus W_{j_k}^{J^k(A)}, \quad (k < n).$$

Natural examples of n-REA operators are given by

(1) Choose j_0 such that $W_{j_0}^A = \{x \mid x \in W_x^A\}$. Then the 1-REA operator J^1 defined by $J^1(A) = A \oplus W_{j_0}^A$ is Turing equivalent to A', the usual Turing jump of A.
(2) If $D = W_i - W_j$ is a *d-r.e. set* (a difference of two r.e. sets), we can define a 2-REA operator J^2 (see p. 1209 of [JSh84] for a detailed verification) by
$$J^2(A) = A \oplus (W_i^A - W_j^A).$$

(One can derive n-REA operators from n-r.e. sets in a similar way.)

We will need the following analogues of the Friedberg completeness and Posner–Robinson cupping theorems:

COMPLETENESS THEOREM FOR n-REA OPERATORS. *If J is an n-REA operator, for each C there is a set X^C such that $C \oplus \emptyset^n \equiv_T J(X^C)$.*

(The proof is a simplification of that of Theorem 2.3 of Jockusch and Shore [JSh84].)

CUPPING THEOREM FOR n-REA OPERATORS. *If J is an n-REA operator derived from an n-r.e. set, then if $D \geq_T \emptyset^{(n)} \oplus X$ and $X \not\leq_T \emptyset^{(n-1)}$, we can find an A such that*

$$X \oplus A \equiv_T D \equiv_T J(A).$$

(The proof is essentially contained in that of Theorem 3.2 of Jockusch and Shore [JSh84].)

In the next section we apply these theorems to a particular 2-REA operator.

§7. Some local degree theory and the definability of the jump.

We first outline how the construction of a d-r.e. degree with special properties yields the required 2-REA operator.

DEFINITION. Given $\mathbf{a}, \mathbf{b}, \mathbf{d}$, we say $\mathbf{d}$ is *unsplittable over* $\mathbf{a}$ *avoiding* $\mathbf{b}$ if and only if $\mathbf{a}, \mathbf{b} \leq \mathbf{d}$, $\mathbf{b} \not\leq \mathbf{a}$, and for all $\mathbf{d}_0, \mathbf{d}_1 < \mathbf{d}$, if $\mathbf{a} < \mathbf{d}_0, \mathbf{d}_1$ then *either* $\mathbf{b} \leq \mathbf{d}_0$ or $\mathbf{d}_1$, or $\mathbf{d} \neq \mathbf{d}_0 \cup \mathbf{d}_1$.

$\mathbf{d}$ is *relatively unsplittable* if and only if $\mathbf{d}$ is unsplittable over $\mathbf{a}$ avoiding $\mathbf{b}$, some $\mathbf{a}, \mathbf{b}$.

It is important to notice that, by the relativised Sacks Splitting Theorem (see [So87], p.124), there is no relatively unsplittable r.e. degree.

THE MAIN THEOREM. *There is a relatively unsplittable d-r.e. degree.*

That is, there is a d-r.e. set $D = W_i - W_j$ (say) and sets $A, B \leq_T D$ such that $\deg(D)$ is unsplittable over $\deg(A)$ avoiding $\deg(B)$.

Before outlining the construction we list some immediate applications of the main theorem.

We first notice that we can use the main theorem to get a 2-REA operator J such that for each B we have $J(B) = B \oplus (W_i^B - W_j^B)$ and $\deg(J(B))$ is unsplittable over some $\mathbf{a} \geq \deg(B)$. Then applying the Completeness Theorem for n-REA Operators to J we get:

THEOREM 12. *There is a cone of relatively unsplittable degrees with base* $\mathbf{0}''$. □

Using J with the Cupping Theorem for n-REA Operators we get:

THEOREM 13 (DEFINABILITY OF $\mathbf{0}'$). $\mathbf{0}'$ *is definable in* $\mathcal{D}$ *as the largest degree satisfying*

$$(\dagger) \qquad \neg(\exists \mathbf{a}, \mathbf{b})[\mathbf{x} \cup \mathbf{a} \textit{ is unsplittable over } \mathbf{a} \textit{ avoiding } \mathbf{b}].$$

Proof. As previously remarked, each r.e. degree, including $\mathbf{0}'$, satisfies $(\dagger)$ by the relativised Sacks Splitting Theorem.

On the other hand, say $X \not\leq_T \emptyset'$. Then if $D \geq_T \emptyset'' \oplus X$ the Cupping Theorem gives us an A_1 for which $X \oplus A_1 \equiv_T D \equiv_T J(A_1)$, and $\deg(X \oplus A_1)$ is unsplittable over some degree $\mathbf{a} \geq \deg(A_1)$ avoiding some $\mathbf{b}$, where $\deg(X) \cup \deg(A_1) = \deg(X) \cup \mathbf{a}$. □

Relativising, this means that for each $\mathbf{a}$, $\mathbf{a}'$ is definable in $\mathcal{D}(\geq \mathbf{a})$, giving:

THEOREM 14 (DEFINABILITY OF THE JUMP). *The Turing jump is definable in $\mathcal{D}$.* □

This of course implies that the jump is order-theoretic, answering the question of Rogers referred to in section 2 above. Since $\mathbf{0}^{(n)}$ is definable for each n, we can get Theorem 6 (definability of the set of arithmetical degrees) as another corollary. And although Theorem 14 adds nothing to the known definability results below $\mathbf{0}'$, we can complement Theorem 4 above with:

THEOREM 15. *All the jump classes $High_n$ and Low_n, $n > 0$, are definable in $\mathcal{D}$.* □

We can use Theorem 14 to translate known results on definability, homogeneity and automorphisms for $\mathcal{D}'$ into dramatically improved results for $\mathcal{D}$. In place of Theorem 3 above we can restate Theorem 2:

THEOREM 2+. *Any relation on $\mathcal{D}(\geq \mathbf{0}^{(3)})$ which is definable in second-order arithmetic is definable in $\mathcal{D}$.*

Instead of Theorem 9 we get from Theorems 7 and 8:

THEOREM 7+. *If $\mathcal{D}(\geq \mathbf{a}) \cong \mathcal{D}(\geq \mathbf{b})$ then $\mathbf{a}^{(3)} = \mathbf{b}^{(3)}$.*

THEOREM 8+. *If $\mathcal{D} \equiv \mathcal{D}(\geq \mathbf{a})$ then $\mathbf{a}^{(3)} = \mathbf{0}^{(3)}$.*

And using Theorem 10 we can replace 'ω' by '3' in Theorem 11:

THEOREM10+. *Let f be an automorphism of $\mathcal{D}$. Then $f(\mathbf{a}) = \mathbf{a}$ for all $\mathbf{a} \geq \mathbf{0}^{(3)}$.*

§8. Proof of the main theorem.

The following sketch can be used as an introduction to the full proof in [Co*ta*l].

Let $(\Theta_k, \Psi_k, \Phi_k, \widehat{\Phi}_k)$, $k \geq 0$, be a standard listing of all 4-tuples of p.r. functionals. We need to construct a d-r.e. set D and sets $A, B \leq_T D$ satisfying the requirements:

$$P_k: \quad B \neq \Theta_k^A,$$
$$Q_k: \quad D = \Psi_k(\Phi_k^D, \widehat{\Phi}_k^D) \Rightarrow B = \Gamma_k(\Phi_k^D, A) \vee B = \Lambda_k(\widehat{\Phi}_k^D, A),$$

$k \geq 0$, where Γ_k, Λ_k are p.r. functionals to be constructed. We also need an overall constraint that $A = \Omega^D$, Ω a p.r. functional to be defined during the construction. The Q-requirements will ensure that $B \leq_T D$.

The basic module is closely related to the Lachlan 'monster construction' [La75] of a r.e. degree which is relatively non-splitting within the r.e. degrees. We consider just two requirements P $(= P_{k'}$, say) and Q $(= Q_k$, say) in relation to each other, Q being of higher priority than P. We follow the convention of writing $\theta_k, \varphi_k, \psi_k, \gamma_k, \lambda_k$ etc. for the respective standard use functions of $\Theta_k, \Phi_k, \Psi_k, \Gamma_k, \Lambda_k$ etc.

The naive P-strategy: Look for an x with $\Theta^A(x)\downarrow$ for which we can *define* $B(x) \neq \Theta^A(x)$ and *restrain* $A \restriction \theta(x)$.

The naive Q-strategy: First try to implement the Γ-strategy: If P requires us to make a $B(x)$-change, try to produce a situation such that *either*
(a) $\gamma(x) > \theta(x)$ (so we can rectify the equation $B(x) = \Gamma(\Phi^D, A)(x)$ following the $B(x)$-change with an A-change bigger than $\theta(x)$), *or*
(b) $\gamma(x) > \psi(y)$, some y, and hope to get a $\Phi^D \restriction \gamma(x)$-change by using a $D(y)$-change to force a $\Phi^D \restriction \psi(y)$-change.

If it looks like we always get a $\widehat{\Phi}^D \restriction \psi(y)$ change in (b), start to implement the Λ-strategy.

We consider in detail some of the problems involved in reconciling the strategies for P and Q:

Some problems: Roughly speaking, our strategy for P and Q together is as follows. If $\ell(D, \Psi(\Phi^D, \widehat{\Phi}^D))$ (the standard length of agreement function at stage $s+1$) grows large, we follow the naive Q-strategy in initially implementing the Γ-strategy for making $B \leq_T \Phi^D \oplus A$. P will have associated with it followers x for which we hope to get $\Theta^A(x)\downarrow \neq B(x)$. This may conflict with the Γ-strategy in that changing $B(x)$ to disagree with $\Theta^A(x)$ at stage $s+1$ may not result in a $\Phi^D \restriction \gamma(x)$ change. This will require the $B(x)$ change to be signalled through a change in $A \restriction \gamma(x) \subset A \restriction \theta(x)$, resulting in a possible reassertion of the equation $B(x) = \Theta^A(x)$ at some later stage.

According to the naive Q-strategy, our first approach to a resolution of this conflict will be to try to make such a $\gamma(x) > \theta(x)$, so that the A-change is above the use of $\Theta^A(x)$. But in general we can only do this by also injuring the existing use of $\Theta^A(x)$, in the hope that our new larger $\gamma(x)$ will be greater than $\theta(x)$ when this becomes defined again. This process (basically Harrington's 'capricious destruction,' although the capriciousness is only apparent) may be repeated using A-changes on larger and larger numbers, along with them an ascending sequence of numbers needed for corresponding D-permissions.

There are various possible outcomes to this. We may succeed in obtaining a suitable relatively small use for Θ^A, make our choice of $B(x)$, and satisfy P. On the other hand, infinite repetition of this process will lead to $\Theta^A(x)\uparrow$ (P satisfied again), but (without further analysis) we will also end up with $\Gamma^{\Phi^D, A}$ not total so that the Γ-strategy fails. There is another possibility for avoiding this. Since we are only concerned about Q if Φ^D is a total function, we may permit the A-changes needed to move $\gamma(x)$ by the enumeration of an *agitator* y into D, but defer actually making those A-changes until at least $\Phi^D \restriction \gamma(x)$ has become redefined. This leaves open the possibility that we may get a completely new $\Phi^D \restriction \gamma(x)$ (that is, not containing as an initial segment any previously defined $\Phi^D \restriction \gamma(x)$) which can be used to permit $x \searrow B$ ('x entering B') via Γ without the need for the A-changes to be made. But then, assuming we have timed our enumeration of y into D to coincide with $\Theta^A(x)\downarrow = B(x) = 0$, we avoid disturbing $A \restriction \theta(x)$. Hence we get $\Theta^A(x) = 0 \neq B(x)$ following the above actions, so satisfying P and in the process leaving the Γ-strategy intact.

We can assist this outcome by using A to increase the likelihood of the new $\Phi^D \upharpoonright \gamma(x)$ being usable to permit $x \searrow B$ via Γ. Following the monster construction we might try to ensure that whenever we define $\gamma(x)$ or $\lambda(x)$ previous to $y \searrow D$ we have $\psi(y) \downarrow\leq \gamma(x)$ or $\lambda(x)$ respectively, so that $y \searrow D$ will at least produce some sort of change in either $\Phi^D \upharpoonright \gamma(x)$ or $\widehat{\Phi}^D \upharpoonright \lambda(x)$. This is attempted via a process similar to Harrington's 'honestification,' whereby if $\gamma(x)$ or $\lambda(x) < \psi(y)$ when we require $y \searrow D$, we first produce an $A \upharpoonright w$ change, with $w \leq \min\{\gamma(x), \lambda(x)\}$, redefining $\gamma(x), \lambda(x) \geq \psi(y)$ when $\psi(y)$ is next defined. In fact honestification is extended by making $w \leq \min\{\gamma(x), \lambda(x)\}$ for all such $\gamma(x), \lambda(x)$ defined since the last occurrence of honestification, thereby ensuring that previously defined $\Phi^D \upharpoonright \gamma(x)$ or $\widehat{\Phi}^D \upharpoonright \lambda(x)$ will not return in tandem with corresponding $A \upharpoonright \gamma(x)$ or $A \upharpoonright \lambda(x)$ to prevent us Φ^D- or $\widehat{\Phi}^D$-permitting B-changes via Γ or Λ respectively following $y \searrow D$.

There is a problem here (apart from that of not knowing whether we get a Φ^D- or $\widehat{\Phi}^D$-change following $y \searrow D$), in that honestification will very likely also involve an $A \upharpoonright \theta(x)$ change, so that $y \searrow D$ will have to wait for $\theta(x)$ to become redefined, by which time the effects of honestification may have worn off, demanding renewed honestification. If this repetition developes into an infinite outcome, we get $\Gamma^{\Phi^D, A}, \Lambda^{\widehat{\Phi}^D, A}$ not total. But Q is then satisfied since we must have $\min\{\gamma(x), \lambda(x)\} < \psi(y)$ infinitely often so that $\Psi(\Phi^D, \widehat{\Phi}^D)(y)\uparrow$ also. And $\theta(x)\uparrow$ infinitely often, so P is satisfied through $\Theta^A(x)\uparrow$.

However, honestification as described will still not be sufficient to supply the ideal conditions for $y \searrow D$. This is because of a new complication resulting from the possibility of returns to strings $\Phi^D \upharpoonright \gamma(x)$ (following $y \searrow D$) which appeared since the last occurrence of honestification for (P, Q). So before allowing $y \searrow D$ we further ask that $\Phi^D \upharpoonright \psi(y), \widehat{\Phi}^D \upharpoonright \psi(y)$ are unchanged at all stages since the previous occurrence of honestification, and if this condition is not met, we again honestify (even if $\psi(y) \leq \gamma(x), \lambda(x)$). If we never get to act on such a y but continue to honestify we still get $\Psi(\Phi^D, \widehat{\Phi}^D)(y)\uparrow$, and Q is again satisfied (along with P since $\Theta^A(x)\uparrow$).

Anticipating any consideration of how all this is to coexist with our actions on other requirements, we should mention one situation where we do need D to be d-r.e., and not just r.e. Say we have a P', of priority intermediate between that of Q and P, and that we get to enumerate y into D, achieving a suitable new $\Phi^D \upharpoonright \gamma(x)$ which we restrain in order to be able to preserve $\Theta^A(x)\downarrow \neq B(x)$ following $x \searrow B$ while maintaining the Γ-strategy. It may happen at a later stage that we act on some y' through P', resulting in a loss of the new $\Phi^D \upharpoonright \gamma(x)$ (replaced by a new $\widehat{\Phi}^D \upharpoonright \lambda(x)$, presumably). It may not be possible now to rectify Γ by a suitable A-change, as this may conflict with the actions for P' (for instance). We then have no alternative but to extract x from B via an extraction from D. (There is still the possibility of the new $\Phi^D \upharpoonright \gamma(x)$ which permitted $x \searrow B$ via Γ reasserting itself at a later stage, but in defining the corresponding $\gamma(x)$ we will have been able to have regard for higher priority P' to the extent that we can avoid having to re-enumerate x into B by making a suitable $A \upharpoonright \gamma(x)$ change.)

It remains to follow through the consequences of infinitely many occurrences of agitators $y \searrow D$ for (P, Q) and producing no appropriate new strings $\Phi^D \restriction \gamma(x)$. In this case we utilise the fact that following each $y \searrow D$ we get a new $\widehat{\Phi}^D \restriction \lambda(x)$ to satisfy P, Q through the Λ-strategy. In fact, this outcome for (P, Q) gives us a successful Λ-strategy for each (P', Q) with P' ($= P_{k''}$ say) of lower priority than P, so we will not assume that the infinite set of y's we act on necessarily relates to P'.

We now assume that the Λ-strategy has its own set of followers $z \geq 0$ each with its own set of agitators $\widehat{y} \geq 0$, disjoint from any other set of followers or agitators. We act on each $\widehat{y}$ with the pre-knowledge that we get infinitely many $A \restriction \gamma(x)$ changes through capricious destruction, and infinitely many usable $\widehat{\Phi}^D \restriction \lambda(y)$ changes (or, more relevant, no usable $\Phi^D \restriction \gamma(y)$ changes). This means we only bother to act in the interests of $B(z) \neq \Theta^A_{k''}(z)$ if $\theta_{k''}(z) < \gamma(x)$. Since $\gamma(x)$ goes to infinity, this will still provide sufficient space in which to satisfy $P_{k''}$.

In order to use an agitator $\widehat{y}$ we also need to obtain $\psi(\widehat{y}) \leq \psi(y)$ and $\psi(\widehat{y}) \leq \lambda(z)$, and to then act simultaneously to obtain $\widehat{y} \searrow D$ and $y \searrow D$ in the interests of obtaining a usable $\widehat{\Phi}^D \restriction \lambda(z)$ change to permit $z \searrow B$ via Λ without the need to injure $\Theta^A_{k''}(z) \neq D(z)$ with an $A \restriction \lambda(z)$ change. This requires its own honestification, which we can time to coincide with the honestification for (P, Q). Again, the honestification takes the stronger form described previously.

The strategies for the different requirements can be harmonised via a tree of outcomes, in a (by now) fairly standard $0'''$-priority context.

We now give a more formal description of the strategies for (P, Q).

The basic module for P confronted with one higher priority Q.

(All statements in the description below are assumed to relate to stage $s+1$ of the construction.)

Let

$$\begin{aligned} \ell(D, \Psi(\Phi^D, \widehat{\Phi}^D)) &= \mu z\,[D(z) \neq \Psi(\Phi^D, \widehat{\Phi}^D; z)] \text{ and} \\ \ell(B, \Theta^A) &= \mu z\,[B(z) \neq \Theta^A(z)] \qquad \text{(at stage } s+1). \end{aligned}$$

We associate with (P, Q) four disjoint infinite recursive sets $\xi, \eta, \widehat{\eta}$ and ζ.

We have two overall constraints on the construction relative to (P, Q):

(a) If $z \leq s$ then we must define $A(z) = \Omega^D(z)$ at all stages $s'+1 > s$.

(b) If $\ell(D, \psi(\Phi^D, \widehat{\Phi}^D)) > z$ then we must define $\Gamma^{\Phi^D, A} \restriction x$ or $\Lambda^{\widehat{\Phi}^D, A} \restriction x$ whenever z is a D-agitator for x with $z \notin D$.

And if y, z are D-, $\widehat{D}$-agitators respectively for x with $y, z \notin D$ (at stage $s+1$) we ask that whenever we redefine $\Gamma^{\Phi^D, A} \restriction x$ or $\Lambda^{\widehat{\Phi}^D, A} \restriction x$ we choose $\gamma(x), \lambda(x)$ so that $\gamma(x) \geq \psi(y)$ or $\lambda(x) \geq \psi(z)$ respectively. Also, whenever we redefine $\Gamma^{\Phi^D, A}(w)$ or $\Lambda^{\widehat{\Phi}^D, A}(w)$, $w \geq 0$, we define $\Gamma^{\Phi^D, A}(w) = B(w)$ or $\Lambda^{\widehat{\Phi}^D, A}(w) = B(w)$, respectively.

Whenever we redefine values of $\Gamma^{\Phi^D,A}$ or of $\Lambda^{\widehat{\Phi}^D,A}$ in such a way that $\Gamma^{\Phi^D,A} \simeq B$ (that is, they agree on all values on which both are defined) or $\Lambda^{\widehat{\Phi}^D,A} \simeq B$, we say that we *rectify* Γ or Λ, respectively.

The basic module consists of the following phases together with the above overall constraints.

1) We *select* the least $x \in \xi - B$ to *follow* (P, Q).
2) We *select* the least $y \in \eta - D$, $y > x$, as a *D-agitator* for x.
3) We *select* the least $w \in \zeta - A$ as an *A-agitator* for x.
4) We *wait* for $\ell(D, \Psi(\Phi^D, \widehat{\Phi}^D))$ to grow bigger than y and *define* $\gamma(x) \geq w$.
5) And we *wait* for $\ell(B, \Theta^A)$ to grow bigger than x.
6) We then *check* if $\gamma(x)\downarrow > \theta(x)$.
   If so, we *enumerate* x into B, *restrain* $A \upharpoonright \theta(x)$ and *rectify* Γ with an $A \upharpoonright \gamma(x)$ change.
   Outcome: P is *satisfied* and ceases to interfere with Q.
7) (Honestification and capricious destruction combined.)
   Otherwise we *change* $A \upharpoonright \gamma(x)$ using $w \searrow A$ (and a corresponding $D \upharpoonright \omega(w)$ change), and *proceed* through phases 3), 4), 5) and then 8).
8) We now *check* if $\gamma(x) \geq \psi(y)$ and if $\Phi^D \upharpoonright \psi(y)$, $\widehat{\Phi}^D \upharpoonright \psi(y)$ are unchanged at all stages since they became redefined following the latest application of phase 7).
   (a) If so, we *enumerate* y into D, and *go to* 9).
   (b) If not, *return* to 3).
9) We *wait* for $\ell(D, \Psi^{\Phi^D, \widehat{\Phi}^D}) > y$, and then:
10) *Check* if $\Gamma^{\Phi^D,A}(x)\uparrow$.
    (a) If so we *define* $\Gamma^{\Phi^D,A}(x) = B(x) \neq \Theta^A(x)$, *restrain* $\Phi^D \upharpoonright \gamma(x)$ and $A \upharpoonright \theta(x)$.
    Outcome: P is *satisfied*, and Γ is *rectified.*
    (b) Otherwise we *return* to 2).

In the case of infinitely many returns to 2) on behalf of (P, Q), we need to describe the Λ-strategy. This is an auxiliary strategy that synchronises its activities with phases 2), 7) and 8) of the Γ-strategy. As mentioned before, it can relate to (P', Q) even if $P' \neq P$.

$\widehat{1}$) We *select* the least $x' \in \xi' - B$ to *follow* (P', Q) with $x' > x$.
$\widehat{2}$) (Simultaneous with 2).) We *select* the least $z \in \widehat{\eta} - D$, $z < y$, as a $\widehat{D}$*-agitator* for x' (if such a z exists).
$\widehat{8}$) (Simultaneous with 8).) We also *check* if $\lambda(x') \geq \psi(z)$.
   (a) If so and y has entered D, *enumerate* z into D, and *go to* $\widehat{9}$).
   (b) Otherwise we *return* to 3) as already described.
$\widehat{9}$) We *wait* for $\ell(D, \Psi^{\Phi^D, \widehat{\Phi}^D}) > z$, and then:
$\widehat{10}$) *Check* if $\gamma(x) > \theta'(x')$ and $\Lambda^{\widehat{\Phi}^D,A}(x')\uparrow$.
   (a) If so we *define* $\Lambda^{\widehat{\Phi}^D,A}(x') = B(x') \neq \Theta'^A(x')$, and *restrain* $\widehat{\Phi}^D \upharpoonright \lambda(x')$ and $A \upharpoonright \theta'(x')$.
   Outcome: P' is *satisfied* and Λ is *rectified.*

(b) Otherwise, we *return* to $\widehat{2}$).

When we consider more than two requirements the sequence of events (for instance the timing of the A-restraints) will need modifying, but the basic framework still holds.

Summary of outcomes of the Γ- *and* Λ-*strategies for* (P,Q), (P',Q).

The finite outcomes:

$\boxed{w_1}$: The strategy halts at 4). Then $D \neq \Psi(\Phi^D, \widehat{\Phi}^D)$ and Q is satisfied and ceases to interfere with P.

$\boxed{w_2}$: The strategy halts at 5). Then $B \neq \Theta^A$ and P is satisfied and ceases to interfere with Q.

$\boxed{s_1}$: $B(x) \neq \Theta^A(x)$, P is satisfied and ceases to interfere with Q, due to phase 6) applying.

$\boxed{w_1'}$: The strategy halts at 9). Outcome as for $\boxed{w_1}$.

$\boxed{s_2}$: Strategy halts at 10). $B(x) \neq \Theta^A(x)$, P is satisfied while maintaining $\Gamma^{\Phi^D,A} = D$ via a Φ^D-change.

$\boxed{\widehat{w}_1}$: The strategy halts at $\widehat{9}$). Outcome as for $\boxed{w_1}$ and $\boxed{w_1'}$.

$\boxed{\widehat{s}_2}$: Strategy halts at $\widehat{10}$). $B(x') \neq \Theta'^A(x')$, P' is satisfied while maintaining $\Lambda^{\widehat{\Phi}^D,A} = D$ via a $\widehat{\Phi}^D$-change.

The infinitary outcomes:

$\boxed{i_1}$: The stategy passes through phase 7) (but not 8)(a)) infinitely often.

Since we infinitely often pass through phases 3) and 4), $\gamma(x)$ goes to infinity. Since we never halt at 6), $\theta(x) \geq \gamma(x)$ infinitely often, so $\Theta^A(x)\uparrow$ and P is satisfied.

Since we go through 8)(b) infinitely often, either $\psi(y) > \gamma(x)$ infinitely often, or $\Phi^D \restriction \psi(y)$ or $\widehat{\Phi} \restriction \psi(y)$ changes infinitely often, so in either case $\Psi^{\Phi^D,\widehat{\Phi}^D}(y)\uparrow$ (possibly with $\psi(y)$ bounded but $\Phi^D(u)$ or $\widehat{\Phi}^D(u)\uparrow$, some $u \leq \psi(y)$), giving Q also satisfied.

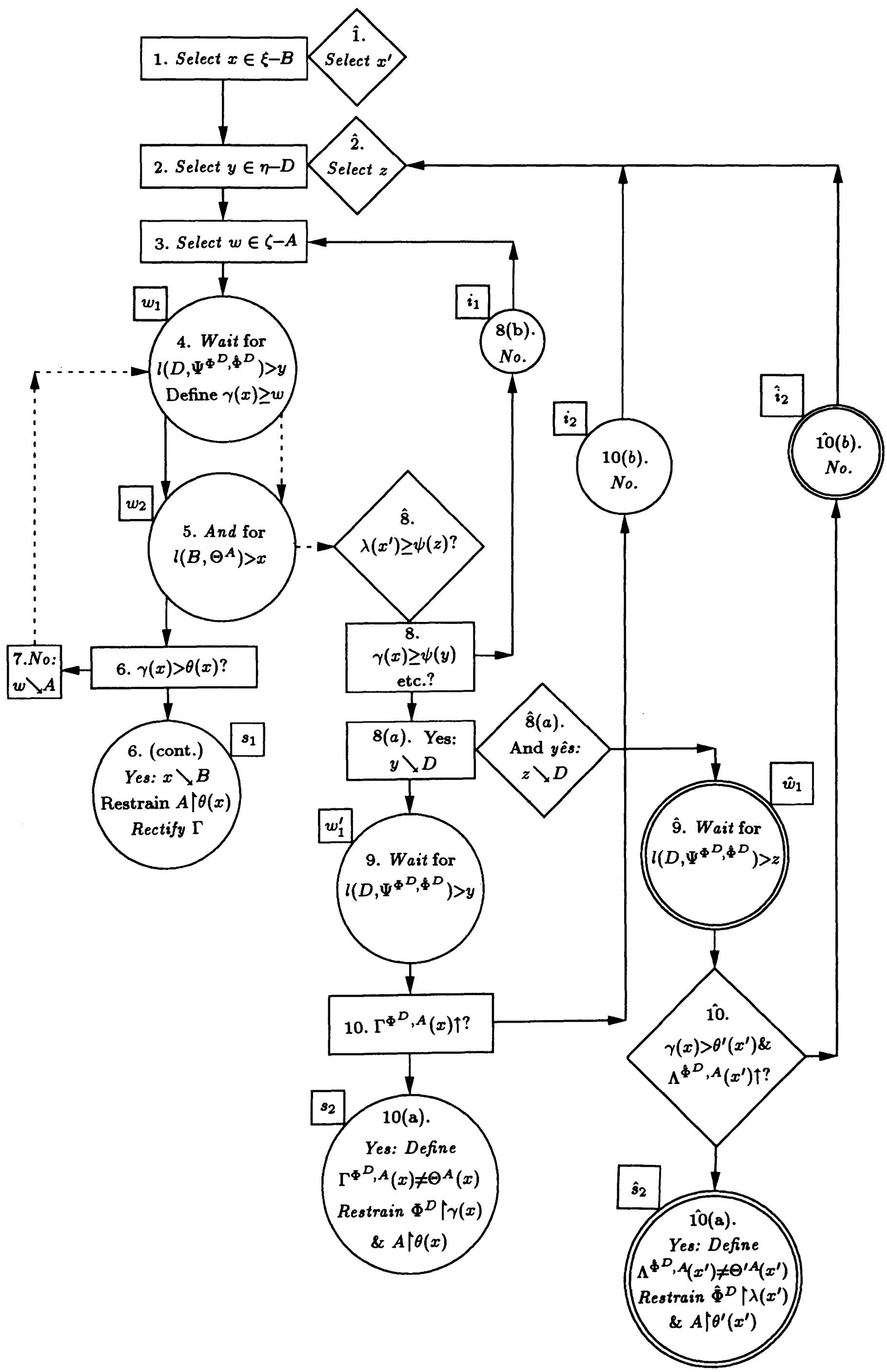
$\boxed{i_2}$: The strategy passes through phase 10)(b) infinitely often. *Outcome:* We implement the Λ-strategy, P' is satisfied as in $\boxed{i_1}$.

$\boxed{\widehat{i}_2}$: The strategy passes through $\widehat{10}$)(b) infinitely often.

As for $\boxed{i_1}$ we get P' satisfied through $\Theta'^A(x')\uparrow$, while the Λ-strategy for (P',Q) is maintained. This is because, by the conditions of 8)(a) and $\widehat{8}$)(a) we must arrive at $\widehat{10}$)(a) with either $\Lambda^{\widehat{\Phi}^D,A}(x')\uparrow$ or $\Gamma^{\Phi^D,A}(x)\uparrow$. Since 10)(a) does not apply, we must have $\Lambda^{\widehat{\Phi}^D,A}(x')\uparrow$, so we get to rectify Λ before returning to 2)/ $\widehat{2}$).

The diagram shown relates the strategies to the outcomes.

The following diagram relates the strategies to the outcomes:



§9. Definability of the recursively enumerable degrees.

Slaman and Woodin showed (see Theorem 5 above) that the r.e. degrees are definable in $\boldsymbol{\mathcal{D}}(\leq \mathbf{0}')$ using a finite number of parameters. The proof used the powerful Slaman–Woodin coding technique [SW86] to define the two sets of low degrees from which Welch [We81] showed the set of all r.e. degrees to be definable below $\mathbf{0}'$. Specifically, by extending the Sacks low splitting theorem, Welch obtained the set of r.e. degrees $\mathbf{a}$ as joins $\mathbf{a} = \mathbf{x} \cup \mathbf{y}$ of r.e. degrees $\mathbf{x}, \mathbf{y}$, $\mathbf{x} \leq \mathbf{c}$ and $\mathbf{y} \leq \mathbf{d}$, $\mathbf{c}, \mathbf{d}$ fixed low r.e. degrees joining to $\mathbf{0}'$. Slaman and Woodin showed that the set of r.e. degrees below $\mathbf{c}$, say, is definable from appropriate parameters $\mathbf{a}, \mathbf{b}$ as the set of minimal solutions $\mathbf{x}$ of $\mathbf{x} \neq (\mathbf{x} \cup \mathbf{a}) \cap (\mathbf{x} \cup \mathbf{b})$.

We now describe how further development of the construction for the main theorem above leads to a positive answer to the question asked at the end of [SW86]: *Is the set of recursively enumerable degrees definable without parameters in* $\boldsymbol{\mathcal{D}}(\leq \mathbf{0}')$?

From:

THEOREM 16. *If* $\mathbf{d} < \mathbf{0}'$ *is not r.e., there exist degrees* $\mathbf{a}, \mathbf{b} < \mathbf{0}'$ *such that* $\mathbf{a} \cup \mathbf{d}$ *is unsplittable over* $\mathbf{a}$ *avoiding* $\mathbf{b}$.

we immediately get:

THEOREM 5+ (DEFINABILITY OF THE RECURSIVELY ENUMERABLE DEGREES). *The set of recursively enumerable degrees is definable in* $\boldsymbol{\mathcal{D}}(\leq \mathbf{0}')$, *and hence in* $\boldsymbol{\mathcal{D}}$.

Proof. If $\mathbf{d} \leq \mathbf{0}'$,

$$\mathbf{d} \text{ is r.e.} \Leftrightarrow (\forall \mathbf{a}, \mathbf{b} \leq \mathbf{0}')[\mathbf{a} \cup \mathbf{d} \text{ is not unsplittable over } \mathbf{a} \text{ avoiding } \mathbf{b}].$$

Since $\mathbf{0}'$ is definable in $\boldsymbol{\mathcal{D}}$ the result follows. □

Rogers (p. 261 of [Ro67]) asked whether the relation *recursively enumerable in* is order-theoretic.

THEOREM 17. *The relation "*$\mathbf{d}$ *is* $\mathbf{b}$*-REA" is definable in* $\boldsymbol{\mathcal{D}}$.

Proof. $\mathbf{d}$ is $\mathbf{b}$-REA $\Leftrightarrow \mathbf{d} \in [\mathbf{b}, \mathbf{b}'] \,\&$

$$[\forall \mathbf{a}, \mathbf{c} \in [\mathbf{b}, \mathbf{b}'])[\mathbf{a} \cup \mathbf{d} \text{ is not unsplittable over } \mathbf{a} \text{ avoiding } \mathbf{c}].$$

Since the jump is definable in the degrees, the result follows.[2] □

Sketch proof of Theorem 16. Let $\mathbf{d} < \mathbf{0}'$ be not r.e., where $D \in \mathbf{d}$. We construct a set $A \in \Delta_2$ satisfying the requirements:

$$P_k : \quad B \neq \Theta_k^A \vee (\exists A^* \text{ co-r.e.})(A^* \equiv_T D),$$

$$Q_k : \quad D = \Psi_k(\Phi_k^{A,D}, \widehat{\Phi}_k^{A,D}) \Rightarrow B = \Gamma_k^{\Phi_k^{A,D}, A} \vee B = \Lambda_k^{\widehat{\Phi}_k^{A,D}, A},$$

[2]Following on from a remark of C. G. Jockusch (August 1991), we note that it is possible to extend the characterisation of "REA" to one of "r.e. in." But to do this one needs the extra information provided by replacing the standard relativisation to an upper-cone of the statement of Theorem 16 with an appropriate relativisation of the proof of the theorem.

$k \geq 0$, where $(\Theta_k, \Psi_k, \Phi_k, \widehat{\Phi}_k)$ is a standard list of all quadruples of p.r. functionals and Γ_k, Λ_k are partial recursive functionals to be constructed. As before, we get $B \leq_T A \oplus D$ from the satisfaction of the Q-requirements.

Roughly speaking, our strategy is as follows. We consider just two requirements $P = P_{k'}$ and $Q = Q_k$ in relation to each other, Q being of higher priority than P.

If (dropping reference to k and k' again) $\ell(D, \Psi(\Phi^{A,D}, \widehat{\Phi}^{A,D}))$ grows large, we first attempt to implement the Γ-strategy for making $B \leq_T \Phi^{A,D} \oplus A$. If carried through without help from $\Phi^{A,D}$ it may turn out that this leads to $B \leq_T A$. Our indication that this is happening will be some lower priority P producing a witness $\Theta^A = B$. We will use $\ell(B, \Theta^A)$ to monitor the extent to which this is happening. If $\ell(B, \Theta^A)$ grows large, we will try to prevent this initial segment of agreement from being disturbed by the enumeration of numbers into A. This happens when we have to enumerate x-traces into A on behalf of Γ (or Λ) following $x \searrow B$, some x's.

To avoid this, we select some x_0, and periodically try to use enumerations into A to move $\gamma(x_0)$ beyond the use of the current $\ell(B, \Theta^A)$. This process (of capricious destruction) may be successful in freeing larger and larger segments of agreement from the possibility of injury through some $y \searrow A$, but at the expense of the destruction of the Γ-strategy for Q through $\Gamma^{\Phi^{A,D}, A}(x_0)\uparrow$. These initial segments may still be injured by extractions from A, and this is unavoidable, but the reason why the extractions are unavoidable is that they are linked to changes in an identifiable set of members of D through an appropriate $B = \Gamma_{k^*}^{A,D}$ or $\Lambda_{k^*}^{A,D}$. At the same time as $\ell(B, \Theta^A)$ grows large we code D into B up to $\ell(B, \Theta^A)$. As a result, if we find $B = \Theta^A$ then we will be able to modify A to a co-r.e. A^* which is $\equiv_T D$ due to $D \leq_T B = \Theta^A$ on the one hand, and due to the linking of extractions from A to D on the other. But then this outcome in which $\Gamma^{\Phi^{A,D}, A}(x_0)\uparrow$ must be a pseudo-outcome. We are left with a result in which not only do we get the probability of $\Gamma^{\Phi^{A,D}, A}(x_0)\uparrow$ but we also get $\Theta^A(z)\uparrow$ for some $z \geq x_0$, so satisfying P.

There is still a way of averting this outcome, the failure of which will provide a suitable replacement for the lost Γ-strategy. This consists in using suitable $\Phi^{A,D}$ changes to release us from the commitment to enumerate traces into A in order to A-permit B-changes via Γ. Since we are only concerned about Q if $\Phi^{A,D}$ is total, we may delay A-permissions via Γ for $B(w)$ changes (say) until at least $\Phi^{A,D} \restriction \gamma(w)$ has become redefined. This leaves open the possibility that we may get a completely new $\Phi^{A,D} \restriction \gamma(w)$ (that is, not containing as an initial segment any previously defined $\Phi^{A,D} \restriction \gamma(w)$) which can be used to permit the $B(w)$ change via Γ without the need for A-changes to be made.

Moreover, we may get $\Phi^{A,D} \restriction \gamma(y)$ to be new for other numbers y for which $B(y)$ changes are possible in the future. This will enable us to replace a trace for y which may need enumerating into A when we get a $B(y)$ change, with a trace (which can be chosen initially as large as we like) which need only be extracted from A to A-permit a $B(y)$ change via Γ. This indicates the possibility of ameliorating the effects of capricious destruction and satisfying P by exhibiting an A^* satisfying P while salvaging the Γ-strategy if we can obtain enough of the above $\Phi^{A,D} \upharpoonright \gamma(y)$ changes, $y \geq 0$.

We can assist this outcome by using A to increase the likelihood of our new $\Phi^{A,D} \upharpoonright \gamma(w)$ being usable to permit the $B(w)$ change via Γ, and, more important, enabling us to choose new traces for numbers y consistent with the A^*-strategy for P.

The main ingredient here is the process of 'honestification' whereby we use A changes to try to ensure that whenever we define $\gamma(w)$, some $w \geq x_0$, previous to a $D \upharpoonright w'$ change leading to a $B \upharpoonright w$ change (where we can assume $w \geq w'$) we have $\psi(w) \downarrow \leq \gamma(w), \lambda(w)$, so that the $D \upharpoonright w'$ change must be accompanied by some sort of change in either $\Phi^{A,D} \upharpoonright \gamma(w)$ or $\widehat{\Phi}^{A,D} \upharpoonright \lambda(w)$. In fact, we try to ensure via our A-changes for capricious destruction that for each y which is not yet accompanied by a 'positive trace' (that is, one $\in A$), we have $\gamma(y), \lambda(y) \geq \psi(w)$ for *all* $w < \ell(D, \Psi(\Phi^{A,D}, \widehat{\Phi}^{A,D}))$. Then if such a w occurs, its accompanying $\Phi^{A,D} \upharpoonright \gamma(w)$ or $\widehat{\Phi}^{A,D} \upharpoonright \lambda(w)$ change is usable by y for positive trace selection. The honestification has to be stronger than in the monster construction, as we are not just dealing with r.e. sets. If the honestification is attempted via an $A \upharpoonright u$ change, say, for the number y, then we ask that $u \leq \min\{\gamma(y), \lambda(y)\}$ for all such $\gamma(y), \lambda(y)$ defined since the last occurrence of honestification, thereby ensuring that previously defined $\Phi^{A,D} \upharpoonright \gamma(y)$ or $\widehat{\Phi}^{A,D} \upharpoonright \lambda(y)$ will not return in tandem with corresponding $A \upharpoonright \gamma(y)$ or $A \upharpoonright \lambda(y)$ following a $B \upharpoonright w$ change, $w < \ell(D, \Psi(\Phi^{A,D}, \widehat{\Phi}^{A,D}))$, to defeat the possibility of a positive trace selection for y.

There is a problem here (apart from that of not knowing whether we get a $\Phi^{A,D}$- or $\widehat{\Phi}^{A,D}$-change following the $A \upharpoonright u$ change, and of knowing whether we keep such a change), in that $B \upharpoonright w$ changes may occur while honestification is in progress, demanding renewed honestification. But this repetition can only develope into an infinite outcome, giving $\Gamma^{\Phi^{A,D},A}$, $\Lambda^{\widehat{\Phi}^{A,D},A}$ not total, if we have some w with $\min\{\gamma(y), \lambda(y)\} < \psi(w)$ infinitely often. In this case Q is satisfied since $\Psi(\Phi^{A,D}, \widehat{\Phi}^{A,D})(w) \uparrow$. And we still have $\theta(y) \uparrow$ infinitely often, so P is satisfied through $\Theta^A(y)\uparrow$.

However, honestification as described will still not be sufficient to supply the ideal conditions for positive trace definition for y. This is because of the possibility of returns to strings $\Phi^{A,D} \upharpoonright \gamma(y)$ (following a $B \upharpoonright w$ change) which appeared since the last occurrence of honestification for (P, Q). So before remitting honestification on behalf of y we further require that $\Phi^{A,D} \upharpoonright \psi(w)$, $\widehat{\Phi}^{A,D} \upharpoonright \psi(w)$ are unchanged at all stages since the previous occurrence of honestification, and if this condition is not met, we again honestify (even if $\psi(w) \leq \gamma(y), \lambda(y)$). If we never get a positive trace for y but continue to honestify we still get $\Psi(\Phi^{A,D}, \widehat{\Phi}^{A,D})(w)\uparrow$ for some w (by the non-recursiveness of D) and Q is again satisfied (along with P since $\Theta^A(x_0)\uparrow$).

There is now the possibility of a continuing successful honestification accompanied by infinitely many suitable $D \upharpoonright w'$ changes, but producing only finitely many new strings $\Phi^{A,D} \upharpoonright \gamma(y)$, $y \geq 0$. In this case we utilise the fact that following each suitable $D \restriction w'$ change we get a new $\widehat{\Phi}^{A,D} \restriction \lambda(y)$ to pursue the A^* strategy for P with while satisfying Q via the Λ-strategy. In fact, this outcome for (P, Q) gives a successful Λ-strategy for each $(P_{k''}, Q)$, $k'' > k'$. The Λ-strategy acts with the pre-knowledge that we get infinitely many $A \restriction \gamma(x_0)$ changes, some x_0, through capricious destruction, and infinitely many usable $\widehat{\Phi}^{A,D} \restriction \lambda(y)$ changes. This means we pursue the A^*-strategy for $P_{k''}$ related to Λ below $\gamma(x_0)$. Since $\gamma(x_0)$ goes to infinity, this will still provide sufficient space in which to satisfy $P_{k''}$. The Λ-strategy links up with the already initiated honestification process, and always gets its positive traces needed for the A^*-strategy for $P_{k''}$, without any need for additional A-changes in case the $\widehat{\Phi}^{A,D}$-changes do not live up to their promise (as they always do). There is still the possibility of the A^*-strategy failing through $\Theta^A_{k''}(w)\uparrow$, some w, but this is not the concern of the Λ-strategy, which is not disrupted, unlike the Γ-strategy.

As for the previous construction, one needs a tree of outcomes on which to reconcile the strategies for different pairs (P', Q'), and one of the standard frameworks for discussing $0'''$-injury priority arguments. It is worth mentioning here some of the special complications arising from the fact that D is not r.e. (or perhaps even d-r.e.) but Δ_2. An inevitable consequence is a certain amount of 'Δ_2-noise' in the construction, and the need for one or two nontrivial adjustments. The extra unpredictability of D- and hence B-changes does not in itself cause too many problems for A. With the help of redefinitions of γ and λ, A can cope with the corresponding demands of the Γ- and Λ-strategies. There are slightly more problems with the consequent lack of control over $\Phi^{A,D}$ and $\widehat{\Phi}^{A,D}$. We relied above, in certain situations, on $\Phi^{A,D}$- or $\widehat{\Phi}^{A,D}$-changes enabling us to avoid certain sorts of A-changes (by positive trace selection) in the interests of the A^*-strategy for the P-requirements. When these changes are in doubt, we will have to fall back on the undesired type of A-changes. However, temporary vacillations in $\Phi^{A,D}$ or $\widehat{\Phi}^{A,D}$ can be matched by a corresponding flexibility in the A-changes, and where ultimate outcomes for $\Phi^{A,D}$- or $\widehat{\Phi}^{A,D}$-changes are assured we will be able to maintain our aims in regard to the A^*-strategies. On the other hand, in reconciling the demands of different P-requirements, the A^*-strategies may demand negative A-changes where the immediate need may seem to be new positive A-changes. The key factor here is, of course, that the success of the A^*-strategy for P lies in producing $B \neq \Theta^A$, not in an infinitary outcome of $D \equiv_T$ a co-r.e. A^*.

See [Co*ta*2] for a more formal description of the basic module for (P, Q) and further discussion of problems in reconciling the strategies.

§10. Questions and further results.

More recently Slaman and Woodin [SW*ta*] (see [Sl91]) have extended their earlier results [SW86] to obtain new proofs (not involving the jump) of a number of global theorems concerning the degrees of unsolvability. In some cases improvements have been obtained. For instance, in a more general context including other common degree structures, they improve Theorem 10+:

THEOREM 10++ (Slaman and Woodin [SW*ta*]). *Let f be an automorphism of $\mathcal{D}$. Then $f(\mathbf{a}) = \mathbf{a}$ for all $\mathbf{a} \geq \mathbf{0}''$.*

Other results are proved making full use of Theorem 17 above. Their most dramatic result is:

THEOREM 18 (Slaman and Woodin [SW*ta*]). *The recursively enumerable degrees form an automorphism base for $\mathcal{D}$.*

This of course reduces the problem of showing that $\mathcal{D}$ is rigid to that of showing $\mathcal{R}$ (the structure of the r.e. degrees) to be rigid. In the other direction, Slaman [Sl91] asks whether every automorphism of the r.e. degrees can be extended to one of $\mathcal{D}(\leq \mathbf{0}')$ or of $\mathcal{D}$. Increasingly, not only does one find the more intractable and technically interesting questions of degree theory at the local level, but therein is seen to lie the key to the main outstanding problems of global degree theory.

We summarise some of the more important remaining open questions.

Homogeneity and automorphisms.

Jockusch [Jo81] has shown that there is a comeager set of degrees which are bases of elementarily equivalent cones of degrees. But:

QUESTION 1. *Do there exist degrees $\mathbf{a}, \mathbf{b}$, $\mathbf{a} \neq \mathbf{b}$, with $\mathcal{D}(\geq \mathbf{a}) \cong \mathcal{D}(\geq \mathbf{b})$?*

QUESTION 2. *How far can Theorems 2+, 7+, 8+ and 10++ be improved? For instance, is any automorphism of $\mathcal{D}$ the identity above $\mathbf{0}'$?*

QUESTION 3. *Do there exist nontrivial automorphisms of $\mathcal{D}(\leq \mathbf{0}')$ or of $\mathcal{R}$?*

Definability.

Shore [Sh88] has shown that all levels of the high/low hierarchy from the three level onwards are definable in $\mathcal{D}(\leq \mathbf{0}')$, and Shore and Slaman [SS*ta*] have succeeded in distinguishing between the high and low degrees within $\mathcal{D}(\leq \mathbf{0}')$.

QUESTION 4. *Are $High_1$, $High_2$, Low_1 or Low_2 definable in $\mathcal{D}(\leq \mathbf{0}')$?*

There are questions concerning the definability of particular degrees. For instance (questions first stated by Slaman and Harrington, respectively, in [Sa85]):

QUESTION 5. *Do there exist r.e. degrees other than $\mathbf{0}$ or $\mathbf{0}'$ definable in $\mathcal{D}(\leq \mathbf{0}')$ or $\mathcal{R}$? Are there any r.e. degrees not definable in $\mathcal{D}(\leq \mathbf{0}')$ or $\mathcal{R}$?*

Two particularly interesting questions concerning the definability of classes of degrees are:

QUESTION 6. *Define the class of n-r.e. degrees in $\mathcal{D}(\leq \mathbf{0}')$ for $n \geq 2$.*

QUESTION 7. *For which r.e. degrees $\mathbf{a} > \mathbf{0}$ can one define the r.e. degrees below $\mathbf{a}$ in $\mathcal{D}(\leq \mathbf{a})$?*

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University of Leeds
Leeds LS2 9JT
England