Preface

The model theory of fields is a fascinating subject stretching from Tarski's work on the decidability of the theories of the real and complex fields to Hrushovksi's recent proof of the Mordell-Lang conjecture for function fields. Our goal in this volume is to give an introduction to this fascinating area concentrating on connections to stability theory.

The first paper *Introduction to the model theory of fields* begins by introducing the method of quantifier elimination and applying it to study the definable sets in algebraically closed fields and real closed fields. These first sections are aimed for beginning logic students and can easily be incorporated into a first graduate course in logic. They can also be easily read by mathematicians from other areas. Algebraically closed fields are an important examples of ω-stable theories. Indeed in section 5 we prove Macintyre's result that that any infinite ω-stable field is algebraically closed. The last section surveys some results on algebraically closed fields motivated by Zilber's conjecture on the nature of strongly minimal sets. These notes were originally prepared for a two week series of lecture scheduled to be given in Bejing in 1989. Because of the Tinnanmen square massacre these lectures were never given.

The second paper *Model theory of differential fields* is based on a course given at the University of Illinois at Chicago in 1991. Differentially closed fields provide a fascinating example for many model theoretic phenomena (Sacks referred to differentially closed fields as the "least misleading example"). This paper begins with an introduction to the necessary differential algebra and elementary model theory of differential fields. Next we examine types, ranks and prime models, proving among other things that differential closures are not minimal and that for $\kappa > \aleph_0$ there are 2^κ non-isomorphic models. We conclude with a brief survey of differential Galois theory including Poizat's model theoretic proof of Kolchin's result that the differential Galois group of a strongly normal extension is an algebraic group over the constants and the Pillay-Sokolovic result that any superstable differential field has no proper strongly normal expansions. Most of this article can be read by a beginning graduate student in model theory. At some points a deeper knowledge of stability theory or algebraic geometry will be helpful.

When this course was given in 1991 there was an annoying gap in our knowledge about the model theory of differentially closed fields. Shelah had proved Vaught's conjecture for ω-stable theories. Thus we knew that there were either $\aleph_0$ or $2^{\aleph_0}$ non-isomorphic countable differentially closed fields, but did not know which. In 1993 Hrushovski and Sokolovic showed there are $2^{\aleph_0}$. The proof used the Hrushovski-Zilber work on Zariski geometries and Buium's work on abelian varieties and differential algebraic groups. This circle of ideas is also crucial to Hrushovski's proof of the Mordell-Lang conjecture for function fields. The third paper, *Differential algebraic groups and the number of countable* *differentially closed fields*, gives a proof that the number of countable models is $2^{\aleph_0}$ which avoids the Zariski geometry machinery.

The final paper, *Some model theory of separably closed fields* , is a survey of the model theory of separably closed fields. For primes $p > 0$ there are separably closed fields which are not algebraically closed. These are the only other known example of stable fields. Separably closed fields play an essential role in Hrushovski's proof of the Mordell-Lang conjecture. This paper is intended as a survey of the background information one needs for Hrushovski's paper.

We would like to thank the following people who gave helpful comments on earlier drafts of some of these chapters: John Baldwin, Elisabeth Bouscaren, Zoe Chatzidakis, Lou van den Dries, Wai Yan Pong, Sean Scroll, Zeljko Sokolivic, Patrick Speissegger and Carol Wood.

Suggestions for Further Reading

There are a number of important topics that we either barely touched on or omitted completely. We conclude by listing a few such topics and some suggested references.

o-minimal theories: Many of the important geometric and structural properties of semialgebraic subsets of $\mathbf{R}^n$ generalize to arbitrary o-minimal structures.

L. van den Dries, *Tame topology and O-minimal structures*, preprint.

J. Knight, A. Pillay and C. Steinhorn, Definable sets in ordered structures II, Trans. AMS 295 (1986), 593-605.

A. Pillay and C. Steinhorn, Definable sets in ordered structures I, Trans. AMS 295 (1986), 565-592.

One of the most exciting recent developments in logic is Wilkie's proof that the theory of the real field with exponentiation is model complete and o-minimal. Further o-minimal expansions can be built by adding bounded subanalytic sets.

A. J. Wilkie, Some model completeness results for expansions of the ordered field of reals by Pfaffian functions and exponentiation, Journal AMS (to appear).

J. Denef and L. van den Dries, p-adic and real subanalytic sets, Ann. Math. 128 (1988), 79-138.

L. van den Dries, A. Macintyre, and D.Marker, The elementary theory of restricted analytic fields with exponentiation, Annals of Math. 140 (1994), 183-205.

p-adic fields: There is also a well developed model theory of the p-adic numbers beginning with the Ax-Kochen proof of Artin's conjecture and leading to Denef's proof of the rationality of p-adic Poincare series. Macintyre's paper is a survey of the area.

S. Kochen, Model theory of local fields, *Logic Colloquium '74*, G. Mueller ed., Springer Lecture Notes in Mathematics 499, 1975, 384-425.

J. Denef, The rationality of the Poincare series associated to the p-adic points on a variety, Inv. Math. 77 (1984), 1-23.

A. Macintyre, Twenty years of p-adic model theory, *Logic Colloquium '84*, J. Paris, A. Wilkie and G. Wilmers ed., North Holland 1986, 121-153.

Pseudofinite fields: Ax showed that the theory of finite fields is decidable. This subject is carefully presented in Fried and Jarden's book. The Chatzidakis-van den Dries-Macintyre paper gives useful properties of definable sets.

M. Fried and M. Jarden, *Field Arithmetic*, Springer-Verlag (1986).

Z. Chatzidakis, L. van den Dries and A. Macintyre, Definable sets over finite fields, J. reine angew. Math 427 (1992), 107-135.

Zariski Geometries and the Mordell-Lang Conjecture: In 1992 Hrushovski gave a model theoretic proof of the Mordell-Lang conjecture for function fields. His work depends on a joint result with Zilber which characterizes the Zariski topology on an algebraic curve. The Bouscaren-Lascar volume is the proceedings of a conference in Manchester devoted to Hrushovski's proof and the model theoretic machinery needed in its proof.

E. Hrushovski and B. Zilber, Zariski Geometries, Journal AMS (to appear).

E. Hrushovski, The Mordell-Lang conjecture for function fields, Journal AMS (to appear).

E. Bouscaren and D. Lascar, *Stability Theory and Algebraic Geometry, an introduction* preprint.