Best possible answer is computable for fuzzy SLD-resolution *

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Summary. This is a direct continuation of a joint work with P. Vojtáš in which we proved soundness and completeness of fuzzy SLD-resolution for arbitrary many-valued logic with two continuous conjunctions. Using this result here we prove that the maximal value of grade for a fuzzy answer is attained during fuzzy SLD-resolution in logic with only one continuous conjunction. Based on this result we prove better characterization of the least fuzzy Herbrand model of fuzzy definite logic program, which allows us to give a refinement of the completeness part.

1. Introduction

In [12] authors (P. Vojtáš and L. Paulík) consider theoretical (mathematical) model of extended logic programming in many valued logic with arbitrary triple of connectives $(\mathrm{seq}, \mathrm{et}_1, \mathrm{et}_2)$, where et_1 evaluates modus ponens containing the implication seq, and et_2 is the conjunction from bodies of clauses. Declarative semantics is based on generalization of P. Hájek's RPL and RQL. Let us make several remarks concerning these logics in more general way.

It is worth mentioning here, that well-known non-classical logics was proposed by Łukasiewicz and Gödel and that main interest was payed on 1-tautologies in these logics. J. Pavelka made further development of Łukasiewicz logic on propositional level ([10]). He considered not only single formula φ but also a numerical value r (grade) connected with the formula and proposed deduction calculus for maintaining graded formulas $(\varphi; r)$ meaning truth value of φ is at least r. He introduced notion of graded proof and norma $|\varphi|_T$ means the supremum of values for graded proof of φ from a theory T. Norma $\|\varphi\|_T$ means the infimum of values which formula φ gets in models of T. J. Pavelka proved completeness theorem of the form $|\varphi|_T = \|\varphi\|_T$. Further development of Pavelka's ideas was done by V. Novák on predicate level [9]. Substantial simplification of these logics was achieved in P. Hájek's RPL and RQL, see [3],[5].

Recently P. Hájek and D. Švejda in [7] proved strong completeness for finitely axiomatized theory in Łukasiewicz logic and as a consequence also

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completeness for finite Pavelka's Rational logic (if T is a finite fuzzy theory and r is a rational number such that $\|\varphi\|_T = r$, then $T \vdash (\varphi; r)$). This means that the supremum in the definition of $|\varphi|_T$ is attained in deduction process.

Let us note that completeness theorem ($|\varphi|_T = \|\varphi\|_T$) does not generally hold for Gödel and product logic (see e.g. [4]).

Complete axiomatization for 1-tautologies in product logic was proposed in [6].

In [12] we introduced a procedural semantics for SLD-resolution with two continuous conjunctions (as we mentioned above) and prove soundness and completeness theorem (in the sense ($|\varphi|_P = \|\varphi\|_P$). But in [12] is not proved that the best possible answer is really attained during computation. We only proved that we can obtain answers with values which are arbitrary close to the best one (see Theorem 11). Now we are able to prove that during computation with one continuous conjunction ($\mathrm{et}^\bullet = \mathrm{et}_1^\bullet = \mathrm{et}_2^\bullet$) the best possible answer is really attained (see Theorem 14 and Theorem 17).

Moreover, we can state some more general remark. If the variant of fuzzy logic uses deduction rules based on t-norms, in the sense that the value of a consequent of a deduction step is t-norm value from values of antecedents, (it is not possible to increase value during deductions), then as a corollary of our Lemma 13 we have: If a fuzzy theory $T : Fml \to [0,1]$ is such that range of T, $T(Fml)$ is finite (even when theory T itself is infinite), then

$$|\varphi|_T = \max\{r; T \vdash (\varphi; r)\}$$

for every formula $\varphi \in Fml$. (We can write *max* instead of *sup* for $|\varphi|_T$.)

For information and references concerning motivations and applications of fuzzy logic programming see e.g. [12].

2. Declarative and procedural semantics

Let us recall several definitions and theorems from [11], [12], which we will need in the next sections.

Let $\mathcal{L}$ be a first order language containing variables, function symbols, predicate symbols, constants, quantifiers and connectives $\neg$, seq and et (intended meaning is that seq is an implication — the leftarrow writing version is qes and et is a conjunction). Connectives usually preserve rationality, i.e. if r, q are rational, then the value $\#^\bullet(r,q)$ is rational. The syntactical level is not touched by many valuedness of semantics.

We base our declarative semantics only on fuzzy Herbrand interpretations (skipping here arbitrary interpretations). Herbrand universe $U_{\mathcal{L}}$ consists of all ground terms, having function symbols we are going to interpret them crisp, Herbrand base $B_{\mathcal{L}}$ consists of all ground atoms. Note, that this step is not touched by fuzziness. An n-ary predicate symbol should be interpreted as a fuzzy subset of $U_{\mathcal{L}}^n$, i.e. as a mapping from $U_{\mathcal{L}}^n$ into the unit interval

$[0,1]$. Gluing together all fuzzy predicates we interpret all of them at once by a mapping $f : B_{\mathcal{L}} \to [0,1]$.

For a connective $\#$ the corresponding truth value function will be denoted by $\#^{\bullet}$ (i.e. a dot over the very connective). For arbitrary $x, y \in [0,1]$ put $\neg^{\bullet} x = 1 - x$ and connective $\mathrm{et}^{\bullet} : [0,1]^2 \to [0,1]$ is arbitrary t-norm, (i.e. commutative, associative, monotone in both coordinates, and with 1 as a neutral element). Let us make a notational agreement: for a conjunction et which is binary we often harm the arity, using associativness denoting multiple composition. The implication $\mathrm{seq}^{\bullet} : [0,1]^2 \to [0,1]$ is coupled with $\mathrm{et}^{\bullet}$ in such a way that modus ponens

$$\frac{(B,x),(B \;\mathrm{seq}\; A,y)}{(A,\mathrm{et}^{\bullet}(x,y))}$$

is a sound rule (see [2] and [3, 5]). This means that whenever $f(\varphi) \geq x$ and $f(\varphi \;\mathrm{seq}\; \psi) \geq y$, then $f(\psi) \geq \mathrm{et}^{\bullet}(x,y)$; denote this by MP(seq, et). Recall that for seq there is the largest $\mathrm{et}^{\bullet}_{\mathbf{seq}}(x,y) = \inf\{z; \mathrm{seq}^{\bullet}(x,z) \geq y\}$ for which is modus ponens sound and $\mathrm{seq}^{\bullet}(x,y) = \sup\{z : \mathrm{et}^{\bullet}_{\mathbf{seq}}(x,z) \leq y\}$ holds ([2]). So our assumption that $\mathrm{et}^{\bullet}$ evaluates modus ponens with $\mathrm{seq}^{\bullet}$ in a sound way means that $\mathrm{et}^{\bullet} \leq \mathrm{et}^{\bullet}_{\mathbf{seq}}$ holds.

Let $f : B_{\mathcal{L}} \to [0,1]$ be a fuzzy Herbrand interpretation. The truth value for ground atoms $A \in B_{\mathcal{L}}$ is defined to be $f(A)$. For arbitrary formula φ and an evaluation of variables $e : Var \to U_{\mathcal{L}}$ the truth value $f(\varphi)[e]$ is calculated using following rules along the complexity of formulas:

$$\begin{aligned}
f(p(t_1,\ldots,t_n))[e] &= f(p(t_1[e],\ldots,t_n[e])) \\
f(\neg\varphi)[e] &= \neg^{\bullet}(f(\varphi)[e]) \\
f(\varphi \;\mathrm{seq}\; \psi)[e] &= \mathrm{seq}^{\bullet}(f(\varphi)[e], f(\psi)[e]) \\
f(\varphi \;\mathrm{et}\; \psi)[e] &= \mathrm{et}^{\bullet}(f(\varphi)[e], f(\psi)[e]) \\
f((\forall x)\varphi)[e] &= \inf\{f(\varphi)[e'] : e' =_x e\}
\end{aligned}$$

where $e' =_x e$ means that e' can differ from e only at x.

Finally let truth value of a formula φ under fuzzy Herbrand interpretation f be same as that of its generalization and does not depend on evaluation:

$$f(\varphi) = f(\forall\varphi) = \inf\{f(\varphi)[e] : e \text{ arbitrary}\}.$$

Definition 2.1. (See [3, 5].) *A* fuzzy theory *is a partial mapping T assigning formulas a rational number . Partiality of the mapping T we understand as of being defined constantly zero outside of the domain* $\mathrm{dom}(T)$. *A fuzzy Herbrand interpretation f is a* model *of a fuzzy theory T if for all formulas $\varphi \in \mathrm{dom}(T)$ we have $f(\varphi) \geq T(\varphi)$.*

A (seq, et)-definite program clause is a formula $\forall((B_1 \,\mathrm{et} \cdots \mathrm{et}\, B_n) \;\mathrm{seq}\; A)$, where $A, B_1, \ldots, B_n$ are atoms. We often write it in the leftarrow form as A qes $B_1, \ldots, B_n$, where qes is the leftarrow writing of seq, commas in the antecedent denote conjunction et. Similarly we define (seq, et)-facts and goals. The empty clause is denoted by $\Box$.

Let the symbol $\approx$ denote the following equivalence on the set of all formulas: $\varphi \approx \psi$ if φ is a variant of ψ.

Definition 2.2. *A fuzzy theory P is called a* fuzzy (seq, et)-definite program, *if*

1. $\mathrm{dom}(P)$ *is a set of* (seq, et)*-definite program clauses or facts,*
2. $\mathrm{dom}(P)/_{\approx}$ *is finite*
3. *for* $\varphi \approx \psi$ *and* $\varphi \in \mathrm{dom}(P)$ *we have* $\psi \in \mathrm{dom}(P)$ *and* $P(\varphi) = P(\psi) > 0$.

Let us recall several notions and facts concerning procedural semantics ([11], [12]). Following P. Hájek ([3],[5]) we define a graded formula being a pair $(\varphi; r)$, where φ is a formula and $r \in [0,1]$ is a rational number. Especially, $(A \text{ qes}; r)$, $(A \text{ qes } B_1, \dots, B_n; r)$, and $(\text{qes } B_1, \dots, B_n; r)$ are a graded fact, a graded clause, and a graded goal, respectively.

Definition 2.3. *Let* $G = (\text{qes } A_1, \dots, A_m, \dots, A_k; r)$ *and* $C = (A \text{ qes } B_1, \dots, B_l;$
$q)$ *be a graded goal and a graded clause, respectively. Then a graded goal G' is f-derived from G and C using mgu θ if the following conditions hold:*

1. A_m *is an atom, called the selected atom in* G
2. θ *is a mgu of* A_m *and* A
3. $G' = (\text{qes}(A_1, \dots, A_{m-1}, B_1, \dots, B_l, A_{m+1}, \dots, A_k)\theta; r \,\text{et}^{\bullet}\, q)$.

Definition 2.4. *Let P be a fuzzy* (seq, et)*-definite program and let H be a* (seq, et)*-definite goal. A pair $(\theta; r)$ consisting of a substitution θ and a rational number r is a* graded computed answer *(GCA) for P and H if there is a sequence $G_0, \dots, G_n$ of graded goals, a sequence $D_1, \dots, D_n$ of suitable variants of clauses from the domain of P and a sequence $\theta_1, \dots, \theta_n$ of mgu's such that*

1. $G_0 = (H; 1)$
2. G_{i+1} *is f-derived from G_i and* $(D_{i+1}; P(D_{i+1}))$
3. $\theta = \theta_1 \circ \dots \circ \theta_n$ *restricted to variables of H*
4. $G_n = (\Box; r)$

($G_0, \dots, G_n$ is called a graded SLD-refutation*).*

Definition 2.5. *A pair $(x; \theta)$ consisting of a real number r and a substitution θ is a* fuzzy Herbrand correct answer *for a fuzzy* (seq, et)*-definite program P and* et*-goal* $H = \neg\exists(A_1\text{et} \cdots \text{et}A_n)$ *if for all fuzzy Herbrand interpretations* $f : B_{\mathcal{L}} \to [0,1]$ *which is a model of P we have* $f(\forall((A_1 \,\text{et} \cdots \text{et}\, A_n)\theta)) \geq x$.

Observation. Let us observe the connection between procedural semantics defined above and that of [12]. In [12] we used two different conjunctions et_1 and et_2, so we could not use commutativity and associativity law in full scope. We had to store values for atoms in the bodies of clauses (connected by et_2) until we have all of them and only then we used et_1 for evaluation of modus ponens. Now we have only one conjunction et, used for evaluation in the body of clause as well as for evaluation of modus ponens. Hence, from now on we can use several theorems, (which were fully proved in [12]) also for procedural semantics used here, as it is just the special case for $\mathrm{et}_1^{\bullet} = \mathrm{et}_2^{\bullet}$.

To finish this part (most of material is a slight modification of that in [11], [12]), let us state the Soundness Theorem (cf. [11],[12]) in the form :

Theorem 2.1 (Soundness for fuzzy (seq, et)-SLD-resolution). *Assume* MP(seq, et). *Let P be a fuzzy* (seq, et)*-definite program and H an* et*-goal. Let $(r;\theta)$ be a graded computed answer for P and H. Then $(r;\theta)$ is a fuzzy Herbrand correct answer.*

3. Approximate completeness of fuzzy SLD-resolution

In our proof we follow [11],[12] with a fuzzy analogy of classical crisp fixpoint approach of [1],[8]. Let us recall some notations (see [8],[1],[12]). Let $(L, \leq, \bot, \top)$ be a partial order with smallest ($\bot$) and largest ($\top$) element. L is a complete lattice if for all $X \subseteq L$ the least upper bound $\mathrm{lub}(X)$ and greatest lower bound $\mathrm{glb}(X)$ exists. A set $X \subseteq L$ is directed if every finite subset of X has an upper bound in X. A mapping $T : L \to L$ is monotone, if $x \leq y$ implies $T(x) \leq T(y)$ and moreover it is continuous if $T(\mathrm{lub}(X)) = \mathrm{lub}(T(X))$ holds for every directed subset X of L. Note that for monotone mappings $T(\mathrm{lub}(X)) \geq \mathrm{lub}(T(X))$ holds for all X. We say $a \in L$ is the least fixpoint of T if

$$a = \mathrm{lfp}(T) = \mathrm{glb}\{x : T(x) = x\} = \mathrm{glb}\{x : T(x) \leq x\}.$$

There is another characterization of $\mathrm{lfp}(T)$. Denote by transfinite induction

$$\begin{aligned} T{\uparrow}0 &= \bot \\ T{\uparrow}\alpha &= T(T{\uparrow}(\alpha - 1)) \text{ for } \alpha \text{ successor} \\ T{\uparrow}\alpha &= \mathrm{lub}\{T{\uparrow}\beta; \beta < \alpha\} \text{ for } \alpha \text{ limit} \end{aligned}$$

Then for a complete lattice L and a continuous mapping $T : L \to L$ holds true that $\mathrm{lfp}(T) = T{\uparrow}\omega$.

Denote

$$\mathcal{F}_P = \{f : f \text{ is a mapping from } B_P \text{ into } [0,1]\} = [0,1]^{B_P}.$$

Let functions 0_{B_P} be constantly zero and 1_{B_P} constantly one on B_P and for $f, g \in \mathcal{F}_P$ let $f \leq g$ holds if for all $A \in B_P$ is $f(A) \leq g(A)$. Then $(\mathcal{F}_P, \leq, 0_{B_P}, 1_{B_P})$ is a complete lattice where for $X \subseteq [0,1]^{B_P}$ $\mathrm{lub}(X)(A) = \sup\{f(A); A \in X\}$ and $\mathrm{glb}(X)(A) = \inf\{f(A) : A \in X\}$. Moreover X is directed if for $f_1, \ldots, f_n \in X$ is $\max\{f_1, \ldots, f_n\} \in X$.

Definition 3.1. (Definition 7 in [12]) *Let P be a* (seq, et)*-definite program. The mapping $T_P : \mathcal{F}_P \to \mathcal{F}_P$ defined bellow we call the* (seq, et, P)*-operator (if the context is doubtless simply called operator).*

For $f \in \mathcal{F}_P$, let $T_P(f)$ is a mapping from B_P into $[0,1]$ defined $T_P(f)(A) = \sup\{r :$ *there is* $(A \text{ qes } A_1, \ldots, A_n)$ *a ground instance of* $C \in \mathrm{dom}(P)$ *and* $r = P(C) \,\mathrm{et}^{\bullet}\, f(A_1) \,\mathrm{et}^{\bullet} \cdots \mathrm{et}^{\bullet}\, f(A_n)\}$.

Note, that for a fact $(A \text{ qes.}) \in \mathrm{dom}(P)$ the list $A_1, \ldots, A_n$ is empty and we understand $r = 1 \,\mathrm{et}^{\bullet}\, P(C) = P(C)$ in the previous definition.

Let us also observe that range of $f \in \mathcal{F}_P$ can have infinitely many values and there can be infinitely many values for different ground instances of $A_1, \ldots, A_n$. So we cannot write *max* instead of *sup* in the previous definition.

Theorem 3.1 (Fixpoint character of the least fuzzy Herbrand model). (Theorem 10 in [12]) *Assume* $\mathrm{et}^{\bullet} = \mathrm{et}^{\bullet}_{\mathrm{seq}}$ *is continuous. Let P be a fuzzy* (seq, et)*-definite program and T_P is the corresponding* (seq, et, P) *operator. Then*

$$\mathrm{lfp}(T_P) = T_P{\uparrow}\omega = M_P$$

where M_P is the $([0,1]^{B_P}, \leq)$ *least fuzzy Herbrand model of P.*

Definition 3.2. (Definition 11 in [12]) *Define success-fuzzy Herbrand interpretation of P as $f_{s(P)} : B_P \to [0,1]$ by*

$$f_{s(P)}(A) = \sup\{r : (r, \mathrm{id}) \textit{ is GCA for } P \textit{ and } A\}.$$

Theorem 3.2. (Theorem 12 in [12]) *Assume* $\mathrm{et}^{\bullet} = \mathrm{et}^{\bullet}_{\mathrm{seq}}$ *is continuous. Let P be a fuzzy* (seq, et)*-definite program. Then*

$$f_{s(P)} = M_P,$$

i.e. the success-fuzzy Herbrand interpretation of P is equal to least fuzzy Herbrand model of P.

In [12] we proved completeness theorem in the form that during SLD-resolution we can obtain answers which values are arbitrary close to the best one.

Theorem 3.3. (Theorem 13 in [12]) *Assume* $\mathrm{et}^{\bullet} = \mathrm{et}^{\bullet}_{\mathrm{seq}}$ *is continuous. Let P be a fuzzy* (seq, et)*-definite program and G an* et*-definite goal. For every $(x; \theta)$ a fuzzy Herbrand correct answer for P and G and for every $\epsilon > 0$ there exists a* (seq, et)*-graded computed answer $(q; \sigma)$ for P and G such that $x - \epsilon \leq q$ and $\theta = \sigma\gamma$ for some γ.*

4. Attaining supremum value

Before we formulate additional statement to the Fixpoint characterization of the least fuzzy Herbrand model, let us observe that elements in $T_P(f)(A)$ are values of expressions like

$$q_1 \,\mathrm{et}^{\bullet}\, q_2 \,\mathrm{et}^{\bullet} \ldots \mathrm{et}^{\bullet}\, q_k,$$

where q_i are some of confidence factors of definite program clauses. Assume in general case, that $\mathrm{et}^{\bullet} = \mathbf{t}$, where $\mathbf{t}$ is an arbitrary t-norm. Observe that $x \,\mathbf{t}\, y \leq x \,\mathbf{t}\, 1 = x$ and $x \,\mathbf{t}\, y \leq 1 \,\mathbf{t}\, y = y$ for all $x, y \in [0,1]$.

Definition 4.1. *Let* $\mathbf{t}$ *be an arbitrary t-norm and* $U \subset [0,1]$. *Denote by* $V(U)$ *a set of all values of all terms (expressions)*

$$q_1 \,\mathbf{t}\, q_2 \,\mathbf{t} \cdots \mathbf{t}\, q_k,$$

for $q_1, q_2, \ldots, q_k \in U$ *and* $k = 1, 2, \ldots$

Our key lemma is the following

Lemma 4.1. *If a set* $U \subset [0,1]$ *is finite then the set* $V(U)$ *does not contain any infinite strictly increasing sequence.*

Proof. Let n be the number of elements of a set U. The proof goes by induction along n, the number of values used ($n = |U|$).

I. Let $n = 1$, i.e. $U = \{q_1\}$. In this case it is possible to form only terms like

$$q_1, \quad q_1 \,\mathbf{t}\, q_1, \quad q_1 \,\mathbf{t}\, q_1 \,\mathbf{t}\, q_1, \ldots$$

Because of monotonicity of $\mathbf{t}$ it is obvious that a longer expression cannot have bigger value than a shorter one. Hence, it is impossible to form any infinite strictly increasing sequence.

II. Let the statement of the lemma holds for all $n \leq k$. We have to prove it for $n = k+1$. Let $U = \{q_1, \ldots, q_k, q_{k+1}\}$. Thanks to commutativity and associativity of t-norm $\mathbf{t}$ we can divide the expression

$$q_{i_1} \,\mathbf{t} \cdots \mathbf{t}\, q_{i_l}, \qquad q_{i_1}, \ldots q_{i_l} \in U,\ l \in N,$$

into two parts. The first one (initial) contains only elements from the set $\{q_1, \ldots,$
$q_k\}$ and the second one (terminal) contains only several occurrences of the element q_{k+1}. Of course, every part can be empty, but not both of them in the same time.

Assume by the contrary, that the set $V(U)$ contains infinite strictly increasing sequence of elements. This sequence contains subsequence, for which the lengths of expressions are also increasing. In this subsequence we can select another one, in which at least one of the two parts must be built from subexpressions of increasing length. Let us consider the following cases:

1. There is a maximal length of the first parts, (only the second parts have increasing lengths of subexpressions). Then there is at least one instance of the initial part, for which there are infinitely many continuations in the second parts. We know that if $x \mathbf{t} y < x \mathbf{t} z$ then $y < z$. (For, if $y \geq z$ then from the definition of t-norm we have $x \mathbf{t} y \geq x \mathbf{t} z$.) Hence the second parts in these continuations form strictly increasing infinite sequence. This is a contradiction.

2. There is a maximal length of the second parts, (only the first parts have increasing lengths of subexpressions). Similarly, as in case 1, we can find infinite strictly increasing sequence of values of subexpressions in the first parts, what is again a contradiction.

3. It remains the case, when the lengths of both parts are increasing. If the first parts contain infinite strictly increasing subsequence we have an contradiction to induction hypothesis immediately. If the first parts contain infinite constant subsequence, then the second parts form an infinite strictly increasing sequence (similarly as in case 1 and 2). Again, we get a contradiction. Finally, assume that the first parts contain infinite strictly decreasing subsequence

$$b_1 > b_2 > \ldots > b_i > \ldots$$

In connection with values $c_1, c_2, \ldots, c_i, \ldots$ of the second parts we have by our assumption that

$$b_1 \mathbf{t}\, c_1 < b_2 \mathbf{t}\, c_2 < \cdots < b_i \mathbf{t}\, c_i < \ldots$$

If we assume that $c_i \geq c_{i+1}$ for some i then $b_i \mathbf{t}\, c_i \geq b_{i+1} \mathbf{t}\, c_i \geq b_{i+1} \mathbf{t}\, c_{i+1}$. We get contradiction, so $c_i < c_{i+1}$ for all $i = 1, 2, \ldots$ We have found infinite strictly increasing sequence in the second parts, what is a contradiction to induction hypothesis. Hence the lemma is proved. □

On the base of previous Lemma 13 we can state

Theorem 4.1. *For every $A \in B_P$ there is a number n_0 such that for every $n \geq n_0$*

$$(T_P{\uparrow}n_0)(A) = (T_P{\uparrow}n)(A) = (T_P{\uparrow}\omega)(A),$$

i.e. every element of $T_P \uparrow \omega$ attains his maximal value.

Proof. Let $A \in B_P$. Every element of $T_P \uparrow 0$ has value 0, hence $(T_P \uparrow 0)(A) = 0$. If the program P contains a fact Cqes., where $P(C) = r$, and A is a ground instance of C, then after first step of iteration we have $(T_P \uparrow 1)(A) = r$; otherwise $(T_P \uparrow 1)(A) = 0$. From the definition of T_P follows that every element of $T_P \uparrow \omega$ is a value of an expression $q_1 \,\mathrm{et}^{\bullet} \cdots \mathrm{et}^{\bullet}\, q_k$, where $q_1, \ldots, q_k$ are in the range of P. Recall that $\mathrm{dom}(P)/_{\approx}$ is finite, so the range of P is finite as well. Hence from our Lemma 13 we have that there is no infinite strictly increasing sequence of elements of $T_P \uparrow \omega$. It means, that after a finite number n_0 of steps of iteration, the value $(T_P \uparrow n_0)(A)$ attains the maximal value, which cannot increase during further iterations. (Of course, this value can remain 0.) □

Let us recall two lemmas from [8]:

Lemma 4.2 (Mgu Lemma). (Lemma 8.1 in [8], p. 47) *Let P be a definite program and G a definite goal. Suppose that $P \cup \{G\}$ has an unrestricted SLD-refutation, i.e. the unifiers used need not be most general. Then $P \cup \{G\}$ has an SLD-refutation of the same length such that, if $\theta_1, \ldots, \theta_n$ are the unifiers from the unrestricted SLD-refutation and $\theta'_1, \ldots, \theta'_n$ are mgu's from the SLD-refutation, then there exists a substitution γ such that $\theta_1 \ldots \theta_n = \theta'_1 \ldots \theta'_n \gamma$.*

Lemma 4.3 (Lifting Lemma). (Lemma 8.2 in [8], p. 47) *Let P be a definite program, G a definite goal and θ a substitution. Suppose there exists an SLD-refutation of $P \cup \{G\theta\}$. Then there exists an SLD-refutation of $P \cup \{G\}$ of the same length such that, if $\theta_1, \ldots, \theta_n$ are the mgu's from the SLD-refutation of $P \cup \{G\theta\}$ and $\theta'_1, \ldots, \theta'_n$ are the mgu's from the SLD-refutation of $P \cup \{G\}$, then there exists a substitution γ such that $\theta_1 \ldots \theta_n = \theta'_1 \ldots \theta'_n \gamma$.*

Now we are in the position to give promised refinement of Completeness Theorem (we can drop all things concerning ϵ).

Theorem 4.2 (Tight completeness of fuzzy (seq, et)**-SLD-resolution).** *Assume* $\mathrm{et}^{\bullet} = \mathrm{et}^{\bullet}_{\mathrm{seq}}$ *is continuous. Let P be a fuzzy* (seq, et)*-definite program and G an* et*-definite goal. For every $(x; \theta)$ a fuzzy Herbrand correct answer for P and G there exists a* (seq, et)*-combined graded computed answer $(q; \sigma)$ for P and G such that $x \leq q$ and $\theta = \sigma\gamma$ for some γ.*

Proof. (Simplification of that from [12].) Let us observe that Theorem 10 implies completeness result for ground atoms: Assume $(x; \theta)$ is a correct answer for P and a goal consisting of ground atom A then $x \leq M_P(A\theta) = M_P(A) = f_{s(P)}(A)$. From the Theorem 14 follows that $f_{s(P)}(A)$ is the maximum of computed answers (maximum is attained), we are done. (Note that wlog we can assume that our goal consists of one atom.)

Now let A be an atom (not necessarily ground) and $(x; \theta)$ a fuzzy Herbrand correct answer for P and A. Let $\{X_1, \ldots, X_n\}$ be variables of $A\theta$ and let $\{a_1, \ldots, a_n\}$ be constants distinct from everything appearing in P and A. Denote $\delta = \{X_1/a_1, \ldots, X_n/a_n\}$. Then $A\theta\delta$ is ground and $x \leq M_P(\forall(A\theta)) \leq M_P(A\theta\delta)$. So by the result for the ground atom $A\theta\delta$ there is a GCA $(q; \mathrm{id})$ witnessed by a derivation $G_0, \ldots, G_l$ for P and $A\theta\delta$. Replacing a_i's in G_i by X_i's we get a successful derivation G'_i for P and $A\theta$ giving the same computed answer $(q; \mathrm{id})$. Now arguing in the same way as in Lemma 15 (Mgu Lemma) and Lemma 16 (Lifting Lemma) of Lloyd ([8],p. 47) we can find a sequence of mgu's $\theta'_1 \ldots \theta'_n = \sigma$ witnessing that the derivation G''_i obtained from G'_i only changing substitutions is a successful derivation, $\theta = \sigma\gamma$ and the computed answer $(q; \sigma)$ gives the same numerical value, because clauses and facts of P used along the derivation G''_i are the same as in G'_i (and same as in G_i). □

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