Decidability of the $\exists^*\forall^*$-Class in the Membership Theory NWL *

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Summary. Let NWL be the theory having the obvious axioms for the existence of the empty set and of the result of adding or removing an element from a set. The problem of establishing whether sentences of the form $\exists x_1 \ldots \exists x_n \forall y_1 \ldots y_m F$, with F quantifier free, are satisfiable with respect to NWL is decidable.

1. Introduction

The basic role set theoretic notions play in mathematics, especially in its foundation, makes it quite natural to enquire which 'fragment' of set theory Gödel's incompleteness and undecidability results apply to. Already in [22], Tarski addressed this problem stating that the *small axiomatic fragment* - to use Tarski's wording - formulated in the language $=, \in$, endowed with classical first order logic, whose axioms are

N:	$(\forall x)(x \notin \emptyset)$	(Null-set Axiom),
W:	$(\forall x)(\forall y)(\forall z)(x \in y \,\mathbf{w}\, z \leftrightarrow x \in y \vee x = z)$	(With Axiom),
E:	$(\forall x)(\forall y)((\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y))$	(Extensionality Axiom),

was sufficiently strong to interpret Robinson's Arithmetic **Q** (see also [6], [23]); a result recently improved by dropping the axiom E [11]. Later on R. Vaught establishes with a different method the essential undecidability of NW [21].

On the other hand, already in the early time of automated deduction, the *problem of handling the membership relation in an efficient way, so that the problems involving set theoretic notions can be treated* was brought to evidence by J. Robinson in [20]. Awareness of the importance of that problem has steadily increased and it is at present particularly evident in connection with the enhancement of declarative programming [1], [7], [8], [10]. Earlier, throughout the eighties, a project led by J.Schwartz at the Courant Institute (NYU), aiming at the development of a proof verifier for 'elementary set theory', conceived as a particularly important subdomain of mathematics, led to the discovery of various decision procedure to establish whether in the 'intended' model there are sets satisfying given constraints expressed with (mainly) unquantified formulae over some of the most basic set theoretic constructors [5].

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First limitations to what could be accomplished in that direction were established in [19]. As a matter of fact by a (standard) development of Gödel's incompleteness results combined with suitable coding, [14] establishes the essential undecidability of the theory NWLE obtained from NWL by adding the axiom

$$\mathrm{L} : (\forall x)(\forall y)(\forall z)(z \in x\,\mathrm{l}\,y \leftrightarrow z \in x \wedge z \neq y) \quad \text{(Less Axiom)}$$

with respect to existential closures of a restricted subclass of the formulae on the language $=, \in$, involving only the restricted quantifiers $\forall x \in y, \exists x \in y$ (Δ_0-formulae). Following [14] significant improvements of the limitative results, concerning both the theories and the class of sentences involved, have been obtained, noticeable the undecidability of the (logical) satisfiability of $\forall^*\exists$ sentences with respect to NWLE [3].

On the positive side some decision results (for membership theories) had been obtained in [25] as well as [9]. In particular [9] establishes by a model theoretic argument the completeness of ZFC with respect to $\exists^*\forall$ sentences. [13] improves this result by showing that the theory NWLER, where

$$\mathrm{R} : (\forall x)(x \neq \emptyset \rightarrow (\exists y)(y \in x \wedge (\forall z)(z \in y \rightarrow z \notin x))) \quad \text{(Regularity Axiom)},$$

is already complete with respect to $\exists^*\forall$ sentences. Furthermore [13] yields a number of positive results for such a class of sentences providing decision procedures (for their satisfiability) with respect to the theories NWLE,NWLR, and [18] extends such decidability results to the class of $\exists^*\forall\forall$-sentences with respect to the theory NWL.

In this work we establish the decidability of the full Bernays-Schönfinkel class, namely the class of $\exists^*\forall^*$-sentences with respect to the theory NWL. In the special case of the satisfiability of $\forall$ sentences the decision procedure we obtain does not differ significantly from the one already obtained in [13]. On the contrary for the special case of the $\forall\forall$-formulae our decision procedure is quite different from the one provided in [18] as it reduces to a set-(of finite graphs)-inclusion test, while the procedure presented in [18] consists in a number of attempts to build an Herbrand's model, searching for a construction which proceeds for sufficiently many steps. More importantly the method we present here works for the full Bernays-Schönfinkel class of which the $\forall\forall$ case is merely a special one, and it can be applied also to the Theory NWLR. The addition of the Extensionality Axiom makes matters combinatorially much harder, as it is suggested also by the existence of $\forall\forall$-formulae which are satisfiable but not finitely satisfiable when both the Extensionality and the Regularity Axiom are assumed and of $\forall\forall\forall$ formulae of this kind when the Extensionality Axiom alone is assumed (see [15], [16], [17]). As a matter of fact the decision problem for the full Bernays-Schönfienkel-class when the Extensionality Axiom is assumed is still open.

2. Reduction

A $\forall^m$-formula with free variables $x_1, \ldots, x_\ell$,

$$\forall y_1, \ldots, y_m A(x_1, \ldots, x_\ell, y_1, \ldots, y_m)$$

with A quantifier free in the language $\{\in, =\}$, is logically equivalent to a disjunction of formulae of the following form

$$B(x_1, \ldots, x_n) \wedge \forall y_1, \ldots, y_m (y_1 \neq x_1 \wedge \ldots \wedge y_m \neq x_n \rightarrow C(x_1, \ldots, x_n, y_1, \ldots, y_m))$$

to be abbreviated as $B(\mathbf{x}) \wedge \forall \mathbf{y} \neq \mathbf{x}\, C(\mathbf{x}, \mathbf{y})$, where $B(x_1, \ldots, x_n)$ has the form

$$\bigwedge_{1 \leq i \neq j \leq n} x_i \neq x_j \wedge \bigwedge_{1 \leq i,j \leq n} x_i \in_{i,j} x_j$$

with $\in_{i,j}$ either $\in$ or $\notin$, and all the literals appearing in C contain at least one of the variables $y_1, \ldots, y_m$.

Therefore the satisfiability problem for $\exists^*\forall^*$-sentences in the language $\in, =$ is reducible to the satisfiability problem for sentences of the form

$$B(\mathbf{c}) \wedge \forall \mathbf{y} \neq \mathbf{c}\, C(\mathbf{c}, \mathbf{y}) \tag{2.1}$$

where $\mathbf{c} = (c_1, \ldots c_n)$ is an n-tuple of distinct new constants replacing the free variables $x_1, \ldots, x_n$. $B(\mathbf{c})$ naturally induces a graph G over $\{1, \ldots, n\}$ by letting $(i, j) \in G$ iff $c_i \in c_j$ is a conjunct in $B(\mathbf{c})$.

In the sequel we will let F denote a sentence over $c_1, \ldots, c_n, \in, =$ of the form 2.1 above. We now reduce the satisfiability of such a sentence F to the set inclusion between two finite collections of graphs over $\{1, \ldots, n, n+1, \ldots, n+m\}$.

Definition 2.1. *$\Gamma_m(F)$ is the collection of $n+m$-graphs, i.e. graphs over $\{1, \ldots, n+m\}$ which satisfy F when $c_1, \ldots, c_n$ are interpreted with $1, \ldots, n$ respectively, $\in$ with the graph's relation and $=$ with the identity relation.*

Definition 2.2. *Let M be an interpretation of $\{c_1, \ldots, c_n, \in, =\}$ such that $=_M$ is an equivalence relation congruent with respect to $\in_M$ having more than $n+m$ $=_M$-equivalence classes and the interpretations $e_1, \ldots, e_n$ of $c_1, \ldots, c_n$ are not $=_M$ related. Then $\Gamma_m(M)$ is the collection of the $n+m$-graphs induced over $\{1, \ldots, n+1, \ldots, n+m\}$ by $\in_M$ restricted to $\{e_1, \ldots, e_n, a_1, \ldots, a_m\}$ where $a_1, \ldots, a_m$ is an m-tuple of elements of M not $=_M$-related to each other nor with any of $e_1, \ldots, e_n$.*

Proposition 2.1. *Let M be an interpretation of $\{c_1, \ldots, c_n, \in, =\}$ such that $=_M$ is an equivalence relation congruent with respect to $\in_M$ having more than $n+m$ $=_M$-equivalence classes and the interpretations of $c_1, \ldots, c_n$ are distinct. Then*

$$M \models F \quad \textit{iff} \quad \Gamma_m(M) \subseteq \Gamma_m(F) .$$

Proof. Let $M/=_M$ be the quotient structure of M with respect to $=_M$. Then $M \models F$ iff $M/=_M \models F$. Since $|M/=_M| \geq n+m$ $\Gamma_m(M)$ and $\Gamma_m(M/=_M)$ are both defined and $\Gamma_m(M) = \Gamma_m(M/=_M)$. Furthermore for any normal interpretation N of $\{c_1, \dots, c_n, \in, =\}$, as it is straightforward to check, N $\models F$ iff $\Gamma_m(N) \subseteq \Gamma_m(F)$. □

Proposition 2.2. *If M is an interpretation of $c_1, \dots, c_n, \in, =$ which is a model of NWL then*

$$M \models F \quad \textit{iff} \quad \Gamma_m(M) \subseteq \Gamma_m(F) .$$

Proof. It follows immediately from the previous proposition since if $M \models NWL$ then $=_M$ determines infinitely many $=_M$-equivalence classes. □

From now on we will refer to the skolemized version of NWL.

Definition 2.3. *H_n is the Herbrand's preinterpretation of the language $\emptyset, c_1, \dots, c_n, \mathbf{w}, \mathbf{l}$, namely the collection of closed terms of $\emptyset, c_1, \dots, c_n, \mathbf{w}, \mathbf{l}$, with the canonical interpretation of the constants and of the functional symbols* $\mathbf{w}, \mathbf{l}$.

Since F is a universal sentence, F is satisfiable with respect to NWL iff there is an Herbrand's model of NWL over H_n in which F is true. Therefore, by the previous Proposition 2.2, the satisfiability problem for F with respect to NWL is reducible to the problem of determining whether there exists an Herbrand's model M, over H_n, of NWL such that $\Gamma_m(M) \subseteq \Gamma_m(F)$. The latter problem is reducible to the problem of determining whether a family of finite set of graphs $\mathit{\Gamma}$ contains an element Γ such that $\Gamma \subseteq \Gamma_m(F)$ provided $\mathit{\Gamma}$ fulfills the following two conditions:

1. for every $\Gamma \in \mathit{\Gamma}$ there exists an Herbrand's model M of NWL such that $\Gamma_m(M) \subseteq \Gamma$;
2. for every Herbrand's model M of NWL there exists $\Gamma \in \mathit{\Gamma}$ such that $\Gamma_m(M) \subseteq \Gamma$.

In fact from the previous proposition, given such a π, F is satisfiable in a model of NWL iff there is $\Gamma \in \mathit{\Gamma}$ such that $\Gamma \subseteq \Gamma_m(F)$. Thus our decision problem is reduced to the problem of effectively determining a family $\mathit{\Gamma}$ which fulfills the above conditions.

3. The Decision Test

3.1 Defining $\mathit{\Gamma}$

Let $M = (H_n, \in_M, =_M)$ be an Herbrand's model of NWL. A term t in M has the form $c_h \bullet_1 t_1 \dots \bullet_\ell t_\ell$, where $\ell \in \ltimes$, c_h is a constant in $\{c_0 = \emptyset, c_1, \dots c_n\}$

and $\bullet_i \in \{\mathbf{w}, \mathbf{l}\}$. We will call c_h the 'seed' of t. Let t', t'' be terms in H_n; we define by induction on the construction of terms the binary relations 'to be added in' and 'to be removed in' (with respect to $=_M$) as follows:

if t'' is a constant then t' is not added nor removed in t''
if $t'' = t_1 \bullet t_2$ then t' is added in t'' iff t' is added in t_1 and $t' \neq_M t_2$ or $\bullet = \mathbf{w}$ and $t' =_M t_2$,
t' is removed in t'' iff t' is removed in t_1 and $t' \neq_M t_2$ or $\bullet = \mathbf{l}$ and $t' =_M t_2$.

We say that t' is added (removed) syntactically in t'' if t' is added (removed) in t'' with respect to the syntactical identity.

Due to the form of F we can restrict our attention to Herbrand's models M over H_n such that $c_i \neq_M c_j$, for $i \neq j$. Let S_M be the function that maps a term t in the set

$$S_M(t) = \{i : t \in_M c_i, 1 \leq i \leq n\} \ .$$

Since $M \models$ NWL, given an m-tuple of terms $\mathbf{t} = (t_1, \dots, t_m)$ in M not $=_M$ related to each other nor with any of the c_i's, the $n+m$-graph induced over $\{1, \dots, n+m\}$ is uniquely determined by:

1. the restriction G of $\in_M$ to $\{c_1, \dots, c_n\}$;
2. the m-tuple $\sigma_M(\mathbf{t}) = (sit_M(t_1), \dots, sit_M(t_m))$ where $sit_M(t)$ is the triple (i, J, I) where c_i is the seed of t, $J = \{j : c_j \in_M t\}$, $I = S_M(t)$;
3. the map $B_M(\mathbf{t}) : \{1, \dots, m\}^2 \to \{\mathbf{w}, \mathbf{l}, \times\}$ defined by letting

$$B_M(\mathbf{t})(i,j) \;=\; \left[\begin{array}{ll} \mathbf{w} & \text{if } t_i \text{ is added in } M \text{ to } t_j \\ \mathbf{l} & \text{if } t_i \text{ is removed in } M \text{ from } t_j \\ \times & \text{otherwise} \end{array}\right.$$

We will call m-situation of $\mathbf{t} = (t_1, \dots, t_m)$ in M the tern $(G, \sigma_M(\mathbf{t}), B_M(\mathbf{t}))$.

Abstracting from any given interpretation we give the following further definition:

Definition 3.1. *Let $\mathcal{G}$ be the family of n-graph. An m-situation is a tern of the form (G, σ, B) where*
$G \in \mathcal{G}$;
σ is an m-tuple $s_1, \dots, s_m$, with $s_h = (i_h, I_h, J_h)$, $1 \leq i_h \leq n$, $I_h \subseteq \{0, 1, \dots, n\}$, $J_h \subseteq \{1, \dots, n\}$;
$B : \{1, \dots, m\}^2 \to \{\mathbf{w}, \mathbf{l}, \times\}$.

Given an m-situation (G, σ, B) and an m-tuple of terms $\mathbf{t}$ in a model M which induces the graph G over $\{1, \dots, n\}$, we say that $\mathbf{t}$ realizes σ, B if $\sigma = (sit_M(t_1), \dots, sit_M(t_m))$ and B is the map from $\{1, \dots, m\}^2$ into $\{\mathbf{w}, \mathbf{l}, \times\}$ induced by $(t_1, \dots, t_m)$.

An m-situation (G, σ, B), with $\sigma = (s_1, \dots, s_m)$, $s_h = (i_h, I_h, J_h)$ determines an $n+m$-graph $Ghp(G, \sigma, B) = (\{1, \dots, n, n+1, \dots, n+m\}, \in_E)$ by letting

$i \in_E j$ if $i, j \leq n$ and $i \in_G j$,
$i \in_E h$ if $i \leq n < h \leq n+m$ and $i \in J_h$,
$h \in_E i$ if $i \leq n < h \leq n+m$ and $i \in I_h$,
$h \in_E k$ if $n < h, k \leq n{+}m$ and $B_{h-n,k-n} = \mathbf{w}$ or $c_{i_k} \in I_h$ and $B_{h-n,k-n} \neq \mathbf{l}$.

It is immediate that the $n+m$-graph induced by $t_1, \ldots, t_m$ is precisely the graph induced in that way by its situation. The m-situation (G, σ, B) which induces the $n+m$-graph $(\{1, \ldots, n+m\}, \in_E)$, unless B takes only values in $\{\mathbf{w}, \mathbf{l}\}$ is by no means unique, as a matter of fact it is immediate to verify that the following holds:

Proposition 3.1. *If the m-situation (G, σ, B) induces the relation E over $\{1, \ldots, n+m\}$, and B' is obtained from B by changing $(i, j, \times)$ with $(i, j, \mathbf{w})$, if $(i \in_E j)$ or with $(i, j, \mathbf{l})$, if $(i \notin_E j)$, then $Ghp(G, \sigma, B') = Ghp(G, \sigma, B)$.*

Definition 3.2. *Let $\mathcal{G}$ be the family of n-graph;*
$\mathcal{I}$ the family of maps $I : \{0, 1, \ldots, n\} \times \mathcal{P}(\{1, \ldots, n\}) \to \mathcal{P}(\{1, \ldots, n\})$;
$\mathcal{B}_m$ the family of maps $B : \{1, \ldots, m\}^2 \to \{\mathbf{w}, \mathbf{l}, \times\}$ such that $B(i, j) \in \{\mathbf{w}, \mathbf{l}\}$ if $i < j$, $B(i, j) = \times$ otherwise;
For π a permutation of $\{1, \ldots, m\}$ and $B \in \mathcal{B}_m$, B^π-the permutation of B under π- is defined by letting, for $1 \leq i, j \leq m$, $B^\pi(i, j) = B(\pi(i), \pi(j))$.
For G an n-graph and $I \in \mathcal{I}$,
$\Gamma_m(G, I) = \{Ghp(G, \sigma, B^\pi) : \sigma \in I^m, B \in \mathcal{B}_m, \pi$ permutation of $\{1, \ldots, m\}\}$.

The family Γ having the desired properties, as we will show, is the collection of the finite set of $n+m$-graphs $\{\Gamma_m(G, I) : G \in \mathcal{G}, I \in \mathcal{I}\}$.

Example 3.1. By way of illustration we offer a few examples of how the method could be used to determine the satisfiability/unsatisfiability of specific formulae.

Let $m = n = 1$ and

$$\begin{aligned} F_1 &= \forall y(y \notin c) \Leftrightarrow c \notin c \wedge \forall y \neq c\,(y \notin c) \\ F_2 &= c \notin c \wedge \forall y \neq c\,(y \in c) \\ F_3 &= c \notin c \wedge \forall y \neq c\,(y \in c \leftrightarrow c \in y). \end{aligned}$$

Let I_1 and I_2 be the constant maps on $\{0, 1\} \times \mathcal{P}(\{1\})$ with value $\emptyset$ and $\{1\}$ respectively, and

$$I_3 = \{(0, \emptyset, \emptyset), (0, \{1\}, \{1\}), (1, \emptyset, \emptyset), (1, \{1\}, \{1\})\} .$$

Since the family $\mathcal{B}_1$ consists of the single map $B = \{((1, 1), \times)\}$, $\Gamma_1(G, I_i)$ is determined by four 1-situations of the form: $\{(G, \sigma, B) : \sigma \in I_i\}$. We have that

$\Gamma_1(G, I_1) = \{\emptyset, \{(1, 2)\}\}$
$\Gamma_1(G, I_2) = \{\{(2, 1)\}, \{(1, 2), (2, 1)\}, \{(2, 1), (2, 2)\}, \{(1, 2), (2, 1), (2, 2)\}\}$;
$\Gamma_1(G, I_3) = \{\emptyset, \{(1, 2), (2, 1)\}, \{(1, 2), (2, 1), (2, 2)\}\}$;

where we have omitted to explicitly indicate the common domain $\{1,2\}$. As it is easy to see, for $i = 1,2,3$, $\Gamma_1(G,I) \subseteq \Gamma_1(F_i)$ iff $I = I_i$. That makes it easy to establish the unsatisfiability of the following formula

$$F_3' = F_3 \wedge \forall x \neq c(c \notin x \vee x \notin c \vee x \notin x) \ .$$

In fact, since $F_3' \to F_3$ the only $\Gamma \in \Gamma_1$ that contains models of F_3' is $\Gamma_1(G,I_3)$, but $\Gamma_1(G,I_3) \not\subseteq \Gamma(F_3')$ since $(\{1,2\},\{(1,2),(2,1),(2,2)\}) \not\models F_3'$.

For $n = 1$ $m = 2$, we do not explicitly list any $\Gamma_2(G,I)$, since 128 different 2-situation have to be taken into account to determine $\Gamma_2(G,I)$. Consider however the formula:

$$F_4 = c \notin c \wedge \forall x_2, x_3 \neq c(x_2 \in c \wedge x_3 \in c) \ .$$

Clearly the only $\Gamma(G,I)$ that is contained in $\Gamma(F_4)$ is $\Gamma_2(G,I_4)$, where I_4 is the constant map with value $\{1\}$. In particular F_4 is satisfiable.

$\Gamma_2(G,I_4)$ must contains, for instance, the graphs induced by the following two 2-situations:

$$(G,((0,\emptyset,\{1\}),(1,\{1\},\{1\})),B^w) \quad \text{and}$$
$$(G,((0,\emptyset,\{1\}),(1,\{1\},\{1\})),B_l),$$

where $B^w = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \times & \mathbf{w} \\ 2 & \times & \times \end{array}$ and $B_l = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \times & \times \\ 2 & \ell & \times \end{array}$.

They are respectively the following:

$$(\{1,2,3\},\{(1,2),(2,1),(1,1),(3,1),(2,3),(3,2)\})$$
$$(\{1,2,3\},\{(1,2),(2,1),(1,1),(3,1)\})$$

That suffices to establish the unsatisfiability of the following formula:

$$F_4' = F_4 \wedge \forall x_2, x_3 \neq c((c \in x_2 \wedge c \notin x_3) \to (x_2 \notin x_2 \vee x_2 \in x_3 \vee x_3 \in x_2)) \ .$$

3.2 Proving the Conditions on Γ

Proposition 3.2. *For every graph G over $\{1,\ldots,n\}$ and function $I \in \mathcal{I}$ there is a normal Herbrand's model $H(G,I)$ over H_n such that*

$$\Gamma_m(H(G,I)) \subseteq \Gamma_m(G,I) \ .$$

Proof. Let $H(G,I)$ be the Herbrand's model over H_n, defined by letting:

$$c_i \in_{H(G,I)} c_j \quad \text{iff} \quad (i,j) \in G$$

For $r, s \in H_n$ having seeds c_h and c_k respectively:

$$r \in_{H(G,I)} s \quad \text{iff} \quad r \text{ is added sintactically to } s \text{ or } k \in I(h, J_r) \ ,$$

where $J_r = \{i \,:\, c_i \in_{H(G,I)} c_h \text{ or } (c_i \text{ is added sintactically to } r)\}$.
$H(G,I)$ is a normal Herbrand model of NWL such that for every term $t \notin \{c_1, \ldots, c_n\}$, if c_h is the seed of t and $J_t = \{i : c_i \in_{H(G,I)} t\}$ then $S_{H(G,I)}(t) = I(h, J_t)$.

Given $(t_1, \ldots, t_m)$, m-tuple of distinct terms in $H(G,I)$ all distinct from $c_1, \ldots, c_n$, we have to show that $Ghp_{H(G,I)}(t_1, \ldots, t_m) \in \Gamma_m(G,I)$, namely to find $\sigma \in I^m, B \in \mathcal{B}_m, \pi$ permutation of $\{1, \ldots, n\}$ such that the m-situation (G, σ, B^π) induces $Ghp_{H(G,I)}(t_1, \ldots, t_m)$.
Clearly $\sigma = (sit_{H(G,I)}(t_1), \ldots, sit_{H(G,I)}(t_m)) \in I^m$.

Let $B' : \{1, \ldots, m\}^2 \to \{\mathbf{w}, \mathbf{l}, \times\}$ be defined as follows:

$$B' \;=\; \left[\begin{array}{ll} \mathbf{w} & \text{if } t_i \text{ is added syntactically to } t_j \\ \mathbf{l} & \text{if } t_i \text{ is removed syntactically from } t_j \\ \times & \text{otherwise} \end{array}\right.$$

Since B' is defined with reference to the notion of being added or removed syntactically, the relation R defined by letting $R(i,j)$ iff $B(i,j) \in \{\mathbf{w}, \mathbf{l}\}$ is well-founded and by R-recursion we can define a permutation λ of $\{1, \ldots, m\}$ such that if $B^\lambda(i,j) \in \{\mathbf{w}, \mathbf{l}\}$ then $i < j$.

Let B be the uniquely determined - according to Proposition 3.1 - function $B : \{1, \ldots, m\}^2 \to \{\mathbf{w}, \mathbf{l}, \times\}$ such that:

if $B^\lambda(i,j) \in \{\mathbf{w}, \mathbf{l}\}$ then $B(i,j) = B^\lambda(i,j)$,
$B(i,j) \in \{\mathbf{w}, \mathbf{l}\}$ iff $i < j$

and that induces the same $n+m$-graph as $(G, \lambda\sigma, B")$,
where $\lambda\sigma = (sit(t_{\lambda(1)}), \ldots, sit(t_{\lambda(m)}))$. Letting $\pi = \lambda^{-1}$it is easy to verify that
$Ghp(G, (t_1, \ldots, t_m), B') = Ghp(G, \sigma, B^\pi)$. Since $B \in \mathcal{B}_m$, that proves that $Ghp(G, (t_1, \ldots, t_m), B') \in \Gamma_m(G,I)$. □

Lemma 3.1. *Let M be an Herbrand's model of NWL. Then for every $m \geq 1$ there exists $I_m \in \mathcal{I}$ such that*
for every $1 \leq h \leq m$, $\sigma \in (I_m)^h$ and $B \in \mathcal{B}_h$, there are infinitely many disjoint h-tuples of distinct elements of M which realize (σ, B).

Proof. Let $\{h_k\}_{k\in\aleph}$ be an increasing sequence on numerals distinct from c_0, $c_1, \ldots, c_n$. For every $i \in \{0, 1, \ldots, n\}$ and $J \subset \{0, 1, \ldots, n\}$ let:

$$\begin{array}{rcl} t^1_{i,J}(r) & = & c_i \cup J^c \setminus (\{c_1, \ldots c_n\} \setminus J^c) \\ & & \bullet_0 h_0 \ldots \bullet_n h_n \\ & & \mathbf{l}\; h_{n+1} \mathbf{w}\; h_{n+2} \\ & & \bullet_{n+1} h_{n+3+r} \end{array}$$

where $J^c = \{c_j : j \in J\}$ and for every $0 \leq j \leq n$, $\bullet_j$ is $\mathbf{w}$ if $h_j \notin_M c_j$, $\mathbf{l}$ otherwise and $\bullet_{n+1}$ is $\mathbf{w}$ if $h_{n+3+r} \notin_M c_i$, $\mathbf{l}$ otherwise.

Assuming $m > 1$, for $0 \leq h < m$, σ h-tuple of terns in $SIT_n = \{0, \ldots, n\} \times \mathcal{P}(\{0, \ldots, n\}) \times \mathcal{P}(\{0, \ldots, n\})$, $B \in \mathcal{B}_h$ such that there are infinitely many disjoint h-tuple of distinct terms in M which realize (σ, B), let en_σ^B be an enumeration of infinitely many such h-tuples. Without loss of generality we may suppose that as h, σ and B vary, the ranges of the enumerations en_σ^B are disjoint from each other, from $\{c_0, c_1, \ldots, c_n\}$, from $\{h_k\}_{k\in\aleph}$ and from a further increasing sequences of numerals $\{h'_k\}_{k\in\aleph}$ in its turn disjoint from $\{h_k\}_{k\in\aleph}$.

Let

$$\Lambda_m(r) = \bigcup\{range(en_\sigma^B(r+j)) : 1\le j\le 2^h,\ 1\le h< m,\ B \in \mathcal{B}_h,\ \sigma \in (SIT_n)^h\}$$

and

$$\Delta_m(r)=\Lambda_m(r)\cup\{t : t \text{ is added or removed syntactically in some term in } \Lambda_m(r)\}$$

$\Delta_m(r)$ is a finite set and let $\delta_m(r)$ be its cardinality; furthermore let $\delta_m^{<r} = \Sigma_{i<r}\delta_m(i)$.

Let O_j^h be the operation that assigns to a h-tuple $(t_1, \ldots, t_h)$ of terms the string $\bullet_1 t_1 \ldots \bullet_h\, t_h$ where $(\bullet_1 \ldots \bullet_h)$ is the j-th h-tuple in $\{\mathbf{w}, \mathbf{l}\}^h$, in any fixed ordering whatsoever, and $\circ X$, where X is a set of strings, be the concatenation of the strings in X in any fixed ordering. Let

$$\begin{array}{rcl} t_{i,J}^m(r) & = & t_{i,J}^1(r) \\ & & \circ\{O_j^h en_\sigma^B(r+j) : 1\le j\le 2^h,\ 1\le h<m,\ \sigma \in (SIT_n)^m,\ B \in \mathcal{B}_h\} \\ & & \bullet'_1 h'_{\delta_m^{<r}+1} \cdots \bullet'_{\delta_m(r)}\, h'_{\delta_m^{<r}+\delta_m(r)} \end{array}$$

where for a given enumeration $a_1, \ldots, a_{\delta_m(r)}$ of $\Delta_m(r)$, $\bullet'_i$ is $\mathbf{w}$ if $h'_{\delta_m^{<r}+i} \notin_M a_i$, $\mathbf{l}$ otherwise. Then the following hold:

1. every term added or removed in $t_{i,J}^m(r)$ is added or removed only once; as a consequence it can't happen that a term added is successively removed in $t_{iJ}^m(k)$ or vice versa;
2. the seed of $t_{i,J}^m(r)$ is c_i and the set of predecessor of $t_{iJ}^m(r)$ among the constants $c_0, c_1, \ldots, c_n$ is J^c.
3. $t_{iJ}^m(r)$ is distinct from any constant c_j since $h_j \in_M t_{iJ}^m(r)$ iff $h_j \notin_M c_j$; furthermore it is distinct from any numeral since $h_{n+2} \in_M t_{iJ}^m(r)$ while $h_{n+1} \notin_M t_{iJ}^m(r)$ and we have assumed $h_{n+1} < h_{n+2}$;
4. If $r \neq r'$ then $t_{iJ}^m(r) \neq t_{iJ}^m(r')$: in fact $h_{n+3+r} \in_M t_{iJ}^m(r)$ iff $h_{n+3+r} \notin_M c_i$, furthermore h_{n+3+r} is not added nor removed in $t_{iJ}^m(r')$, hence $h_{n+3+r} \in_M t_{iJ}^m(r)$ iff $h_{n+3+r} \notin_M t_{iJ}^m(r')$;
5. $t_{i,J}^m(r)$ is distinct from any term in $\Delta_m(r)$ since for every $1 \le j \le \delta_m(r)$ $h'_j \in_M t_{i,J}^m(r)$ iff $h'_j \notin_M a_j$
6. $t_{i,J}^m(r)$ is not added or removed in itself: it follows from 3 and 4 since the elements added or removed in $t_{iJ}^m(r)$ are either constants in $\{c_0, \ldots, c_n\}$ or numerals or terms in $\Delta_m(r)$;
7. $t_{i,J}^m(r)$ is not added or removed in any of the terms in $\Lambda_m(r)$ otherwise we would have that $t_{i,J}^m(r) =_M a_j$ for some $a_j \in \Delta_m(r)$ which is impossible by 5 above.

Since the terms $t^m_{i,J}(r)$ are all distinct form each other and there are only 2^n subsets of $\{1,\ldots,n\}$ there must be at least one such subset say $I_{i,J}$ such that for infinitely many $r's$, $S_M(t^m_{iJ}(r)) = I_{i,J}$. Let

$$I_m(i,J) = S_M(t^m_{i,J}(r_0))$$

where r_0 is the least natural number such that for infinitely many $r's$ $S_M(t^m_{iJ}(r)) = S_M(t^m_{i,J}(r_0))$.

By induction on h we now prove that if $1 \leq h \leq m$, $\sigma \in (I_m)^h$ and $B \in \mathcal{B}_h$ then there are infinitely many disjoint h-tuples of distinct terms in M which realize σ, B.

Base case: $h = 1$. Because of 6 above the infinitely many terms of the form $t^m_{i,J}(r)$ such that $S_M(t^m_{i,J}(r)) = S_M(t^m_{i,J}(r_0))$ realize $(i, J, I_m(i,J))$, $B_\times$.

Inductive step: assume the property holds for h and that $h+1 \leq m$. Assume $(s_1,\ldots,s_h,s_{h+1}) \in (I_m)^{h+1}$ and $B \in \mathcal{B}_{h+1}$. Let $s_{h+1} = (i, J, S_M(t^m_{i,J}(r_0))$ and $(B(1,h+1),\ldots,B(h,h+1))$ be the j-th h-tuple in $\{\mathrm{w},\mathrm{l}\}^h$, namely $O^h_j = (B(1,h+1),\ldots,B(h,h+1))$. Let B' be the restriction of B to $\{1,\ldots,h\}^2$. By inductive hypothesis there are infinitely many disjoint h-tuples of distinct terms in M which realize $(s_1,\ldots,s_h), B'$. Therefore $O^h_j en^{B'}_{(s_1,\ldots,s_n)}(r+j)$ is a substring of $t^m_{i,J}(r)$. If $en^{B'}_{(s_1,\ldots,s_n)}(r+j) = (t_{r,1},\ldots,t_{r,h})$ and $S_M(t^m_{i,J}(r)) = S_M(t^m_{i,J}(r_0))$ it is easy to check by using 5 above that $(t_{r,1},\ldots,t_{r,h},t^m_{i,J}(r))$ realizes $(s_1,\ldots,s_h,s_{h+1}), B$.

As r varies the h-tuples $(t_{r,1},\ldots,t_{r,h})$ are disjoint from each other by the assumption on $en^{B'}_{(s_1,\ldots,s_n)}$, furthermore the terms $t^m_{i,J}(r)$ are distinct from each other and from the terms $t_{r,1},\ldots,t_{r,h}$ as well, because of 4 and 5 above. Therefore the $h+1$-tuples $(t_{r,1},\ldots,t_{r,h},t^m_{i,J}(r))$ are infinitely many disjoint $h+1$-tuples of distinct terms of M which realize $(s_1,\ldots,s_h,s_{h+1})$, B and we are done. □

Proposition 3.3. *Let M be an Herbrand's model of NWL. Then for every $m \geq 1$ there exists $I_m \in \mathcal{I}$ such that for every $\sigma \in (I_m)^m$, $B \in \mathcal{B}_m$ and permutation π of $\{1,\ldots,m\}$ σ, B^π is realized in M, therefore $\Gamma_m(G,I) \subseteq \Gamma_m(M)$*

Proof. When π is the identity the proposition follows immediately from the previous lemma. Otherwise let σ' be defined by $\sigma'(i) = \sigma(\pi^{-1}(i))$. By the previous lemma 3.1 there is an m-tuple $(t'_1,\ldots,t'_m)$ of distinct terms in M which realizes σ', B. It is immediate that the m-tuple $(t_1,\ldots,t_m)$ such that $t_i = t'_{\pi(i)}$ realizes σ, B^π. □

Theorem 3.1. *Let F be a $\forall^m$-sentence in normal form over $\{c_1,\ldots,c_n,\in,=\}$. Then the following are equivalent:*

1. *F is satisfiable with respect to NWL*
2. *F is true in a normal Herbrand's model of NWL of the form $H(G,I)$ with $G \in \mathcal{G}_n$ and $I \in \mathcal{I}$*

3. for some $G \in \Gamma$ *and* $I \in \mathcal{I}$ $\quad \Gamma_m(G,I) \subseteq \Gamma_m(F)$

Proof. If F is satisfiable with respect to NWL then there is an Herbrand's model M over H_n of NWL such that $M \models F$. Then by Proposition 2.2, $\Gamma_m(M) \subseteq \Gamma_m(F_0)$. By Proposition 3.3, letting G be the graph induced on $\{1,\ldots,m\}$ by $\in_M$ on $\{c_1,\ldots,c_n\}$ there is a function $I \in \mathcal{I}$ such that $\Gamma_m(G,I) \subseteq \Gamma_m(G,I)$. Thus $\Gamma_m(G,I) \subseteq \Gamma_m(F)$. Hence 1) entails 3). Assuming 3), let $G \in \mathcal{G}$ and $I \in \mathcal{I}$ be such that $\Gamma_m(G,I) \subseteq \Gamma_m(F)$. By Proposition 3.2 $\Gamma_m(H(G,I)) \subseteq \Gamma_m(G,I)$, therefore $\Gamma_m(H(G,I)) \subseteq \Gamma_m(F)$. Thus, by Proposition 2.2, $H(G,I)' \models F$. Hence 3) entails 2). Since, obviously 2) entails 1), the proof is completed. □

As an immediate consequence of the reduction established in Section 2. and of the previous theorem we have our main decidability result:

Theorem 3.2. *The satisfiability problem for* $\exists^*\forall^*$*-sentences in* $\{\in, =\}$ *with respect to NWL is decidable.*

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