1 What are the reals, anyway?

Definitions. Let $\omega = \{0, 1, \ldots\}$ and let ω^ω (*Baire space*) be the set of functions from ω to ω. Let $\omega^{<\omega}$ be the set of all finite sequences of elements of ω. $|s|$ is the length of s, $\langle\rangle$ is the empty sequence, and for $s \in \omega^{<\omega}$ and $n \in \omega$ let $s\hat{\ }n$ denote the sequence which starts out with s and has one more element n concatenated onto the end. The basic open sets of ω^ω are the sets of the form:

$$[s] = \{x \in \omega^\omega : s \subseteq x\}$$

for $s \in \omega^{<\omega}$. A subset of ω^ω is open iff it is the union of basic open subsets. It is *separable* (has a countable dense subset) since it is *second countable* (has a countable basis). The following defines a complete metric on ω^ω:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n+1} & \text{if } x \upharpoonright n = y \upharpoonright n \text{ and } x(n) \neq y(n) \end{cases}$$

Cantor space 2^ω is the subspace of ω^ω consisting of all functions from ω to $2 = \{0, 1\}$. It is compact.

Theorem 1.1 *(Baire [4]) ω^ω is homeomorphic to the irrationals $\mathbb{P}$.*

proof:

First replace ω by the integers $\mathbb{Z}$. We will construct a mapping from $\mathbb{Z}^\omega$ to $\mathbb{P}$. Enumerate the rationals $\mathbb{Q} = \{q_n : n \in \omega\}$. Inductively construct a sequence of open intervals $\langle I_s : s \in \mathbb{Z}^{<\omega}\rangle$ satisfying the following:

1. $I_{\langle\rangle} = \mathbb{R}$, and for $s \neq \langle\rangle$ each I_s is a nontrivial open interval in $\mathbb{R}$ with rational endpoints,
2. for every $s \in \mathbb{Z}^{<\omega}$ and $n \in \mathbb{Z}$ $\;I_{s\hat{\ }n} \subseteq I_s$,
3. the right end point of $I_{s\hat{\ }n}$ is the left end point of $I_{s\hat{\ }n+1}$,
4. $\{I_{s\hat{\ }n} : n \in \mathbb{Z}\}$ covers all of I_s except for their endpoints,
5. the length of I_s is less than $\frac{1}{|s|}$ for $s \neq \langle\rangle$, and
6. the n^{th} rational q_n is an endpoint of I_t for some $|t| \leq n+1$.

Define the function $f : \mathbb{Z}^\omega \to \mathbb{P}$ as follows. Given $x \in \mathbb{Z}^\omega$ the set

$$\bigcap_{n\in\omega} I_{x\upharpoonright n}$$

must consist of a singleton irrational. It is nonempty because

$$\text{closure}(I_{x\upharpoonright n+1}) \subseteq I_{x\upharpoonright n}.$$

It is a singleton because their diameters shrink to zero.

So we can define f by

$$\{f(x)\} = \bigcap_{n\in\omega} I_{x\restriction n}.$$

The function f is one-to-one because if s and t are incomparable then I_s and I_t are disjoint. It is onto since for every $u \in \mathbb{P}$ and $n \in \omega$ there is a unique s of length n with $u \in I_s$. It is a homeomorphism because

$$f([s]) = I_s \cap \mathbb{P}$$

and the sets of the form $I_s \cap \mathbb{P}$ form a basis for $\mathbb{P}$.

■

Note that the map given is also an order isomorphism from $\mathbb{Z}^\omega$ with the lexicographical order to $\mathbb{P}$ with it's usual order.

We can identify 2^ω with $P(\omega)$, the set of all subsets of ω, by identifying a subset with its characteristic function. Let $F = \{x \in 2^\omega : \forall^\infty n\ x(n) = 0\}$ (the quantifier $\forall^\infty$ stands for "for all but finitely many n"). F corresponds to the finite sets and so $2^\omega \setminus F$ corresponds to the infinite subsets of ω which we write as $[\omega]^\omega$.

Theorem 1.2 *ω^ω is homeomorphic to $[\omega]^\omega$.*

proof:

Let $f \in \omega^\omega$ and define $F(f) \in 2^\omega$ to be the sequence of 0's and 1's determined by:

$$F(f) = 0^{f(0)}{}^\frown 1^\frown 0^{f(1)}{}^\frown 1^\frown 0^{f(2)}{}^\frown 1^\frown \cdots$$

where $0^{f(n)}$ refers to a string of length $f(n)$ of zeros. The function F is a one-to-one onto map from ω^ω to $2^\omega \setminus F$. It is a homeomorphism because $F([s]) = [t]$ where $t = 0^{s(0)}{}^\frown 1^\frown 0^{s(1)}{}^\frown 1^\frown 0^{s(2)}{}^\frown 1^\frown \cdots {}^\frown 0^{s(n)}{}^\frown 1$ where $|s| = n+1$. Note that sets of the form $[t]$ where t is a finite sequence ending in a one form a basis for $2^\omega \setminus F$.

■

I wonder why ω^ω is called Baire space? The earliest mention of this I have seen is in Sierpiński [97] where he refers to ω^ω as the 0-dimensional space of Baire. Sierpiński also says that Frechet was the first to describe the metric d given above. Unfortunately, Sierpiński [97] gives very few references.[2]

The classical proof of Theorem 1.1 is to use "continued fractions" to get the correspondence. Euler [19] proved that every rational number gives rise to a finite continued fraction and every irrational number gives rise to an infinite continued fraction. Brezinski [13] has more on the history of continued fractions.

My proof of Theorem 1.1 allows me to remain blissfully ignorant[3] of even the elementary theory of continued fractions.

Cantor space, 2^ω, is clearly named so because it is homeomorphic to Cantor's middle two thirds set.

[2] I am indebted to John C. Morgan II for supplying the following reference and comment. "Baire introduced his space in Baire [3]. Just as coefficients of linear equations evolved into matrices the sequences of natural numbers in continued fraction developments of irrational numbers were liberated by Baire's mind to live in their own world."

[3] It is impossible for a man to learn what he thinks he already knows.-Epictetus