Uniform Interpolation and Layered Bisimulation *

Albert Visser

Department of Philosophy, University of Utrecht, Heidelberglaan 8, 3584CS Utrecht

Summary. In this paper we give perspicuous proofs of Uniform Interpolation for the theories IPC, K, GL and S4Grz, using bounded bisimulations. We show that the uniform interpolants can be interpreted as propositionally quantified formulas, where the propositional quantifiers get a semantics with bisimulation extension or bisimulation reset as the appropriate accessibility relation. Thus, reversing the conceptual order, the uniform interpolation results can be viewed as quantifier elimination for bisimulation extension quantifiers.

1. Introduction

Bisimulation and bounded bisimulation can be used to 'visualize' proofs. The aim of this paper is to present proofs for uniform interpolation results as clearly and perspicuously as possible using bounded bisimulation. Ordinary interpolation for a given theory T says that if $\mathsf{T} \vdash A \rightarrow B$, then there is a formula $I(A, B)$ in the language containing only the shared propositional variables, say $\mathbf{q}$, such that $\mathsf{T} \vdash A \rightarrow I$ and $\mathsf{T} \vdash I \rightarrow B$. Uniform interpolation is a strengthening of ordinary interpolation in which the data in terms of which the interpolant is to be specified are weaker: the interpolant can be found from either A and $\mathbf{q}$ or from $\mathbf{q}$ and B. Thus, if uniform interpolation holds, there is, for every A and $\mathbf{q}$, a 'post-interpolant' $I(A, \mathbf{q})$ such that $\mathsf{T} \vdash A \rightarrow I(A, \mathbf{q})$ and, for all B such that $\mathsf{T} \vdash A \rightarrow B$ and such that the shared propositional variables of A and B are among $\mathbf{q}$, we have $\mathsf{T} \vdash I(A, \mathbf{q}) \rightarrow B$. Similarly for the 'pre-interpolant'. As we will see, uniform interpolation can be viewed as quantifier elimination for certain propositional quantifiers in T: the quantifiers that correspond to the trans-model accessibility relation *bisimulation extension* (or: *bisimulation reset*).

In this paper we prove Uniform Interpolation for IPC (Intuitionistic Propositional Calculus), for K, for GL (Löb's Logic) and for S4Grz.[1] Uni-

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* The present paper is in its final form and no similar paper has been or is being submitted elsewhere.

[1] Uniform Interpolation for IPC was first proved by Pitts using proof theoretical methods. It was proved by the present method by Ghilardi and Zawadowski and, independently but later, by the author. Uniform Interpolation for K is due to Ghilardi. Uniform Interpolation for GL was first proved by Shavrukov. It was proved by the present method by the author. To give the due credit it should be pointed out that the method here is similar to the one used by Ghilardi and Zawadowski and, independently, the author, to prove the result for IPC. The result for S4Grz is, as far as I know, new in this paper.

form interpolation for S4Grz is rather surpising, since it fails for the closely related theory S4, as was shown by Ghilardi and Zawadowski in their paper [4].

2. Models

We start with introducing the notion of Kripke model and specifying some notations. A (Kripke) model is a structure $\mathbb{K} = \langle K, \prec, \models, \mathcal{P}\rangle$. Here:

- K is a non-empty set of nodes
- $\prec$ is a binary relation on K
- $\mathcal{P}$ is a (possibly empty) set of propositional variables[2]
- $\models$ is a relation between K and $\mathcal{P}$

We can, alternatively, view a model $\mathbb{K}$ as a function that assigns to a fixed set of pairwise disjoint labels $\{\underline{K}, \underline{\prec}, \underline{\models}, \underline{\mathcal{P}}\}$ the appropriate objects. In this style we will write e.g. $\mathcal{P}_{\mathbb{K}}$ for $\mathbb{K}(\underline{\mathcal{P}})$. We will say that $\mathbb{K}$ is a $\mathcal{P}$-model if $\mathcal{P}_{\mathbb{K}} = \mathcal{P}$. Similarly for $K, \models$-model, etcetera. Similar conventions will be employed for other kinds of models. Define: $\mathsf{PV}_{\mathbb{K}}(k) := \{p \in \mathcal{P}_{\mathbb{K}} \mid k \models_{\mathbb{K}} p\}$. Note that $\models$ and PV are interdefinable. $\mathbf{p}, \mathbf{q}, \mathbf{r}$ will range over finite sets of propositional variables. A model $\mathbb{K}$ is finite if both $K_{\mathbb{K}}$ and $\mathcal{P}_{\mathbb{K}}$ are finite. We will call the class of models Mod.

It is often pleasant to think in terms of *a node in a model.* It is worthwile to make this notion explicit. A *pointed* model is a structure $\mathbb{K} = \langle \mathbb{K}_0, k\rangle$, where $\mathbb{K}_0$ is a model, and k is a node of $\mathbb{K}_0$. A pointed model $\langle \mathbb{K}, b\rangle$ is called *rooted* if for all $k \in K$: $b \prec^* k$[3] b is called the *root.* We can confuse a class of models with its disjoint union, taking as new nodes the pointed models corresponding to the models of the class. We define, e.g., $\langle \mathbb{K}, k\rangle \prec \langle \mathbb{K}', k'\rangle :\Leftrightarrow \mathbb{K} = \mathbb{K}'$ and $k \prec_{\mathbb{K}} k'$. Thus, we can confuse a pointed model $\langle \mathbb{K}, k\rangle$ with a 'free floating' node k. Note that the disjoint union of all models is not strictly speaking a model in our sense. The set of popositional variables that is declared to be present need not be constant in different 'nodes'. It is essential for our purposes for this to be so, since we want to study transitions between nodes in different models that do not leave the set of variables present constant. The totality of pointed models will be called Pmod and the totality of rooted models Rmod.

Suppose $\mathbb{K}$ is a —possibly pointed— $\mathcal{P}$-model. Then $\mathbb{K}[\mathcal{Q}]$ is the $\mathcal{P} \cap \mathcal{Q}$-model obtained by restricting $\models_{\mathbb{K}}$ to $\mathcal{P} \cap \mathcal{Q}$. For any $k \in K$, $\mathbb{K}[k]$ is the rooted model $\langle K', k, \prec', \models', \mathcal{P}\rangle$, where $K' := \uparrow k := \{k' \in K \mid k \prec^* k'\}$ and where $\prec'$ and $\models'$ are the restrictions of $\prec$ respectively $\models$ to K'. (We will often

[2] We take the set of propositional variables as 'internal' to the models (and the languages), because we want to think about model extensions, which involve changing the set of variables of the model.

[3] $\prec^*$ is the transitive reflexive closure of $\prec$.

simply write $\prec$ and $\models$ for $\prec'$ and $\models'$.) In case we are using the convention of confusing a node k with its pointed model, $\langle \mathbb{K}, k\rangle$, we will, e.g., write $k[\mathcal{Q}]$ for $\langle \mathbb{K}[\mathcal{Q}], k\rangle$.

We will consider several properties of models. $\mathbb{K}$ will be said to be *transitive* if $\prec_{\mathbb{K}}$ is transitive, etcetera. $\mathbb{K}$ is *persistent* if $\mathsf{PV}_{\mathbb{K}}$ is monotonic w.r.t. $\prec_{\mathbb{K}}$ and $\subseteq$.

It will be convenient to extend the natural numbers ω with an extra element ∞. Let ω^{∞} be $\omega \cup \{\infty\}$. We let $\alpha, \beta, \ldots$ range over ω^{∞}. ω^{∞} is equipped with the obvous ordering $\leq$. We extend addition by: $\infty + \alpha = \alpha + \infty = \infty$. We extend cut-off substraction in our structure by: $\infty - n = \infty$. We will avoid the question of what $\infty - \infty$ is.

Transitive models are going to play a special role in this paper so we will need some some special notions concerned with transitive models. Consider any *transitive* model $\mathbb{K}$. Define:

- $k \prec^{+} k' :\Leftrightarrow k \prec k'$ and not $k' \prec k$
- $k \approx k' :\Leftrightarrow k = k'$ or ($k \prec k'$ and $k \prec k'$). So $\approx$ means *being in the same cluster.*
- $d_{\mathbb{K}}(k) := \sup(\{(d_{\mathbb{K}}(k')+1) \in \omega^{\infty} \mid k' \succ^{+} k\})$
- If $\mathbb{K}$ is pointed with designated node k, we put: $d(\mathbb{K}) := d_{\mathbb{K}}(k)$

Note that if $k \prec^{+} k'$, then $d_{\mathbb{K}}(k') < d_{\mathbb{K}}(k)$. k is a top node if it is a top node w.r.t. $\prec^{+}$. Note that k is a top node precisely if $d_{\mathbb{K}}(k) = 0$.

3. Layered Bisimulation

In this section we introduce bisimulation and bounded bisimulation. To avoid formulating most definitions and theorems twice —once for bounded and once for ordinary bisimulation— we make use of a portmanteau notion: *layered bisimulation.*[4]

Consider $\mathcal{P}$-models $\mathbb{K}$ and $\mathbb{M}$. We write $K := K_{\mathbb{K}}$ and $M := K_{\mathbb{M}}$. A *layered bisimulation* or *ℓ-bisimulation* $\mathcal{Z}$ between $\mathbb{K}$ and $\mathbb{M}$ is a ternary relation between K, ω^{∞} and M, satisfying the conditions specified below. We will consider $\mathcal{Z}$ also as an ω^{∞}-indexed set of binary relations between K and M writing $k\mathcal{Z}_{\alpha}m$ for $\langle k, \alpha, m\rangle \in \mathcal{Z}$. We often write $k\mathcal{Z}m$ for $k\mathcal{Z}_{\infty}m$. We give the conditions:

1. $k\mathcal{Z}_{\alpha}m \Rightarrow \mathsf{PV}_{\mathbb{K}}(k) = \mathsf{PV}_{\mathbb{M}}(m)$
2. $k' \succ_{\mathbb{K}} k\mathcal{Z}_{\alpha+1}m \Rightarrow$ there is an m' with $k'\mathcal{Z}_{\alpha}m' \succ_{\mathbb{M}} m$;
   i.o.w. $\succ_{\mathbb{K}} \circ \mathcal{Z}_{\alpha+1} \subseteq \mathcal{Z}_{\alpha} \circ \succ_{\mathbb{M}}$.

[4] Bisimulation is used both in computer science and modal logic. See e.g. the papers in [11] for an impression. In model theory bisimulation and bounded bisimimulation appears in the guise of Ehrenfeucht games and back-and-forth equivalence. See [7].

3. $k\mathcal{Z}_{\alpha+1}m \prec_{\mathbb{M}} m' \Rightarrow$ there is a k' with $k \prec_{\mathbb{K}} k'\mathcal{Z}_{\alpha}m'$;
i.o.w. $\mathcal{Z}_{\alpha+1} \circ \prec_{\mathbb{M}} \subseteq \prec_{\mathbb{K}} \circ \mathcal{Z}_{\alpha}$

Note that we allow ℓ-bisimulations to be undefined on some nodes. They may even be empty. Note also that ℓ-bisimulations occur only between models for the same set of variables. We call (2) the *zig*$_{\alpha+1}$*-property* (see Fig. 3.1) and (3) the *zag*$_{\alpha+1}$*-property*. If $\alpha = \infty$ we simply speak of the zig- and the zag-property. A binary relation $\mathcal{Z}$ between $\mathbb{K}$ and $\mathbb{M}$ is a *bisimulation* between

$$\begin{tikzcd} k' \arrow[r, dash, dashed, "\mathcal{Z}_\alpha"] & m' \\ k \arrow[u, "\prec_{\mathbb{K}}"] \arrow[r, dash, "\mathcal{Z}_{\alpha+1}"] & m \arrow[u, dashed, "\prec_{\mathbb{M}}"'] \end{tikzcd}$$

Fig. 3.1. *The* zig$_{\alpha+1}$*-property*

$\mathbb{K}$ and $\mathbb{M}$ iff $\{\langle k, \infty, m\rangle \mid k\mathcal{Z}m\}$ is an ℓ-bisimulation. We will simply confuse bisimulations $\mathcal{Z}$ with the corresponding ℓ-bisimulations. An ℓ-bisimulation $\mathcal{Z}$ is a *bounded bisimulation* if for some natural number n: $k\mathcal{Z}_{\alpha}m \Rightarrow \alpha \leq n$.

Let $\mathsf{ID}_{\mathbb{K}} := \{\langle k, \alpha, k\rangle \mid k \in K,\ \alpha \in \omega^{\infty}\}$. Suppose $\mathcal{Z}$ is an ℓ-bisimulation between $\mathbb{K}$ and $\mathbb{M}$ and that $\mathcal{U}$ is an ℓ-bisimulation between $\mathbb{M}$ and $\mathbb{N}$. We define $\mathcal{Z} \circ \mathcal{U}$ by: $(\mathcal{Z} \circ \mathcal{U})_{\alpha} := \mathcal{Z}_{\alpha} \circ \mathcal{U}_{\alpha}$, and $\widehat{\mathcal{Z}}$ by $(\widehat{\mathcal{Z}})_{\alpha} := \widehat{(\mathcal{Z}_{\alpha})}$, where $\widehat{(.)}$ is the usual inverse on binary relations. $\mathcal{Z}^{\alpha}$ is the relation given by: $\mathcal{Z}^{\alpha}_{\beta} := \mathcal{Z}_{\alpha+\beta}$. We say that $\mathcal{Z}$ is *downward closed* if for all $\alpha \prec \beta$: $\mathcal{Z}_{\beta} \subseteq \mathcal{Z}_{\alpha}$, The *downward closure* $\mathcal{Z}{\downarrow}$ of $\mathcal{Z}$ is the smallest downwards closed relation extending $\mathcal{Z}$. In the following theorem we collect the necessary elementary facts.

Theorem 3.1. *1.* $\mathsf{ID}_{\mathbb{K}}$ *is an ℓ-bisimulation.*
2. $\mathcal{Z} \circ \mathcal{U}$ *is an ℓ-bisimulation between $\mathbb{K}$ and $\mathbb{N}$.*
3. $\widehat{\mathcal{Z}}$ *is an ℓ-bisimulation between $\mathbb{M}$ and $\mathbb{K}$.*
4. $\mathcal{Z}^{\alpha}$ *is an ℓ-bisimulation.*
5. The downward closure of $\mathcal{Z}$ is an ℓ-bisimulation.
6. Suppose Z is a set of ℓ-bisimulations between $\mathbb{K}$ and $\mathbb{M}$. Then $\bigcup \mathsf{Z}$ is again an ℓ-bisimulation between $\mathbb{K}$ and $\mathbb{M}$. It follows that there is always a maximal ℓ-bisimulation, $\simeq^{\mathbb{K},\mathbb{M}}$ between two models. (1)-(5) imply that for any α:
– $\mathsf{ID}_{\mathbb{K}} \subseteq \simeq^{\mathbb{K},\mathbb{M}}$
– $\simeq^{\mathbb{K},\mathbb{M}} \circ \simeq^{\mathbb{M},\mathbb{N}} \subseteq \simeq^{\mathbb{K},\mathbb{N}}$
– $\widehat{\simeq^{\mathbb{K},\mathbb{M}}} \subseteq \simeq^{\mathbb{M},\mathbb{K}}$
– $\simeq^{\mathbb{K},\mathbb{M}}$ *is downward closed.*
Note that, by the above, each of the $\simeq^{\mathbb{K},\mathbb{M}}_{\alpha}$ is an equivalence relation.

7. *Consider $k \in K$ and $m \in M$. Let $\mathcal{Z}[k,m]$ be the restriction of $\mathcal{Z}$ to $\uparrow k \times \uparrow m$. Then $\mathcal{Z}[k,m]$ is an ℓ-bisimulation between $\mathbb{K}[k]$ and $\mathbb{M}[m]$.*
8. *Consider two transitive models $\mathbb{K}$ and $\mathbb{M}$. Consider the relation $\mathcal{W}$, given by:*

$$k\mathcal{W}_\alpha m :\Leftrightarrow \text{for some } k', m' : \ k \approx k'\mathcal{Z}_\alpha m' \approx m \text{ and } k \simeq_0 m.$$

We have: $\mathcal{W}$ is an ℓ-bisimulation. It follows, e.g., taking $\mathbb{M} := \mathbb{K}$ and $\mathcal{Z} := \mathsf{ID}_K$, that $\approx \cap \simeq_0$ is an ℓ-bisimulation on $\mathbb{K}$.

We will often drop the superscript of $\simeq^{\mathbb{K},\mathbb{M}}$ In case $\alpha = \infty$, we will drop the subscript of $\simeq^{\mathbb{K},\mathbb{M}}_\alpha$ (if no confusion is possible). We will say that k and m (considered as pointed models) *n-simulate* if $k \simeq_n m$ and that k and m *bisimulate* if $k \simeq m$. The following theorem tells us that the number of $\simeq_n$-equivalence classes on a model has a fixed finite bound that only depends on n.

Theorem 3.2. *Define $F(N,0) := 2^N$, $F(N,n+1) := 2^{F(N,n)+N}$. Suppose $|\mathcal{P}| = \mathcal{N}$, then the number of possible $\simeq_n$ equivalence classes is smaller or equal to $F(N,n)$.*

Proof. By a simple induction on n, noting that the $n+1$-equivalence class of a node k is fully determined by the atoms forced in k and the n-equivalence classes of the nodes 'seen' by k. □

In this paper we are particularly interested in things like extending or even changing the forcing of the propositional variables on nodes. We introduce the relevant notions. Let $k, k', m, m' \ldots$ be pointed models.

- $k \simeq_{\alpha,\mathcal{Q}} m :\Leftrightarrow \mathcal{P}_k \cap \mathcal{Q} = \mathcal{P}_m \cap \mathcal{Q}$ and $k[\mathcal{Q}] \simeq_\alpha m[\mathcal{Q}]$. So, roughly, this means that k and m α-bisimulate w.r.t. the variables in $\mathcal{Q}$.
- $k \simeq_{\alpha,[\mathcal{Q}]} m :\Leftrightarrow k \simeq_{\alpha,\mathcal{Q}^c} m$ and $\mathcal{Q} \subseteq \mathcal{P}_m$. So, roughly, this means that k differs from m *modulo* $\simeq_\alpha$ only at $\mathcal{Q}$ and m is at least a $\mathcal{Q}$-node. We will say that m is a *$\mathcal{Q}, \alpha$-bisimulation reset* of k. In case $\alpha = \infty$, we will speak of a *$\mathcal{Q}$-bisimulation reset.*
- $k \sqsubseteq_{\alpha,\mathcal{Q}} m :\Leftrightarrow k \simeq_{\alpha,\mathcal{P}_k} m$ and $\mathcal{Q} \cap \mathcal{P}_k = \emptyset$ and $\mathcal{Q} \cup \mathcal{P}_k = \mathcal{P}_m$. We will say that m is a *$\mathcal{Q}, \alpha$-bisimulation extension* of k. In case $\alpha = \infty$, we will speak of a *$\mathcal{Q}$-bisimulation extension.*

If we are studying persistent models it is often more natural to think in terms of certain orderings related to layered bisimulation, than in terms of layered bisimulation itself. We can think of these orderings as a kind of extension of the ordering in the model. For the rest of this section we think about persistent pointed $\mathcal{P}$-models. We let $k, k', m, m' \ldots$ range over such models.

- $k \preceq_0 m :\Leftrightarrow \mathsf{PV}(k) \subseteq \mathsf{PV}(m)$
- $k \preceq_{\alpha+1} m :\Leftrightarrow \mathsf{PV}(k) \subseteq \mathsf{PV}(m),\ \forall m' \succ m\, \exists k' \succ k'\ k' \simeq_\alpha m'$

In case $\alpha = \infty$, we will drop the subscript.

Theorem 3.3. *1. $\preceq_\alpha$ is a partial preordering on pointed, persistent $\mathcal{P}$-models.*
2. $k \prec k' \Rightarrow k \preceq_\alpha k'$.
3. $\alpha \leq \beta \Rightarrow \preceq_\beta \subseteq \preceq_\alpha$.
4. $k \simeq_\alpha m \Leftrightarrow k \preceq_\alpha m$ and $m \preceq_\alpha k$.
5. $k \preceq_\infty m \Leftrightarrow$ for some $k' \succeq k$ $k' \simeq m$.

Proof. We prove (4). For $\alpha = 0$ this is easy. Suppose $\alpha > 0$. "$\Rightarrow$" Easy. "$\Leftarrow$" Suppose $k \preceq_\alpha m$ and $m \preceq_\alpha k$. We show that $\mathcal{U} :=\simeq \cup \{\langle k, \alpha, m\rangle\}$ is an ℓ-bisimulation, and, hence, that $k \simeq_\alpha m$. Clearly $\mathsf{PV}(k) = \mathsf{PV}(m)$. The zig-property for $\mathcal{U}$ follows from the fact that $m \preceq_\alpha k$. The zag-property for $\mathcal{U}$ follows from the fact that $k \preceq_\alpha m$. □

4. Basic Facts for IPC

In this section, $\vdash$ will stand for derivability in IPC. Consider any set of propositional variables, $\mathcal{P}$. We define $\mathcal{L}^i(\mathcal{P})$ as the smallest set such that:

- $\mathcal{P} \subseteq \mathcal{L}^i(\mathcal{P})$, $\bot, \top \in \mathcal{L}^i(\mathcal{P})$
- if $A, B \in \mathcal{L}^i(\mathcal{P})$, then $(A \wedge B), (A \vee B), (A \to B) \in \mathcal{L}^i(\mathcal{P})$.

$\mathsf{PV}(A)$ is the set of propositional variables occurring in A. $\mathsf{Sub}(A)$ is the set of subformulas of A. A model is an IPC-*model* if it is transitive, reflexive, antisymmetric and persistent. In this section all models will be IPC models. Consider a $\mathcal{P}$-model $\mathbb{K}$ we take $\models_i$ to be the smallest relation between K and $\mathcal{L}^i(\mathcal{P})$ such that:

- $k \models_i p :\Leftrightarrow k \models p$, $k \models_i \top$
- $k \models_i A \wedge B :\Leftrightarrow k \models_i A$ and $k \models_i B$
- $k \models_i A \vee B :\Leftrightarrow k \models_i A$ or $k \models_i B$
- $k \models_i A \to B :\Leftrightarrow \forall k' \succ k\ (k' \models_i A \Rightarrow k' \models_i B)$

We will omitt the subscript i, as long as it is sufficiently clear from the context that the persistent case is intended. Note that, by transitivity, the persistence for $\mathcal{P}$ extends to the persistence for $\mathcal{L}^i(\mathcal{P})$. Define further:

- $k \models \Gamma :\Leftrightarrow$ for all $A \in \Gamma : k \models A$
- $\mathbb{K} \models A :\Leftrightarrow$ for all $k \in K$ $k \models A$

A set X is $\mathcal{P}$-*adequate* if $X \subseteq \mathcal{L}^i(\mathcal{P})$ and X is closed under subformulas. A set Γ is X-saturated (for IPC) if for any subset Y of X: $\Gamma \vdash \bigvee Y \Rightarrow Y \cap \Gamma \neq \emptyset$. Note that it follows that Γ is consistent (the case that Y is $\emptyset$) and that Γ is closed under X-consequences (the case that Y is a singleton).

We describe the Henkin construction for IPC. To lighten our notational burdens we will assume in this section that we work with some fixed $\mathcal{P}$. Consider a $\mathcal{P}$-adequate set X. The Henkin model for X is the model $\mathbb{H} := \mathbb{H}_X$, where:

- $K_{\mathbb{H}} := \{\Delta \mid \Delta \text{ is } X\text{-saturated}\}$
- $\Gamma \prec \Delta :\Leftrightarrow \Gamma \subseteq \Delta$
- $\mathcal{P}_{\mathbb{H}} := \mathcal{P} \cap X$
- $\Gamma \models p :\Leftrightarrow p \in \Gamma$

It is easily verified that $\mathbb{H}$ is an IPC-model.

Theorem 4.1. *for all* $A \in X$: $\Gamma \models_{\mathbb{H}} A \Leftrightarrow A \in \Gamma$.

If X is finite, then $\mathbb{H}_X$ is finite. We say that $\mathbb{M}$ is a *rooted Henkin model* if it is of the form $\mathbb{H}_X[\Delta]$ for some X-saturated Δ. We have:

Theorem 4.2 (Kripke Completeness). *For* $\Gamma \subseteq \mathcal{L}^i(\mathcal{P})$ *and* $A \in \mathcal{L}^i(\mathcal{P})$*:*

$$\Gamma \vdash_{\mathcal{P}} A \Leftrightarrow \text{ for all } \mathcal{P}\text{-models } \mathbb{K} : \Gamma \models_{\mathbb{K}} A.$$

In case Γ *is finite, we can improve this to:*

$$\Gamma \vdash_{\mathcal{P}} A \Leftrightarrow \text{ for all finite } \mathcal{P}\text{-models } \mathbb{K} : \Gamma \models_{\mathbb{K}} A.$$

For IPC we have a distinctive result involving downward extensions of models. We first introduce the necessary machinery. Let K be a set of IPC-models. $\mathbb{M} := \mathsf{M}(\mathsf{K})$ is the IPC-model with :

- $M := \{\langle k, \mathbb{K}\rangle \mid k \in K_{\mathbb{K}} \text{ and } \mathbb{K} \in \mathsf{K}\}$
- $\langle k, \mathbb{K}\rangle \prec \langle m, \mathbb{M}\rangle :\Leftrightarrow \mathbb{K} = \mathbb{M} \text{ and } k \prec_{\mathbb{K}} m$
- $\mathcal{P}_{\mathbb{M}} := \bigvee\{\mathcal{P}_{\mathbb{K}} \mid \mathbb{K} \in \mathsf{K}\}$.
- $\langle k, \mathbb{K}\rangle \models p :\Leftrightarrow k \models_{\mathbb{K}} p$

In practice we will forget the second components of the new nodes, pretending the domains to be disjoint already. Let $\mathbb{K}$ be a IPC $\mathcal{P}$-model. $\mathsf{B}(\mathbb{K})$ is the (rooted) IPC $\mathcal{P}$-model obtained by adding a new bottom $\mathfrak{b}$ to $\mathbb{K}$ and by taking: $\mathfrak{b} \models p :\Leftrightarrow \mathbb{K} \models p$. Finally we define $\mathsf{Glue}(\mathsf{K}) := \mathsf{B}(\mathsf{M}(\mathsf{K}))$.

Theorem 4.3 (Push Down Lemma). *Let* X *be adequate. Suppose* Δ *is* X*-saturated and* $\mathbb{K}$ *is an* IPC*-model with* $\mathbb{K} \models \Delta$*. Then* $\mathsf{Glue}(\mathbb{H}_X[\Delta], \mathbb{K}) \models \Delta$.

Proof. We show by induction on $A \in X$ that $\mathfrak{b} \models A \Leftrightarrow A \in \Delta$. The cases of atoms, conjunction and disjunction are trivial. If $(B \to C) \in X$ and $\mathfrak{b} \models (B \to C)$, then $\Delta \models (B \to C)$ and, hence, $(B \to C) \in \Delta$. Conversely suppose $(B \to C) \in \Delta$. If $\mathfrak{b} \not\models B$, we are easily done. If $\mathfrak{b} \models B$, then, by the Induction Hypothesis: $B \in \Delta$, hence $C \in \Delta$ and, by the induction hypothesis: $\mathfrak{b} \models C$. □

Instead of using the Push Down Lemma we could have employed the Kleene slash. We say that Δ is $\mathcal{P}$*-prime* if it is consistent and for every $(C \vee D) \in \mathcal{L}^i(\mathcal{P})$: $\Delta \vdash (C \vee D) \Rightarrow \Delta \vdash C$ or $\Delta \vdash D$. A formula A is $\mathcal{P}$*-prime* if $\{A\}$ is $\mathcal{P}$-prime. As usual, we will suppress the $\mathcal{P}$.

Theorem 4.4. *Suppose* X *is adequate and* Δ *is* X*-saturated. then* Δ *is prime.*

Proof. Δ is consistent by definition. Suppose $\Delta \vdash C \vee D$ and $\Delta \nvdash C$ and $\Delta \nvdash D$. Suppose $\mathbb{K} \models \Delta$, $\mathbb{K} \not\models C$, $\mathbb{M} \models \Delta$ and $\mathbb{M} \not\models D$. Consider $\mathsf{Glue}(\mathbb{H}_X(\Delta), \mathbb{K}, \mathbb{M})$. By the Push Down Lemma (Theorem 4.3) we have: $\mathfrak{b} \models \Delta$. On the other hand by persistence: $\mathfrak{b} \not\models C$ and $\mathfrak{b} \not\models D$. Contradiction. □

We turn to the consideration of fragments and model descriptions for IPC. Define $\mathfrak{i} : \mathcal{L}^i(\mathcal{P}) \to \omega$, by:

- $\mathfrak{i}(p) := \mathfrak{i}(\bot) := \mathfrak{i}(\top) := 0$
- $\mathfrak{i}(A \wedge B) := \mathfrak{i}(A \vee B) := max(\mathfrak{i}(A), \mathfrak{i}(B))$
- $\mathfrak{i}(A \to B) := max(\mathfrak{i}(A), \mathfrak{i}(B)) + 1$
- $I_n(\mathcal{P}) := \{A \in \mathcal{L}^i(\mathcal{P}) \mid \mathfrak{i}(A) \leq n\}$
- $I_\infty(\mathcal{P}) := \mathcal{L}^i(\mathcal{P})$

By an easy induction on n we may prove the following theorem.

Theorem 4.5. *$I_n(\mathbf{p})$ is finite modulo IPC-provable equivalence.*

Define for $X \subseteq \mathcal{L}^i(\mathcal{P})$:

- $\mathsf{Th}_X(k) := \{A \in X \mid k \models A\}$
- For $\mathbb{K}$ pointed with point k: $\mathsf{Th}_X(\mathbb{K}) := \mathsf{Th}_X(k)$
- $\mathsf{Th}(k) := \mathsf{Th}_{\mathcal{L}^i(\mathcal{P})}(k)$

Theorem 4.6. *Suppose that $\mathcal{Z}$ is an ℓ-simulation between the $\mathcal{P}$-models $\mathbb{K}$ and $\mathbb{M}$. Then: $k\mathcal{Z}_\alpha m \Rightarrow \mathsf{Th}_{I_\alpha(\mathcal{P})}(k) = \mathsf{Th}_{I_\alpha(\mathcal{P})}(m)$.*

Proof. By induction on A in I_α. Suppose $k\mathcal{Z}_\alpha m$. The cases of atoms, conjunction and disjunction are trivial. Suppose, e.g., $k \not\models (B \to C)$. Then, for some $k' \succ k$, $k' \models B$ and $k' \not\models C$. There is an $m' \succ m$, such that $k'\mathcal{Z}_{\alpha-1}m'$ and hence by the induction hypothesis (applied for $\alpha - 1$, noting that if $A \in I_\alpha(\mathcal{P})$, then $B, C \in I_{\alpha-1}(\mathcal{P})$): $m' \models B$ and $m' \not\models C$. Ergo $m \not\models (B \to C)$. □

Theorem 4.7. *$k \preceq_\alpha m \Rightarrow \mathsf{Th}_{I_\alpha(\mathcal{P})}(k) \subseteq \mathsf{Th}_{I_\alpha(\mathcal{P})}(m)$, for $\mathcal{P}$-nodes k and m.*

Proof. In case $\alpha = 0$, this is trivial. Suppose $\alpha > 0$ and $k \preceq_\alpha m$. The proof is a simple induction on $A \in I_\alpha(\mathcal{P})$. The cases of atoms, $\wedge$, $\vee$ are trivial. Suppose $A = (B \to C)$ and $m \not\models (B \to C)$. Then for some $m' \succeq m$: $m' \models B$ and $m' \not\models C$. There is a $k' \succeq k$, such that $k' \simeq_{\alpha-1} m'$ and, hence, by Theorem 4.6: $k' \models B$ and $k' \not\models C$. Ergo $k \not\models (B \to C)$. □

We formulate a partial converse for Theorem 4.7. It is well known that the converse for the case of ∞, i.e. for the case where one would like to infer bisimulation from the relation of forcing the same formulas of the full language, does not go through. There is a lot of work (for the analogous case of modal logic) on better converses than the one given here. We refer the reader to [6] and [8].

Theorem 4.8. *$\mathsf{Th}_{I_n(\mathbf{p})}(k) \subseteq \mathsf{Th}_{I_n(\mathbf{p})}(m) \Rightarrow k \preceq_n m$, for $\mathbf{p}$-nodes k and m.*

Proof. Suppose k and m are p-nodes, and $\mathsf{Th}_{I_n(\mathbf{p})}(k) \subseteq \mathsf{Th}_{I_n(\mathbf{p})}(m)$. We want to prove: $k \preceq_n m$. In case $n = 0$ this is trivial. Suppose $n > 0$. Define, for k' in the model corresponding to k and m' in the model corresponding to m:

$$k'\mathcal{Z}_i m' :\Leftrightarrow \mathsf{Th}_{I_i(\mathbf{p})}(k') = \mathsf{Th}_{I_i(\mathbf{p})}(m').$$

We check that $\mathcal{Z}$ is an ℓ-simulation and that for every $k' \succeq k$ there is an $m' \succeq m$ with $k'\mathcal{Z}_n m'$.

Suppose $i > 0$ and $k'\mathcal{Z}_i m'$. Clearly k' and m' force the same atoms. We verify e.g. the zig-property. Suppose $k' \preceq k''$. Let:

$$\eta_i(k'') := (\bigwedge\{B{\in}I_{i-1}(\mathbf{p}) \mid k'' \models B\} \to \bigvee\{C{\in}I_{i-1}(\mathbf{p}) \mid k'' \not\models C\}).$$

Clearly $k' \not\models \eta_i(k'')$ and $\eta_i(k'') \in I_i(\mathbf{p})$. Ergo $m' \not\models \eta_i(k'')$. But then for some $m'' \geq m'$:

$$m'' \models \bigwedge\{B{\in}I_{i-1}(\mathbf{p}) \mid k'' \models B\} \text{ and } m'' \not\models \bigvee\{C{\in}I_{i-1}(\mathbf{p}) \mid k' \not\models C\}.$$

It follows that $k''\mathcal{Z}_{i-1}m''$.

To show that for any $m' \succeq m$ there is a $k' \succeq k$ with $k'\mathcal{Z}_n m'$. Note that $m \not\models \eta_n(m')$, ergo $k \not\models \eta_n(m')$, and, thus, for some k':

$$k' \models \bigwedge\{B{\in}I_{n-1}(\mathbf{p}) \mid m' \models B\} \text{ and } k' \not\models \bigvee\{C{\in}I_{n-1}(\mathbf{p}) \mid m' \not\models C\}.$$

Hence: $k\mathcal{Z}_{n-1}m$. □

Let k be a p-node. Define:

- $\mathsf{Y}_{n,k} := \mathsf{Y}_{n,k}(\mathbf{p}) := \bigwedge\{C{\in}I_n(\mathbf{p}) \mid k \models C\}$
- $\mathsf{N}_{n,k} := \mathsf{N}_{n,k}(\mathbf{p}) := \bigvee\{D{\in}I_n(\mathbf{p}) \mid k \not\models D\}$

Theorem 4.9. *$k \models \mathsf{Y}_{n,k}$ and $k \not\models \mathsf{N}_{n,k}$.*

Let m be a p-node. We have:

Theorem 4.10. $k \preceq_n m \Leftrightarrow m \models \mathsf{Y}_{n,k} \Leftrightarrow k \not\models \mathsf{N}_{n,m}$.

Theorem 4.11. *For $n \leq n'$: $\mathsf{IPC} \vdash \mathsf{Y}_{n',k} \to \mathsf{Y}_{n,k}$ and $\mathsf{IPC} \vdash \mathsf{N}_{n,k} \to \mathsf{N}_{n',k}$.*

Theorem 4.12. $k \preceq_n m \overset{1}{\Leftrightarrow} \mathsf{IPC} \vdash \mathsf{Y}_{n,m} \to \mathsf{Y}_{n,k} \overset{2}{\Leftrightarrow} \mathsf{IPC} \vdash \mathsf{N}_{n,m} \to \mathsf{N}_{n,k}$

Proof. (1) "$\Rightarrow$" Suppose $k \preceq_n m$. Let r be any p-node with $r \models \mathsf{Y}_{n,m}$. It follows that $m \preceq_n r$ and, hence, $k \preceq_n r$. Ergo, $r \models \mathsf{Y}_{n,k}$. "$\Leftarrow$" Suppose $\mathsf{IPC} \vdash \mathsf{Y}_{n,m} \to \mathsf{Y}_{n,k}$. Since $m \models \mathsf{Y}_{n,m}$, it follows that $m \models \mathsf{Y}_{n,k}$, and, hence, $k \preceq_n m$.

(2) "$\Rightarrow$" Suppose $k \preceq_n m$. Let r be any p-node with $r \not\models \mathsf{N}_{n,k}$. It follows that $r \preceq_n k$ and, hence, $r \preceq_n m$. Ergo: $r \not\models \mathsf{N}_{n,m}$. "$\Leftarrow$" Suppose $\mathsf{IPC} \vdash \mathsf{N}_{n,m} \to \mathsf{N}_{n,k}$. Since $k \not\models \mathsf{N}_{n,k}$, it follows that $k \not\models \mathsf{N}_{n,m}$ and hence: $k \preceq_n m$. □

Theorem 4.13. *$\mathsf{Y}_{n,k}$ is a prime formula.*

Proof. It is easily seen that $\mathsf{Y}_{n,k}$ is $I_n(\mathbf{p})$-saturated. Apply Theorem 4.4. □

5. Uniform Interpolation for IPC

Uniform Interpolation was proved for GL by V. Shavrukov (see: [12]). Shavrukov used the method of characters as developed by Z. Gleit and W. Goldfarb, who proved the Fixed Point Theorem of Provability Logic and the ordinary Interpolation Theorem employing characters (see: [5]). The methods of Gleit & Goldfarb and later of Shavrukov can be viewed as model theoretical. For IPC, A. Pitts proved Uniform Interpolation by proof theoretical methods, using proof systems allowing efficient cut-elimination (see: [10]), developed, independently, by J. Hudelmaier (see: [9]) and R. Dyckhoff (see: [1]). Later S. Ghilardi and M. Zawadowski (see: [3]), and, independently but later, A. Visser, found a model theoretical proof for Pitt's result using bounded bisimulations.

In this section, we will use $\preceq$ for the weak partial orderings and $\prec$ for the associated strict orderings. We prove an amalgamation lemma.

Lemma 5.1. *Consider disjoint sets of propositional variables $\mathcal{Q}$, $\mathbf{p}$ and $\mathcal{R}$. Let $X \subseteq \mathcal{L}^i(\mathcal{Q},\mathbf{p})$ be a finite adequate set. Let $\langle \mathbb{K}, k_0\rangle \in \mathsf{Pmod}(\mathcal{Q},\mathbf{p})$, $\langle \mathbb{M}, m_0\rangle \in \mathsf{Pmod}(\mathbf{p},\mathcal{R})$. Let:*

$$\nu := |\{C \in X \mid C \text{ is a propositional variable or an implication}\}|.$$

Suppose that $k_0 \simeq_{2.\nu+1,\mathbf{p}} m_0$. Then there is a $\mathcal{Q}$-extension $\langle \mathbb{N}, n_0\rangle$ of $\langle \mathbb{M}, m_0\rangle$ such that $\mathsf{Th}_X(n_0) = \mathsf{Th}_X(k_0)$.

Proof. Let $\mathcal{Z}$ be a downwards closed witness of $k_0 \simeq_{2.\nu+1,\mathbf{p}} m_0$. Define Φ_X from $\mathbb{K}$ to the Henkin model $\mathbb{H} := \mathbb{H}_X$ as follows: $\Phi_X(k) := \Delta(k) := \{B \in X \mid k \models B\}$. Define further for k in $\mathbb{K}$: $d_X(k) := d_{\mathbb{H}}(\Delta(k))$. Note that: $d_X(k) \le \nu$.

Consider a pair $\langle \Delta, m\rangle$ for Δ in $\mathbb{H}$ and m in $\mathbb{M}$. We say that k', k, m' is a *witnessing triple* for $\langle \Delta, m\rangle$ if:

$$\Delta = \Delta(k) = \Delta(k'),\ k' \preceq k,\ m' \preceq m,\ k'\mathcal{Z}_{2.d_X(k')+1}m',\ k\mathcal{Z}_{2.d_X(k')}m.$$

$$\begin{tikzcd}
\Delta \arrow[r, dash, "\Phi_X"] & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \\
\Delta \arrow[u, "="] \arrow[r, dash, "\Phi_X"] & k' \arrow[u, "\preceq"] \arrow[r, dash, "\mathcal{Z}_{2d_X(k')+1}"] & m' \arrow[u, "\preceq"']
\end{tikzcd}$$

Define:

- $N := \{\langle \Delta, m\rangle \mid$ there is a witnessing triple for $\langle \Delta, m\rangle\}$
- $n_0 := \langle \Delta(k_0), m_0\rangle$
- $\langle \Delta, m\rangle \preceq_{\mathbb{N}} \langle \Gamma, n\rangle :\Leftrightarrow \Delta \preceq_{\mathbb{H}} \Gamma$ and $m \preceq_{\mathbb{M}} n$
- $\langle \Delta, m\rangle \models_{\mathbb{N}} s :\Leftrightarrow \Delta \models_{\mathbb{H}} s$ or $m \models_{\mathbb{M}} s$

Note that by assumption $k_0 \mathcal{Z}_{2\nu+1} m_0$. Moreover: $2.d_X(k_0)+1 \leq 2.\nu+1$. Hence: $k_0 \mathcal{Z}_{2d_X(k_0)+1} m_0$. So we can take k_0, k_0, m_0 as witnessing triple for n_0. Let k', k, m' be a witnessing triple for $\langle \Delta, m \rangle$. Note that for $p \in \mathbf{p} \cap X$: $\Delta \models p \Leftrightarrow k \models p \Leftrightarrow m \models p$, and hence: $\langle \Delta, m \rangle \models p \Leftrightarrow \Delta \models p \Leftrightarrow m \models p$. We claim:

Claim 1 $n_0 \simeq_{\mathbf{p},\mathcal{R}} m_0$.
Claim 2 For $B \in X : \langle \Delta, m \rangle \models B \Leftrightarrow B \in \Delta$.

Evidently the lemma is immediate from the claims.

We prove Claim 1. Take as bisimulation $\mathcal{B}$ with $\langle \Delta, m \rangle \mathcal{B} m$. It is evident that $\mathsf{Th}_{\mathbf{p},\mathcal{R}}(\langle \Delta, m \rangle) = \mathsf{Th}_{\mathbf{p},\mathcal{R}}(m)$. Moreover, $\mathcal{B}$ has the zig-property. We check that $\mathcal{B}$ has the zag-property. Suppose $\langle \Delta, m \rangle \mathcal{B} m \preceq n$. We are looking for a pair $\langle \Gamma, n \rangle$ in N such that $\Delta \preceq \Gamma$. Let k', k, m' be a witnessing triple for $\langle \Delta, m \rangle$. Since $k' \mathcal{Z}_{2.d_X(k')+1} m' \preceq n$, there is a h such that $k' \preceq h \mathcal{Z}_{2.d_X(k')} n$. We take $\Gamma := \Delta(h)$. We need a witnessing triple k'^*, k^*, m'^* for $\langle \Gamma, n \rangle$ We distinguish two possibilities. First, $\Delta = \Gamma$. In this case we can take: $k'^* := k'$, $k^* := h$, $m'^* := m'$.

$$\begin{tikzcd}
\Gamma = \Delta \arrow[rr, dash, dashed, "\Phi_X"] & & k^* = h \arrow[rr, dash, dashed, "\mathcal{Z}_{2d_X(k')}"] & & n \\
& \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, "\preceq"'] \\
\Delta \arrow[uu, dashed, "="] \arrow[ur, "="] \arrow[rr, dash, "\Phi_X"] & & k'^* = k' \arrow[uu, dashed, "\preceq"] \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m'^* = m' \arrow[u, "\preceq"']
\end{tikzcd}$$

Secondly, $\Delta \neq \Gamma$. In this case we can take: $k'^* := h$, $k^* := h$, $m'^* := n$. To see this, note that, since $k' \preceq h$, we have: $\Delta = \Delta(k') \prec \Gamma$. Ergo $d_X(h) < d_X(k')$. It follows that: $2.d_X(h) + 1 \leq 2.d_X(k')$. So, $h \mathcal{Z}_{2.d_X(k')+1} n$ (and by downward closure also $h \mathcal{Z}_{2.d_X(k')} n$).

$$\begin{tikzcd}
\Gamma \arrow[rr, dash, dashed, "\Phi_X"] & & k'^* = k^* = h \arrow[rr, dash, dashed, "\mathcal{Z}_{2d_X(h)+1}"] & & m'^* = n \\
& \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, "\preceq"'] \\
\Delta \arrow[uu, dashed, "\prec"] \arrow[ur, "="] \arrow[rr, dash, "\Phi_X"] & & k' \arrow[uu, dashed, "\preceq"] \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"']
\end{tikzcd}$$

Finally, clearly, $b_{\mathrm{N}} \mathcal{B} b_{\mathrm{M}}$.

We prove Claim 2. The proof is by induction on X. The cases of atoms, conjunction and disjunction are trivial. We treat the case of implication. Suppose $(C \to D) \in X$. Consider the node $\langle \Delta, m\rangle$ with witnessing triple k', k, m'.

Suppose $(C \to D) \notin \Delta$. In case $C \in \Delta$ and $D \notin \Delta$, by the Induction Hypothesis, $\langle \Delta, m\rangle \models C$ and $\langle \Delta, m\rangle \not\models D$. So, $\langle \Delta, m\rangle \not\models (C \to D)$. Suppose $C \notin \Delta$. Clearly, $k \not\models (C \to D)$, so there is an $h \succeq k$ with $h \models C$ and $h \not\models D$. Let $\Gamma := \Delta(h)$. Since, $C \notin \Delta$, we find: $\Delta \prec \Gamma$ and, thus, $k \prec h$. Note that it follows that $2.d_X(k') \geq 2$. Since $k \mathcal{Z}_{2.d_X(k')} m$ and $k \preceq h$, there is an $n \succeq m$ with $h \mathcal{Z}_{2.d_X(k')-1} n$. Moreover: $2.d_X(h) + 1 \leq 2.d_X(k') - 1$. Ergo: $h \mathcal{Z}_{2.d_X(h)+1} n$. So h, h, n is a witnessing triple for $\langle \Gamma, n\rangle$. Clearly, $\langle \Delta, m\rangle \preceq \langle \Gamma, n\rangle$. By the Induction Hypothesis: $\langle \Gamma, n\rangle \models C$ and $\langle \Gamma, n\rangle \not\models D$. Hence, $\langle \Delta, m\rangle \not\models (C \to D)$.

$$
\begin{tikzcd}
 & \Gamma \arrow[r, dash, "\Phi_X"] & k'^* = k^* = h \arrow[r, dash, dashed, "\mathcal{Z}_{2d_X(h)+1}"] & m'^* = n \\
 & \Delta \arrow[u, "\prec"] \arrow[r, dash, "\Phi_X"] & k \arrow[u, "\preceq"] \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, dashed, "\preceq"] \\
\Delta \arrow[ur, "="] \arrow[r, dash, "\Phi_X"] & k' \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"]
\end{tikzcd}
$$

Suppose $\langle \Delta, m\rangle \not\models (C \to D)$. There is a $\langle \Gamma, n\rangle$ in N with $\langle \Delta, m\rangle \preceq \langle \Gamma, n\rangle$ and $\langle \Gamma, n\rangle \models C$ and $\langle \Gamma, n\rangle \not\models D$. Clearly $\Delta \preceq \Gamma$. By the Induction Hypothesis $C \in \Gamma$ and $D \notin \Gamma$. Ergo $(C \to D) \notin \Delta$. Thus we have proved Claim 2. □

Theorem 5.1 (Pitts' Uniform Interpolation Theorem). *Here is our version of Pitts' Uniform Interpolation Theorem.*

1. *Consider any formula A and any finite set of variables* $\mathbf{q}$. *Let*

   $$\nu := |\{C \in \mathsf{Sub}(A) \mid C \textit{ is a propositional variable or an implication}\}|$$

   There is a formula $\exists \mathbf{q}.A$ such that:
   a) $\mathsf{PV}(\exists \mathbf{q}.A) \subseteq \mathsf{PV}(A) \setminus \mathbf{q}$
   b) $\mathsf{i}(\exists \mathbf{q}.A) \leq 2.\nu + 2$
   c) For all $B \in \mathcal{L}^i$ with $\mathsf{PV}(B) \cap \mathbf{q} = \emptyset$, we have:

   $$\mathsf{IPC} \vdash A \to B \Leftrightarrow \mathsf{IPC} \vdash \exists \mathbf{q}.A \to B.$$

2. *Consider any formula B and any finite set of variables* $\mathbf{q}$. *Let* $\nu := \nu_{\mathsf{Sub}(B)}$. *There is a formula $\forall \mathbf{q}.B$ such that:*

a) $\mathsf{PV}(\forall \mathbf{q}.B) \subseteq \mathsf{PV}(B) \setminus \mathbf{q}$
b) $\mathrm{i}(\forall \mathbf{q}.B) \leq 2.\nu + 1$
c) For all $A \in \mathcal{L}^i$ *with* $\mathsf{PV}(A) \cap \mathbf{q} = \emptyset$, *we have:*

$$\mathsf{IPC} \vdash A \to B \Leftrightarrow \mathsf{IPC} \vdash A \to \forall \mathbf{q}.B.$$

Proof. (1) Consider A and $\mathbf{q}$. Let $\mathbf{p} := \mathsf{PV}(A) \setminus \mathbf{q}$. Take:

$$\exists \mathbf{q}.A := \bigwedge \{C \in I_{2.\nu+2}(\mathbf{p}) \mid \mathsf{IPC} \vdash A \to C\}.$$

Clearly $\exists \mathbf{q}.A$ satisfies (a) and (b). Moreover, $\mathsf{IPC} \vdash A \to \exists \mathbf{q}.A$. Hence all we have to prove is that for all B with $\mathsf{PV}(B) \cap \mathbf{q} = \emptyset$:

$$\mathsf{IPC} \vdash A \to B \Rightarrow \mathsf{IPC} \vdash \exists \mathbf{q}.A \to B.$$

Suppose, to the contrary, that for some B: $\mathsf{PV}(B) \cap \mathbf{q} = \emptyset$ and $\mathsf{IPC} \vdash A \to B$ and $\mathsf{IPC} \nvdash \exists \mathbf{q}.A \to B$. Take $\mathbf{r} := \mathsf{PV}(B) \setminus \mathbf{p}$. Note that $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are pairwise disjoint, $\mathsf{PV}(A) \subseteq \mathbf{q} \cup \mathbf{p}$ and $\mathsf{PV}(B) \subseteq \mathbf{p} \cup \mathbf{r}$.

Let m be any $\mathbf{p}, \mathbf{r}$-node with $m \models \exists \mathbf{q}.A$ and $m \not\models B$. Let $\mathsf{Y} := \mathsf{Y}_{2.\nu+1,m[\mathbf{p}]}$ and $\mathsf{N} := \mathsf{N}_{2.\nu+1,m[\mathbf{p}]}$ (see Sect. 4.). We claim that: $A, \mathsf{Y} \nvdash \mathsf{N}$. If it did, we would have: $A \vdash \mathsf{Y} \to \mathsf{N}$. And hence by definition: $\exists \mathbf{q}.A, \mathsf{Y} \vdash \mathsf{N}$. Quod non, since $m \models \exists \mathbf{q}.A, \mathsf{Y}$ and $m \not\models \mathsf{N}$. Let k be any $\mathbf{q}, \mathbf{p}$-node such that: $k \models A, \mathsf{Y}$ and $k \not\models \mathsf{N}$. We find that $k \simeq_{2.\nu+1,\mathbf{p}} m$. Apply Lemma 5.1 with $\mathsf{Sub}(A)$ in the role of X to find a $\mathbf{q}, \mathbf{p}, \mathbf{r}$-node n with: $m \simeq_{\mathbf{p},\mathbf{r}} n$ and $\mathsf{Th}_{\mathsf{Sub}(A)}(k) = \mathsf{Th}_{\mathsf{Sub}(A)}(n)$. It follows that $n \not\models B$, but $n \models A$. A contradiction.

(2) Consider B and $\mathbf{q}$. Let $\mathbf{p} := \mathsf{PV}(B) \setminus \mathbf{q}$. Take:

$$\forall \mathbf{q}.B := \bigvee \{D \in I_{2.\nu+1}(\mathbf{p}) \mid \mathsf{IPC} \vdash D \to B\}.$$

Clearly $\forall \mathbf{q}.B$ satisfies (a) and (b). Moreover $\mathsf{IPC} \vdash \forall \mathbf{q}.B \to B$. Hence all we have to prove is that for all A with $\mathsf{PV}(A) \cap \mathbf{q} = \emptyset$:

$$\mathsf{IPC} \vdash A \to B \Rightarrow \mathsf{IPC} \vdash A \to \forall \mathbf{q}.B.$$

Suppose that, to the contrary, for some A: $\mathsf{PV}(A) \cap \mathbf{q} = \emptyset$ and $\mathsf{IPC} \vdash A \to B$ and $\mathsf{IPC} \nvdash A \to \forall \mathbf{q}.B$. Take $\mathbf{r} := \mathsf{PV}(A) \setminus \mathbf{p}$. Note that $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are pairwise disjoint, $\mathsf{PV}(B) \subseteq \mathbf{q}, \mathbf{p}$ and $\mathsf{PV}(A) \subseteq \mathbf{p}, \mathbf{r}$.

Let m be any $\mathbf{p}, \mathbf{r}$-node with $m \models A$ and $m \not\models \forall \mathbf{q}.B$. Let $\mathsf{Y} := \mathsf{Y}_{2.\nu+1,m[\mathbf{p}]}$ and $\mathsf{N} := \mathsf{N}_{2.\nu+1,m[\mathbf{p}]}$. We claim that: $\mathsf{Y} \nvdash \mathsf{N} \vee B$. Note that, by Theorem 4.13, Y is prime. So if $\mathsf{Y} \vdash \mathsf{N} \vee B$, then $\mathsf{Y} \vdash \mathsf{N}$ or $\mathsf{Y} \vdash B$. Since $\mathsf{Y} \nvdash \mathsf{N}$, it follows that $\mathsf{Y} \vdash B$. But then by definition: $\mathsf{Y} \vdash \forall \mathbf{q}.B$. Quod non, since $m \models \mathsf{Y}$ and $m \not\models \forall \mathbf{q}.B$. Let k be any $\mathbf{q}, \mathbf{p}$-node such that: $k \models \mathsf{Y}$ and $k \not\models \mathsf{N} \vee B$. We find that $k \simeq_{2.\nu+1,\mathbf{p}} m$. Apply Lemma 5.1 with $\mathsf{Sub}(B)$ in the role of X to find a $\mathbf{q}, \mathbf{p}, \mathbf{r}$-node n with: $m \simeq_{\mathbf{p},\mathbf{r}} n$ and $\mathsf{Th}_{\mathsf{Sub}(B)}(k) = \mathsf{Th}_{\mathsf{Sub}(B)}(n)$. It follows that $n \models A$, but $n \not\models B$. A contradiction. □

Note that in (1) of the above theorem we have estimate $2.\nu + 2$ and in (2) $2.\nu + 1$. With some extra work one can get the marginal improvement to $2.\nu + 1$ also for (1). We will not derive the sharper estimate here.

Theorem 5.2 (Semantics of Pitts' Quantifiers). *Consider a node m. Suppose $A \in \mathcal{L}^i$. We have:*

1. $m \models \exists q.A \Leftrightarrow \exists n\ m \simeq_{[q]} n$ *and* $n \models A$.
2. $m \models \forall q.A \Leftrightarrow$ *for all* n *with* $m \simeq_{[q]} n,\ n \models A$.

Proof. (1) "$\Leftarrow$" Trivial. "$\Rightarrow$" Let $\mathbf{p} := \mathsf{PV}(A) \setminus \{q\}$ and $\nu := \nu_{\mathsf{Sub}(A)}$. Suppose $m \models \exists q.A$, where m is an $\mathcal{R}$-node with $\mathbf{p} \subseteq \mathcal{R}$. Let $\mathsf{Y} := \mathsf{Y}_{2.\nu+1,m[\mathbf{p}]}$ and $\mathsf{N} := \mathsf{N}_{2.\nu+1,m[\mathbf{p}]}$. As in Theorem 5.1(2), $A, \mathsf{Y} \not\vdash \mathsf{N}$. Let k be any $q,\mathbf{p}$-node such that: $k \models A, \mathsf{Y}$ and $k \not\models \mathsf{N}$. We find that $k \simeq_{2.\nu+1,\mathbf{p}} m$. Apply Lemma 5.1 to k and $m[\mathcal{R} \setminus \{q\}]$ with $\mathsf{Sub}(A)$ in the role of X, $\{q\}$ in the role of $\mathcal{Q}$, $\mathbf{p}$ in the role of $\mathbf{p}$, $\mathcal{R} \setminus (\mathbf{p} \cup \{q\})$ in the role of $\mathcal{R}$, to find a $q,\mathbf{p},\mathcal{R}$-node n with: $m \simeq_{[q]} n$ and $\mathsf{Th}_{\mathsf{Sub}(A)}(k) = \mathsf{Th}_{\mathsf{Sub(A)}}(n)$, and, thus, $n \models A$. The proof of (2) is similar. □

Theorem 5.1 is not formulated entirely in terms of ℓ-simulations. The reason is that such a form does not provide a very sharp estimate on uniform interpolants. But if we do not want to worry about precise complexities a watered down version can be pleasant to have. By applying Theorem 5.1 to $X := I_n(\mathbf{p}, \mathbf{q})$ we find:

Corollary 5.1. *For all disjoint $\mathbf{q}, \mathbf{p}$ and numbers s, there is an N (multi-exponential in $|\mathbf{q}, \mathbf{p}| + s$), such that: for all $k \in \mathsf{Pmod}(\mathbf{q}, \mathbf{p})$, and all $m \in \mathsf{Pmod}$ with $\mathbf{q} \cap \mathcal{P}_{\mathsf{M}} = \emptyset$ and $\mathbf{p} \subseteq \mathcal{P}_m$, we have:*

$$k \simeq_{N,\mathbf{p}} m \Rightarrow \text{ there is an } n \in \mathsf{Pmod}(\mathbf{q}, \mathcal{P}_m) \text{ with } n \simeq_{s,\mathbf{q},\mathbf{p}} k \text{ and } n \simeq_{\mathcal{P}_m} m.$$

We repeat a result from [13]. We illustrate that the increase of implicational complexity in going to a uniform interpolant is unavoidable. It is an interesting problem to find both better upper and lower bounds.

Theorem 5.3. *Every formula of $\mathcal{L}^i$ is equivalent to an I_2-formula preceded by existential quantifiers and to an I_3-formula preceded by universal quantifiers.*

Proof. Suppose $A \in \mathcal{L}^i(\mathbf{p})$. Let $\mathbf{q}$ be a set of variables disjoint from $\mathbf{p}$ that is in 1-1 correspondence with the subformulas of the form $(B \to C)$ of A. Let the correspondence be q. We define a mapping $\mathcal{T}$ as follows:

- $\mathcal{T}$ commutes with atoms, conjunction and disjunction
- $\mathcal{T}(B \to C) := \mathsf{q}(\mathfrak{B} \to \mathfrak{C})$

Define:

- $\mathsf{EQ} := \bigwedge \{\mathsf{q}(\mathfrak{B} \to \mathfrak{C}) \leftrightarrow (\mathcal{T}(\mathfrak{B}) \to \mathcal{T}(\mathfrak{C})) \mid (\mathfrak{B} \to \mathfrak{C}) \in \mathsf{Sub}(\mathfrak{A})\}$

Note that EQ is I_2. Finally we put:

- $A^{\#} := \exists \mathbf{q}(\mathsf{EQ} \wedge \mathcal{T}(A))$, $A^{\$} := \forall \mathbf{q}(\mathsf{EQ} \rightarrow \mathcal{T}(A))$

By elementary reasoning in second order propositional logic we find: $\vdash A \leftrightarrow A^{\#}$ and $\vdash A \leftrightarrow A^{\$}$. □

We end this section by verifying semantically a striking principle (present in Pitts' paper) valid for the Pitts interpretation.

Theorem 5.4. *$k \models \forall p(B \vee C) \Rightarrow k \models \forall p.B$ or $k \models \forall p.C$.*

Proof. We reason by contraposition. Suppose $k \not\models \forall p.B$ and $k \not\models \forall p.C$. It follows that there are nodes m and n, such that $k \simeq_{[p]} m \not\models B$ and $k \simeq_{[p]} n \not\models C$. Let $\mathbb{M}$ and $\mathbb{N}$ be the models of, respectively, m and n. Let $\mathbb{P} := \mathsf{Glue}(\mathbb{M}[m], \mathbb{N}[n])$. Let b be the new root. It is easily seen that $k \simeq_{[p]} b$ and $b \not\models (B \vee C)$ □

6. Uniform Interpolation for K

We first survey the connection between modal propositional formulas and bounded bisimulations. Since these facts are similar to, but simpler than the corresponding facts for IPC, we just state the results without the proofs. Let $\mathfrak{b}(A)$ be the box-depth of a formula. $B_k(\mathbf{p})$ is the set of formulas in the variables $\mathbf{p}$ with box-depth $\leq k$. $B_k(\mathbf{p})$ is finite *modulo provable equivalence.* Consider $\mathbf{p}$-nodes k and m. Then: $k \simeq_n m \Leftrightarrow \mathsf{Th}_{B_n(\mathbf{p})}(k) = \mathsf{Th}_{B_n(\mathbf{p})}(m)$. Define: $\mathsf{Y}_{n,k} := \bigwedge \mathsf{Th}_{B_n(\mathbf{p})}(k)$. Clearly, $k \simeq_n m \Leftrightarrow m \models \mathsf{Y}_{n,k} \Leftrightarrow \mathsf{K} \vdash \mathsf{Y}_{n,m} \leftrightarrow \mathsf{Y}_{n,k}$.

Before considering uniform interpolation for more complicated modal systems like S4Grz, we do the relatively easy proof for K. This theorem was first proved by Silvio Ghilardi, see [2]. Uniform interpolation for K follows from the amalgamation lemma below.

Lemma 6.1. *Consider pairwise disjoint sets of propositional variables $\mathcal{Q}$, $\mathbf{p}$ and $\mathcal{R}$. Let $\langle \mathbb{K}, k_0 \rangle \in \mathsf{Pmod}(\mathcal{Q}, \mathbf{p})$ and $\langle \mathbb{M}, m_0 \rangle \in \mathsf{Pmod}(\mathbf{p}, \mathcal{R})$. Suppose that $k_0 \simeq_{\alpha,\mathbf{p}} m_0$. Then there is a $\mathcal{Q}$-extension $\langle \mathbb{N}, n_0 \rangle$ of $\langle \mathbb{M}, m_0 \rangle$ such that $n_0 \simeq_\alpha k_0$.*

Proof. Let $\mathcal{Z}$ be a downwards closed witness of $k_0 \simeq_{\alpha,\mathbf{p}} m_0$. We add a 'virtual top' $\top$ to $\mathbb{K}$ and stipulate that $\top$ satisfies no atoms. Let's call the new model $\mathbb{K}^\top$. We extend ω^∞ with a new bottom $\bot$ to $\omega^{\infty,\bot}$. Define $\mathsf{Pd}(n+1) := n$, $\mathsf{Pd}(0) := \mathsf{Pd}(\bot) := \bot$, $\mathsf{Pd}(\infty) = \infty$. Now define the following model $\mathbb{N}$:

- $N := \mathcal{Z} \cup \{\langle \top, \bot, m \rangle \mid m \in M\}$
- $\langle k, \alpha, m \rangle \prec_{\mathbb{N}} \langle k', \alpha', m' \rangle :\Leftrightarrow k \prec_{\mathbb{K}^\top} k'$ and $\alpha' = \mathsf{Pd}(\alpha)$ and $m \prec_{\mathbb{M}} m'$
- $\langle k, \alpha, m \rangle \models s :\Leftrightarrow k \models_{\mathbb{K}} s$ or $m \models_{\mathbb{M}} s$

We claim:

Claim 1 $n_0 \simeq_{\mathbf{p},\mathcal{R}} m_0$.
Claim 2 $n_0 \simeq_{\alpha,(\mathcal{Q},\mathbf{p})} k_0$.

We prove Claim 1. Take as bisimulation $\mathcal{B}$, with $\langle k,\alpha,m\rangle \mathcal{B} m' :\Leftrightarrow m = m'$. Clearly, if $n\mathcal{B}m$ then $\mathsf{Th}_{\mathbf{p},\mathcal{R}}(n) = \mathsf{Th}_{\mathbf{p},\mathcal{R}}(m)$. Moreover, $\mathcal{B}$ trivially has the zig-property. We check that $\mathcal{B}$ has the zag-property. Suppose $\langle k,\alpha,m\rangle\mathcal{B}m \prec m'$. If $\alpha \in \{0,\bot\}$, we can finish the diagram with $\langle \top,\bot,m'\rangle$. If $\alpha = \alpha' + 1$ for $\alpha' \in \omega^\infty$, we have $k\mathcal{Z}_\alpha m$ and, hence, there is a k' such that $k \prec_{\mathbb{K}} k'$ and $k'\mathcal{Z}_{\alpha'}m'$. So we can finish the diagram with $\langle k',\alpha',m'\rangle$.

We prove Claim 2. Take as layered bisimulation $\mathcal{S}$, with $\langle k,\alpha,m\rangle\mathcal{S}_\alpha k' \Leftrightarrow k = k'$ (for $\alpha \in \omega^\infty$). Clearly, if $n\mathcal{S}_\alpha k$ then $\mathsf{Th}_{\mathcal{Q},\mathbf{p}}(n) = \mathsf{Th}_{\mathcal{Q},\mathbf{p}}(k)$. We check that $\mathcal{S}$ has the zag-property. The zig-property is analogous. Suppose $\langle k,\alpha + 1,m\rangle\mathcal{S}_{\alpha+1}k \prec k'$. Since $k\mathcal{Z}_{\alpha+1}m$, there exists $m' \succ m$ such that $k'\mathcal{Z}_\alpha m'$. Hence $\langle k',\alpha,m'\rangle \succ \langle k,\alpha+1,m\rangle$, and $\langle k',\alpha,m'\rangle\mathcal{S}_\alpha k'$. □

Theorem 6.1 (Uniform Interpolation). *We prove uniform interpolation for* K

1. *Consider any formula A and any finite set of variables* $\mathbf{q}$. *Let* $\nu := \mathfrak{b}(A)$. *There is a formula* $\exists\mathbf{q}.A$ *such that:*
   a) $\mathsf{PV}(\exists\mathbf{q}.A) \subseteq \mathsf{PV}(A) \setminus \mathbf{q}$
   b) $\mathfrak{b}(\exists\mathbf{q}.A) \leq \nu$
   c) *For all* $B \in \mathcal{L}^m$ *with* $\mathsf{PV}(B) \cap \mathbf{q} = \emptyset$, *we have:*
   $$\mathsf{K} \vdash A \to B \Leftrightarrow \mathsf{K} \vdash \exists\mathbf{q}.A \to B.$$
2. *Consider any formula B and any finite set of variables* $\mathbf{q}$. *Let* $\nu := \mathfrak{b}(B)$. *There is a formula* $\forall\mathbf{q}.B$ *such that:*
   a) $\mathsf{PV}(\forall\mathbf{q}.B) \subseteq \mathsf{PV}(B) \setminus \mathbf{q}$
   b) $\mathfrak{b}(\forall\mathbf{q}.B) \leq \nu$
   c) *For all* $A \in \mathcal{L}^m$ *with* $\mathsf{PV}(A) \cap \mathbf{q} = \emptyset$, *we have:*
   $$\mathsf{K} \vdash A \to B \Leftrightarrow \mathsf{K} \vdash A \to \forall\mathbf{q}.B.$$

Proof. We just prove (1). The proof of (2) is analogous. (Alternatively, we may take $(\forall\mathbf{q}.B) := (\neg\exists\mathbf{q}\neg B)$.) Consider A and $\mathbf{q}$. Let $\mathbf{p} := \mathsf{PV}(A) \setminus \mathbf{q}$. Take:

$$\exists\mathbf{q}.A := \bigwedge\{C \in I_\nu(\mathbf{p}) \mid \mathsf{K} \vdash A \to C\}.$$

Clearly $\exists\mathbf{q}.A$ satisfies (a) and (b). Moreover, $\mathsf{K} \vdash A \to \exists\mathbf{q}.A$. Hence, all we have to prove is that for all B with $\mathsf{PV}(B) \cap \mathbf{q} = \emptyset$:

$$\mathsf{K} \vdash A \to B \Rightarrow \mathsf{K} \vdash \exists\mathbf{q}.A \to B.$$

Suppose, to the contrary, that for some B: $\mathsf{PV}(B) \cap \mathbf{q} = \emptyset$ and $\mathsf{K} \vdash A \to B$ and $\mathsf{K} \nvdash \exists\mathbf{q}.A \to B$. Take $\mathbf{r} := \mathsf{PV}(B) \setminus \mathbf{p}$. Note that $\mathbf{p},\mathbf{q},\mathbf{r}$ are pairwise disjoint, $\mathsf{PV}(A) \subseteq \mathbf{q} \cup \mathbf{p}$ and $\mathsf{PV}(B) \subseteq \mathbf{p} \cup \mathbf{r}$.

Let m be any $\mathbf{p},\mathbf{r}$-node with $m \models \exists \mathbf{q}.A$ and $m \not\models B$. Let $\mathsf{Y} := \mathsf{Y}_{\nu,m[\mathbf{p}]}$ and We claim that: A, Y is consistent. If it were not, we would have: $A \vdash \neg\mathsf{Y}$. And, hence, by definition: $\exists \mathbf{q}.A \vdash \neg\mathsf{Y}$. Quod non, since $m \models \exists \mathbf{q}.A, \mathsf{Y}$ and $\mathfrak{d}(\neg\mathsf{Y}) = \nu$. Let k be any $\mathbf{q},\mathbf{p}$-node such that: $k \models A, \mathsf{Y}$. We find that $k \simeq_{\nu,\mathbf{p}} m$. Apply Lemma 6.1 to find a $\mathbf{q},\mathbf{p},\mathbf{r}$-node n with: $m \simeq_{\mathbf{p},\mathbf{r}} n$ and $m \simeq_{\nu,(\mathbf{p},\mathbf{r})} n$. It follows that $n \not\models B$, but $n \models A$. A contradiction. □

The proof of the following theorem is fully analogous to the the proof of its twin for the case of IPC.

Theorem 6.2. *Consider a node m. Suppose $A \in \mathcal{L}^m$. We have:*

1. $m \models \exists q.A \Leftrightarrow \exists n\ m \simeq_{[q]} n$ *and* $n \models A$.
2. $m \models \forall q.A \Leftrightarrow$ *for all* n *with* $m \simeq_{[q]} n$, $n \models A$.

7. Uniform Interpolation for GL

In this section we prove Uniform Interpolation for GL. It is well known that GL is sound and complete for upward wellfounded Kripke models and that it has the finite model property. Since GL-models are irreflexive we use '$\prec$' for their accessibility relation and '$\preceq$' for the corresponding weak partial order. '$\vdash$' will stand for GL-derivability.

Let X be a finite, adequate set of formulas. *Adequate* means: closed under subformulas. The GL Henkin model $\mathbb{H}_X$ for X is constructed in the following way.

- The nodes are the subsets Δ of X that are *X-saturated*, i.e. if Δ proves some finite disjunction of elements of X then some disjunct is in Δ.
- $\Delta \prec \Delta'$ iff $\Box A \in \Delta \Rightarrow A, \Box A \in \Delta'$

Note that this model may contain *non-trivial loops!* and, thus is not a GL-model. (It is easy to remove these loops, but for the present purposes, we need to keep them.) The height of a model is the maximal depth. The height of the Henkin model is $\leq 2.|\{C \in X \mid C \text{ is boxed}\}|$. To see this, consider $\Delta_0 \prec^+ \Delta_1 \prec^+ \Delta_2$. Clearly, going up the set of boxed formulas in the Δ_i increases. Suppose we had the same boxed formulas in Δ_0, Δ_1 and Δ_2. Suppose $\Box A \in \Delta_2$. Then, ex hypothesi, $\Box A \in \Delta_0$. Hence, $A, \Box A \in \Delta_1$. We may conclude that $\Delta_2 \prec \Delta_1$. Quod non. So, necessarily, the boxed formulas increase by at least one in going from Δ_0 to Δ_2. It follows that if we have a strictly ascending chain of length $2.n$, then there are at least n boxed subformulas.

As in for IPC and K we start with an amalgamation lemma. Consider disjoint sets of propositional variables $\mathcal{Q}$, $\mathbf{p}$ and $\mathcal{R}$. Let $\langle \mathbb{K}, k_0 \rangle \in \mathsf{Pmod}(\mathcal{Q}, \mathbf{p})$ and $\langle \mathbb{M}, m_0 \rangle \in \mathsf{Pmod}(\mathbf{p}, \mathcal{R})$ be pointed GL-models.

Lemma 7.1. *Let $X \subseteq \mathcal{L}^m(\mathcal{Q}, \mathbf{p})$ be a finite adequate set. Let:*

$$\nu := 2.|\{C \in X \mid C \textit{ is boxed}\}|.$$

Suppose that $k_0 \simeq_{2.\nu+1,\mathbf{p}} m_0$. *Then there is a* $\mathcal{Q}$*-extension* $\langle \mathbb{N}, n_0 \rangle$ *of* $\langle \mathbb{M}, m_0 \rangle$ *such that* $\mathbb{N}$ *is a* GL*-model and* $\mathsf{Th}_X(n_0) = \mathsf{Th}_X(k_0)$.

Proof. Let $\mathcal{Z}$ be a downwards closed witness of $k_0 \simeq_{2.\nu+1,\mathbf{p}} m_0$. Define Φ_X from $\mathbb{K}$ to the Henkin model $\mathbb{H} := \mathbb{H}_X$ as follows: $\Phi_X(k) := \Delta(k) := \{B \in X \mid k \models B\}$. Define further for k in $\mathbb{K}$: $d_X(k) := d_{\mathbb{H}}(\Delta(k))$. Note that: $d_X(k) \leq \nu$.

Consider a pair $\langle \Delta, m \rangle$ for Δ in $\mathbb{H}$ and m in $\mathbb{M}$. Consider k', k, m'. Let $\Delta' := \Phi_X(k')$. We say that k', k, m' is a *witnessing triple* for $\langle \Delta, m \rangle$ if:

$$\Delta' \approx \Delta,\ k' \preceq k,\ m' \preceq m,\ k' \mathcal{Z}_{2.d_X(k')+1} m',\ k \mathcal{Z}_{2.d_X(k')} m.$$

$$\begin{tikzcd}
 & \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')}"] & & m \\
\Delta' \arrow[ur, "\approx"] \arrow[rr, dash, "\Phi_X"] & & k' \arrow[ur, "\preceq"] \arrow[rrr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & & m' \arrow[u, "\preceq"]
\end{tikzcd}$$

Define:

- $N := \{\langle \Delta, m \rangle \mid$ there is a witnessing triple for $\langle \Delta, m \rangle\}$
- $n_0 := \langle \Delta(k_0), m_0 \rangle$
- $\langle \Delta, m \rangle \prec_{\mathbb{N}} \langle \Gamma, n \rangle :\Leftrightarrow \Delta \prec_{\mathbb{H}} \Gamma$ and $m \prec_{\mathbb{M}} n$
- $\langle \Delta, m \rangle \models_{\mathbb{N}} s :\Leftrightarrow \Delta \models_{\mathbb{H}} s$ or $m \models_{\mathbb{M}} s$

Note that by assumption $k_0 \mathcal{Z}_{2\nu+1} m_0$. Moreover: $2.d_X(k_0)+1 \leq 2.\nu+1$. Hence: $k_0 \mathcal{Z}_{2d_X(k_0)+1} m_0$. So we can take k_0, k_0, m_0 as witnessing triple for n_0. Let k', k, m' be a witnessing triple for $\langle \Delta, m \rangle$. Note that for $p \in \mathbf{p} \cap X$: $\Delta \models p \Leftrightarrow k \models p \Leftrightarrow m \models p$, and hence: $\langle \Delta, m \rangle \models p \Leftrightarrow \Delta \models p \Leftrightarrow m \models p$. It is easy to see that $\mathbb{N}$ is a GL-model (even if $\mathbb{H}_X$ need not be one). We claim:

Claim 1 $n_0 \simeq_{\mathbf{p},\mathcal{R}} m_0$.
Claim 2 For $B \in X : \langle \Delta, m \rangle \models B \Leftrightarrow B \in \Delta$.

Evidently the lemma is immediate from the claims.

We prove Claim 1. Take as bisimulation $\mathcal{B}$ with $\langle \Delta, m \rangle \mathcal{B} m$. It is evident that $\mathsf{Th}_{\mathbf{p},\mathcal{R}}(\langle \Delta, m \rangle) = \mathsf{Th}_{\mathbf{p},\mathcal{R}}(m)$. Moreover, $\mathcal{B}$ has the zig-property. We check that $\mathcal{B}$ has the zag-property. Suppose $\langle \Delta, m \rangle \mathcal{B} m \prec n$. We are looking for a pair $\langle \Gamma, n \rangle$ in N such that $\Delta \prec \Gamma$. Let k', k, m' be a witnessing triple for $\langle \Delta, m \rangle$. We write $\Delta' := \Delta(k')$. Since $k' \mathcal{Z}_{2.d_X(k')+1} m' \prec n$, there is a h such that $k' \prec h \mathcal{Z}_{2.d_X(k')} n$. We take $\Gamma := \Delta(h)$. Clearly $\Delta \prec \Gamma$. We need a witnessing triple k'^*, k^*, m'^* for $\langle \Gamma, n \rangle$ We distinguish two possibilities. First, $\Delta \approx \Gamma$. In this case we can take: $k'^* := k'$, $k^* := h$, $m'^* := m'$.

$$
\begin{tikzcd}
\Gamma \arrow[rr, dash, dashed, "\Phi_X"] & & k^* = h \arrow[rr, dash, dashed, "\mathcal{Z}_{2d_X(k')}"] & & n \\
& \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, "\prec"'] \\
\Delta' \arrow[uu, dashed, "\approx"] \arrow[ur, "\approx"] \arrow[rr, dash, "\Phi_X"] & & k'^* = k' \arrow[uu, dashed, "\prec"] \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m'^* = m' \arrow[u, "\preceq"']
\end{tikzcd}
$$

Secondly, $\Delta \not\approx \Gamma$. In this case we can take: $k'^* := h$, $k^* := h$, $m'^* := n$. To see this, note that, since $k' \prec h$, we have: $\Delta \approx \Delta' \prec \Gamma$ and, hence, $\Delta \prec^+ \Gamma$. Ergo $d_X(h) < d_X(k')$. It follows that: $2.d_X(h) + 1 \leq 2.d_X(k')$. So, $h\mathcal{Z}_{2.d_X(k')+1}n$.

$$
\begin{tikzcd}
\Gamma \arrow[rr, dash, dashed, "\Phi_X"] & & k'^* = k^* = h \arrow[rr, dash, dashed, "\mathcal{Z}_{2d_X(h)+1}"] & & m'^* = n \\
& \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, "\prec"'] \\
\Delta' \arrow[uu, dashed, "\prec^+"] \arrow[ur, "\approx"] \arrow[rr, dash, "\Phi_X"] & & k' \arrow[uu, dashed, "\prec"] \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"']
\end{tikzcd}
$$

Finally, clearly, $b_{\mathrm{N}}\mathcal{B}b_{\mathrm{M}}$.

We prove Claim 2. The proof is by induction on X. The cases of atoms, conjunction and disjunction are trivial. We treat the only non-trivial case: the left-to-right case of the box. Consider $\Box C \in X$ and consider the node $\langle \Delta, m \rangle$ with witnessing triple k', k, m'. Suppose $\Box C \notin \Delta$. Clearly, $k \not\models \Box C$, so there is an $h' \succ k$ with $h' \not\models C$. Let h be maximal in $\mathbb{K}$ with $h \succ k$ and $h \not\models C$. By maximality, we find: $h \models \Box C$. Let $\Gamma := \Delta(h)$. Since, $\Box C \notin \Delta$ and $\Box C \in \Gamma$, we find: $\Delta \prec^+ \Gamma$. Note that it follows that $d_X(k') \geq 1$. Since, $k\mathcal{Z}_{2.d_X(k')}m$ and $k \prec h$, there is an $n \succ m$ with $h\mathcal{Z}_{2.d_X(k')-1}n$. Moreover: $2.d_X(h) + 1 \leq 2.d_X(k') - 1$. Ergo: $h\mathcal{Z}_{2.d_X(h)+1}n$. So we can take $k'^* := h$, $k^* := h$, $m'^* := n$ to witness $\langle \Gamma, n \rangle$. Clearly, $\langle \Delta, m \rangle \prec \langle \Gamma, n \rangle$. By the Induction Hypothesis: $\langle \Gamma, n \rangle \not\models C$. Hence, $\langle \Delta, m \rangle \not\models \Box C$.

$$\begin{tikzcd}
 & \Gamma \arrow[rr, dash, "\Phi_X"] & & k'^* = k^* = h \arrow[r, dash, dashed, "\mathcal{Z}_{2d_X(h)+1}"] & m'^* = n \\
 & \Delta \arrow[u, "\prec^+"] \arrow[rr, dash, "\Phi_X"] & & k \arrow[u, "\prec"] \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, dashed, "\prec"] \\
\Delta \arrow[ur, "\approx"] \arrow[rr, dash, "\Phi_X"] & & k' \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"]
\end{tikzcd}$$

Thus we have proved Claim 2. □

We formulate Uniform Interpolation for GL. Its proof is fully analogous to the one of Uniform Interpolation for K.

Theorem 7.1 (Uniform Interpolation). *We state uniform interpolation for* GL

1. *Consider any formula A and any finite set of variables $\mathbf{q}$. Let*

$$\nu := 2.|\{C \in \mathsf{Sub}(A) \mid C \textit{ is boxed}\}|.$$

There is a formula $\exists\mathbf{q}.A$ such that:
a) $\mathsf{PV}(\exists\mathbf{q}.A) \subseteq \mathsf{PV}(A) \setminus \mathbf{q}$
b) $\mathfrak{d}(\exists\mathbf{q}.A) \leq 2.\nu + 1$
c) For all $B \in \mathcal{L}^m$ *with* $\mathsf{PV}(B) \cap \mathbf{q} = \emptyset$*, we have:*

$$\mathsf{GL} \vdash A \to B \Leftrightarrow \mathsf{GL} \vdash \exists\mathbf{q}.A \to B.$$

2. *Consider any formula B and any finite set of variables $\mathbf{q}$. Let*

$$\nu := 2.|\{C \in \mathsf{Sub}(B) \mid C \textit{ is boxed}\}|.$$

There is a formula $\forall\mathbf{q}.B$ such that:
a) $\mathsf{PV}(\forall\mathbf{q}.B) \subseteq \mathsf{PV}(B) \setminus \mathbf{q}$
b) $\mathfrak{d}(\forall\mathbf{q}.B) \leq 2.\nu + 1$
c) For all $A \in \mathcal{L}^m$ *with* $\mathsf{PV}(A) \cap \mathbf{q} = \emptyset$*, we have:*

$$\mathsf{GL} \vdash A \to B \Leftrightarrow \mathsf{GL} \vdash A \to \forall\mathbf{q}.B.$$

The semantical interpretation of the propositional quantifiers is fully analogous to the case of K.

8. Uniform Interpolation for S4Grz

S4Grz, a logic called after Andrzej Gregorczyk, is K extended with:

T $\vdash \Box A \to A$
4 $\vdash \Box A \to \Box\Box A$
Grz $\vdash \Box(\Box(A \to \Box A) \to A) \to A$

It is easy to see that T is superfluous. Note also that over KT4 (= S4), *Grz* is equivalent to:

Grz' $\Box(\Box(A \to \Box A) \to A) \to \Box A$

The logic is sound for weak partial orderings such that the associated strict ordering is upward wellfounded. We will show that the completeness of the logic in finite partial orderings. Since we deal with reflexive structures in this section, we will use '$\preceq$' for these relations. In case our relation is a weak partial ordering we write '$\prec$' for the associated strict ordering. For weak partial preorderings we will use $\prec^+$ for the associated strict version to stress the fact that also non-trivial loops are removed. '$\vdash$' will stand for S4Grz-provability.

Let X be a finite adequate set. We construct a Henkin model $\mathbb{J}_X$ as follows. Let

$$X^+ := X \cup \{(B \to \Box B), \Box(B \to \Box B) \mid \Box B \in X\}.$$

Clearly, X is again adequate. Define:

- The domain J is the set of X^+-saturated sets Δ.
- $\Delta \preceq \Delta' :\Leftrightarrow \Delta = \Delta'$ or (for all $\Box C \in \Delta$, $\Box C \in \Delta'$ and for some $\Box D \in \Delta'$, $\Box D \notin \Delta$)
- $\Delta \models p :\Leftrightarrow p \in \Delta$

It is easily seen that $\mathbb{J}_X$ is a finite partial order. We show that for all A in X, $\Delta \models A \Leftrightarrow A \in \Delta$. The crucial feature here is that we do *not* prove this fact for all A in X^+! The proof is by induction on A. We consider the only interesting case. Suppose that A is $\Box B$ and that $\Box B \notin X$. We show $\Delta \not\models \Box B$. We have to produce a Δ' with $\Delta' \succeq \Delta$ and $\Delta' \not\models B$. In case $B \notin \Delta$, and, hence, by the Induction Hypothesis, $\Delta \not\models B$, we are immediately done. So suppose $B \in \Delta$. Note that $\Box(B \to \Box B)$ cannot be in Δ, since, if it were, $\Box B$ would be in Δ. We claim: $\{\Box C \mid \Box C \in \Delta\} \cup \{\Box(B \to \Box B)\} \nvdash B$. If it were otherwise, it would follow by S4-reasoning that: $\{\Box C \mid \Box C \in \Delta\} \vdash \Box(\Box(B \to \Box B) \to B)$. Hence by *Grz'*, $\{\Box C \mid \Box C \in \Delta\} \vdash \Box B$, and, thus $\Delta \vdash \Box B$. Quod non. By the usual methods we can construct an X^+-saturated set Δ' such that $\{\Box C \mid \Box C \in \Delta\} \cup \{\Box(B \to \Box B)\} \subseteq \Delta'$ and $B \notin \Delta'$. It follows that $\Delta \preceq \Delta'$ (with $\Box(B \to \Box B)$ in the role of the D of the definition). Since $B \notin \Delta'$, we have, by the Induction Hypothesis, $\Delta' \not\models B$.

For our proof of Uniform Interpolation we will use a different Henkin model $\mathbb{H}_X$, which is defined like $\mathbb{J}_X$, dropping the clause involving D, which excludes non-trivial loops. The height of $\mathbb{H}_X$ is estimated by the number of boxed formulas in X^+, which is two times the number of boxed formulas in X. We start with an amalgamation lemma. Consider disjoint sets of propositional variables $\mathcal{Q}$, $\mathbf{p}$ and $\mathcal{R}$. Let $\langle \mathbb{K}, k_0 \rangle \in \mathsf{Pmod}(\mathcal{Q}, \mathbf{p})$ and $\langle \mathbb{M}, m_0 \rangle \in \mathsf{Pmod}(\mathbf{p}, \mathcal{R})$ be S4Grz-models.

Lemma 8.1. *Let $X \subseteq \mathcal{L}^m(\mathcal{Q}, \mathbf{p})$ be a finite adequate set. Let:*

$$\nu := 2.|\{C \in X \mid C \text{ is boxed}\}|.$$

Suppose that $k_0 \simeq_{2.\nu+1,\mathbf{p}} m_0$. Then there is a $\mathcal{Q}$-extension $\langle \mathbb{N}, n_0 \rangle$ of $\langle \mathbb{M}, m_0 \rangle$ such that $\mathbb{N}$ is a S4Grz-model and $\mathsf{Th}_X(n_0) = \mathsf{Th}_X(k_0)$.

Proof. Let $\mathcal{Z}$ be a downwards closed witness of $k_0 \simeq_{2.\nu+1,\mathbf{p}} m_0$. Define Φ_X from $\mathbb{K}$ to the Henkin model $\mathbb{H} := \mathbb{H}_X$ as follows: $\Phi_X(k) := \Delta(k) := \{B \in X^+ \mid k \models B\}$. Define further for k in $\mathbb{K}$: $d_X(k) := d_{\mathbb{H}}(\Delta(k))$. Note that: $d_X(k) \leq \nu$.

Consider a pair $\langle \Delta, m \rangle$ for Δ in $\mathbb{H}$ and m in $\mathbb{M}$. Consider k', k, m'. Let $\Delta' := \Phi_X(k')$. We say that k', k, m' is a *witnessing triple* for $\langle \Delta, m \rangle$ if:

$$\Delta' \approx \Delta,\ k' \preceq k,\ m' \preceq m,\ k' \mathcal{Z}_{2.d_X(k')+1} m',\ k \mathcal{Z}_{2.d_X(k')} m.$$

$$\begin{tikzcd}
 & \Delta \arrow[r, dash, "\Phi_X"] & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \\
\Delta' \arrow[ur, "\approx"] \arrow[r, dash, "\Phi_X"'] & k' \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"'] & & m' \arrow[u, "\preceq"']
\end{tikzcd}$$

Define:

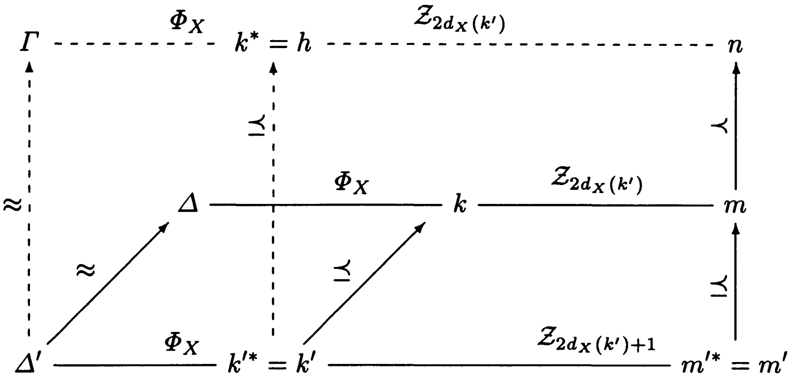
- $N := \{\langle \Delta, m \rangle \mid$ there is a witnessing triple for $\langle \Delta, m \rangle\}$
- $n_0 := \langle \Delta(k_0), m_0 \rangle$
- $\langle \Delta, m \rangle \preceq_{\mathrm{N}} \langle \Gamma, n \rangle :\Leftrightarrow \langle \Delta, m \rangle = \langle \Gamma, n \rangle$ or $(\Delta \preceq_{\mathrm{H}} \Gamma$ and $m \prec_{\mathrm{M}} n)$ or $(\Delta \prec_{\mathrm{H}}^{+} \Gamma$ and $m \preceq_{\mathrm{M}} n)$
- $\langle \Delta, m \rangle \models_{\mathrm{N}} s :\Leftrightarrow \Delta \models_{\mathrm{H}} s$ or $m \models_{\mathrm{M}} s$

Note that by assumption $k_0 \mathcal{Z}_{2\nu+1} m_0$. Moreover: $2.d_X(k_0)+1 \leq 2.\nu+1$. Hence: $k_0 \mathcal{Z}_{2d_X(k_0)+1} m_0$. So we can take k_0, k_0, m_0 as witnessing triple for n_0. Let k', k, m' be a witnessing triple for $\langle \Delta, m \rangle$. Note that for $p \in \mathbf{p} \cap X$: $\Delta \models p \Leftrightarrow k \models p \Leftrightarrow m \models p$, and hence: $\langle \Delta, m \rangle \models p \Leftrightarrow \Delta \models p \Leftrightarrow m \models p$. It is easy to see that $\mathbb{N}$ is a S4Grz-model (even if $\mathbb{H}_X$ need not be one). We claim:

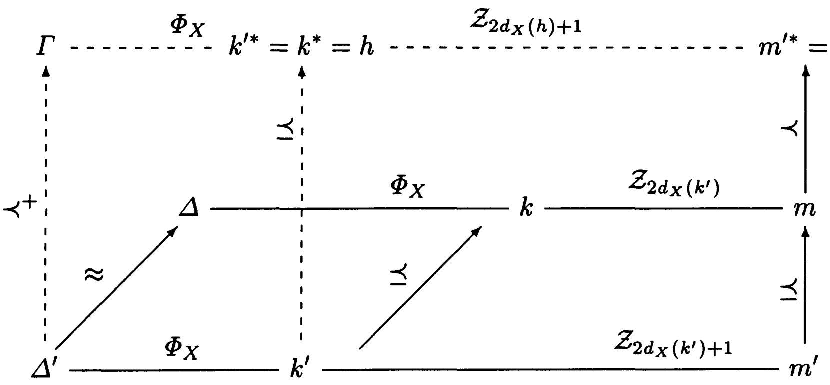
Claim 1 $n_0 \simeq_{\mathbf{p},\mathcal{R}} m_0$.
Claim 2 For $B \in X : \langle \Delta, m \rangle \models B \Leftrightarrow B \in \Delta$.

Evidently the lemma is immediate from the claims.

We prove Claim 1. Take as bisimulation $\mathcal{B}$ with $\langle\Delta, m\rangle\mathcal{B}m$. It is evident that $\mathsf{Th}_{\mathbf{p},\mathcal{R}}(\langle\Delta, m\rangle) = \mathsf{Th}_{\mathbf{p},\mathcal{R}}(m)$. Moreover, $\mathcal{B}$ has the zig-property. We check that $\mathcal{B}$ has the zag-property. Suppose $\langle\Delta, m\rangle\mathcal{B}m \preceq n$. We are looking for a pair $\langle\Gamma, n\rangle$ in N such that $\Delta \preceq \Gamma$. In case $m = n$, we take $\langle\Gamma, n\rangle := \langle\Delta, m\rangle$. Suppose $m \neq n$ and, hence, $m \prec n$. Let k', k, m' be a witnessing triple for $\langle\Delta, m\rangle$. We write $\Delta' := \Delta(k')$. Since $k'\mathcal{Z}_{2.d_X(k')+1}m' \preceq n$, there is a h such that $k' \prec h\mathcal{Z}_{2.d_X(k')}n$. We take $\Gamma := \Delta(h)$. Clearly $\Delta \preceq \Gamma$. We need a witnessing triple k'^*, k^*, m'^* for $\langle\Gamma, n\rangle$ We distinguish two possibilities. First, $\Delta \approx \Gamma$. In this case we can take: $k'^* := k'$, $k^* := h$, $m'^* := m'$.



Secondly, $\Delta \not\approx \Gamma$. In this case we can take: $k'^* := h$, $k^* := h$, $m'^* := n$. To see this, note that, since $k' \preceq h$, we have: $\Delta \approx \Delta' \preceq \Gamma$ and, hence, since $\Delta \not\approx \Gamma$, $\Delta \prec^+ \Gamma$. Ergo $d_X(h) < d_X(k')$. It follows that: $2.d_X(h) + 1 \leq 2.d_X(k')$. So, $h\mathcal{Z}_{2.d_X(k')+1}n$.



Finally, clearly, $n_0\mathcal{B}m_0$.

We prove Claim 2. The proof is by induction on X. The cases of atoms, conjunction and disjunction are trivial. We treat the only non-trivial case:

the right-to-left case for the box. Consider $\Box C \in X$ and consider the node $\langle \Delta, m\rangle$ with witnessing triple k', k, m'. Suppose $\Box C \notin \Delta$.

In case $C \notin \Delta$, we have, by the Induction Hypothesis, $\langle \Delta, m\rangle \not\models C$ and, hence, $\langle \Delta, m\rangle \not\models \Box C$.

Suppose $C \in \Delta$. It follows that $\Box(C \to \Box C)$ is not in Δ, since, otherwise, $\Box C$ would be in Δ. Clearly, $k \not\models \Box C$, so there is an $h' \succeq k$ with $h' \not\models C$. Let h be maximal in $\mathbb{K}$ with $h \succeq k$ and $h \not\models C$. By maximality, we find: $h \models \Box(C \to \Box C)$. Let $\Gamma := \Delta(h)$. Since, $\Box(C \to \Box C) \notin \Delta$ and $\Box(C \to \Box C) \in \Gamma$, we find: $\Delta \prec^+ \Gamma$. Note that it follows that $d_X(k') \geq 1$. Since, $k\mathcal{Z}_{2.d_X(k')}m$ and $k \preceq h$, there is an $n \succeq m$ with $h\mathcal{Z}_{2.d_X(k')-1}n$. Moreover: $2.d_X(h)+1 \leq 2.d_X(k')-1$. Ergo: $h\mathcal{Z}_{2.d_X(h)+1}n$. So we can take $k'^* := h$, $k^* := h$, $m'^* := n$ to witness $\langle \Gamma, n\rangle$. Clearly, $\langle \Delta, m\rangle \preceq \langle \Gamma, n\rangle$. By the Induction Hypothesis: $\langle \Gamma, n\rangle \not\models C$. Hence, $\langle \Delta, m\rangle \not\models \Box C$.

$$\begin{tikzcd}
& \Gamma \arrow[rr, no head, "\Phi_X"] & & k'^* = k^* = h \arrow[r, no head, dashed, "\mathcal{Z}_{2d_X(h)+1}"] & m'^* = n \\
& \Delta \arrow[u, "\prec^+"] \arrow[rr, no head, "\Phi_X"] & & k \arrow[u, "\preceq"'] \arrow[r, no head, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, dashed, "\preceq"'] \\
\Delta \arrow[ur, "\approx"] \arrow[rr, no head, "\Phi_X"] & & k' \arrow[ur, "\preceq"] \arrow[rr, no head, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"']
\end{tikzcd}$$

Thus we have proved Claim 2. □

The statement of uniform interpolation and the semantical interpretation of the propositional quantifiers are fully analogous to the case of GL.

We show that Uniform Interpolation for S4Grz implies Uniform Interpolation for IPC. By itself this is not so important, since we proved Uniform Interpolation for IPC directly. I feel, however, that the methodology of such transfers is interesting by itself.

Define $\mathsf{Nec}(A) := \bigwedge\{\Box(p \to \Box p) \mid p \in \mathsf{PV}(A)\}$. The Gödel Translation $(.)^*$ from $\mathcal{L}^i$ to $\mathcal{L}^m$ is specified as follows.

- $(.)^*$ commutes with atoms, $\wedge$ and $\vee$
- $(A \to B)^* := \Box(A^* \to B^*)$

Lemma 8.2. *1.* $\mathsf{IPC} \vdash A \Leftrightarrow \mathsf{S4Grz} \vdash \mathsf{Nec}(A) \to A^*$.
2. $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A) \to \Box A \Rightarrow$ *for some* $A^i \in \mathcal{L}^i$, $\mathsf{S4Grz} \vdash \mathsf{Nec}(A) \to (A \leftrightarrow A^{i*})$.

Proof. (1) and (2) are a well know facts. (1) is due to Gödel. (2) is probably first due to Rybakov. We prove (2). The proof is by induction on the length of A. Suppose $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A) \rightarrow \Box A$. We rewrite A to conjunctive normal form treating the boxed formulas as atoms. Schematically, this form is: $\bigwedge\{\bigvee\{\Box B, \neg\Box C, p, \neg q\}\}$. We find, in $\mathsf{S4Grz} + \mathsf{Nec}(A)$:

$$\begin{aligned} A &\leftrightarrow \bigwedge\{\bigvee\{\Box B, \neg\Box C, p, \neg q\}\} \\ &\leftrightarrow \Box\bigwedge\{\bigvee\{\Box B, \neg\Box C, p, \neg q\}\} \\ &\leftrightarrow \bigwedge\{\Box\bigvee\{\Box B, \neg\Box C, p, \neg q\}\} \\ &\leftrightarrow \bigwedge\{\Box(\bigwedge\{\Box C, q\} \rightarrow \bigvee\{\Box B, p\})\} \\ &\leftrightarrow \bigwedge\{\Box(\bigwedge\{(\Box C)^{i*}, q\} \rightarrow \bigvee\{(\Box B)^{i*}, p\})\} \end{aligned}$$

So we can take $A^i := \bigwedge\{(\bigwedge\{(\Box C)^i, q\} \rightarrow \bigvee\{(\Box B)^i, p\})\}$. □

Theorem 8.1. *Uniform Interpolation for* S4Grz *implies Uniform Interpolation for* IPC

Proof. Consider A in $\mathcal{L}^i$. Let $\mathbf{q}$ be some subset of $\mathsf{PV}(A)$. Let $\tilde{A}$ be the post-interpolant w.r.t. $\mathbf{q}$ of $\mathsf{Nec}(A) \wedge A^*$ in S4Grz. Note that: $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \rightarrow \Box\tilde{A}$. Hence, by the properties of the post-interpolant: $\mathsf{S4Grz} \vdash \tilde{A} \rightarrow \Box\tilde{A}$. Thus, we can find an $\mathcal{L}^i$-formula $\tilde{A}^i$, such that $\mathsf{S4Grz} \vdash \mathsf{Nec}(\tilde{A}) \rightarrow (\tilde{A} \leftrightarrow \tilde{A}^{i*})$. We show that $\tilde{A}^i$ is the desired post-interpolant. Note that, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \rightarrow \tilde{A}^{i*}$. We may conclude: $\mathsf{IPC} \vdash A \rightarrow \tilde{A}^i$.

Suppose $\mathsf{IPC} \vdash A \rightarrow B$, where the shared variables of A and B are in $\mathbf{q}$. It follows that: $\mathsf{S4Grz} \vdash \mathsf{Nec}(A \rightarrow B) \rightarrow (A^* \rightarrow B^*)$. Hence, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \rightarrow (\mathsf{Nec}(B) \rightarrow B^*)$. Thus: $\mathsf{S4Grz} \vdash \tilde{A}^{i*} \rightarrow (\mathsf{Nec}(B) \rightarrow B^*)$. And so, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(\tilde{A}^i \rightarrow B) \wedge \tilde{A}^{i*}) \rightarrow B^*$. Ergo, $\mathsf{IPC} \vdash \tilde{A}^i \rightarrow B$.

We turn to pre-interpolants. Consider B in $\mathcal{L}^i$. Let $\mathbf{q}$ be some subset of $\mathsf{PV}(B)$. Let B' be the pre-interpolant w.r.t. $\mathbf{q}$ of $\mathsf{Nec}(B) \rightarrow B^*$ in S4Grz. Take $\breve{B} := \Box B'$. We can find an $\mathcal{L}^i$-formula $\breve{B}^i$, such that $\mathsf{S4Grz} \vdash \mathsf{Nec}(\breve{B}) \rightarrow (\breve{B} \leftrightarrow \breve{B}^{i*})$. We show that $\breve{B}^i$ is the desired pre-interpolant. Note that, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(B) \wedge \breve{B}^{i*}) \rightarrow B^*$. We may conclude: $\mathsf{IPC} \vdash \breve{B}^i \rightarrow B$.

Suppose $\mathsf{IPC} \vdash A \rightarrow B$, where the shared variables of A and B are in $\mathbf{q}$. It follows that: $\mathsf{S4Grz} \vdash \mathsf{Nec}(A \rightarrow B) \rightarrow (A^* \rightarrow B^*)$. Hence, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \rightarrow (\mathsf{Nec}(B) \rightarrow B^*)$. Thus: $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \rightarrow B'$. And so, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \rightarrow \breve{B}$ (since $(\mathsf{Nec}(A) \wedge A^*)$ is self-necessitating). So, finally, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \rightarrow \breve{B}^{i*}$. Ergo, $\mathsf{IPC} \vdash A \rightarrow \breve{B}^i$. □

It would be interesting to find a similar argument to prove Uniform Interpolation for S4Grz from Uniform Interpolation for GL. In their paper [4] Ghilardi and Zawadowski show that S4 does not satisfy uniform interpolation. In fact, the following formula $A(p, q, r)$,

$$p \wedge \Box(p \rightarrow \Diamond q) \wedge \Box(q \rightarrow \Diamond p) \wedge \Box(p \rightarrow r) \wedge \Box(q \rightarrow \neg r)$$

does not have a post-interpolant w.r.t. r.

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