8 Boolean algebras

In this section we consider the length of Borel hierarchies generated by a subset of a complete boolean algebra. We find that the generators of the complete boolean algebra associated with α-forcing generate it in exactly $\alpha+1$ steps. We start by presenting some background information.

Let $\mathbb{B}$ be a *cBa*, i.e, complete boolean algebra. This means that in addition to being a boolean algebra, infinite sums and products, also exist; i.e., for any $C \subseteq \mathbb{B}$ there exists b (denoted $\sum C$) such that

1. $c \leq b$ for every $c \in C$ and
2. for every $d \in \mathbb{B}$ if $c \leq d$ for every $c \in C$, then $b \leq d$.

Similarly we define $\prod C = -\sum_{c\in C} -c$ where $-c$ denotes the complement of c in $\mathbb{B}$.

A partial order $\mathbb{P}$ is *separative* iff for any $p, q \in \mathbb{P}$ we have

$$p \leq q \text{ iff } \forall r \in \mathbb{P}(r \leq p \text{ implies } q, r \text{ compatible}).$$

Theorem 8.1 *(Scott, Solovay see [43]) A partial order $\mathbb{P}$ is separative iff there exists a cBa $\mathbb{B}$ such that $\mathbb{P} \subseteq \mathbb{B}$ is dense in $\mathbb{B}$, i.e. for every $b \in \mathbb{B}$ if $b > 0$ then there exists $p \in \mathbb{P}$ with $p \leq b$.*

It is easy to check that the α-forcing $\mathbb{P}$ is separative (as long as $\mathcal{B}$ is infinite): If $p \not\leq q$ then either

1. t_p does not extend t_q, so there exists s such that $t_q(s) = B$ and either s not in the domain of t_p or $t_p(s) = C$ where $C \neq B$ and so in either case we can find $r \leq p$ with r, q incompatible, or
2. F_p does not contain F_q, so there exists $(s, x) \in (F_q \setminus F_p)$ and we can either add $(s\hat{\ }n, x)$ for sufficiently large n or add $t_r(s\hat{\ }n) = B$ for some sufficiently large n and some $B \in \mathcal{B}$ with $x \in B$ and get $r \leq p$ which is incompatible with q.

The elegant (but as far as I am concerned mysterious) approach to forcing using complete boolean algebras contains the following facts:

1. for any sentence θ in the forcing language

   $$[\![\theta]\!] = \sum\{b \in \mathbb{B} : b \Vdash \theta\} = \sum\{p \in \mathbb{P} : p \Vdash \theta\}$$

   where $\mathbb{P}$ is any dense subset of $\mathbb{B}$,
2. $p \Vdash \theta$ iff $p \leq [\![\theta]\!]$,
3. $[\![\neg\theta]\!] = -[\![\theta]\!]$,
4. $[\![\theta \wedge \psi]\!] = [\![\theta]\!] \wedge [\![\psi]\!]$,

5. $[\![\theta \vee \psi]\!] = [\![\theta]\!] \vee [\![\psi]\!]$,

6. for any set X in the ground model,

$$[\![\forall x \in \check{X}\ \theta(x)]\!] = \prod_{x \in X} [\![\theta(\check{x})]\!].$$

Definitions. For $\mathbb{B}$ a cBa and $C \subseteq \mathbb{B}$ define:
$\underset{\sim}{\Pi}^0_0(C) = C$ and
$\underset{\sim}{\Pi}^0_\alpha(C) = \{\prod \Gamma : \Gamma \subseteq \{-c : c \in \bigcup_{\beta<\alpha} \underset{\sim}{\Pi}^0_\beta(C)\}\}$ for $\alpha > 0$.
$\text{ord}(\mathbb{B}) = \min\{\alpha : \exists C \subseteq \mathbb{B}$ countable with $\underset{\sim}{\Pi}^0_\alpha(C) = \mathbb{B}\}$.

Theorem 8.2 *(Miller [73]) For every $\alpha \leq \omega_1$ there exists a countably generated ccc cBa $\mathbb{B}$ with* $\text{ord}(\mathbb{B}) = \alpha$.

proof:
Let $\mathbb{P}$ be α-forcing and $\mathbb{B}$ be the cBa given by the Scott-Solovay Theorem 8.1. We will show that $\text{ord}(\mathbb{B}) = \alpha + 1$.
Let

$$C = \{p \in \mathbb{P} : F_p = \emptyset\}.$$

C is countable and we claim that $\mathbb{P} \subseteq \underset{\sim}{\Pi}^0_\alpha(C)$. Since $\mathbb{B} = \underset{\sim}{\Sigma}^0_1(\mathbb{P})$ this will imply that $\mathbb{B} = \underset{\sim}{\Sigma}^0_{\alpha+1}(C)$ and so $\text{ord}(\mathbb{B}) \leq \alpha + 1$.
First note that for any $s \in T$ with $r(s) = 0$ and $x \in X$,

$$[\![x \in U_s]\!] = \sum \{p \in C : \exists B \in \mathcal{B}\ t_p(s) = B \text{ and } x \in B\}.$$

By Lemma 7.3 we know for generic filters G that for every $x \in X$ and $s \in T^{>0}$

$$x \in U_s \iff \exists p \in G\ (s, x) \in F_p.$$

Hence $[\![x \in U_s]\!] = \langle \emptyset, \{(s, x)\} \rangle$ since if they are not equal, then

$$b = [\![x \in U_s]\!]\ \Delta\ \langle \emptyset, \{(s, x)\} \rangle > 0,$$

but letting G be a generic ultrafilter with b in it would lead to a contradiction. We get that for $r(s) > 0$:

$$\langle \emptyset, \{(s, x)\} \rangle = [\![x \in U_s]\!] = [\![x \in \bigcap_{n \in \omega} \sim U_{s\frown n}]\!] = \prod_{n \in \omega} -[\![x \in U_{s\frown n}]\!].$$

Remembering that for $r(s\frown n) = 0$ we have $[\![x \in U_{s\frown n}]\!] \in \underset{\sim}{\Sigma}^0_1(C)$, we see by induction that for every $s \in T^{>0}$ if $r(s) = \beta$ then

$$\langle \emptyset, \{(s, x)\} \rangle \in \underset{\sim}{\Pi}^0_\beta(C).$$

For any $p \in \mathbb{P}$

$$p = \langle t_p, \emptyset \rangle \wedge \prod_{(s,x) \in F_p} \langle \emptyset, \{(s, x)\} \rangle.$$

So we have that $p \in \underset{\sim}{\Pi}^0_\alpha(C)$.

Now we will see that ord($\mathbb{B}$) $> \alpha$. We use the following Lemmas.
$\mathbb{B}^+$ are the nonzero elements of $\mathbb{B}$.

Lemma 8.3 *If $r : \mathbb{P} \to$ OR is a rank function, i.e. it satisfies the Rank Lemma 7.4 and in addition $p \leq q$ implies $r(p) \leq r(q)$, then if $\mathbb{P}$ is dense in the cBa $\mathbb{B}$ then r extends to r^* on $\mathbb{B}^+$:*

$$r^*(b) = \min\{\beta \in \mathrm{OR} : \exists C \subseteq \mathbb{P} : b = \sum C \textit{ and } \forall p \in C\ r(p) \leq \beta\}$$

and still satisfies the Rank Lemma.

proof:
Easy induction.
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Lemma 8.4 *If $r : \mathbb{B}^+ \to$ ord is a rank function and $E \subseteq \mathbb{B}$ is a countable collection of rank zero elements, then for any $a \in \underset{\sim}{\Pi}^0_\gamma(E)$ and $a \neq 0$ there exists $b \leq a$ with $r(b) \leq \gamma$.*

proof:
To see this let $E = \{e_n : n \in \omega\}$ and let $\overset{\circ}{Y}$ be a name for the set in the generic extension

$$Y = \{n \in \omega : e_n \in G\}.$$

Note that $e_n = [\![\, n \in \overset{\circ}{Y} \,]\!]$. For elements b of $\mathbb{B}$ in the complete subalgebra generated by E let us associate sentences θ_b of the infinitary propositional logic $L_\infty(P_n : n \in \omega)$ as follows:

$$\theta_{e_n} = P_n$$

$$\theta_{-b} = \neg\theta_b$$

$$\theta_{\prod R} = \bigwedge\!\!\!\bigwedge_{r \in R} \theta_r$$

Note that $[\![\, Y \models \theta_b \,]\!] = b$ and if $b \in \underset{\sim}{\Pi}^0_\gamma(E)$ then θ_b is a Π_γ-sentence. The Rank and Forcing Lemma 7.5 gives us (by translating $p \Vdash Y \models \theta_b$ into $p \leq [\![\, Y \models \theta_b \,]\!] = b$) that:

> For any $\gamma \geq 1$ and $p \leq b \in \underset{\sim}{\Pi}^0_\gamma(E)$ there exists a $\hat{p}$ compatible with p such that $\hat{p} \leq b$ and $r(\hat{p}) \leq \gamma$.

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Now we use the lemmas to see that ord($\mathbb{B}$) $> \alpha$.

Given any countable $E \subseteq \mathbb{B}$, let $Q \subseteq X$ be countable so that for any $e \in E$ there exists $H \subseteq \mathbb{P}$ countable so that $e = \sum H$ and for every $p \in H$ we have rank$(p, Q) = 0$. Let $x \in X \setminus Q$ be arbitrary; then we claim:

$$[\![\, x \in U_{\langle\rangle} \,]\!] \notin \underset{\sim}{\Sigma}^0_\alpha(E).$$

We have chosen Q so that $r(p) = \text{rank}(p, Q) = 0$ for any $p \in E$ so the hypothesis of Lemma 8.4 is satisfied. Suppose for contradiction that $[\![x \in U_{\langle\rangle}]\!] = b \in \Sigma^0_\alpha(E)$. Let $b = \sum_{n \in \omega} b_n$ where each b_n is $\Pi^0_{\gamma_n}(C)$ for some $\gamma_n < \alpha$. For some n and $p \in \mathbb{P}$ we would have $p \leq b_n$. By Lemma 8.4 we have that there exists $\hat{p}$ with $\hat{p} \leq b_n \leq b = [\![x \in U_{\langle\rangle}]\!]$ and $\text{rank}(\hat{p}, Q) \leq \gamma_n$. But by the definition of $\text{rank}(\hat{p}, Q)$ the pair $(\langle\rangle, x)$ is not in $F_{\hat{p}}$, but this contradicts $\hat{p} \leq b_n \leq b = [\![x \in U_{\langle\rangle}]\!] = \langle \emptyset, \{(\langle\rangle, x)\}\rangle$.

This takes care of all countable successor ordinals. (We leave the case of $\alpha = 0, 1$ for the reader to contemplate.) For λ a limit ordinal take α_n increasing to λ and let $\mathbb{P} = \sum_{n<\omega} \mathbb{P}_{\alpha_n}$ be the direct sum, where $\mathbb{P}_{\alpha_n}$ is α_n-forcing. Another way to describe essentially the same thing is as follows: Let $\mathbb{P}_\lambda$ be λ-forcing. Then take $\mathbb{P}$ to be the subposet of $\mathbb{P}_\lambda$ such that $\langle\rangle$ doesn't occur, i.e.,

$$\mathbb{P} = \{p \in \mathbb{P}_\lambda : \neg \exists x \in X \ \ (\langle\rangle, x) \in F_p\}.$$

Now if $\mathbb{P}$ is dense in the cBa $\mathbb{B}$, then $\text{ord}(\mathbb{B}) = \lambda$. This is easy to see, because for each $p \in \mathbb{P}$ there exists $\beta < \lambda$ with $p \in \Pi^0_\beta(C)$. Consequently, $\mathbb{P} \subseteq \bigcup_{\beta<\lambda} \Pi^0_\beta(C)$ and so since $\mathbb{B} = \Sigma^0_1(\mathbb{P})$ we get $\mathbb{B} = \Sigma^0_\lambda(C)$. Similarly to the other argument we see that for any countable E we can choose a countable $Q \subseteq X$ such for any $s \in T$ with $2 \leq r(s) = \beta < \lambda$ (so $s \neq \langle\rangle$) we have that $[\![x \in U_s]\!]$ is not $\Sigma^0_\beta(E)$. Hence $\text{ord}(\mathbb{B}) = \lambda$.

For $\text{ord}(\mathbb{B}) = \omega_1$ we postpone until section 12.

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