Infinite-valued Gödel Logics with 0-1-Projections and Relativizations *

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Summary. Infinite-valued Gödel logic, i.e., Dummett's **LC**, is extended by projection modalities and relativizations to truth value sets. An axiomatization for the corresponding propositional logic (sound and complete relative to any infinite set of truth values) is given. It is shown that certain simple infinite sets of truth values correspond to first-order Gödel logics which are not recursively axiomatizable.

1. Introduction

One of Gödel's main contributions to the study of nonstandard, in particular, many-valued and intuitionistic logics was his [4]. In that paper, he introduced a sequence of finite-valued propositional logics $\mathbf{G}_n$ intermediate in strength between classical and intuitionistic propositional logic. The definition of $\mathbf{G}_n$ is uniform, i.e., makes no explicit reference to the number of truth values. The only restrictions on the set of truth values V are that V is a (linearly ordered) subset of $[0, 1]$ and that $0, 1 \in V$. Dummett [3] subsequently showed that the infinite-valued Gödel logics are axiomatized by intuitionistic propositional calculus plus the axiom schema $(A \supset B) \vee (B \supset A)$. We extend this result to infinite-valued Gödel logics with the projection modalities on 0 and 1:

$$\triangledown(A) = \begin{cases} 1 & \text{if } A \neq 0 \\ 0 & \text{if } A = 0 \end{cases} \qquad \triangle(A) = \begin{cases} 1 & \text{if } A = 1 \\ 0 & \text{if } A \neq 1 \end{cases}$$

Only the addition of $\triangle$ is of interest, as $\triangledown$ may be defined by $\triangledown(A) \equiv \neg\neg A$.

The main result of the first part of this paper is the completeness theorem for all infinite sets of truth values V for the axiomatization consisting of the axiom schemas of intuitionistic propositional logic and of modal logic **S4** for $\triangle$ (including the necessitation rule $A/\triangle A$), plus the following schemas:

$$(A \supset B) \vee (B \supset A)$$
$$\triangle A \vee \neg\triangle A$$
$$\triangle(A \vee B) \supset \triangle A \vee \triangle B$$

* This paper is in its final form and no similar paper has been or is being submitted elsewhere.

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The completeness result will be extended to relativizations to arbitrary truth value sets: A relativization to a subset $W \subseteq V$ is obtained by adding a new connective R_W with truth function

$$\mathsf{R}_W A = \begin{cases} 1 & \text{if } A \in W \\ 0 & \text{otherwise} \end{cases}$$

If V is closed under least upper and greatest lower bounds, we arrive naturally at first-order versions of the corresponding infinite-valued logics, taking as $\forall$ the inf and as $\exists$ the sup of the corresponding truth-value distributions. It is worth pointing out right away that, in contrast to the propositional case, which logic we get depends crucially on the order type of the set of truth values. So the Gödel logic based on $[0,1]$ is axiomatizable while the one based on $\{1/k : k \in \omega \setminus \{0\}\} \cup \{0\}$ is not. The main result of the second part of the present paper is that *none* of the infinite-valued first-order Gödel logics with projection modalities (i.e., independent of the order type of the set of truth values) are recursively axiomatizable.

2. Propositional Gödel logics

We work in the language L_p of propositional logic containing a countably infinite set Var of propositional variables (X, Y, Z, ...), the constants $\top$ (true) and $\bot$ (false), as well as the connectives $\wedge$, $\vee$, $\supset$, and $\triangle$. We introduce $\neg$ and ∇ as abbreviations: $\neg A \equiv A \supset \bot$ and $\nabla A \equiv \neg\neg A$. The set of formulas of L_p is denoted $\mathrm{Frm}(L_p)$.

Definition 2.1. *Let* $V \subseteq [0,1]$ *be some set of truth values which contains* 0 *and* 1. *A* valuation $\mathfrak{V}$ based on V *is a function from* Var *to* V.. *The valuation for formulas is defined as follows:*

1. $A \equiv \top$: $\mathfrak{V}(A) = 1$.
2. $A \equiv \bot$: $\mathfrak{V}(A) = 0$.
3. $A \equiv B \wedge C$: $\mathfrak{V}(A) = \min(\mathfrak{V}(B), \mathfrak{V}(C))$.
4. $A \equiv B \vee C$: $\mathfrak{V}(A) = \max(\mathfrak{V}(B), \mathfrak{V}(C))$.
5. $A \equiv B \supset C$:
$$\mathfrak{V}(A) = \begin{cases} \mathfrak{V}(C) & \text{if } \mathfrak{V}(B) > \mathfrak{V}(C) \\ 1 & \text{if } \mathfrak{V}(B) \leq \mathfrak{V}(C). \end{cases}$$
6. $A \equiv \triangle B$:
$$\mathfrak{V}(A) = \begin{cases} 1 & \text{if } B = 1 \\ 0 & \text{if } B \neq 1 \end{cases}$$

$\mathfrak{V}$ satisfies *a formula* A, $\mathfrak{V} \models A$, *if* $\mathfrak{V}(A) = 1$. *The* propositional Gödel logic based on V, $\mathbf{GP}(V)$, *is the set of formulas* A *s.t.* $\mathfrak{V}(A) = 1$ *for every* $\mathfrak{V}$ *based on* V. *We write* $\mathbf{GP}(V) \models A$ *for* $A \in \mathbf{GP}(V)$.

It is easily verified that, for any $\mathfrak{V}$,

$$\mathfrak{V}(\neg B) = \begin{cases} 0 & \text{if } \mathfrak{V}(B) \neq 0 \\ 1 & \text{otherwise.} \end{cases} \qquad \mathfrak{V}(\nabla B) = \begin{cases} 1 & \text{if } B \neq 0 \\ 0 & \text{if } B = 0 \end{cases}$$

Also, Δ cannot be defined by the other connectives. (To see this, suppose that $B(X)$ takes on only 0 or 1. Then, if $\mathfrak{V}(X) \neq 0 \neq \mathfrak{V}(Y)$, $\mathfrak{V}(B(X)) = \mathfrak{V}(B(Y))$ *independently* of whether $\mathfrak{V}(X) = 1$.)

Definition 2.2. *Let* **LGP** *be the calculus obtained by adding to the calculus for intuitionistic propositional logic [5] the following axioms*

$$\top$$
$$(A \supset B) \vee (B \supset A)$$
$$\Delta A \vee \neg \Delta A$$
$$\Delta(A \vee B) \supset \Delta A \vee \Delta B$$
$$\Delta A \supset A$$
$$\Delta A \supset \Delta\Delta A$$
$$\Delta(A \supset B) \supset \Delta A \supset \Delta B$$

as well as the rule

$$\frac{A}{\Delta A}$$

Remark 2.1. Maehara [6, Ch. 1] gave a sequent calculus for intuitionistic logic where the restriction to at most formula in the succedent applies not generally but only in the case of applications of $\supset$:right. If we use such a calculus, the axioms and rules involving Δ may be subsumed under the two rules:

$$\frac{A, \Gamma \to \varDelta}{\Delta A, \Gamma \to \varDelta} \qquad \frac{\Delta\Gamma \to \varDelta}{\Delta\Gamma \to \Delta\varDelta}$$

Proposition 2.1. **LGP** *is sound for* **GP**(V), *i.e., if* **LGP** $\vdash S$ *then* **GP**$(V) \models S$.

Proof. By induction on length of derivations.

Proposition 2.2. *The deduction theorem obtains in the form: If* $A_1, \ldots, A_n \vdash B$, *then* $\vdash \Delta A_1 \wedge \ldots \wedge \Delta A_n \supset B$.

Note that the usual deduction theorem, which is true in ordinary (infinite-valued) Gödel logic, is false here: In general, $A \supset \Delta A$ is not valid.

Proposition 2.3. **LGP** *proves the following formula:*

$$(A \supset B) \wedge (B \supset A) \vee$$
$$(A \supset B) \wedge ((B \supset A) \supset A) \vee$$
$$(A \supset B) \wedge ((A \supset B) \supset B)$$

Proof. Derive $(B \supset A) \vee ((B \supset A) \supset A)$ from $(B \supset (B \supset A)) \vee ((B \supset A) \supset B)$, $B \supset (B \supset A) \supset B \supset A$, $((B \supset A) \supset B), (B \supset A) \supset (B \supset A) \supset ((B \supset A) \supset A)$, and similarly $(A \supset B) \vee ((A \supset B) \supset B)$. The result follows from $(A \supset B) \vee (B \supset A)$.

3. Completeness of propositional Gödel logics with projections

To prove completeness of **LGP** we first show that an ordering of the variables w.r.t. $\supset$ induces an ordering on all formulas containing these variables only.

Lemma 3.1. *Let* $U = \{X_1, \ldots, X_n, \Delta X_1, \ldots, \Delta X_n, \top, \bot\}$, *and let* G *contain*

(a) $\{A \supset B, B \supset A\}$ *or* $\{A \supset B, (B \supset A) \supset A\}$ *or* $\{(A \supset B) \supset B, B \supset A\}$ *for all* $A, B \in U$.
(b) $\{\top \supset \Delta A\}$ *or* $\{\Delta A \supset \bot\}$ *for all* $\Delta A \in U$.

Then (a), (b) are derivable from **G** *for all formulas containing only* $X_1, \ldots,$ X_n *as propositional variables.*

Proof. By induction on the complexity of formulas:

1. $A \equiv C \vee D$: (a) By induction hypothesis, $\mathbf{G} \vdash C \supset D$ or $\mathbf{G} \vdash D \supset C$, i.e., $\mathbf{G} \vdash A \leftrightarrow D$ or $\mathbf{G} \vdash A \leftrightarrow C$. Apply induction hypothesis.
   (b) By (a) we have $\mathbf{G} \vdash \Delta A \leftrightarrow \Delta C$ or $\mathbf{G} \vdash \Delta A \leftrightarrow \Delta D$. Apply induction hypothesis.
2. $A \equiv C \wedge D$: Similarly.
3. $A \equiv C \supset D$: (a) By induction hypothesis, $\mathbf{G} \vdash C \supset D$ or $\mathbf{G} \vdash (C \supset D) \supset D$, so $\mathbf{G} \vdash A \leftrightarrow \top$ or $\mathbf{G} \vdash A \leftrightarrow D$. Apply induction hypothesis.
   (b) By (a), we have $\mathbf{G} \vdash \Delta A \leftrightarrow \top$ or $\mathbf{G} \vdash \Delta A \leftrightarrow \Delta D$.
4. $A \equiv \Delta C$: (a) By induction hypothesis (b) for C, $\mathbf{G} \vdash \Delta C \leftrightarrow \top$ or $\mathbf{G} \Delta C \leftrightarrow \bot$.
   (b) $\mathbf{G} \vdash \Delta\Delta C \leftrightarrow \Delta A$, so by (a), $\mathbf{G} \vdash \top \supset \Delta C$ or $\mathbf{G} \vdash \Delta C \supset \bot$, hence $\mathbf{G} \vdash \top \supset \Delta A$ or $\mathbf{G} \vdash \Delta A \supset \bot$. □

Definition 3.1. *Let* $U = \{X_1, \ldots, X_n, \Delta X_1, \ldots, \Delta X_n, \top, \bot\}$, *and let* G *be a set of formulas of the form* $A \supset B$ *or* $(B \supset A) \supset A$ *with* $A, B \in U$. G *is called a* complete $\supset$-order *if (a) and (b) of Lemma 3.1 are satisfied,* $\{A \supset \top, \bot \supset B, A \supset A\} \subseteq G$ *for all* $A, B \in U$, *and* $(\Delta X_i \supset X_i) \in G$ *for all* $i = 1, \ldots, n$.

Definition 3.2. *Let* G *be a complete* $\supset$*-order. The* stratification $[G^*, H^*]$ *of* G *is defined as follows: Let* G' *be the least set* $G' \supseteq G$ *s.t., if* $\{A \supset C, C \supset B\} \subseteq G'$, *then also* $(A \supset B) \in G'$ *(for all* A, B, $C \in U$*), and s.t., if* $(\top \supset X_i) \in G'$ *then also* $(\top \supset \Delta X_i) \in G'$ *(for all* $\Delta X_i \in U$*). Then*

$$\begin{aligned} G^* &= G' \setminus \{(A \supset B) \supset B : (A \supset B) \in G'\}, \text{ and} \\ H^* &= \{B : \{(A \supset B) \supset B, A \supset B\} \subseteq G'\} \end{aligned}$$

Proposition 3.1. *Let* G *be a complete* $\supset$*-order, and* $[G^*, H^*]$ *be its stratification. Then* $G \equiv \bigwedge G^* \wedge \bigwedge H^*$

Proof. Note that $\vdash (A \supset B) \supset (((A \supset B) \supset B) \leftrightarrow B)$.

Definition 3.3. *Let* $V = \{X_1, \ldots, X_n, \Delta X_1, \ldots, \Delta X_n, \top, \bot\}$, *let* G *be a complete* $\supset$*-order on* V, *and let* D, E, E' *and* F *be formulas in the variables* $X_1, \ldots, X_n$. *The* right reduction $R(E \supset D)$ *is defined by*

$$R(E \supset D) = \begin{cases} E \supset D_1 & \text{if } G \vdash C_1 \\ E \supset D_2 & \text{if } G \vdash C_2; \end{cases}$$

the left reduction $L(E \wedge D \wedge E' \supset F)$ *is defined by*

$$L(E \wedge D \wedge E' \supset F) = \begin{cases} E \wedge D_1 \wedge E' \supset F & \text{if } G \vdash C_1 \\ E \wedge D_2 \wedge E' \supset F & \text{if } G \vdash C_2, \end{cases}$$

where D_i, C_i *are as follows:*

D	D_1	C_1	D_2	C_2
$A \vee B$	B	$A \supset B$	A	$B \supset A$
$A \wedge B$	A	$A \supset B$	B	$B \supset A$
$A \supset B$	$\top$	$A \supset B$	B	$(A \supset B) \supset B$
ΔA	$\top$	$\top \supset \Delta A$	$\bot$	$\Delta A \supset \bot$.

Proposition 3.2. $G \vdash S \leftrightarrow R(S)$ *and* $G \vdash S \leftrightarrow L(S)$.

Proposition 3.3. *Let* G *be a complete and stratified* $\supset$*-order on* $U = \{X_1, \ldots, X_n, \Delta X_1, \ldots, \Delta X_n, \top, \bot\}$, *and let* $A \in U$, $H \subseteq U$. *Suppose that (a)* $(\top \supset \bot) \notin G$ *and (b)* $(B \supset A) \notin G$ *if* $B \in H$. *Then there is a valuation* $\mathfrak{V}$ *on* $V_{n+2} = \{1/k : 1 \leq k \leq n+1\} \cup \{0\}$ *s.t.* $\mathfrak{V}(G) = 1$ *and* $\mathfrak{V}(H) > \mathfrak{V}(A)$.

Proof. G determines an equivalence relation on U by: $A \in [B]$ iff $\{A \supset B, B \supset A\} \subseteq G$, and an order on the equivalence classes by: $[A] < [B]$ iff $(A \supset B) \in G$ but $(B \supset A) \notin G$. We have $[\top] \neq [\bot]$ and $[\bot] < [\top]$ by (a). ΔA is in $[\top]$ or $[\bot]$, and if $A \in [\top]$, then also $\Delta A \in [\top]$. Define $\mathfrak{V}$ according to the (at most $n+2$) equivalence classes on U, with $\mathfrak{V}([\top]) = 1$ and $\mathfrak{V}([\bot]) = 0$. Then $\mathfrak{V}(G) = 1$ and $\mathfrak{V}(H) > \mathfrak{V}(A)$ by (b).

Proposition 3.4. *Let* G, H, A *be as in the previous proposition. Then* $G \vdash H \supset A$ *if* $(\top \supset \bot) \in G$ *or* $(B \supset A) \in G$ *for some* $B \in H$.

Theorem 3.1. **LGP** *is complete for* **GP**(V) *for all infinite* D.

Proof. Suppose **LGP** $\nvdash H \supset A$. Let $X_1, \ldots, X_n$ be the variables in H, A, and let $U = \{X_1, \ldots, X_n, \Delta X_1, \ldots, \Delta X_n, \top, \bot\}$. Let $G_1, \ldots, G_r$ be all combinations of the sets $\{A \supset B, B \supset A\}$, $\{A \supset B, (B \supset A) \supset A\}$, $\{(A \supset B) \supset B, B \supset A\}$, and of $\{\top \supset \Delta X_i\}$, $\{\Delta X_i \supset \bot\}$, for all A, B, $\Delta X_i \in U$. Finally, let $G' = \{\Delta X_i \supset X_i : 1 \leq i \leq n\} \cup \{(A \supset \top), (\bot \supset A) : A \in U\}$. By Proposition 2.3, $G_i, G' \nvdash H \supset A$. Let $[G^*, H^*]$ be the stratification of $G_i \cup G'$. Then, by Propositions 2.2 and 3.1, $G^* \vdash \Delta H^* \wedge H \supset A$. The reduction rules yield H' and A' s.t. $H' \cup \{A'\} \subseteq U$ and $\mathfrak{V}(\Delta H^* \wedge H \supset A) = \mathfrak{V}(\Delta H^* \wedge H' \supset$

A') whenever $\mathfrak{V}(G^*) = 1$. By Proposition 3.3, there is such a valuation $\mathfrak{V}$ on V_{n+2} s.t. $\mathfrak{V}(G^*) = 1$ and $\mathfrak{V}(\Delta H^* \wedge H') > \mathfrak{V}(A')$. $\mathfrak{V}$ may naturally be taken to be an interpretation on any infinite set of truth values making $H \supset A$ not true.

Corollary 3.1. $\mathbf{GP}(V) = \bigcap_{n \in \omega} \mathbf{GP}(V_n)$ *for any infinite* V.

$\mathbf{GP}(V)$ for finite V is axiomatizable using suitable sequent calculi [1]. If $|V| = n + 2$ one may obtain an axiomatization also directly by adding the schema

$$(\bigvee_{1 \leq i \leq n} Z \leftrightarrow X_i) \vee Z \leftrightarrow \top \vee Z \leftrightarrow \bot$$

to **LGP**. To see this, take a formula A valid in $\mathbf{GP}(V)$ and replace each variable occurring in it by one of $X_1, \ldots, X_n$, $\top$, $\bot$. All formulas thus obtained are valid in $\mathbf{GP}(V)$ and have $\leq n$ variables, hence, are provable in **LGP**. The schema then yields A.

We now proceed to extend the completeness result to relativizations to arbitrary subsets of the set of truth values.

Definition 3.4. *Let* $W \subseteq V$. *The language* L_p^W *is* L_p *plus a monadic operator* R_W. *The logic* $\mathbf{GP}(V, W)$ *is defined just like* $\mathbf{GP}(V)$ *with the truth function for* R_W *given by*

$$\mathfrak{V}(\mathsf{R}_W A) = \begin{cases} 1 & \text{if } \mathfrak{V}(A) \in W \\ 0 & \text{otherwise} \end{cases}$$

Corollary 3.2. $\mathbf{GP}(V, W)$ *is axiomatizable for arbitrary* W.

Proof. We distinguish cases according to whether W is s.t.

(a) there are $\{d_i : i \in \omega\} \subseteq W$, $\{e_i : i \in \omega\} \subseteq V \setminus W$ s.t. $d_i < e_i < d_{i+1}$ for all $i \in \omega$,

or there is a maximal k s.t. there are $\{d_1, \ldots, d_k\} \subseteq W$ and $\{e_1, \ldots, e_k\} \subseteq V \setminus W$ and

(b) $d_1 < e_2 < \ldots < e_{k-1} < d_k$,
(c) $d_1 < e_2 < \ldots < d_{k-1} < e_k$,
(d) $e_1 < d_2 < \ldots < e_{k-1} < d_k$,
(e) $e_1 < d_2 < \ldots < d_{k-1} < e_k$.

We extend **LGP** by

$$(A \leftrightarrow B) \supset (\mathsf{R}_W A \leftrightarrow \mathsf{R}_W B)$$
$$\mathsf{R}_W A \vee \neg \mathsf{R}_W A$$
$$\mathsf{R}_W U \; (\neg \mathsf{R}_W U) \text{ if } U \in W \; (U \notin W) \text{ for } U = \top, \bot.$$

plus the following formulas for cases (b)–(e):

(b) $\neg(\neg \mathsf{R}_W A_0 \wedge \mathsf{R}_W A_1 \wedge \neg \mathsf{R}_W A_2 \wedge \ldots \wedge \mathsf{R}_W A_k \wedge \bigwedge \neg \Delta(A_{i+1} \supset A_i))$ and
$\neg(\mathsf{R}_W A_1 \wedge \neg \mathsf{R}_W A_2 \wedge \ldots \wedge \mathsf{R}_W A_k \wedge \neg \mathsf{R}_W A_{k+1} \wedge \bigwedge \neg \Delta(A_{i+1} \supset A_i))$

(c) $\neg(\neg \mathsf{R}_W A_0 \wedge \mathsf{R}_W A_1 \wedge \neg \mathsf{R}_W A_2 \wedge \ldots \wedge \neg \mathsf{R}_W A_k \wedge \bigwedge \neg \Delta(A_{i+1} \supset A_i))$ and $\neg(\mathsf{R}_W A_1 \wedge \neg \mathsf{R}_W A_2 \wedge \ldots \wedge \neg \mathsf{R}_W A_k \wedge \mathsf{R}_W A_{k+1} \wedge \bigwedge \neg \Delta(A_{i+1} \supset A_i))$

(d) $\neg(\mathsf{R}_W A_0 \wedge \neg \mathsf{R}_W A_1 \wedge \mathsf{R}_W A_2 \wedge \ldots \wedge \mathsf{R}_W A_k \wedge \bigwedge \neg \Delta(A_{i+1} \supset A_i))$ and $\neg(\neg \mathsf{R}_W A_1 \wedge \mathsf{R}_W A_2 \wedge \ldots \wedge \mathsf{R}_W A_k \wedge \neg \mathsf{R}_W A_{k+1} \wedge \bigwedge \neg \Delta(A_{i+1} \supset A_i))$

(e) $\neg(\mathsf{R}_W A_0 \wedge \neg \mathsf{R}_W A_1 \wedge \mathsf{R}_W A_2 \wedge \ldots \wedge \neg \mathsf{R}_W A_k \wedge \bigwedge \neg \Delta(A_{i+1} \supset A_i))$ and $\neg(\neg \mathsf{R}_W A_1 \wedge \mathsf{R}_W A_2 \wedge \ldots \wedge \neg \mathsf{R}_W A_k \wedge \mathsf{R}_W A_{k+1} \wedge \bigwedge \neg \Delta(A_{i+1} \supset A_i))$

Now suppose $\nvdash A$. Let $[G^*, H^*]$ be constructed as in the proof of Theorem 3.1 (i.e., disregarding R_W) and let $Y_1, \ldots, Y_\ell$ be representatives of the equivalence classes other than those of $\bot$ and $\top$. Let $\{\mathsf{R}_W(Y_1)^{n_1}, \ldots, \mathsf{R}_W(Y_\ell)^{n_\ell}\}$ (where $n_i \in \{0,1\}$ and $A^0 \equiv \neg A$, $A^1 \equiv A$) be a restriction on the $\mathsf{R}_W(Y_i)$ consistent with the order of $Y_1, \ldots, Y_\ell$ and with W. Since $G^* \cup H^* \vdash B \leftrightarrow C$ for $C \in \{Y_1, \ldots, Y_\ell, \top, \bot\}$ for all formulas B containing only the original variables and $\vee, \wedge, \supset, \top, \bot$, R_W can be eliminated step-by-step using the additional axioms. The construction of a counterexample then works as before.

Conversely, if for every restriction, $\{\mathsf{R}_W(Y_1)^{n_1}, \ldots, \mathsf{R}_W(Y_\ell)^{n_\ell}\} \cup G^* \cup H^* \vdash A$ holds, we get $G^* \cup H^* \vdash A$ by Proposition 3.4 and $\mathsf{R}_W Y_i \vee \neg \mathsf{R}_W Y_i$.

4. First-order Gödel logics

In considering first-order infinite valued logics, care must be taken in choosing the set of truth values. In order to define the semantics of the quantifier we must restrict the set of truth values to those which are closed under infima and suprema. (Note that in propositional infinite valued logics this restriction is not required.) For instance, the *rational* interval $[0,1] \cap \mathbb{Q}$ will not give a satisfactory set of truth values. The following, however, do:

$$\begin{aligned} V_R &= [0,1] \\ V^0 &= \{1/k : k \in \omega \setminus \{0\}\} \cup \{0\} \\ V^1 &= \{1 - 1/k : k \in \omega \setminus \{0\}\} \cup \{1\} \end{aligned}$$

We work in a usual first-order language L extending L_p by individual variables x, y, z, …, predicate symbols P, Q, …, function symbols f, g, …, and the quantifiers $\forall$ and $\exists$.

Definition 4.1. *Let $V \subseteq [0,1]$ be some set of truth values which contains* 0 *and* 1 *and is closed under supremum and infimum. An* interpretation $\mathfrak{I} = \langle D, \mathrm{s}\rangle$ based on V *is given by the* domain D *and the* valuation function s *where* s *maps atomic formulas in* $\mathrm{Frm}(L^{\mathfrak{I}})$ *into* V *and* n*-ary function symbols to functions from* D^n *to* D.

s can be extended in the obvious way to a function on all terms. The valuation for formulas is defined as follows:

(1) $A \equiv \top$: $\mathfrak{I}(A) = 1$.

(2) $A \equiv \bot$: $\mathfrak{I}(A) = 0$.
(3) $A \equiv P(t_1, \dots, t_n)$ *is atomic:* $\mathfrak{I}(A) = \mathrm{s}(P)(\mathrm{s}(t_1), \dots, \mathrm{s}(t_n))$.
(4) $A \equiv B \wedge C$: $\mathfrak{I}(A) = \min(\mathfrak{I}(B), \mathfrak{I}(C))$.
(5) $A \equiv B \vee C$: $\mathfrak{I}(A) = \max(\mathfrak{I}(A), \mathfrak{I}(B))$.
(6) $A \equiv B \supset C$:

$$\mathfrak{I}(A) = \begin{cases} \mathfrak{I}(C) & \text{if } \mathfrak{I}(B) > \mathfrak{I}(C) \\ 1 & \text{if } \mathfrak{I}(B) \leq \mathfrak{I}(C). \end{cases}$$

(7) $A \equiv \triangle B$:

$$\mathfrak{I}(A) = \begin{cases} 1 & \text{if } \mathfrak{I}(B) = 1 \\ 0 & \text{if } \mathfrak{I}(B) \neq 1 \end{cases}$$

The set $\{\mathfrak{I}(A(d)) : d \in D\}$ *is called the* distribution *of* $A(x)$, *we denote it by* $\mathrm{Distr}_{\mathfrak{I}}(A(x))$. *The quantifiers are, as usual, defined by infimum and supremum of their distributions.*

(8) $A \equiv (\forall x)B(x)$: $\mathfrak{I}(A) = \inf \mathrm{Distr}_{\mathfrak{I}}(B(x))$.
(9) $A \equiv (\exists x)B(x)$: $\mathfrak{I}(A) = \sup \mathrm{Distr}_{\mathfrak{I}}(B(x))$.

$\mathfrak{I}$ satisfies *a formula* A, $\mathfrak{I} \models A$, *if* $\mathfrak{I}(A) = 1$.

The first-order Gödel logic based on V, $\mathbf{G}(V)$, *is the set of all formulas* $A(\bar{x})$ *s.t.* $\mathfrak{I} \models A(\bar{a})$ *for every interpretation based on* V *and every* $\bar{a} \in V^{<\omega}$.

While the set of tautologies of propositional infinite-valued Gödel logic is independent of the set of truth values, this is not the case in the first-order case. Here, the infinite-valued systems need not be equivalent.

Proposition 4.1. *Let*

$$\begin{array}{rcl} C & = & (\exists x)(A(x) \supset (\forall y)A(y)) \text{ and} \\ C' & = & (\exists x)((\exists y)A(y) \supset A(x)) \end{array}$$

C' *is valid in both* $\mathbf{G}(V^0)$ *and* $\mathbf{G}(V^1)$. C *is valid in* $\mathbf{G}(V^1)$ *but not in* $\mathbf{G}(V^0)$. *Neither* C *nor* C' *are valid in* $\mathbf{G}(V_R)$.

Proof. See [2].

If $V \subseteq V'$, then $\mathbf{G}(V') \subseteq \mathbf{G}(V)$, i.e., $\mathbf{G}(V_R)$ is the logic with the fewest valid formulas. The next proposition shows that there are infinitely many infinite-valued first-order Gödel logics.

Proposition 4.2. *Let* $V_k = \{\frac{x}{k} + \frac{1}{2yk} : 0 \leq x < k, y \in \omega \setminus \{0\}\}$. *Then* $\mathbf{G}(V_k) \subset \mathbf{G}(V_\ell)$ *if* $k > \ell$.

Proof. Since V_ℓ can be embedded in V_k preserving the order structure if $k > \ell$, we have $\mathbf{G}(V_k) \subseteq \mathbf{G}(V_\ell)$.

Let

$$\begin{aligned} F_k^* \equiv \quad & \bigwedge_{0 \le i < j < k} (\forall x)\big[P_i(x) \ll (\forall x)P_j(x)\big] \wedge \\ \wedge \quad & \bigwedge_{0 \le i < k} (\exists x)\big[P_i(x) \supset (\forall y)P(y)\big] \supset (\forall y)P_i(y) \wedge \\ \wedge \quad & \bigwedge_{0 \le i < k} (\forall x)P_i(x) \ll R \wedge (\forall x)Q(x) \ll R \wedge (\forall x)P_i(x) \ll (\forall x)Q(x), \end{aligned}$$

where $A \ll B \equiv (A \supset B) \wedge ((B \supset A) \supset A)$, and let

$$F_k \equiv F_k^* \supset (\exists x)(Q(x) \supset (\forall y)Q(y)) \vee R$$

If one of the conditions in F_k^* is not satisfied, then the value of F_k equals 1, since then every conjunct in F_k^* gets a value $\le R$. If all conditions are satisfied, and the value of R is < 1, then F_k expresses: $(\forall x)Q(x)$ is an infimum which is different from k distinct infima none of which is a minimum. In $\mathbf{G}(V_k)$, it then must be a minimum, and so the value of $(\exists x)(Q(x) \supset (\forall y)Q(y)) \vee R$ equals 1. This need not be the case in general in $\mathbf{G}(V_\ell)$ with $\ell > k$.

In terms of complexity, there may be significant differences as well. It was shown, e.g., that $\mathbf{G}(V_R)$ is axiomatizable [7], but that $\mathbf{G}(V^0)$ is not [2].

As in the propositional case, we may extend $\mathbf{G}(V)$ to $\mathbf{G}(V, W)$ by adding an operator R_W. The definition of interpretation is extended by adding the clause

(10) $A \equiv \mathsf{R}_W B$:

$$\mathfrak{I}(A) = \begin{cases} 1 & \text{if } \mathfrak{I}(B) \in W \\ 0 & \text{otherwise} \end{cases}$$

and the other definitions amended accordingly.

5. Incompleteness of first-order Gödel logics with 0-1-projections and relativizations

In order to prove the main theorem of this section we need some tools from recursion theory.

Definition 5.1. *Let ψ be an effective recursive enumeration of the set PR_1^1 of all primitive recursive functions from ω to ω. We define a two place function φ (which enumerates a subclass of PR_1^1):*

$$\varphi_k(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0 & \text{if } \psi_k(y) = 0 \text{ for } 1 \le y \le x \\ 1 & \text{otherwise} \end{cases}$$

The index set O_φ is defined as $\{k : (\forall y)\varphi_k(y) = 0\}$.

Proposition 5.1. *The index set O_φ is not recursively enumerable.*

Proof. By definition of φ, $\{k : (\forall y)\varphi_k(y) = 0\} = \{k : (\forall y)\psi_k(y) = 0\}$. But for every $g \in \mathrm{PR}_1^1$ the index set $\{k : (\forall y)\psi_k = g\}$ is Π_1-complete. Therefore O_φ is Π_1-complete and thus not recursively enumerable.

The essence of the incompleteness proof is represented by a sequence of formulas $(A_k)_{k\in\omega}$ constructed via φ s.t.

$$\mathbf{G}(V, W) \models A_k \iff k \in O_\varphi$$

i.e. O_φ is m-reducible to the validity problem of $\mathbf{G}(V, W)$.

We prove the incompleteness result separately for W which have a cumulation point which is approached from above and those with one approached from below. The idea is to write down axioms which express that the values of $P(s^n(0))$ form a decreasing sequence in W as long as $\varphi_k(n) = 0$, and that $P(s^n(0))$ gets the value of $P(0)$ if $\varphi_k(n) \neq 0$ Using Δ, we can force $\mathrm{Distr}(P(x))$ to have no minimum iff the decreasing sequence is infinite. Thus, the value of $(\exists x)\Delta(P(x) \supset (\forall y)P(y))$ will equal 1 if $\varphi_k \equiv 0$ and equal 0 otherwise. For W with a cumulation point approached from below the argument is similar with an increasing sequence.

Theorem 5.1. *Suppose V is a set of truth values and $W \subseteq V$ is infinite and there are only finitely many elements of W between any two elements of W. Then $\mathbf{G}(V, W)$ is incomplete.*

Proof. Obviously W cannot have both a minimum and a maximum. Suppose first that it has no minimum, i.e., contains an infinite descending sequence with no lower bound in W.

Let P be a one-place predicate symbol, s be the function symbol for the successor function and $\bar{0}$ be the constant symbol representing 0 (in particular, we choose a signature containing this symbol and all symbols from Robinson's arithmetic Q).

Let A_1 be a conjunction of axioms strong enough to represent every recursive function (e.g. the axioms of Q) and a defining axiom for the function φ and write Δ in front of every positively occurring atomic formula. This ensures that these formulas behave essentially as classical formulas. We define the formulas A_2^k, A_3^k, A_4^k for $k \in \omega$; for formulas representing the equality $\varphi_k(x) = 0$ we write $[\varphi_k(x) = 0]$ (these also contain Δ in front of atomic formulas).

$A_2^k \equiv (\forall x, y)(\neg[\varphi_k(x) = 0] \wedge \Delta(x \leq y) \supset \neg[\varphi_k(y) = 0])$

$A_3^k \equiv (\forall x)[\neg[\varphi_k(x){=}0] \supset \Delta(P(\bar{0}) \supset P(s(x))) \wedge \Delta(P(s(x)) \supset P(0)) \wedge \mathsf{R}_W P(s(x))]$

$A_4^k \equiv (\forall x)[[\varphi_k(s(x)) = 0] \supset \neg\Delta(P(x) \supset P(s(x))) \wedge \mathsf{R}_W P(s(x))]$

$A_5 \equiv \mathsf{R}_W P(0)$

Finally we set

$$B_k \equiv A_1 \wedge A_2^k \wedge A_3^k \wedge A_4^k \wedge A_5$$

and

$$A_k \equiv B_k \supset (\neg(\exists x)\Delta[P(x) \supset (\forall y)P(y)].\square$$

Provided $\mathfrak{I}(B_k) = 1$, the sequence given by $\mathfrak{I}(P(s^n(0)))$ lies in W, hence cannot contain a cumulation point. Hence, the implication A_k is true iff this sequence is infinite, i.e., iff $\varphi_k(n) = 0$ for all $n \in \omega$.

Now suppose W contains an infinite increasing sequence without upper bound in W. We have to replace

$$A_4^k \quad \text{by} \quad (\forall x)[[\varphi_k(x) = 0] \supset \neg\Delta(P(s(x)) \supset P(x)) \wedge \mathsf{R}_W P(s(x))]$$

and set $A_k \equiv B_k \supset \neg(\exists x)\Delta((\exists y)P(y) \supset P(x))$.

6. Conclusion

The results of this paper establish that no extension of the infinite-valued first-order Gödel logic is recursively enumerable, if

1. the projection function Δ (or by a negation symmetric to $\neg$, i.e.,

$$\mathfrak{I}(\sim A) = \begin{cases} 0 & \text{if } \mathfrak{I}(A) = 1 \\ 1 & \text{if } \mathfrak{I}(A) \neq 1, \end{cases}$$

which would make ΔA definable as $\sim\sim A$) and

2. a relativization operator based on a subset $W \subseteq V$ s.t. there are only finitely many elements of W between any two elements of W.

are present. This leaves open the question whether the result holds for all relativizations, and in particular, whether $\mathbf{G}([0,1],[0,1])$ (= $\mathbf{G}(V_R)$ extended by 0-1-projections) and $\mathbf{G}([0,1],[0,1] \cap \mathbb{Q})$ are r.e. Further investigations of 0-1-projections promise to shed light on these problems.

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