§11. The Construction

At last we are in a position to construct our extender sequence $\vec{E}$. We will construct the sequence $\vec{E}$ inside of V_θ where θ is least such that $L(V_\theta)$ satisfies that θ is Woodin. Note that every bounded subset of θ in $L(\vec{E})$ is in $L_\theta[\vec{E}]$ since θ is inaccessible.

The construction of $\vec{E}$ will differ from that for sequences of measures in that we do not simply define E_α by induction on α. The reason is that we want the construction to provide each E_α with an ancestry tracing back (by inverting certain collapses) to an extender on V having a certain amount of strength. The illustrious ancestry of the extenders which lie on $\vec{E}$ guarantees that all levels of $L[\vec{E}]$ are ω-iterable.

Let us call a premouse $\mathcal{M}$ *reliable* iff for all $k \leq \omega$, $\mathfrak{C}_k(\mathcal{M})$ exists and is k-iterable. We shall simply assume in this section that the premice we produce in our construction are reliable, and discharge our obligation to show this in §12.

We now define by induction on ξ a reliable coremouse $\mathcal{M}_\xi$. Simultaneously, we verify an induction hypothesis A_ξ describing the agreement between $\mathcal{M}_\xi$ and the $\mathcal{M}_\alpha$ for $\alpha < \xi$:

(A_ξ) $\mathcal{J}_\eta^{\mathcal{M}_\alpha} = \mathcal{J}_\eta^{\mathcal{M}_\xi}$ for all $\alpha < \xi$ and $\kappa \leq \inf\{\rho_\omega(\mathcal{M}_\nu) : \alpha < \nu \leq \xi\}$, where $\eta = (\kappa^+)^{\mathcal{M}_\alpha}$.

In the formulation of A_ξ, we understand that $\omega\eta = \mathrm{OR}^{\mathcal{M}_\alpha}$ in the case that $\mathcal{M}_\alpha \models \kappa^+$ doesn't exist.

We begin by setting $\mathcal{M}_0 = (V_\omega, \in, \varnothing)$. Now suppose that $\mathcal{M}_\xi$ is given and that A_ξ holds. We define $\mathcal{M}_{\xi+1}$ and verify $A_{\xi+1}$.

Case 1. $\mathcal{M}_\xi = (J_\alpha^{\vec{E}}, \varepsilon, \vec{E})$ is a passive premouse, and there are an extender F^* over V, an extender F over $\mathcal{M}_\xi$, and an ordinal $\nu < \alpha$ such that

$$V_{\nu+\omega} \subseteq \mathrm{Ult}(V, F^*)$$

and

$$F \restriction \nu = F^* \cap ([\nu]^{<\omega} \times J_\alpha^{\vec{E}})$$

and

$$\mathcal{N}_{\xi+1} = (J_\alpha^{\vec{E}}, \in, \vec{E}, \tilde{F})$$

is a 1-small, reliable premouse, with $\nu = \nu^{\mathcal{N}_{\xi+1}}$.

In this case we choose F^*, F, ν, and $\mathcal{N}_{\xi+1}$ as above with ν, the natural length of F, minimal among all such F^*. Let

$$\mathcal{M}_{\xi+1} = \mathfrak{C}_\omega(\mathcal{N}_{\xi+1}).$$

Case 2. Otherwise.

In this case, let $\omega\alpha = \mathrm{OR}^{\mathcal{M}_\xi}$, and set

$$\mathcal{N}_{\xi+1} = \left(J_{\alpha+1}^{\dot{E}^{\mathcal{M}_\xi} \frown \dot{F}^{\mathcal{M}_\xi}}, \in, \dot{E}^{\mathcal{M}_\xi} \frown \dot{F}^{\mathcal{M}_\xi}\right).$$

(Of course, $\dot{F}^{\mathcal{M}_\xi} = \varnothing$ is possible.) Thus $\mathcal{N}_{\xi+1}$ is a passive premouse. If $\mathcal{N}_{\xi+1}$ is not reliable, stop the construction. Otherwise,

$$\mathcal{M}_{\xi+1} = \mathfrak{C}_\omega(\mathcal{N}_{\xi+1}).$$

We must verify $A_{\xi+1}$. Now Theorem 8.1 tells us that $\mathcal{N}_{\xi+1}$ agrees with $\mathcal{M}_{\xi+1} = \mathfrak{C}_\omega(\mathcal{N}_{\xi+1})$ below $(\rho_\omega^+)^{\mathcal{N}_{\xi+1}} = (\rho_\omega^+)^{\mathcal{M}_{\xi+1}}$. The obvious agreement between $\mathcal{M}_\xi$ and $\mathcal{N}_{\xi+1}$, together with our induction hypothesis A_ξ, easily gives $A_{\xi+1}$.

Now suppose λ is a limit ordinal. Let

$$\eta = \liminf_{\xi \to \lambda} \, (\rho_\omega^+)^{\mathcal{M}_\xi}$$

(where again we set $(\rho_\omega^+)^{\mathcal{M}_\xi}$ = unique α s.t. $\omega\alpha = \mathrm{OR}^{\mathcal{M}_\xi}$ in case $\mathcal{M}_\xi \models \rho_\omega^{\mathcal{M}_\xi}$ has no successor cardinal, or $\rho_\omega^{\mathcal{M}_\xi} = \mathrm{OR}^{\mathcal{M}_\xi}$.) Then we let $\mathcal{N}_\lambda$ be the passive premouse $\mathcal{P} = \mathcal{J}_\eta^{\mathcal{P}}$, where for all $\beta < \eta$ we set $\mathcal{J}_\beta^{\mathcal{P}}$ equal to the eventual value of $\mathcal{J}_\beta^{\mathcal{M}_\xi}$ as $\xi \to \lambda$.

$\mathcal{N}_\lambda$ exists since A_ξ holds for all $\xi < \lambda$. Now suppose $\mathcal{N}_\lambda$ is reliable; if not we stop the construction. Set

$$\mathcal{M}_\lambda = \mathfrak{C}_\omega(\mathcal{N}_\lambda).$$

It is easy, using 8.1 and the induction hypothesis, to verify A_λ.

This completes the inductive definition of the $\mathcal{M}_\xi$'s. For the moment, let us assume:

Lemma 11.1. *The construction above never stops; $\mathcal{M}_\xi$ is defined for all ordinals ξ.*

Promise of Proof. We have to show $\mathcal{N}_\xi$ is reliable for all ξ. We will prove as theorem 12.1 that $\mathfrak{C}_k(\mathcal{N}_\xi)$ is k-iterable, for all $k \leq \omega$, provided that $\mathfrak{C}_k(\mathcal{N}_\xi)$ exists. Given this it follows from theorem 8.1 that $\mathcal{N}_\xi$ is reliable. □

Lemma 11.2. *Suppose α_0 and ξ are ordinals such that $\alpha_0 < \xi$ and $\kappa = \rho_\omega^{\mathcal{M}_\xi} \leq \rho_\omega^{\mathcal{M}_\alpha}$ for all $\alpha \geq \alpha_0$. Then $\mathcal{M}_\xi$ is an initial segment of $\mathcal{M}_\eta$, for all $\eta \geq \xi$. Moreover, $\mathcal{M}_{\xi+1} \models$ every set has cardinality at most κ.*

Proof. We may assume $\kappa < \mathrm{OR}^{\mathcal{M}_\xi}$. We claim $\mathcal{M}_{\xi+1}$ is defined by Case 2. For suppose not; let $\mathcal{M}_\xi = (J_\alpha^{\vec{E}}, \in, \vec{E})$ and let F be as in Case 1. Then $\mathcal{M}_\xi$ is a proper initial segment of $\mathrm{Ult}_0(\mathcal{M}_\xi, F)$, and $\mathrm{Ult}_0(\mathcal{M}_\xi, F) \models \alpha$ is a cardinal. As there is a map from κ onto α which is $\Sigma_n^{\mathcal{M}_\xi}$ for some n, we have a contradiction.

Let $\mathcal{M}_\xi = (J_\alpha^{\vec{E}}, \in, \vec{E}, \tilde{F})$, where $F = \varnothing$ if $\mathcal{M}_\xi$ is passive. Let $\mathcal{N}_{\xi+1} = (J_{\alpha+1}^{\vec{E}^\frown F}, \in, \vec{E}^\frown F)$ be as in Case 2. Then the $\Sigma_n^{\mathcal{M}_\xi}$ map from κ onto α guarantees $\mathcal{N}_{\xi+1} \models$ every set has card $= \kappa$, and $\rho_1^{\mathcal{N}_{\xi+1}} \leq \kappa$. Thus $\rho_\omega^{\mathcal{N}_{\xi+1}} = \kappa$. Theorem 8.1 implies that $\mathfrak{C}_\omega(\mathcal{N}_{\xi+1}) = \mathcal{N}_{\xi+1}$. Thus $\mathcal{M}_{\xi+1} = \mathcal{N}_{\xi+1}$, and the claim holds for $\eta = \xi + 1$. For $\eta > \xi + 1$, the claim follows easily from the induction hypothesis A_η. □

The claim implies that $\liminf_{\xi \to \mathrm{OR}} \rho_\omega^{\mathcal{M}_\xi} = \mathrm{OR}$. So we can define our desired $\vec{E}$ by

$$\mathcal{J}_\beta^{\vec{E}} = \text{eventual value of } \mathcal{J}_\beta^{\mathcal{M}_\xi}, \text{ all sufficiently large } \xi \in \mathrm{OR}.$$

Clearly this determines $\vec{E}$, and we have that every level $\mathcal{J}_\beta^{\vec{E}}$ of $L[\vec{E}]$ is an ω-sound, ω-iterable 1-small mouse.

We can think of the construction as producing, in increasing order, the cardinals of $L[\vec{E}]$ together with the levels of $L[\vec{E}]$ whose ωth projectum is a cardinal of $L[\vec{E}]$. Namely, let

$$\kappa_0 = \omega, \quad \xi_0 = 1,$$

and now suppose we have κ_γ and ξ_γ for $\gamma < \alpha$. Set

$$\kappa_\alpha = \inf \{\rho_\omega^{\mathcal{M}_\beta} \mid \beta \geq \sup \{\xi_\gamma \mid \gamma < \alpha\}\}$$

and

$$\xi_\alpha = \text{least } \beta \geq \sup \{\xi_\gamma \mid \gamma < \alpha\} \text{ such that } \rho_\omega^{\mathcal{M}_\beta} = \kappa_\alpha .$$

One can check easily that $\langle \kappa_\alpha \mid \alpha \in \mathrm{OR} \rangle$ enumerates in non-decreasing order the cardinals of $L[\vec{E}]$, that $\rho_\omega^{\mathcal{M}_{\xi_\alpha}} = \kappa_\alpha$, and that $\mathcal{M}_{\xi_\alpha}$ is a level of the eventual $L[\vec{E}]$. In fact, for κ a cardinal of $L[\vec{E}]$, the $\mathcal{M}_{\xi_\alpha}$ for $\kappa_\alpha = \kappa$ are precisely those levels $\mathcal{J}_\beta^{\vec{E}}$ of $L[\vec{E}]$ whose ωth projectum is κ.

We now show that $L[\vec{E}] \models$ there is a Woodin cardinal. Once again, certain iterability assumptions will crop up during the proof. We shall verify these assumptions in §12.

Theorem 11.3. *Suppose there is a Woodin cardinal. Let $\vec{E}$ be the extender sequence constructed above. Then $L[\vec{E}] \models$ there is a Woodin cardinal.*

PROOF. Let θ be least such that $L(V_\theta) \models$ "θ is Woodin." We show that θ is Woodin in $L[\vec{E}]$.

So fix $f : \theta \to \theta$ such that $f \in L[\vec{E}]$. Define $g : \theta \to \theta$ in V by

$$g(\alpha) = \text{2nd strongly inaccessible (of } V) > f(\alpha).$$

(The strong inaccessible is just a security blanket.) As θ is Woodin in V there is an extender F^* over V, $F^* \in V_\theta$, crit $F^* = \kappa$, such that if $j^* : V \to \mathrm{Ult}(V, F^*)$ is the canonical embedding, then $g''\kappa \subseteq \kappa$ and

$$V_{j^*(g)(\kappa)+1} \subseteq \mathrm{Ult}(V, F^*)$$

and

$$\vec{E} \restriction j^*(g)(\kappa) = j^*(\vec{E}) \restriction j^*(g)(\kappa).$$

Let

$$F = F^* \cap ([\mathrm{lh}\, F^*]^{<\omega} \times L[\vec{E}]).$$

Notice that $L[\vec{E}]$ agrees with $\mathrm{Ult}(L[\vec{E}], F)$ below $j^*(g)(\kappa)$, where the ultrapower is computed using functions in $L[\vec{E}]$. That is, F "coheres" with $\vec{E}$ sequence out to $j^*(g)(\kappa)$. Notice $j^*(g)(\kappa)$ is a strongly inaccessible cardinal of V, hence of $L[\vec{E}]$. We now show that for $\rho < j^*(g)(\kappa)$, the trivial completion of $F \restriction \rho$ is on $\vec{E}$, or an ultrapower thereof.

Let $(\kappa^+)^{L[\vec{E}]} \le \rho < j^+(g)(\kappa)$. Let

$$i : L[\vec{E}] \to \mathrm{Ult}(L[\vec{E}], F \restriction \rho)$$

be the canonical embedding, and let

$$\gamma = (\rho^+)^{\mathrm{Ult}(L[\vec{E}], F\restriction\rho)},$$
$$G = \{(a, x) \mid a \in [\gamma]^{<\omega} \wedge x \subseteq [\kappa]^{\mathrm{card}(a)} \wedge x \in L[\vec{E}] \wedge a \in i(x)\}.$$

Thus G is the trivial completion of $F \restriction \rho$. The generators of G are of course just those generators of F which are less than ρ, and $G \restriction \rho = F \restriction \rho$.

Lemma 11.4. *Let $(\kappa^+)^{L[\vec{E}]} \le \rho < j^*(g)(\kappa)$, and suppose that ρ is the natural length of $F \restriction \rho$. Let G be the trivial completion of $F \restriction \rho$, and $\gamma = \mathrm{lh}\, G$. Then $E_\gamma = G = F \restriction \rho$ unless ρ is a limit ordinal greater than $(\kappa^+)^{L[\vec{E}]}$, and is itself a generator of F. In this case*

$$G = \begin{cases} E_\gamma & \text{if } \gamma \notin \mathrm{dom}\, \vec{E} \\ (i^{E_\rho}(\vec{E}))_\gamma & \text{if } \gamma \in \mathrm{dom}\, \vec{E} \end{cases},$$

where $i^{E_\rho} : J^{\vec{E}}_\rho \to \mathrm{Ult}_0(J^{\vec{E}}_\rho, E_\rho)$ is the canonical embedding.

Proof (modulo §12). The proof proceeds by induction on ρ, and is divided into a number of cases. In those cases where ρ is not a cardinal we will apply theorem 10.1, and in the other cases we will be able to use bicephali.

Case A. ρ is a successor. In this case $\rho - 1$ must be a generator of F. Let

$$\sigma \colon \mathrm{Ult}(L[\vec{E}], F \restriction \rho) \to \mathrm{Ult}(L[E], F)$$

be the canonical embedding. From the case hypothesis we see that $\sigma(\rho) = \rho$ and hence $\sigma \restriction \gamma = \mathrm{id}$. Also, $G = F \restriction \gamma$ as $(a, x) \in G \Leftrightarrow a \in i_{F\restriction\rho}(x) \Leftrightarrow a \in \sigma(i_{F\restriction\rho}(x)) \Leftrightarrow a \in i_F(x) \Leftrightarrow (a, x) \in F$, for all $a \in [\gamma]^{<\omega}$ and appropriate x.

We claim there is a stage η of the construction such that

$$\mathcal{M}_\eta = (J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma).$$

For let δ be the largest cardinal of $L[\vec{E}]$ which is $\leq \rho$. (So $\delta < \rho$.) Now $i_{F\restriction\rho}(\vec{E}) \restriction \gamma = i_F(\vec{E}) \restriction \gamma = \vec{E} \restriction \gamma$, and by the definition of γ, we have $J_\gamma^{i_{F\restriction\rho}(\vec{E})} \models$ every set has cardinality $\leq \rho$, so $J_\gamma^{\vec{E}} \models$ every set has cardinality $\leq \rho$. On the other hand, δ is the largest cardinal $\leq \rho$ in $L[\vec{E}]$, hence in $J_{\sigma(\gamma)}^{i_F(\vec{E})} = J_{\sigma(\gamma)}^{\vec{E}}$, (as $\sigma(\gamma)$ is a cardinal of $L[\vec{E}]$) hence in $J_\gamma^{\vec{E}}$. So

$$J_\gamma^{\vec{E}} \models \text{every set has cardinality } \delta.$$

Let $\langle \xi_\alpha \mid \alpha < (\delta^+)^{L[\vec{E}]} \rangle$ enumerate in increasing order those ordinals ξ such that $\rho_\omega(\mathcal{M}_\xi) = \delta$ and $\rho_\omega(\mathcal{M}_\beta) \geq \delta$ for all $\beta \geq \xi$. We observed earlier that the $\mathcal{M}_{\xi_\alpha}$ are precisely those levels of $L[\vec{E}]$ whose ωth projectum is δ. It is clear that γ is a limit of such levels. So letting $\eta = \sup \{\xi_\alpha \mid \mathrm{OR} \cap \mathcal{M}_{\xi_\alpha} < \gamma\}$, we have that η is a limit and $\mathcal{M}_\eta = (J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma)$. This proves our claim.

Now clearly $(J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma, \tilde{F} \restriction \gamma)$ is a type II premouse. It is also 1-small, since otherwise $J_\kappa^{\vec{E}}$ satisfies that some ordinal $\alpha < \kappa$ is Woodin. But then since κ is a cardinal of $L[\vec{E}]$, α is Woodin in $L[\vec{E}]$, and $\alpha < \kappa < \theta$, contrary to our initial assumption that no ordinal $\alpha < \theta$ is Woodin in $L[\vec{E}]$.

Let us assume until §12:

Sublemma 11.4.1. *$(J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma, \tilde{F} \restriction \gamma)$ is reliable.*

It follows that $\mathcal{M}_{\eta+1}$ is defined by Case 1 in our construction. That is, $\mathcal{N}_{\eta+1} = (J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma, H)$ for some H, and $\mathcal{M}_{\eta+1} = \mathfrak{C}_\omega(\mathcal{N}_{\eta+1})$. But $\rho_\omega(\mathcal{N}_{\eta+1}) \geq \delta$ since δ is the largest cardinal of $L[\vec{E}]$ and hence $\mathcal{M}_{\eta+1} = \mathcal{N}_{\eta+1}$ since $J_\gamma^{\vec{E}}$ satisfies that every set has cardinality at most δ. Moreover $\mathcal{M}_{\eta+1}$ is an initial segment of the eventual $L[\vec{E}]$, and $H = E_\gamma$. Thus it is enough to show $E_\gamma = F \restriction \gamma$.

Notice that γ is a generator of F, as otherwise $\sigma(\gamma) = \gamma$, so that γ is a cardinal of $L[\vec{E}]$, contrary to $\gamma \in \mathrm{dom}\,\vec{E}$. Let G' be the trivial completion of $F \restriction \gamma + 1$. Arguing as above, with $\xi = \mathrm{lh}\, G'$, we see that $G' = F \restriction \xi$, and that $\vec{E} \restriction \xi^\frown F \restriction \xi$ satisfies conditions 1-4 of "good at ξ". We now show that $(J_\xi^{\vec{E}}, \in, \vec{E} \restriction \xi, F \restriction \xi)$ is a psuedo-premouse.

Since it is easy to see that δ remains the largest cardinal of $J_\xi^{\vec{E}}$, as γ is not a cardinal of $J_\xi^{\vec{E}}$, we need to verify that the trivial completion of $F \restriction \delta$ is on $\vec{E}$. Now either $\delta = (\kappa^+)^{L[\vec{E}]}$ or δ is a limit of generators of F. [Otherwise, let $\tau < \delta$ be such that $\tau = \bigcup\{\xi < \delta \mid \xi = (\kappa^+)^{L[\vec{E}]}$ or ξ is a generator of $F\}$. By our inductive hypothesis the trivial completion of $F \restriction \tau$ is on $\vec{E}$ - it falls under either (b) or (e) of the lemma. But from $F \restriction \tau$ we easily construct a collapse of δ.] By our inductive hypothesis, as $\delta < \rho$, the trivial completion of $F \restriction \delta$ is on $\vec{E}$. (Note here that clause (d) of the lemma cannot apply as $\delta \notin \operatorname{dom} \vec{E}$ as δ is a cardinal of $L[\vec{E}]$.) That is, if $\beta = (\delta^+)^{\mathrm{Ult}(L[\vec{E}], F\restriction\delta)}$, then E_β is the trivial completion of $F \restriction \delta$. Clearly $\beta \leq \gamma < \xi$. Thus $(J_\xi^{\vec{E}}, \in, \vec{E} \restriction \xi, F \restriction \xi)$ satisfies the initial segment condition on psuedo-premice, as desired.

We now borrow from §12:

Sublemma 11.4.2. $(J_\xi^{\vec{E}}, \in, \vec{E} \restriction \xi, F \restriction \xi)$ *is iterable.*

Granted 11.4.2, Theorem 10.1 tells us that $(J_\xi^{\vec{E}}, \in, \vec{E} \restriction \xi, F \restriction \xi)$ satisfies the full initial segment condition, so that $F \restriction \gamma = E_\gamma$.

Remark. We can't use bicephali here because E_γ might be of type III, while $F \restriction \gamma$ is of type II.

Case B. ρ is a limit of generators of F, but not itself a generator of F.

Let $\sigma : \mathrm{Ult}(L[\vec{E}], F \restriction \rho) \to \mathrm{Ult}(L[\vec{E}], F)$ by the canonical embedding. As ρ is not a generator of F, $\sigma \restriction \gamma = \mathrm{id}$ and $G = F \restriction \gamma$. Note ρ is a cardinal of $J_\gamma^{\vec{E}}$, hence of $L[\vec{E}]$ because σ exists.

Arguing exactly as in Case A we find a stage η of the construction such that

$$\mathcal{M}_\eta = (J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma)$$

and

$$(J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma, F \restriction \gamma)$$

is a premouse of type III. In §12 we prove:

Sublemma 11.4.3. $(J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma, F \restriction \gamma)$ *is reliable.*

Thus $\mathcal{M}_{\eta+1}$ is defined through Case 1 of our construction. Let H be the set such that $\mathcal{N}_{\eta+1} = (J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma, H)$. Now ρ is the largest cardinal of $(J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma)$, and we chose H so as to minimize $\nu^{\mathcal{N}_{\eta+1}}$, the sup of the generators of H. Thus $\nu^{\mathcal{N}_{\eta+1}} = \rho$ and $\mathcal{N}_{\eta+1}$ is of type III or type I. Drawing on §12, we get

Sublemma 11.4.4. *The structure $(J_\gamma^{\vec{E}}, \in, \vec{E} \upharpoonright \gamma, F \upharpoonright \gamma, H)$ is an iterable type III bicephalus.*

It follows from Theorem 9.2 that $H = F \upharpoonright \gamma$. Also, as in Case A, $\mathcal{M}_{\eta+1} = \mathfrak{C}_\omega(\mathcal{N}_\eta) = \mathcal{N}_\eta$, and $\mathcal{M}_{\eta+1}$ (note here $\rho = \rho_\omega(\mathcal{N}_\eta)$ is the largest cardinal of $\mathcal{N}_\eta$, and ρ is a cardinal of $L[\vec{E}]$) is an initial segment of $L[\vec{E}]$. Thus $\gamma \in \operatorname{dom} \vec{E}$ and $E_\gamma = F \upharpoonright \gamma$.

Case C. ρ is a limit of generators of F, and is itself a generator of F, and $\rho \notin \operatorname{dom} \vec{E}$.

Again, let

$$\sigma : \operatorname{Ult}(L[\vec{E}], F \upharpoonright \rho) \to \operatorname{Ult}(L[\vec{E}], F)$$

be the canonical embedding. This time we have $\rho = \operatorname{crit} \sigma$, and thus it is not obvious that G "coheres" with $\vec{E}$ up to γ. Nevertheless, Theorem 8.2 implies that this is true.

Claim 1. $\operatorname{Ult}(L[\vec{E}], F \upharpoonright \rho)$ agrees with $L[\vec{E}]$ below γ.

Proof. Let η be any ordinal such that $\rho < \eta < \gamma$ and $\rho_\omega^{\mathcal{H}} = \rho$ where $\mathcal{H} = \mathcal{J}_\eta^{\operatorname{Ult}(L[\vec{E}], F \upharpoonright \rho)}$. Since γ is the successor cardinal of ρ in $\operatorname{Ult}(L[\vec{E}], F \upharpoonright \rho)$, there are arbitrarily large such ordinals $\eta < \gamma$. It will thus be enough to see that $\mathcal{H}$ is an initial segment of $L[\vec{E}]$. But now $\sigma \upharpoonright \mathcal{H}$ is a fully elementary map from $\mathcal{H}$ into $\sigma(\mathcal{H})$; moreover $\operatorname{crit}(\sigma \upharpoonright \mathcal{H}) = \rho_\omega^{\mathcal{H}}$ and $\rho_\omega^{\mathcal{H}} \notin \operatorname{dom} \vec{E}$ (so $\rho_\omega^{\mathcal{H}} \notin \operatorname{dom} \dot{E}^{\sigma(\mathcal{H})}$). Thus Theorem 8.2 implies that $\mathcal{H}$ is an initial segment of $\sigma(\mathcal{H})$. But $\sigma(\mathcal{H})$ agrees with $L[\vec{E}]$ below $\operatorname{lh} F$, hence below γ, and thus $\mathcal{H}$ is an initial segment of $L[\vec{E}]$.

Claim 2. $(J_\gamma^{\vec{E}}, \in, \vec{E} \upharpoonright \gamma, G)$ is a 1-small type III premouse.

Proof. For coherence, we use Claim 1. The initial segment condition follows from our induction hypothesis on ρ. We get 1-smallness as in Case A.

We now consider two subcases.

Subcase C 1. ρ is a cardinal of $L[\vec{E}]$. In this case we have, just as in Case A, that there is a stage η of our construction such that

$$\mathcal{M}_\eta = (J_\gamma^{\vec{E}}, \in, \vec{E} \upharpoonright \gamma).$$

(Here η is the sup of all ξ_α s.t. $\kappa_\alpha = \rho$ and $\rho < \mathrm{OR}^{\mathcal{M}_{\xi_\alpha}} < \gamma$.) Granted this, we will proceed just as in Case B:

Sublemma 11.4.5. *$(J_\gamma^{\vec{E}}, \in, \vec{E} \upharpoonright \gamma, G)$ is reliable.*

PROOF. In §12.

So $\mathcal{M}_{\eta+1}$ is defined via Case 1 in our construction. Let

$$\mathcal{N}_{\eta+1} = (J_\gamma^{\vec{E}}, \in, \vec{E} \upharpoonright \gamma, H).$$

As we chose H to minimize $\nu^{\mathcal{N}_{\eta+1}}$, $\mathcal{N}_{\eta+1}$ is either type I or type III, and $\nu^{\mathcal{N}_{\eta+1}} = \rho$.

Sublemma 11.4.6. *The structure* $(J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma, G, H)$ *is an iterable type* III *bicephalus.*

PROOF. In §12.

From Theorem 9.2 we get that $G = H$. As in Case B, Theorem 8.1 and Lemma 11.2 guarantee that $\mathcal{M}_{\eta+1} = \mathcal{N}_{\eta+1}$ and $\mathcal{M}_{\eta+1}$ is an initial segment of $L[\vec{E}]$. Thus $G = E_\gamma$.

Subcase C 2. ρ is not a cardinal of $L[\vec{E}]$. We use the argument from Case A. Let δ be the largest cardinal of $L[\vec{E}]$ which is $< \rho$. Let G' be the trivial completion of $F \restriction \rho + 1$, and $\xi = \mathrm{lh}\, G'$. Thus $G' = F \restriction \xi$, and $(J_\xi^{\vec{E}}, \in, \vec{E} \restriction \xi, \tilde{F} \restriction \xi)$ satisfies conditions 1 through 4 of goodness at ξ. Using σ we see that δ is the largest cardinal of $J_\gamma^{\vec{E}}$ which is less than ρ. It follows that δ is the largest cardinal of $J_\xi^{\vec{E}}$ (note ρ is not a cardinal in $J_\xi^{\vec{E}}$ since the natural embedding from $\mathrm{Ult}(L[\vec{E}], F \restriction \rho + 1)$ into $\mathrm{Ult}(L[\vec{E}], F)$ fixes ρ, and ρ is not a cardinal of $\mathrm{Ult}(L[\vec{E}], F)$). Our induction hypothesis guarantees that the trivial completion of $F \restriction \delta$ is on $\vec{E}$, and hence on $\vec{E} \restriction \xi$. Thus $(J_\xi^{\vec{E}}, \in, \vec{E} \restriction \xi,\ \tilde{F} \restriction \xi)$ is a psuedo-premouse.

Sublemma 11.4.7. $(J_\xi^{\vec{E}}, \in, \vec{E} \restriction \xi, \tilde{F} \restriction \xi)$ *is iterable.*

PROOF. In §12.

Theorem 10.1 implies that $(J_\xi^{\vec{E}}, \in, \vec{E} \restriction \xi, \tilde{F} \restriction \xi)$ satisfies the full initial segment condition on premice, so that $\gamma \in \mathrm{dom}\, \vec{E}$ and $E_\gamma = F \restriction \gamma$, as desired.

Remark. We do not seem to get that there is a stage η of the construction such that $\mathcal{M}_\eta = (J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma)$ in Subcase C2.

CASE D. ρ is a limit of generators of F, a generator of F itself, and $\rho \in \mathrm{dom}\, \vec{E}$.

As $\rho \in \mathrm{dom}\, \vec{E}$, ρ is not a cardinal of $L[\vec{E}]$. We can now just repeat the argument from Subcase C2. Letting ξ be the length of the trivial completion of $F \restriction \rho + 1$, $(J_\xi^{\vec{E}}, \in, \vec{E} \restriction \xi, \tilde{F} \restriction \xi)$ is a psuedo-premouse and, borrowing from §12, is iterable. By Theorem 10.1, $(J_\xi^{\vec{E}}, \in, \vec{E} \restriction \xi, \tilde{F} \restriction \xi)$ satisfies the full initial segment condition on premice. As $\rho \in \mathrm{dom}\, \vec{E}$, this means G is on the sequence of $\mathrm{Ult}((J_\rho^{\vec{E}}, \in, E \restriction \rho), E_\rho)$, as desired.

Case E. $\rho = (\kappa^+)^{L[\vec{E}]}$.

The proof is the same as that in Case B. We omit further detail.

This completes the proof of Lemma 11.4. □

We can now easily finish the proof of Theorem 11.3. Let

$$\rho = \text{ least strongly inaccessible cardinal of } L[\vec{E}] \; > j^*(f)(\kappa)\,.$$

Let G be the trivial completion of $F \restriction \rho$, and $\gamma = \mathrm{lh}\, G$. By the choice of j^* we know that ρ is definable in $\mathrm{Ult}(L[\vec{E}], F)$ from $j^*(f)(\kappa) < \rho$ and hence is not a generator of F. Thus lemma 11.4 implies that $\gamma \in \mathrm{dom}\, \vec{E}$ and $E_\gamma = F \restriction \gamma$. We have the diagram

$$\begin{array}{ccc} L[\vec{E}] & \xrightarrow{\;j^*\;} & \mathrm{Ult}(L[\vec{E}], F^*) \\ & \searrow^{\mathrm{i}} & \uparrow k \\ & & \mathrm{Ult}(L[\vec{E}], F \restriction \gamma) \end{array}$$

where the upper ultrapower is computed using functions in V, and the lower using functions in $L[\vec{E}]$. The function k is defined by $k([a,h]^{L[\vec{E}]}_{F\restriction\gamma}) = [a,h]^{V}_{F^*}$. Since $k \restriction \gamma = \mathrm{id}$, $i(f)(\kappa) < \rho$. By coherence, $J^{\vec{E}}_{\rho} = J^{\mathrm{Ult}(L[\vec{E}], F\restriction\gamma)}_{\rho}$, and thus

$$L[\vec{E}] \models V_\rho \subseteq \mathrm{Ult}(L[\vec{E}], F \restriction \gamma)\,.$$

So $F \restriction \gamma$ witnesses the Woodin property for the function f. □