§1. Good Extender Sequences

Definition 1.0.1. Let $\kappa < \nu$ and suppose that M is transitive and rudimentarily closed. We call E a (κ, ν)-*extender over* M iff there is a nontrivial Σ_0-elementary embedding $j : M \to N$, with N transitive and rudimentarily closed, such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \nu$, and

$$E = \{(a, x) \mid a \in [\nu]^{<\omega} \wedge x \subseteq [\kappa]^{\mathrm{card}\, a} \wedge x \in M \wedge a \in j(x)\}.$$

We write $\kappa = \mathrm{crit}\, E$, $\nu = \mathrm{lh}\, E$.

Remark. If the requirement that N be transitive is weakened to $\nu + 1 \subseteq \mathrm{wfp}(N)$, where $\mathrm{wfp}(N)$ is the wellfounded part of N, then we call E a (κ, ν) *pre-extender over* M.

We are interested in this weakening for the purely technical reason that if $\nu \leq \mathrm{OR}^M$, then pre-extenderhood is expressible by a simple first order sentence about $(M, \in, E)$.

Definition 1.0.2. Let $S \subseteq \mathrm{OR}$ and suppose $\vec{E} = \langle E_\alpha \mid \alpha \in S\rangle$ is a sequence of extenders (E_α over some M_α). Then

$$J_\alpha^{\vec{E}} = J_\alpha^A$$

where

$$A = \{(\alpha, a, x) \mid \alpha \in S \wedge (a, x) \in E_\alpha\}.$$

Remark. So $J_\alpha^{\vec{E}} = J_\alpha^{\vec{E}\restriction\alpha}$. The ordinals in S are the stages at which extenders are activated.

Let $\vec{E} = \langle E_\alpha \mid \alpha \in S\rangle$ be given. Let $\alpha \in S$ and E_α be a (κ, γ) pre-extender. We put

$$\begin{aligned}(a, b, \delta) \in \tilde{E}_\alpha \quad \text{iff} \quad & [\delta < \gamma \wedge b \quad \text{is a function} \\ & \wedge \mathrm{dom}\, b = \kappa \wedge b \in J_\alpha^{\vec{E}\restriction\alpha} \\ & \wedge a = E_\alpha \cap ([\delta]^{<\omega} \times \mathrm{ran}\, b)].\end{aligned}$$

We then define

$$\mathcal{J}_\alpha^{\vec{E}} = \begin{cases} (J_\alpha^{\vec{E}}, \in, \vec{E}\restriction\alpha) & \text{if} \quad \alpha \notin S \\ (J_\alpha^{\vec{E}}, \in, \vec{E}\restriction\alpha, \tilde{E}_\alpha) & \text{if} \quad \alpha \in S. \end{cases}$$

Remarks. (a) In the sequences $\vec{E}$ we shall consider, $\mathrm{lh}\ E_\alpha = \alpha$ for all $\alpha \in S$.

(b) Of course if $\alpha \in S$ then the sets first order definable over $(J_\alpha^{\vec{E}}, \in, \vec{E}\restriction\alpha, E_\alpha)$ are the same as those first order over $\mathcal{J}_\alpha^{\vec{E}}$, but the latter structure is a better starting point for fine structure.

(c) We are interested only in the case that each E_α, $(\alpha \in S)$ is an extender over $J_\alpha^{\vec{E}\restriction\alpha}$. Thus E_α is to measure no sets constructed after it is activated, a very useful idea due to Baldwin and Mitchell. As a consequence, the subsets of $J_\beta^{\vec{E}}$ in $J_{\beta+1}^{\vec{E}}$ are just those first order definable over $\mathcal{J}_\beta^{\vec{E}}$. In earlier setups one needed also "measure quantifiers" coming from the E_α, $\alpha \leq \beta$, to define the new sets. This complicated the fine structure substantially.

One consequence of the Baldwin-Mitchell idea is that we can work entirely with structures which are strongly acceptable in the sense of Dodd and Jensen [DJ1].

DEFINITION 1.0.3. A structure $(J_\alpha^A, \in, \ldots)$ is *strongly acceptable* iff whenever $\beta < \alpha$ and

$$P(\kappa) \cap (J_{\beta+1}^A - J_\beta^A) \neq \varnothing$$

then $J_{\beta+1}^A \models^f \mathrm{card}(J_\beta^A) \leq \kappa$.

Notice that if J_α^A is strongly acceptable and $J_\alpha^A \models$ "κ^+ exists" then $J_\alpha^A \models$ "$P(\kappa)$ exists and $P(\kappa) \subseteq J_{\kappa^+}^A$". In particular, GCH holds in strongly acceptable structures.

It is a basic fact in the fine structure of L that $(J_\alpha^\varnothing, \in)$ is always strongly acceptable. On the other hand, in the usual stratification of $L[\mu]$, $J_{\kappa+2}^\mu$ is not strongly acceptable (for $\kappa = \mathrm{crit}\ \mu$).

Let E be a (κ, ν) extender over M. For $\kappa \leq \xi < \nu$, we say ξ is a *generator of* E iff whenever $a \in [\xi]^n$ and $f \in M$ and $f : [\kappa]^n \to \kappa$, $\xi \neq [a, f]_E^M$ (that is, $\{\langle u_1 \cdots u_n, u_{n+1}\rangle \mid f(u_1 \cdots u_n) = u_{n+1}\} \notin E_{a\cup\{\xi\}}$). (Equivalently, ξ is a generator of E iff ξ is the critical point of the natural embedding from $\mathrm{Ult}(M, E \restriction \xi)$ into $\mathrm{Ult}(M, E)$.)

Thus κ is the least generator of E. All other generators are strictly greater than $(\kappa^+)^M$. Note that the property of being a generator of E depends only on E and $P(\kappa)^M$, and E determines $P(\kappa)^M$.

Let η be the larger of $(\kappa^+)^M$ and $\sup\{\xi+1 : \xi \text{ is a generator of } E\}$. Then $\eta \leq \nu$, since $M \models \kappa^+$ exists in the models of interest. We call η the *natural length* of E. Suppose $\mathrm{Ult}(M, E)$ is wellfounded, where the ultrapower is formed using functions in M. Regarding $\mathrm{Ult}(M, E)$ as transitive, let $i \colon M \to \mathrm{Ult}(M, E)$ be the canonical embedding. Then $\mathrm{Ult}(M, E) \models$ "η^+ exists" since $\eta < i(\kappa)$. We will use the ordinal $(\eta^+)^{\mathrm{Ult}(M,E)}$ as the index of E in sequences of extenders. The *trivial completion* of E is the $(\kappa, (\eta^+)^{\mathrm{Ult}(M,E)})$ pre-extender G consisting of pairs (a, x) such that

$$a \in [(\eta^+)^{\mathrm{Ult}(M,E)}]^{<\omega} \wedge x \subseteq [\kappa]^{\mathrm{card}\, a} \wedge x \in M \wedge a \in i(x)\,.$$

Then $E \restriction \alpha = G \restriction \alpha$, where $\alpha = \inf(\mathrm{lh}\, E, \mathrm{lh}\, G)$.

We now record some of the main properties of the extender sequences we shall consider:

DEFINITION 1.0.4. A sequence $\langle E_\beta \mid \beta \in S\rangle$ is *good at* α if it satisfies the following five clauses:

(1) $J_\alpha^{\vec{E}}$ is strongly acceptable,

and if $\alpha \in S$, then

(2) E_α is a (κ, α) pre-extender over $J_\alpha^{\vec{E}\restriction\alpha}$ for some κ such that $J_\alpha^{\vec{E}\restriction\alpha} \models \kappa^+$ exists,

(3) (bounded generators) E_α is the trivial completion of $E_\alpha \restriction \nu$, where ν is the natural length of E_α.

(4) (coherence) $i(\vec{E} \restriction \alpha) \restriction \alpha + 1 = \vec{E} \restriction \alpha$ where $i : J_\alpha^{\vec{E}\restriction\alpha} \to \mathrm{Ult}(J_\alpha^{\vec{E}\restriction\alpha}, E_\alpha)$ is the canonical embedding, and

(5) (closure under initial segment) Let ν be the natural length of E_α. If η is an ordinal such that $(\kappa^+)^{J_\alpha^{\vec{E}}} \leq \eta < \nu$ and η is the natural length of $E_\alpha \restriction \eta$, then one of (a) or (b) below holds:

(a) There is $\gamma < \alpha$ such that E_γ is the trivial completion of $E_\alpha \restriction \eta$.

(b) $\eta \in S$, and there is a $\gamma < \alpha$ such that $\pi(\vec{E} \restriction \eta)_\gamma$ is the trivial completion of $E_\alpha \restriction \eta$, where $\pi \colon \mathcal{J}_\eta^{\vec{E}\restriction\eta} \to \mathrm{Ult}(\mathcal{J}_\eta^{\vec{E}\restriction\eta}, E_\eta)$ is the canonical embedding.

DEFINITION 1.0.5. A potential premouse (ppm) is a structure of the form $\mathcal{J}_\beta^{\vec{E}}$, where $\vec{E}$ is good at all $\alpha \leq \beta$.

DEFINITION 1.0.6. A ppm $\mathcal{J}_\beta^{\vec{E}}$ is active if $\beta \in \operatorname{dom} \vec{E}$; otherwise it is passive.

DEFINITION 1.0.7. If $\mathcal{M} = \mathcal{J}_\beta^{\vec{E}}$ is active then $\nu^{\mathcal{M}}$ is the natural length of E_β.

Remarks. (a) Activity is determined by the similarity type of the ppm.

(b) Condition 3 implies every $\beta < \alpha$ is represented mod E_α by a function with support $\subseteq \nu^{\mathcal{M}}$, so that $E_\alpha \restriction \nu^{\mathcal{M}}$ determines all of E_α. We include the extra coordinates just so that the functions witnessing coherence will be trivial (essentially projections on a coordinate) which helps show coherence is preserved by Σ_0 ultrapowers. (Here $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E}}$.)

(c) The ultrapowers in the definition are "Σ_0 ultrapowers", that is, formed using functions belonging to the model in question. Note $J_\alpha^{\vec{E}\restriction\alpha}$ is always passive, hence amenable, so that we can move its predicate.

Notice that since E_α is only a pre-extender, $\mathrm{Ult}(J_\alpha^{\vec{E}\restriction\alpha}, E_\alpha)$ may not be well-founded. However, $\alpha + 1 \subseteq \mathrm{wfp}\,(\mathrm{Ult})$, which is enough to make sense of conditions (3) and (4).

(d) Let $\vec{E}$ be good at α and $i : J_\alpha^{\vec{E}\restriction\alpha} \to \mathrm{Ult}(\mathcal{J}_\alpha^{\vec{E}\restriction\alpha}, E_\alpha)$ the canonical embedding. Let $\nu < \alpha$ be the natural length of E_α. By coherence, $J_\alpha^{i(\vec{E}\restriction\alpha)} = J_\alpha^{\vec{E}}$. Since $\alpha = \nu^+$ in $\mathrm{Ult}(J_\alpha^{\vec{E}\restriction\alpha}, E_\alpha)$, which is strongly acceptable, there are no cardinals

$> \nu$ in $J_\alpha^{i(\vec{E}\restriction\alpha)}$. So there are no cardinals $> \nu$ in $J_\alpha^{\vec{E}}$. The ordinal ν itself may be a successor ordinal. It is easy to see that if ν is a limit ordinal, then in fact ν is a cardinal, both in $J_\alpha^{\vec{E}}$ and $\mathrm{Ult}(J_\alpha^{\vec{E}\restriction\alpha}, E_\alpha)$.

(e) Let $\kappa = \mathrm{crit}\ E_\alpha$. By (3) there is a map of $(P(\kappa) \cap J_\nu^{\vec{E}}) \times [\nu]^{<\omega}$ onto α, the map being in $J_{\alpha+1}^{\vec{E}}$. Thus α is not a cardinal in $J_{\alpha+1}^{\vec{E}}$.

(f) For the good $\vec{E}$ we construct, E_α is an extender over $L[\vec{E} \restriction \alpha]$, which is strongly acceptable, and $\alpha = \nu^+$ in both $L[\vec{E} \restriction \alpha]$ and $\mathrm{Ult}(L[\vec{E} \restriction \alpha], E_\alpha)$. This in fact follows from the definition of goodness if we can iterate from $J_\alpha^{\vec{E}}$ via E_α and its images OR times (and preserve wellfoundedness).

(h) It might be hoped that alternative (b) of the initial segment condition could be dropped, but we suspect that if $L[\vec{E}]$ is to have a Woodin cardinal, or even lots of strong cardinals, one cannot demand this stronger version of the initial segment condition. The initial segment condition is crucial in the proof that the comparison process terminates (cf. §7). We need some form of it as an axiom on our extender sequences in order to get a decent theory going. Sy Friedman has suggested that it might be possible to eliminate this clause if the sequences are indexed by letting an extender be E_α where α is the double successor of the natural length of E in the ultrapower by E, instead of using the single successor as in this paper. We do not know whether this idea can be made to work.

Notice passive ppm are amenable. For active ppm we have a weaker property, which we call weak amenability.

DEFINITION 1.0.8. Let $J_\alpha^{\vec{E}}$ be an active ppm, and $\kappa = \mathrm{crit}\ E_\alpha$. We say $J_\alpha^{\vec{E}}$ is *weakly amenable* iff whenever $\langle A_\beta \mid \beta < \kappa\rangle \in J_\alpha^{\vec{E}}$ and $\forall\beta\exists n < \omega(A_\beta \subseteq [\kappa]^n)$ and $\eta < \alpha$, then

$$E_\alpha \cap ([\eta]^{<\omega} \times \{A_\beta \mid \beta < \kappa\}) \in J_\alpha^{\vec{E}}.$$

The proof of the next lemma is well-known (due to K. Kunen?).

Lemma 1.1. *Every active ppm is weakly amenable.*

PROOF. Let $\langle A_\beta \mid \beta < \kappa\rangle$ be as in the definition of weak amenability, and

$$i : J_\alpha^{\vec{E}\restriction\alpha} \to \mathrm{Ult}(J_\alpha^{\vec{E}\restriction\alpha}, E_\alpha) = \mathrm{Ult}$$

the canonical embedding. Let $F = E_\alpha \cap ([\eta]^{<\omega} \times \{A_\beta \mid \beta < \kappa\})$ where $\eta < \alpha$. Now

$$\langle i(A_\beta) \mid \beta < \kappa\rangle \in \mathrm{Ult}$$

so as $A_\beta = i(A_\beta) \cap [\kappa]^{<\omega}$,

$$F = \{(a, A_\beta) \mid a \in [\eta]^{<\omega} \wedge a \in i(A_\beta)\} \in \mathrm{Ult}\ .$$

By strong acceptability, since α is a cardinal in Ult, $F \in J_\alpha^{i(E\upharpoonright\alpha)}$. By coherence, $F \in J_\alpha^{\vec{E}}$. □

Remark. Let ν be the natural length of E_α. It is easy to see that if $A \subset \nu$ and $A \in \mathrm{Ult}(J_\alpha^{\vec{E}}, E_\alpha)$, then A can be computed from $E_\alpha \cap ([\nu]^{<\omega} \times \{A_\beta \mid \beta < \kappa\})$ for some sequence $\langle A_\beta \mid \beta < \kappa \rangle \in J_\nu^{\vec{E}}$ of subsets of $[\kappa]^n$, $n < \omega$. Thus in fact α is the least ordinal γ such that $E_\alpha \cap ([\nu]^{<\omega} \times \{A_\beta \mid \beta < \kappa\}) \in J_\gamma^{\vec{E}}$ for all such $\langle A_\beta \mid \beta < \kappa \rangle \in J_\nu^{\vec{E}}$. This is the motivation for condition (3) of good at α: we don't add an extender until we have weak amenability.

Remark. For $\xi < (\kappa^+)^{J_\alpha^{\vec{E}}}$ define γ_ξ to be the least ordinal γ such that $E_\alpha \cap \left([\nu]^{<\omega} \times J_\xi^{\vec{E}}\right) \in J_\gamma^{\vec{E}}$. The ordinals γ_ξ's are cofinal in α. To see this, let $A \in \mathrm{Ult}(J_\alpha^{\vec{E}}, E_\alpha)$ be any subset of the natural length ν of E_α, and let $A = [a, f]_{E_\alpha}$. Then A can be computed from $(E_\alpha)_a \cap \{\, A_\eta : \eta < \kappa \,\}$ where $A_\eta = \{\, \bar{u} : \eta \in f(\bar{u}) \,\}$, so that if $f \in J_\xi^{\vec{E}}$ for $\xi < \kappa^+$ of $J_\alpha^{\vec{E}}$ then $A \in J_{\gamma_\xi + 1}^{\vec{E}}$.

Since the sequence $(\, \gamma_\xi : \xi < (\kappa^+)^{J_\alpha^{\vec{E}}} \,)$ is in $J_{\alpha+1}^{\vec{E}}$ it follows that $J_{\alpha+1}^{\vec{E}} \models \mathrm{cf}(\alpha) = \mathrm{cf}((\kappa^+)^{J_\alpha^{\vec{E}}})$.