§7. Some applications

In this section, we use the theory developed in §1 - §6 to show that various propositions imply the existence of an inner model with a Woodin cardinal.

A. Saturated ideals

Shelah has shown, in unpublished work, that Con (ZFC + There is a Woodin cardinal) implies Con (ZFC + There is an ω_2-saturated ideal on ω_1). (For earlier results in this direction, see [Ku2], [SVW], [W], and [FMS].) Here we shall prove what is very nearly a converse to Shelah's result. We shall show that Con (ZFC + There is an ω_2-saturated ideal on ω_1+ There is a measurable cardinal) implies Con (ZFC + There is a Woodin cardinal).

The best lower bound on the consistency strength of the existence of an ω_2-saturated ideal on ω_1 known before our work is due to Mitchell ([M?]). He obtained Con (ZFC + $\exists\kappa(o(\kappa) = \kappa^{++})$), which of course is as far as the models studied in [M?] could go.

Actually, our proof does not require that the given ideal be on ω_1, nor does it require ω_2-saturation in full. A generic almost-huge embedding will suffice.

Theorem 7.1. *Let Ω be measurable, and let G be V-generic/ $\mathbb{P}$ for some $\mathbb{P} \in V_\Omega$. Suppose that in $V[G]$ there is a transitive class M and an elementary embedding*

$$j : V \to M \subseteq V[G]$$

with critical point κ such that

$$\forall\, \alpha < j(\kappa)(^{\alpha}M \cap V[G] \subseteq M)\,.$$

Then $K^c \models$ There is a Woodin cardinal.

Proof. Suppose toward contradiction that K^c has no Woodin cardinals. This supposition puts the theory of §1- §6 at our disposal. In particular, by 5.18 we have that $K^V = K^{V[G]}$. Moreover, by 6.15, the agreement between M and $V[G]$ implies that if $\mathcal{P}$ is a properly small premouse of cardinality $< j(\kappa)$ in $V[G]$, and $\alpha < j(\kappa)$, then

$$(M \models \mathcal{P} \text{ is } \alpha\text{-strong}) \Leftrightarrow (V[G] \models \mathcal{P} \text{ is } \alpha\text{-strong})\,.$$

It follows that K^M agrees with $K^{V[G]}$ below $j(\kappa)$. That is, $\mathcal{J}_\alpha^{K^M} = \mathcal{J}_\alpha^{K^{V[G]}}$ for all $\alpha < j(\kappa)$.

Since $\kappa = \mathrm{crit}(j)$, κ is a regular cardinal in V, and thus $j(\kappa)$ is a regular cardinal in M. Since $P(\alpha)^M = P(\alpha)^{V[G]}$ for all $\alpha < j(\kappa)$, $j(\kappa)$ is a cardinal of $V[G]$. Thus $j(\kappa)$ is a cardinal of both K^M and $K^{V[G]}$.

We claim κ is inaccessible in K^V. For otherwise, we have $\beta < \kappa$ such that $\kappa = (\beta^+)^{K^V}$. This means $j(\kappa) = (\beta^+)^{K^M} = (\beta^+)^{K^{V[G]}} = (\beta^+)^{K^V}$, a contradiction. So κ is inaccessible in K^V. But then $j(\kappa)$ is inaccessible in

K^M, and $j(\kappa)$ is a limit cardinal in $K^{V[G]}$. Also, since K^V and K^M agree below $j(\kappa)$, κ is inaccessible in K^M.

Let E_j be the extender over V derived from j. We shall show that for all $\alpha < j(\kappa)$:

$$E_j \cap ([\alpha]^{<\omega} \times K^V) \in K^V .$$

This is a contradiction, as then these fragments of E_j witness that κ is Shelah in K^V. (That is so because $j(K^V) = K^M$ agrees with K^V below $j(\kappa)$.)

So fix $\alpha < j(\kappa)$, and take α large enough that $(\kappa^+)^{K^V} < \alpha$. Let $W \in V$ be a weasel which witnesses that $\mathcal{J}_\alpha^{K^V}$ is A_0-sound. We may assume α is chosen to be a cardinal of K^V and W. It will be enough to find an extender F on the W sequence such that $\text{crit}(F) = \kappa$, $\nu(F) \geq \alpha$, and for all $A \in P(\kappa) \cap K^V$,

$$i_F^W(A) \cap \alpha = j(A) \cap \alpha .$$

Working in $V[G]$, we shall compare W with $j(W)$. Notice that by 5.12, there is in $V[G]$ a (unique) $\Omega + 1$ iteration strategy Σ for W. We shall show that there is a (unique) Ω+1 iteration strategy Γ for the phalanx $(\langle W, j(W)\rangle, \langle\alpha\rangle)$. Let us assume for now that such a Γ exists, and complete the proof.

Let $\mathcal{T}$ on W and $\mathcal{U}$ on $(\langle W, j(W)\rangle, \langle\alpha\rangle)$ be the iteration trees resulting from a (Σ, Γ) coiteration. (Coiteration was defined only for premice, but it makes obvious sense for phalanxes. Here we start out comparing $j(W)$ with W, iterating the least disagreement, but the tree $\mathcal{U}$, which begins on $j(W)$, goes back to W whenever it uses an extender with critical point $< \alpha$.) Let $\mathcal{M}_\alpha$ be the αth model of $\mathcal{T}$ and $\mathcal{N}_\alpha$ the αth model of $\mathcal{U}$. In order to save a little notation, let us assume $\mathcal{T}$ and $\mathcal{U}$ are "padded", so that $lh\ \mathcal{T} = lh\ \mathcal{U}$. Let $lh\ \mathcal{T} = lh\ \mathcal{U} = \theta + 1$, where $\theta \leq \Omega$.

We claim that $\text{root}^{\mathcal{U}}(\theta) = 1$. For otherwise, $\text{root}^{\mathcal{U}}(\theta) = 0$; that is, $\mathcal{N}_\theta$ is above $W = \mathcal{N}_0$ in $\mathcal{U}$. Now W is universal, and therefore there is no dropping on $[0,\theta]_U$ or $[0,\theta]_T$, so that $i_{0,\theta}^{\mathcal{U}}$ and $i_{0,\theta}^{\mathcal{T}}$ are defined; moreover, $\mathcal{M}_\theta = \mathcal{N}_\theta$. Let

$$\Delta = \{\gamma < \Omega \mid i_{0,\theta}^{\mathcal{U}}(\gamma) = i_{0,\theta}^{\mathcal{T}}(\gamma) = \gamma\} ,$$

so that Δ is thick in W and $\mathcal{M}_\theta$. The construction of $\mathcal{U}$ guarantees crit $i_{0,\theta}^{\mathcal{U}} < \alpha$. It follows that crit $i_{0,\theta}^{\mathcal{U}}$ is the least γ such that $\gamma \notin H^{\mathcal{M}_\theta}(\Delta)$. From this we get crit $i_{0,\theta}^{\mathcal{T}} =$ crit $i_{0,\theta}^{\mathcal{U}}$. Using the hull property for W at crit $i_{0,\theta}^{\mathcal{U}}$, we proceed to the standard contradiction.

So $\mathcal{N}_\theta$ is above $j(W) = \mathcal{N}_1$ on $\mathcal{U}$. Now $j(W)$ is universal (in $V[G]$) since the class of fixed points of j is α-club in Ω for all sufficiently large regular α, so that $j(W)$ computes α^+ correctly for stationary many $\alpha < \Omega$. Thus $\mathcal{N}_\theta = \mathcal{M}_\theta$, and $i_{1,\theta}^{\mathcal{U}}$ and $i_{0,\theta}^{\mathcal{T}}$ and defined.

Let

$$\Gamma = \{\gamma < \Omega \mid i_{0,\theta}^{\mathcal{T}}(\gamma) = i_{1,\theta}^{\mathcal{U}} \circ j(\gamma) = \gamma\} ,$$

so that Γ is thick in W and $\mathcal{M}_\theta = \mathcal{N}_\theta$. Now $\kappa = \text{crit}(i_{1,\theta}^{\mathcal{U}} \circ j)$, so

$$\kappa = \text{least } \eta \text{ s.t. } \eta \notin H^{\mathcal{N}_\theta}(\Gamma) .$$

It follows that $\kappa = \text{crit } i^{\mathcal{T}}_{0,\theta}$. Similarly, using the hull property for W at κ,

$$i^{\mathcal{T}}_{0,\theta}(A) = i^{\mathcal{U}}_{1,\theta} \circ j(A)$$

for all $A \subseteq \kappa$ s.t. $A \in W$.

Let $\eta + 1 \in [0, \theta]_T$ be such that T-pred$(\eta + 1) = 0$. Now all extenders used in $\mathcal{T}$ or $\mathcal{U}$ have length $> \alpha$, and sup of generators $\geq \alpha$. So crit $i^{\mathcal{T}}_{\eta+1,\theta} \geq \alpha$. Also, crit $i^{\mathcal{U}}_{1,\theta} \geq \alpha$ by construction. So for all $A \in P(\kappa)^W$,

$$i^{\mathcal{T}}_{0,\eta+1}(A) \cap \alpha = j(A) \cap \alpha \,.$$

Let F be the trivial completion of $E^{\mathcal{T}}_\eta \restriction \alpha$. Then F is on the sequence of $\mathcal{M}^{\mathcal{T}}_\eta$. It follows (using coherence if $\eta > 0$) that F is on the sequence of $W = \mathcal{M}^{\mathcal{T}}_0$. Moreover, for $A \subseteq \kappa$ s.t. $A \in W$,

$$\begin{aligned} i^{\mathcal{T}}_{0,\eta+1}(A) \cap \alpha &= i^W_{E_\eta}(A) \cap \alpha \\ &= i^W_F(A) \cap \alpha \,. \end{aligned}$$

So F is as desired.

It remains to show that the phalanx $(\langle W, j(W)\rangle, \langle \alpha \rangle)$ is $\Omega + 1$ iterable in $V[G]$. We claim that the strategy of choosing the unique cofinal wellfounded branch is winning in the length $\Omega + 1$ iteration game. If not, then as in 6.14 there are properly small $\mathcal{R} \trianglelefteq W$ and $\mathcal{S} \trianglelefteq j(W)$ such that $\alpha \in \text{OR}^{\mathcal{R}} \cap \text{OR}^{\mathcal{S}}$, and a putative iteration tree on $(\langle \mathcal{R}, \mathcal{S}\rangle, \langle \alpha \rangle)$ which is bad; that is, which has a last, illfounded model, or is of limit length but has no cofinal wellfounded branch. Since Ω is weakly compact, our bad tree has length $< \Omega$, so that its sharp exists. Using this for absoluteness purposes, as in the proof of 6.14, we can find in $V[G]$

$$\begin{aligned} \sigma : \mathcal{P} &\to \mathcal{R} \qquad (\text{where } \mathcal{R} \trianglelefteq W)\,, \\ \tau : \mathcal{Q} &\to \mathcal{S} \qquad (\text{where } \mathcal{S} \trianglelefteq j(W))\,, \end{aligned}$$

such that $\mathcal{P}$ and $\mathcal{Q}$ are of cardinality $\leq \alpha$ and

$$\sigma \restriction \alpha = \tau \restriction \alpha = \text{identity}\,,$$

together with a countable bad tree on $(\langle \mathcal{P}, \mathcal{Q}\rangle, \langle \alpha \rangle)$. Now $\mathcal{P}$ is α-strong in $V[G]$, as witnessed by σ. Also, $\mathcal{Q}$ is α-strong in M, as witnessed by τ; note that $\tau \in M$ as $M^{<j(\kappa)} \subseteq M$ in $V[G]$. Since M and $V[G]$ have the same subsets of α, 6.11 and 6.14 imply that $\mathcal{Q}$ is α-strong in $V[G]$. But then the (1)$\to$(2) direction of 6.11 implies that $(\langle \mathcal{P}, \mathcal{Q}\rangle, \langle \alpha \rangle)$ is $\Omega + 1$ iterable, a contradiction. □

Corollary 7.2. *Let Ω be measurable, and suppose there is a pre-saturated ideal on ω_1; then $K^c \models$ There is a Woodin cardinal.*

We conjecture that the measurable cardinal is not needed in the hypotheses of 7.1 and 7.2.

B. Generic absoluteness

One of the most important consequences of the existence of large cardinals is that the truth values of sufficiently simple statements about the reals cannot be changed by forcing. For example, if there are arbitrarily large Woodin cardinals, then $L(\mathbb{R})^V \equiv L(\mathbb{R})^{V[G]}$ for all G set-generic over V. (This result is due to Hugh Woodin.) We shall show that this generic absoluteness implies that there are inner models with Woodin cardinals.

Hugh Woodin pointed out this application of §1 - §6. The key is the following lemma.

Lemma 7.3. (Woodin) *Let Ω be measurable, and suppose K^c has no Woodin cardinals. Then there is a sentence φ in the language of set theory, and a partial order $\mathbb{P} \in V_\Omega$, such that whenever G is V-generic over $\mathbb{P}$*

$$L_{\omega_1}(\mathbb{R}))^V \models \varphi \textit{ iff } (L_{\omega_1}(\mathbb{R}))^{V[G]} \not\models \varphi .$$

Proof. Here $(L_{\omega_1}(\mathbb{R}))^{V[G]} = L_{\omega_1^{V[G]}}(\mathbb{R}^{V[G]})$. Using the formula ψ described in 6.15 (2) which defines $\langle \mathcal{J}_\delta^K \mid \delta < \omega_1 \rangle$ in all generic extensions of V by posets $\mathbb{P} \in V_\Omega$, we can construct a sentence φ such that (provably in ZFC + "Ω is measurable" + "$K^c \models$ There are no Woodin cardinals") we have

$$L_{\omega_1}(\mathbb{R}) \models \varphi \textit{ iff } \omega_1 \text{ is a successor cardinal of } K .$$

Our hypotheses guarantee $(\alpha^+)^K = \alpha^+$ for some α. If $(L_{\omega_1}(\mathbb{R}))^V \not\models \varphi$, then take $\mathbb{P} = \mathrm{Col}(\omega, \alpha)$; letting G be V-generic / $\mathbb{P}$, we have $(L_{\omega_1}(\mathbb{R}))^{V[G]} \models \varphi$ by 5.18 (3). On the other hand, if $(L_{\omega_1}(\mathbb{R}))^V \models \varphi$, then take $\mathbb{P} = \mathrm{Col}(\omega, < \alpha)$ where $\alpha < \Omega$ is inaccessible; letting G be V-generic/$\mathbb{P}$, we have $(L_{\omega_1}(\mathbb{R}))^{V[G]} \not\models \varphi$. □

Theorem 7.4. (Woodin) *Suppose that Ω is measurable, and that whenever G is V-generic/$\mathbb{P}$ for some $\mathbb{P} \in V_\Omega$, $(L_{\omega_1}(\mathbb{R}))^V \equiv (L_{\omega_1}(\mathbb{R}))^{V[G]}$. Then $K^c \models$ There is a Woodin cardinal.*

It is well known that weak homogeneity can be used to obtain generic absoluteness. We can therefore use 7.4, together with standard arguments, to show

Theorem 7.5. *If every set of reals definable over $L_{\omega_1}(\mathbb{R})$ is weakly homogeneous, then letting K^c be the model constructed in §1 below Ω, where Ω is the least measurable cardinal, we have $K^c \models$ There is a Woodin cardinal.*

Proof. If any set is weakly homogeneous, then there is a measurable cardinal. Let Ω be the least measurable cardinal. For any weakly homogeneous tree T, let T^* be the tree for the complement coming from the Martin-Solovay construction. (The notation assumes the homogeneity measures for T are given somehow.) So

$$p[T^*] = \mathbb{R} - p[T]$$

is true in $V[G]$ whenever G is generic for $\mathbb{P}$ with card $(\mathbb{P})$ < additivity of homogeneity measures for T, hence whenever G is generic for $\mathbb{P} \in V_\Omega$. Let

$$\begin{array}{rcl} S_n & = & \text{universal } \Sigma_n(L_{\omega_1}(\mathbb{R})) \text{ set of reals} \\ P_n & = & \text{universal } \Pi_n(L_{\omega_1}(\mathbb{R})) \text{ set of reals} \end{array}$$

for $1 \leq n < \omega$. Pick weakly homogeneous trees U_n such that $P_n = p[U_n]$ and let T_{n+1} be the canonical weakly homogeneous tree which projects to $\exists^{\mathbb{R}} p[U_n]$ in all $V[G]$

$$p[T_{n+1}] = \exists^{\mathbb{R}} p[U_n] \text{ in all } V[G]\,.$$

(Thus in V, $p[T_{n+1}] = S_{n+1}$.)

Claim. If G is $\mathbb{P}$-generic, where $\mathbb{P} \in V_\Omega$, then for all $n \geq 2$

$$V[G] \models p[U_n] = \mathbb{R} - p[T_n]\,.$$

Proof. Fix $V[G]$. Since $p[U_n] \cap p[T_n] = \emptyset$ in V, this remains true in $V[G]$ by absoluteness of wellfoundedness. On the other hand, if $x \in \mathbb{R}^{V[G]}$ and $x \notin (p[U_n] \cup p[T_n])$, then $x \in p[U_n^*] \cap p[T_n^*]$ because U_n^* and T_n^* project absolutely to the complements of the projections of U_n, T_n. But then $p[U_n^*] \cap p[T_n^*] \neq \emptyset$ in V by absoluteness of wellfoundedness. This is a contradiction as $p[U_n] = \mathbb{R} - p[T_n]$ in V. □

It follows that in all $V[G]$, G generic for $\mathbb{P} \in V_\Omega$,

$$p[U_{n+1}] = \text{universal } \Pi^1_n(A) \text{ set of reals, where } \; A = p[U_1]\,.$$

But now the fact that A is the universal $\Pi_1(L_{\omega_1}(\mathbb{R}))$ set of reals is a Π^1_{20} fact about A. So in all such $V[G]$

$$p[U_{n+1}] = \text{universal } \Pi_{n+1}(L_{\omega_1}(\mathbb{R})) \text{ set of reals}\,.$$

Thus for any sentence φ of the language of set theory

$$(L_{\omega_1}(\mathbb{R}))^V \models \varphi \text{ iff } (L_{\omega_1}(\mathbb{R}))^{V[G]} \models \varphi\,.$$

By Theorem 7.4, $K^c \models$ There is a Woodin cardinal, where K^c is constructed below Ω. □

Woodin has shown (unpublished) that if there is a strongly compact cardinal, then all sets of reals in $L(\mathbb{R})$ are weakly homogeneous. So we have at once:

Theorem 7.6. *Suppose there is a strongly compact cardinal, and let K^c be the model of §1 constructed below Ω, where Ω is the least measurable cardinal. Then $K^c \models$ There is a Woodin cardinal.*

We shall give a more direct proof of Theorem 7.6 in §8, a proof which does not rely on Woodin's work deriving weak homogeneity from strong compactness.

We conclude this section on generic absoluteness by re-proving a slightly weaker version of the following theorem due to Woodin:

$$\text{Con (ZFC} + \Delta^1_2\text{-determinacy)} \Rightarrow \text{Con (ZFC + There is a Woodin cardinal)}.$$

Because the theory of §1 - §6 relies on the measurable cardinal cardinal Ω, we do not see how to use it to prove Woodin's theorem in full, although we believe that should be possible. We can, however, prove the theorem with its hypothesis strengthened to: Con $(ZFC + \Delta^1_2\text{-determinacy} + \forall x \in \mathbb{R}\,(x^\sharp \text{ exists}))$. Modulo the theory of §1 - §6, our proof is simpler than Woodin's.

Our proof relies on the observation that the theory of §1 - §6 uses somewhat less than a measurable cardinal. Namely, suppose A is a set of ordinals and $A^\sharp$ exists. Let c_0 be an indiscernible of $L[A]$, let $j : L[A] \to L[A]$ have critical point c_0, and let $\mathcal{U}$ be the $L[A]$-ultrafilter on c_0 given by: $X \in \mathcal{U} \Leftrightarrow c_0 \in j(X)$. Working in $L[A]$, we can construct $(K^c)^{L[A]}$ below c_0 just as we constructed K^c below Ω in §1. Let us assume that $L[A]$ satisfies: There is no proper class inner model with a Woodin cardinal. We can then conduct our proof of iterability within $L[A]$ (using 2.4 (b) rather than 2.4 (a)), and we have that indeed $(K^c)^{L[A]}$ exists and (by 2.10) is (ω, θ) iterable for all θ. Further, the proof of 1.4 shows that for $\mathcal{U}$ a.e. $\alpha < c_0$, $L[A] \models (\alpha^+)^{K^c} = \alpha^+$. (The main point here is that we don't really need $\mathcal{U} \in L[A]$ to carry out the proof; it is enough that if E_j is the $(c_0, j(c_0))$ extender over $L[A]$ derived from j, and $\mathcal{A} \in L[A]$ and $|\mathcal{A}|^{L[A]} \leq c_0$, then $E_j \cap ([j(c_0)]^{<\omega} \times \mathcal{A}) \in L[A]$. That these fragments of E_j are in $L[A]$ is well known.) This implies that $L[A] \models$ "c_0 is A_0-thick in K^c". We can therefore carry out the arguments of §3 - §6 within $L[A]$, and we get that $K^{L[A]}$ exists, is absolute for forcing over $L[A]$ with posets $\mathbb{P} \in V_{c_0}^{L[A]}$, and inductively definable over $L[A]$ as in §6. (The only serious use of the measurable cardinal Ω in these sections occurs in the proof of 4.8. Once again, it is clear from that proof that we only need the fragments $E_j \cap ([j(c_0)]^{<\omega} \times \mathcal{A})$, for $|\mathcal{A}|^{L[A]} \leq c_0$, to be in $L[A]$.) We also have that for $\mathcal{U}$ a.e. $\alpha < c_0$, $L[A] \models (\alpha^+)^K = \alpha^+$.

Theorem 7.7. (Woodin) *If $\forall x \in {}^\omega\omega$ ($x^\sharp$ exists) and all Δ^1_2 games are determined, then there is a proper class inner model with a Woodin cardinal.*

Proof. According to a theorem of Kechris and Solovay (cf. [KS]), Δ^1_2 determinacy implies that there is a real x such that for all reals $y \geq_T x$, $L[y] \models$ "All ordinal-definable games are determined". Fix such a real x, and let c_0 be the least indiscernible of $L[x]$. We may suppose that $L[x] \models$ "There is no proper class inner model with a Woodin cardinal". As we have observed, this means that $K^{L[x]}$ exists and is absolute for size $< c_0$ forcing over $L[x]$, and that for $\mathcal{U}$-a.e. $\alpha < c_0$, $L[x] \models (\alpha^+)^K = \alpha^+$, where $\mathcal{U}$ is the $L[x]$-ultrafilter on c_0 given by $x^\sharp$. Let $\alpha < c_0$ be such that $L[x] \models (\alpha^+)^K = \alpha^+$, and let

$y = \langle x, z\rangle$ where z is (a real) generic over $L[x]$ for the poset $\mathrm{Col}(\omega, \alpha)$ collapsing α to be countable. Then in $L[y]$, K exists and is inductively definable as in §6, and ω_1 is a successor cardinal of K. Moreover, OD determinacy holds in $L[y]$. Let us work in $L[y]$. Now OD determinacy implies that every OD set $A \subseteq \omega_1$ either contains or is disjoint from a club, and therefore that ω_1^V is measurable in HOD. On the other hand, $K \subseteq \mathrm{HOD}$, so since $\omega_1^V = (\alpha^+)^K$, $\omega_1^V = (\alpha^+)^{\mathrm{HOD}}$. But $\mathrm{HOD} \models \mathrm{AC}$, so $\mathrm{HOD} \models$ all measurable cardinals are inaccessible. This contradiction completes the proof. □

C. Unique branches

The Unique Branches Hypothesis, or UBH, is the assertion that if $\mathcal{T}$ is an iteration tree on V, then $\mathcal{T}$ has at most one cofinal wellfounded branch. Martin and the author showed that the negation of UBH has some logical strength, in that it implies the existence of an inner model with a Woodin cardinal and a measurable above. (Cf. [IT], §5.) Woodin then showed, in unpublished work, that if there is a nontrivial elementary $j : V_{\lambda+1} \to V_{\lambda+1}$, for some λ, then UBH fails. The gap between these two bounds on the consistency strength of -UBH is, of course, enormous. Here we shall improve the lower bound to two Woodin cardinals. (However, we must add "There is a measurable cardinal" to -UBH because the basic theory demands it.) We conjecture that -UBH is equiconsistent with the existence of two Woodin cardinals.

Theorem 7.8. *Let Ω be measurable, and suppose there is a normal iteration tree $\mathcal{T}$ on V such that $\mathcal{T} \in V_\Omega$ and $\mathcal{T}$ has distinct cofinal wellfounded branches. Then there is a proper class inner model satisfying "There are two Woodin cardinals".*

Proof. Assume toward contradiction than there is no such model.

We shall need a slight generalization of the K^c construction in §1. Let X be any transitive set, $X \in V_\Omega$ where Ω is measurable. We can form $K^c(X)$ by relativizing the construction of §1. So $\mathcal{N}_0 = X$, and all hulls used in forming $\mathfrak{C}_\omega(\mathcal{N}_\xi(X)) = \mathcal{M}_\xi(X)$ contain $X \cup \{X\}$, so that $X \in \mathcal{N}_\xi(X)$ for all ξ. We require that all extenders added to the $K^c(X)$ sequence have critical point $> \mathrm{OR} \cap X$. We require that the levels $\mathcal{N}_\xi(X)$ of the construction be "1-small above X", that is, if κ is a critical point of an extender from the $\mathcal{N}_\xi(X)$ sequence, then for no $\delta > \mathrm{OR} \cap X$ do we have $\mathcal{J}_\kappa^{\mathcal{N}_\xi(X)} \models \delta$ is Woodin. By $K^c(X)$ we mean the limit as $\xi \to \Omega$ of the $\mathcal{M}_\xi(X)$. Let us call a structure with the appropriate first order properties of the $\mathcal{M}_\xi(X)$ an X-premouse.

If there is no $\delta > (\mathrm{OR} \cap X)$ such that $K^c(X) \models \delta$ is Woodin, then as in §2 we get that $K^c(X)$ is $(\omega, \Omega+1)$ iterable "above X", that is, via extenders on its sequence and the images thereof. (All such extenders have critical point $> \mathrm{OR} \cap X$, so none of the embeddings move X.) Of course, any two $\Omega+1$ iterable-above-X X-premice have a successful coiteration. As in 1.4, we also have $(\alpha^+)^{K^c(X)} = \alpha^+$ for μ_0 a.e. $\alpha < \Omega$, where μ_0 is a normal measure on Ω. The rest of §3 - §6 adapts in an obvious way. (We shall not need §6.)

Now let $\mathcal{T}$ be our iteration tree on V having distinct cofinal wellfounded branches b and c. We have $\mathcal{T} \in V_\Omega$, where Ω is measurable. Let

$$\delta = \delta(\mathcal{T}) = \sup\{lh\ E^{\mathcal{T}}_\alpha \mid \alpha + 1 < lh\ \mathcal{T}\}.$$

By the results of §2 of [IT], whenever $f : \delta \to \delta$ and $f \in \mathcal{M}^{\mathcal{T}}_b \cap \mathcal{M}^{\mathcal{T}}_c$, then $\mathcal{M}^{\mathcal{T}}_b \models$ "δ is Woodin with respect to f". (Equivalently, $\mathcal{M}^{\mathcal{T}}_c$ satisfies this.)

Notice that $i^{\mathcal{T}}_{ob}(\Omega) = i^{\mathcal{T}}_{oc}(\Omega) = \Omega$. Working in $\mathcal{M}^{\mathcal{T}}_b$ and $\mathcal{M}^{\mathcal{T}}_c$, let us form the models

$$\begin{aligned} R &= K^c(V_\delta)^{\mathcal{M}^{\mathcal{T}}_b}, \\ S &= K^c(V_\delta)^{\mathcal{M}^{\mathcal{T}}_c}. \end{aligned}$$

Notice here that $V_\delta^{\mathcal{M}^{\mathcal{T}}_b} = V_\delta^{\mathcal{M}^{\mathcal{T}}_c}$; setting $X = V_\delta^{\mathcal{M}^{\mathcal{T}}_b}$, we have that both R and S are 1-small above X.

Claim 1. Let $\alpha > \delta$ be a successor cardinal of R such that $\mathcal{J}^R_\alpha \not\models \exists\kappa(\delta < \kappa \wedge \kappa$ is Woodin); then $\mathcal{J}^R_\alpha$ is $\Omega + 1$ iterable above X. Similarly for S.

Proof. Our "proper smallness above X" requirement on α guarantees, as in §6, that no iteration tree on $\mathcal{J}^R_\alpha$ which is above X can have distinct cofinal wellfounded branches. Our standard reflection argument (cf. 2.4 (a)) shows that it is enough to prove the following.

Subclaim. Let $\pi : \mathcal{P} \to \mathcal{J}^R_\alpha$ be elementary, with $\mathcal{P}$ countable, and let $\pi(\bar{X}) = X$. Let $\mathcal{U}$ be a countable putative iteration tree on $\mathcal{P}$; then either $\mathcal{U}$ has a last, wellfounded model, or $\mathcal{U}$ has a cofinal wellfounded branch.

Proof. Since $\mathcal{P}$ is countable, $\mathcal{P} \in \mathcal{M}^{\mathcal{T}}_b$, and of course $\mathcal{J}^R_\alpha \in \mathcal{M}^{\mathcal{T}}_b$. Since $\mathcal{M}^{\mathcal{T}}_b$ is wellfounded, an easy absoluteness argument gives us an embedding $\sigma : \mathcal{P} \to \mathcal{J}^R_\alpha$ such that $\sigma \in \mathcal{M}^{\mathcal{T}}_b$. But also $\mathcal{U} \in \mathcal{M}^{\mathcal{T}}_b$. We can therefore carry out the iterability proof of Theorem 2.5 within $\mathcal{M}^{\mathcal{T}}_b$ using the background extenders given by the construction of $R = K^c(V_\delta)^{\mathcal{M}^{\mathcal{T}}_b}$. □

Claim 2. $P(\delta) \cap R = P(\delta) \cap S$.

Proof. Let $\alpha = (\delta^+)^R$ and $\beta = (\delta^+)^S$. By Claim 1, both $\mathcal{J}^R_\alpha$ and $\mathcal{J}^S_\beta$ are $\Omega+1$ iterable above X. It follows that they have a successful coiteration above X, and since neither can move without dropping, we get $\mathcal{J}^R_\alpha \trianglelefteq \mathcal{J}^S_\beta$ or $\mathcal{J}^S_\beta \trianglelefteq \mathcal{J}^R_\alpha$. Suppose without loss of generality that $\mathcal{J}^R_\alpha \trianglelefteq \mathcal{J}^S_\beta$.

It follows that $P(\delta) \cap R \subseteq S$, so $P(\delta) \cap R \subseteq \mathcal{M}^{\mathcal{T}}_b \cap \mathcal{M}^{\mathcal{T}}_c$, so that $R \models \delta$ is Woodin. We are done if R has another Woodin cardinal above δ, so we assume otherwise. But then, whenever γ is a successor cardinal of R above δ, then $\mathcal{J}^R_\gamma \not\models \exists\kappa(\delta < \kappa \wedge \kappa$ is Woodin). Claim 1 then shows $\mathcal{J}^R_\gamma$ is $\Omega + 1$ iterable, and since this is true for all γ, R itself is $\Omega + 1$ iterable above X.

Theorem 1.4, applied within $\mathcal{M}_b^{\mathcal{T}}$ to $R = K^c(X)$, implies that $\mathcal{M}_b^{\mathcal{T}}$ satisfies "for $i_{ob}^{\mathcal{T}}(\mu_0)$ a.e. $\alpha < \Omega$, $\alpha^+ = (\alpha^+)^{K^c(X)}$". But for μ_0 a.e. $\alpha < \Omega$, $i_{ob}^{\mathcal{T}}(\alpha) = \alpha$ and $i_{ob}^{\mathcal{T}}(\alpha^+) = \alpha^+$. Thus it is true in V that for μ_0 a.e. $\alpha < \Omega$, $\alpha^+ = (\alpha^+)^R$.

Since R and $\mathcal{J}_\beta^S$ are $\Omega + 1$ iterable above X, they have a successful coiteration above X. Since R computes successor cardinals correctly μ_0 a.e., and $\mathcal{J}_\beta^S$ cannot move without dropping, $\mathcal{J}_\beta^S \trianglelefteq R$. This completes the proof of the claim. □

Inspecting the proof of claim 2, we have:

Claim 3. δ is Woodin in both R and S. Both R and S are $\Omega + 1$ iterable. Finally, $(\alpha^+)^R = (\alpha^+)^S = \alpha^+$ for μ_0 a.e. α.

Let us emphasize that R and S are $(\omega, \Omega + 1)$ iterable in V, not just in the models $\mathcal{M}_b^{\mathcal{T}}$ or $\mathcal{M}_c^{\mathcal{T}}$.

We wish to compare R with S, but first we must pass to models for which the comparison will have a large set of fixed points. Working in $\mathcal{M}_b^{\mathcal{T}}$, let R^* come from R by taking ultrapowers by the order zero total measure at each measurable cardinal of R. Thus R^* is $\mathcal{M}_b^{\mathcal{T}}$ definable (from δ and Ω), R^* is a linear iterate of R, and if $\delta < \kappa < \Omega$ and κ is strongly inaccessible in $\mathcal{M}_b^{\mathcal{T}}$, then κ is not the critical point of a total extender on the R^* sequence. Let S^* be obtained from S, working inside $\mathcal{M}_c^{\mathcal{T}}$, in a similar fashion.

Now let $(\mathcal{U}, \mathcal{V})$ be a successful coiteration of R^* with S^*, according to their unique $\Omega + 1$ iteration strategies. Since R^* and S^* compute α^+ correctly for a.e. $\alpha < \Omega$, $\mathcal{U}$ and $\mathcal{V}$ have a common last model Q. Let $j : R^* \to Q$ and $k : S^* \to Q$ be the iteration maps. Let

$$Z = \{\alpha < \Omega \mid j(i_{ob}^{\mathcal{T}}(\alpha)) = k(i_{oc}^{\mathcal{T}}(\alpha)) = \alpha\}$$

be the set of common fixed points of $j \circ i_{ob}^{\mathcal{T}}$ and $k \circ i_{oc}^{\mathcal{T}}$. We have then that $\mu_0(Z) = 1$, and for μ_0 a.e. α, Z is cofinal in α^+ and $\alpha^+ = (\alpha^+)^Q$.

Now let $\alpha_0 \in b - c$, and define

$$\begin{aligned} \beta_n &= \text{least } \gamma \in (c - \alpha_n)\,, \\ \alpha_{n+1} &= \text{least } \gamma \in (b - \beta_n)\,. \end{aligned}$$

Let us assume that α_0 is chosen large enough that $\delta \in \operatorname{ran} i_{\alpha_1,b}^{\mathcal{T}} \cap \operatorname{ran} i_{\beta_1,c}^{\mathcal{T}}$. It follows, of course, that $R^* \in \operatorname{ran} i_{\alpha_1,b}^{\mathcal{T}}$ and $S^* \in \operatorname{ran} i_{\beta_1,c}^{\mathcal{T}}$. Set

$$\kappa = \operatorname{crit} i_{\alpha_1,b}^{\mathcal{T}}$$

and

$$H = \text{transitive collapse of } \mathrm{Hull}^Q(V_\kappa^{\mathcal{M}_b^{\mathcal{T}}} \cup Z \cup \{\delta\})\,.$$

The next claim comes directly from the proof of the uniqueness theorem of §2 of [IT] (see also 6.1 of [FSIT]).

Claim 4. $\mathrm{Hull}^Q(V_\kappa^{\mathcal{M}_b^{\mathcal{T}}} \cup Z \cup \{\delta\}) \cap V_\delta^{\mathcal{M}_b^{\mathcal{T}}} = V_\kappa^{\mathcal{M}_b^{\mathcal{T}}}$.

Proof. (Sketch) For all $i \geq 1$, α_i and β_i are successor ordinals, and

$$\mathrm{crit}(E^{\mathcal{T}}_{\alpha_i-1}) < \mathrm{crit}(E^{\mathcal{T}}_{\beta_i-1}) < \mathrm{str}^{\mathcal{M}^{\mathcal{T}}_{\alpha_i}}(E^{\mathcal{T}}_{\alpha_i-1})$$

and

$$\mathrm{crit}(E^{\mathcal{T}}_{\beta_i-1}) < \mathrm{crit}(E^{\mathcal{T}}_{\alpha_{i+1}-1}) < \mathrm{str}^{\mathcal{M}^{\mathcal{T}}_{\beta_i}}(E^{\mathcal{T}}_{\beta_i-1})\,.$$

Here $\mathrm{str}^M(E)$ is the strength of E in the model M. Now suppose t is a sequence of parameters from $V_\kappa^{\mathcal{M}^{\mathcal{T}}_b} \cup Z \cup \{\delta\}$, and

$$Q \models \exists x \in V_\delta^{\mathcal{M}^{\mathcal{T}}_b}\, \varphi(x,t)\,.$$

Let $\kappa_i = \mathrm{crit}(E^{\mathcal{T}}_{\alpha_i-1})$ and $\nu_i = \mathrm{crit}(E^{\mathcal{T}}_{\beta_i-1})$. Since $\sup\{\kappa_i \mid i \in \omega\} = \sup\{\nu_i \mid i \in \omega\} = \delta$, we can let i be least such that for some $x \in V_{\kappa_i}^{\mathcal{M}^{\mathcal{T}}_b}$, $Q \models \varphi[x,t]$. Fix such an x in $V_{\kappa_i}^{\mathcal{M}^{\mathcal{T}}_b}$. We claim $i = 1$; since $\kappa_1 = \kappa$ this will complete the proof of claim 4. Suppose then $i = e+1$.

Since k is the identity on $V_\delta^{\mathcal{M}^{\mathcal{T}}_b} \cup Z \cup \{\delta\}$, we have $k(\langle x,t\rangle) = \langle x,t\rangle$, so $S^* \models \varphi[x,t]$. Let $\beta = T$-pred (β_e), and let

$$i^{\mathcal{T}}_{\beta,c}(\langle \bar{S},\bar{t}\rangle) = \langle S^*, t\rangle\,.$$

Now $\nu_e = \mathrm{crit}\ i^{\mathcal{T}}_{\beta,c}$, and $\kappa_{e+1} < i^{\mathcal{T}}_{\beta,c}(\nu_e)$. Since $i^{\mathcal{T}}_{\beta,c}$ is elementary and $x \in V_{\kappa_{e+1}}^{\mathcal{M}^{\mathcal{T}}_c}$, we have $x' \in V_{\nu_e}^{\mathcal{M}^{\mathcal{T}}_c}$ such that $\bar{S} \models \varphi[x',\bar{t}]$. But then $S^* \models \varphi[x',t]$, and hence $Q \models \varphi[x',t]$.

We can now go apply the argument of the last paragraph to $i^{\mathcal{T}}_{\alpha,b}$, where $\alpha = T$-pred(α_e), using R^* and j instead of S^* and k. We get $x'' \in V_{\kappa_e}^{\mathcal{M}^{\mathcal{T}}_b}$ such that $Q \models \varphi[x'',t]$. This contradicts the minimality of i, and completes the proof of claim 4. □

Let $\pi : H \to Q$ be the collapse map, so that $\pi(\kappa) = \delta$ and $H \models \kappa$ is Woodin by claim 4. The properties of Z guarantee that $(\alpha^+)^H = \alpha^+$ for μ_0 a.e. $\alpha < \Omega$, and that in fact Ω is A-thick in H, where $A = \{\alpha < \Omega \mid \alpha \text{ is inaccessible}\}$.

Now, working in R, let

$$M = K^c(V_\kappa)^R\,.$$

Claim 5. $M \models \kappa$ is Woodin.

Proof. Assume otherwise; letting $\alpha = (\kappa^+)^M$, we then have that $\mathcal{J}^M_\alpha$ is properly small above $V_\kappa^{\mathcal{M}^{\mathcal{T}}_b}$. We get that $\mathcal{J}^M_\alpha$ is $\Omega+1$ iterable (in V, not just in R) by the same argument we used to prove claim 1. But then H and $\mathcal{J}^M_\alpha$ are $V_\kappa^{\mathcal{M}^{\mathcal{T}}_b}$-premice which are $\Omega+1$ iterable above $V_\kappa^{\mathcal{M}^{\mathcal{T}}_b}$, so they have a successful coiteration above $V_\kappa^{\mathcal{M}^{\mathcal{T}}_b}$. Since $\mathcal{J}^M_\alpha \models \kappa$ is not Woodin, there is a subset of κ which is in $\mathcal{J}_\alpha^M$ but not H. This means $\mathcal{J}_\alpha^M$ must iterate past H. On the other hand, H computes α^+ correctly for μ_0 a.e. $\alpha < \Omega$, so $\mathcal{J}_\alpha^M$ cannot iterate past H. □

Now $\kappa < \delta$, δ is Woodin in R, and $M = K^c(V_\kappa)^R$. A standard argument shows that for some ν such that $\kappa < \nu \leq \delta$, $M \models \nu$ is Woodin. (See the proof of 11.3 of [FSIT]. Thus $M \models$ There are two Woodin cardinals, and the proof of 7.8 is complete. □

D. Σ_3^1 correctness and the size of u_2

We say that a transitive model M is Σ_3^1 correct iff whenever $x \in M \cap {}^\omega\omega$ and P is a nonempty $\Pi_2^1(x)$ set of reals, then $P \cap M \neq \emptyset$. The proof of the following theorem was inspired by, and relies quite heavily upon, an idea due to G. Hjorth.

Theorem 7.9. *Suppose $K^c \models$ "There are no Woodin cardinals", and suppose there is a measurable cardinal $\mu < \Omega$; then K^c (or equivalently, K) is Σ_3^1 correct.*

The remarkable insight that there are theorems along the lines of 7.9, and the proof of the first of them, are due to Jensen (cf. [D]). Jensen's work was later extended by Mitchell ([M2]), and by Steel and Welch ([SW]). The smallness hypotheses on K in these works are, respectively: no inner model with a measurable cardinal, no inner model with a cardinal κ such that $o(\kappa) = \kappa^{++}$, and no inner model with a strong cardinal.

The smallness hypothesis on K in Theorem 7.9 is necessary. For if $K^c \models$ "There is a Woodin cardinal", then K^c is not Σ_3^1 correct. [Let $P = \{x \in {}^\omega\omega \mid x$ codes a countable, Π_2^1 -iterable premouse which is not 1-small$\}$. The existence of the measurable cardinal Ω gives $P \neq \emptyset$. On the other hand $P \cap K^c = \emptyset$, since if $\mathcal{M}$ is coded by a real in P, then $\mathcal{J}_\alpha^{K^c} \trianglelefteq \mathcal{M}$ for $\alpha = \omega_1^{K^c}$. (Cf. [PW], 3.1.)] However, if we liberalize our definition of K^c so as to allow levels which are not 1-small (but still retain some weaker smallness condition, e.g. tameness, which suffices to develop the basic theory of K^c), then we can simply drop the hypothesis that K^c satisfies "There are no Woodin cardinals" from 7.9. This is because if there are arbitrarily large $\alpha < \omega_1^{K^c}$ such that $\mathcal{J}_\alpha^{K^c}$ is not 1-small, then K^c is Σ_3^1 correct. (In fact, if x is a real coding a countable, $\omega_1 + 1$-iterable, non-1-small mouse $\mathcal{M}$ such that $y \in M$, and P is nonempty and $\Pi_2^1(y)$, then $\exists z \in P(z \leq_T x)$. This result is due to Woodin; cf. [PW], §4.)

Where we have assumed in 7.9 that there are two measurable cardinals, [D] requires only that every real has a sharp, and [M2] and [SW] require only the sharps of certain reals. We believe that it should be possible to eliminate the hypothesis that there is a measurable cardinal $< \Omega$ from 7.9. Of course, the need for Ω itself is also problematic, here and elsewhere.

Proof of 7.9. Our proof descends from a proof of Jensen's Σ^1_3 correctness theorem which is much simpler than Jensen's original proof. That simpler proof is due to Magidor.

Suppose that $K^c \models$ "There are no Woodin cardinals", and let $\mu < \Omega$ be measurable. For $\alpha \geq 1$, we let u_α be the αth uniform indiscernible relative to parameters in V_μ, that is

$$\begin{aligned} u_\alpha &= \alpha\text{th ordinal } \beta \text{ such that } \forall x \in V_\mu \\ &\quad (\beta \text{ is an indiscernible of } L[x]) . \end{aligned}$$

Thus $u_1 = \mu$. Magidor's argument is based on the following lemma.

Lemma 7.10. (Magidor) *Suppose $u_2^K = u_2$; then there is a tree $T_2 \in K$ such that $p[T_2]$ is the universal Π^1_2 set of reals, and thus K is Σ^1_3 correct.*

Proof. (Sketch) We first show that for all α, $u_\alpha^K = u_\alpha$. The proof is by induction on α; the cases $\alpha = 1$ and α is a limit are trivial. Let $\alpha = \beta + 1$. Let

$$n(\gamma, x) = \text{least indiscernible of } L[x] \text{ which is } > \gamma .$$

We have

$$u_{\beta+1} = \sup\{n(u_\beta, x) \mid x \in V_\mu\} ,$$

and

$$\begin{aligned} u_2 &= \sup\{n(u_1, x) \mid x \in V_\mu\} \\ &= \sup\{n(u_1, x) \mid x \in V_\mu^K\} , \end{aligned}$$

since $u_2 = u_2^K$. But then for any $x \in V_\mu$, we can find $y \in V_\mu^K$ so that $n(u_1, x) < n(u_1, y)$, and thus $n(u_\beta, x) < n(u_\beta, y)$ by the uniform indiscernibility of the u_η's. It follows that

$$u_{\beta+1} = \sup\{n(u_\beta, x) \mid x \in V_\mu^K\} ,$$

as desired.

It is well known that for any ordinal η, these are an $x \in V_\mu$ and a term τ and uniform indiscernibles $u_{\alpha_0} < \cdots < u_{\alpha_n} \leq \eta$ such that $\eta = \tau^{L[x]}(u_{\alpha_0} \cdots u_{\alpha_n})$. (This result is due to Solovay; the proof is an easy induction on η.) Since $u_\alpha = u_\alpha^K$ for all α, we can take $x \in K$ in the above.

By T_2, we mean the Martin-Solovay tree for Π^1_2 constructed as follows. Let $\mathbf{L} = \bigcup\{L[x] \mid x \in V_\mu\}$. Let S on $\omega \times \omega \times \mu$ be the Shoenfield tree for a Buniversal Σ^1_2 set. For u, $v \in \omega^{<\omega}$ such that $\text{dom}(u) = \text{dom}(v)$, let $S_{(u,v)} = \{w \mid (u, v, w) \in S\}$. We define an ultrafilter on $P(S_{(u,v)}) \cap \mathbf{L}$ as follows. For $X \subseteq \mu$ and $n < \omega$, let $[X]^n = \{\langle \alpha_0 \cdots \alpha_{n-1}\rangle \mid \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} \wedge \forall i < n(\alpha_i \in X)\}$. Letting $n = \text{dom}(u) = \text{dom}(v)$, there is a unique permutation $(i_0, \ldots, i_{n-1})$ of n such that $S_{(u,v)} = \{\langle \alpha_{i_0} \cdots \alpha_{i_{n-1}}\rangle \mid \langle \alpha_0 \cdots \alpha_{n-1}\rangle \in [\mu]^n\}$. For $A \subseteq S_{(u,v)}$ with $A \in \mathbf{L}$, we put

$$\mu_{(u,v)}(A) = 1 \Leftrightarrow \quad \exists C(C \text{ is club in } \mu \wedge \forall \langle \alpha_0 \cdots \alpha_{n-1} \rangle \in [C]^n \\ (\langle \alpha_{i_0} \cdots \alpha_{i_{n-1}} \rangle \in A)) .$$

Since $\forall x \in V_\mu (x^\sharp$ exists), $\mu_{(u,v)}$ is an ultrafilter on $P(S_{(u,v)}) \cap \mathbf{L}$. If $u \subseteq r$ and $v \subseteq s$, then $\mu_{(u,v)}$ is compatible with $\mu_{(r,s)}$, so we have a natural embedding

$$\pi_{(u,v),(r,s)} : \mathrm{Ult}(\mathbf{L}, \mu_{(u,v)}) \to \mathrm{Ult}(\mathbf{L}, \mu_{(r,s)}) .$$

The ultrapowers here are formed using functions in $\mathbf{L}$. The result of Solovay mentioned above yields

$$\langle u_{i_0+1}, \ldots, u_{i_{n-1}+1} \rangle = [\text{identity}]_{\mu_{u,v)}} ,$$

where $(i_0, \ldots, i_{n-1})$ is the permutation of $n = \mathrm{dom}(u)$ used to define $\mu_{(u,v)}$, and the u_i's are the uniform indiscernibles. By convention, $\mu_{(\emptyset,\emptyset)}$ is principal and $\mathrm{Ult}(\mathbf{L}, \mu_{(\emptyset,\emptyset)}) = \mathbf{L}$. We then have:

$$\mu_{(u,v)}(A) = 1 \quad \text{iff} \quad \langle u_{i_0+1}, \ldots, u_{i_{n-1}+1} \rangle \in \pi_{(\emptyset,\emptyset),(u,v)}(A) .$$

Except for the fact that they are not total on V, the measures $\mu_{(u,v)}$ witness the weak homogeneity of S. In particular $x \in p[S]$ iff $\exists y \in {}^\omega\omega$ (the direct limit of the $\mathrm{Ult}(\mathbf{L}, \mu_{(x\restriction n, y\restriction n)})$ under the $\pi_{(x\restriction n,y\restriction n),(x\restriction n+1,y\restriction n+1)}$ is wellfounded). The tree T_2 builds a real x on one coordinate, and proves $x \notin p[S]$ on the other by showing continuously that all associated direct limits are illfounded. More precisely, let $\langle r_i \mid i \in \omega \rangle$ enumerate $\omega^{<\omega}$ so that $r_0 = \emptyset$ and $r_i \subseteq r_j \Rightarrow i \leq j$, and put for $u \in \omega^{<\omega}$ with $\mathrm{dom}(u) = n$,

$$(u, \langle \alpha_0, \ldots, \alpha_{n-1} \rangle) \in T_2 \text{ iff } \alpha_0 = \mu \wedge \forall i < j \leq n-1 \\ (r_i \subsetneq r_j \Rightarrow \pi_{(u\restriction \mathrm{dom}\, r_i, r_i)(u\restriction \mathrm{dom}\, r_j, r_j)}(\alpha_i) > \alpha_j) .$$

Then $p[T_2] = {}^\omega\omega - p[S]$.

Since $K \models \forall x \in V_\mu (x^\sharp$ exists), we can form T_2^K inside K. In order to see that $T_2^K = T_2$, we must see that for any $u, v \in \omega^{<\omega}$ with $\mathrm{dom}(u) = \mathrm{dom}\, v$

$$\pi^K_{(\emptyset,\emptyset),(u,v)}(\mu) = \pi_{(\phi,\phi),(u,v)}(\mu) ,$$

and if $u \subseteq r$ and $v \subseteq s$ and $\mathrm{dom}(r) = \mathrm{dom}\, s$,

$$\pi^K_{(u,v),(r,s)} \restriction \mu^* = \pi_{(u,v),(r,s)} \restriction \mu^*$$

for $\mu^* = \pi_{(\emptyset,\emptyset),(r,s)}(\mu)$. Now clearly, $\mu^K_{(u,v)} = \mu_{(u,v)} \cap K$ for all (u,v). We are done, then, if we show that for any (u,v) and $f : S_{(u,v)} \to \mu$ such that $f \in L$, $[f]_{\mu_{(u,v)}}$ has a representative in K. We may assume that for some $x \in V_\mu$ and term τ, $f(w) = \tau^{L[x]}[w]$ for all $w \in S_{(u,v)}$. Let $w^* = [\text{identity}]_{\mu_{(u,v)}} < K_{U_{n+1}}$, where $n = \mathrm{dom}(v)$, so by the result of Solovay mentioned above, applied inside K, we can find a $y \in V^K_\mu$ and a term σ such that $\sigma^{L[y]}[w^*] = \tau^{L[x]}[w^*]$. It follows that for $\mu_{(u,v)}$ a.e. w, $\sigma^{L[y]}[w] = \tau^{L[x]}[w]$. Letting $g(w) = \sigma^{L[y]}[w]$ for all $w \in S_{(u,v)}$, we have $g \in K$ and $[g]_{\mu_{(u,v)}} = [f]_{\mu_{(u,v)}}$.

This completes the proof of 7.10 □

Now let

$$\mathcal{F} = \{\mathcal{M} \in V_\mu \mid \mathcal{M} \text{ is } \Omega + 1 \text{ iterable and properly small}\}.$$

Recall from 6.12 that a premouse is properly small just in case it satisfies "There are no Woodin cardinals" and "There is a largest cardinal". There can be at most one cofinal wellfounded branch in an iteration tree based on a properly small premouse, so any $\mathcal{M} \in \mathcal{F}$ has a unique $\Omega + 1$ iteration Bstrategy $\Sigma_{\mathcal{M}}$. For $\mathcal{M}$, $\mathcal{N} \in \mathcal{F}$, let $\mathcal{P}$ and $\mathcal{Q}$ be the last models of $\mathcal{T}$ and $\mathcal{U}$, where $(\mathcal{T},\mathcal{U})$ is the unique successful $(\Sigma_{\mathcal{M}}, \Sigma_{\mathcal{N}})$ coiteration of $\mathcal{M}$ with $\mathcal{N}$. We define $\mathcal{M} \leq^* \mathcal{N}$ iff $\mathcal{P} \trianglelefteq \mathcal{Q}$. Thus $\leq^*$ is just the usual mouse order, restricted to $\mathcal{F}$. The Dodd-Jensen lemma implies that $\leq^*$ is a prewellorder. Set

$$\delta = \text{order type of } (\mathcal{F}, \leq^*).$$

Also, for $\mathcal{M} \in \mathcal{F}$, let $|\mathcal{M}|_{\leq^*}$ be the rank of $\mathcal{M}$ in the prewellorder $\leq^*$.

The following lemma is part of the folklore.

Lemma 7.11. $\delta \leq u_2^K$.

Proof. It is easy to see that if U is any normal ultrafilter on μ, then $(\alpha^+)^K = \alpha^+$ for U a.e. $\alpha < \mu$. B(We prove this as part of the proof of lemma 8.15 in the next section.) It follows that $K \cap \mathcal{F}$ is $\leq^*$-cofinal in $\mathcal{F}$. For let $\mathcal{M} \in \mathcal{F}$, and let $(\mathcal{T},\mathcal{U})$ be the successful coiteration of $\mathcal{M}$ with $\mathcal{J}_\mu^K$ determined by $\Omega + 1$ iteration strategies for the two mice. Since $\mathcal{J}_\mu^K$ computes successor cardinals correctly almost everywhere, $\max(lh\,\mathcal{T}, lh\,\mathcal{U}) < \mu$, and the last model $\mathcal{P}$ of $\mathcal{T}$ is an initial segment of the last model of $\mathcal{U}$. Let $\alpha < \mu$ be a successor cardinal of K and such that $lh\ E_\xi^{\mathcal{U}} < \alpha$ for all $\xi + 1 < lh\,\mathcal{U}$; then we can regard $\mathcal{U}$ as tree on $\mathcal{J}_\alpha^K$, so that $(\mathcal{T},\mathcal{U})$ demonstrates that $\mathcal{M} \leq^* \mathcal{J}_\alpha^K$.

It suffices then to show that if $\mathcal{M} \in K \cap \mathcal{F}$, then $|\mathcal{M}|_{\leq^*} < u_2^K$. Fix $\mathcal{M}$, and let G be V-generic for $\mathrm{Col}(\omega, < \mu)$, and let x_0 be a real coding $\mathcal{M}$ in $V[G]$. Choose x_0 to be generic over $L[\mathcal{M}]$, so that $(\mu^+)^{L[x_0]} = (\mu^+)^{L[\mathcal{M}]} < u_2^K$. For x and y reals in $V[G]$, let

$R(x,y)$ iff (x and y code properly small premice$\mathcal{M}_x$ *and* $\mathcal{M}_y$, and there is a successful coiteration $(\mathcal{T},\mathcal{U})$ of $\mathcal{M}_x$with $\mathcal{M}_y$ such that $\mathcal{T}$ and $\mathcal{U}$ are simple, and the last model of $\mathcal{T}$ is a proper initial segment of that of $\mathcal{U}$),

and let

$$S(x,y) \text{ iff } R(x,y) \cap R(y,x_0).$$

It is easy to check that S is a $\Sigma_2^1(x_0)$ relation on the reals in $V[G]$.

Claim. S is wellfounded.

Proof. Suppose not, and let $S = \dot{S}_G$, where $\dot{S}$ represents the natural definition of S from $\mathcal{M}$ over $V[G]$. Working in V, we can construct a countable, transitive P and an elementary $\pi : P \to V_\Omega$ with $\pi(\bar{\mathcal{M}}) = \mathcal{M}$ and $\pi(\bar{\mu}) = \mu$ for some $\bar{\mathcal{M}}$, $\bar{\mu}$. Let $\bar{G}$ in V be P-generic for $\mathrm{Col}(\omega, < \bar{\mu})$. Then $\dot{S}_{\bar{G}}$ is illfounded, and this implies that the mouse order below $\mathcal{M}$ is illfounded. Since $\pi \restriction \bar{\mathcal{M}} : \bar{\mathcal{M}} \to M$, and $\mathcal{M} \in \mathcal{F}$, this is a contradiction.

Since S is wellfounded and $\Sigma^1_2(x_0)$, its rank is $< (\mu^+)^{L[x_0]}$ by the Kunen-Martin theorem. Clearly, $(|\mathcal{M}|_{\leq^*})^V$ is less than or equal to the rank of S. This proves 7.11. □

In view of 7.10 and 7.11, we would like to show that $\delta = u_2$. The key idea for doing this is due to Greg Hjorth.

Lemma 7.13. (Hjorth) *Suppose $\delta < u_2$; then there is a set $M \in V_\mu$ such that $\mathcal{F} \cap L[M]$ is $\leq^*$-cofinal in $\mathcal{F}$.*

Proof. Since $\delta < u_2$, we have an $x \in V_\mu$ and a term τ such that $\delta = \tau^{L[x]}[x, \mu]$. Now let $x \in V_\eta$ where $\eta < \mu$, and let $(Z, \epsilon) < (V_\Omega, \epsilon)$ be such that $\mathrm{card}(Z) < \mu$, $V_\eta \subseteq Z$, and $\mu \in Z$. Let M be the transitive collapse of Z, $\pi : M \to V_\Omega$ the collapse map, and $\pi(\bar{\mu}) = \mu$. Let $\bar{U}$ be such that $M \models \bar{U}$ is a normal ultrafilter on $\bar{\mu}$, let N be the μth (linear) iterate of M by $\bar{U}$ and its images, and let $i : M \to N$ be the iteration map. By an argument due to Jensen, there is an embedding $\sigma : N \to V_\Omega$ such that $\pi = \sigma \circ i$. Now $i(\bar{\mu}) = \mu$, so $\sigma(\mu) = \sigma(i(\bar{\mu})) = \pi(\bar{\mu}) = \mu$. We also have $\pi \restriction V_\eta = i \restriction V_\eta =$ identity, so $\sigma \restriction V_\eta =$ identity, so $\sigma(x) = x$. Thus $\sigma(\tau^{L[x]}[x, \mu]) = \tau^{L[x]}[x, \mu]$; that is, $\sigma(\delta) = \delta$. It follows that $(\mathcal{F}^N, \leq^{*\,N})$ has order type δ. Now if $\mathcal{P} \in \mathcal{F}^N$, then $\sigma \restriction \mathcal{P} : \mathcal{P} : \mathcal{P} \to \sigma(\mathcal{P})$ and $\sigma(\mathcal{P}) \in \mathcal{F}$, and thus $\mathcal{P}$ is $\Omega + 1$-iterable. Thus $\mathcal{F}^N \subseteq \mathcal{F}$. It is easy to see that $(\leq^*)^N = \leq^* \cap N$. Since $N \in L[M]$, $\mathcal{F}^N \subseteq \mathcal{F} \cap L[M]$, and we are done. □

We will actually use the proof of 7.12, rather than the lemma itself.

So far we haven't worked with K above μ, and indeed Hjorth formulated his lemma with $\mu = \Omega$. But now let M and N be as in the proof of 7.12. We would be done if we could find $\mathcal{P} \in \mathcal{F}$ such that $\forall Q \in \mathcal{F}^N (Q \leq^* \mathcal{P})$. There is a natural candidate for such a $\mathcal{P}$, namely K^M. (K^M is not actually properly small, but this problem is easily finessed.) Of course, the iteration map $i : K^M \to K^N$ comes from an "external" iteration of all of M, but suppose we could absorb its action into an internal iteration of K^M. We'd be done. Since $\mathrm{crit}(i) = \bar{\mu}$, we must use the part of K^M above $\bar{\mu}$ to do this. So we must work with $\mu < \Omega$.

The following lemma is the key to absorbing the map from K^M to K^N into an iteration of K^M. Its proof borrows Lemma 8.2 from §8, a lemma we originally proved as part of the proof of 7.9.

Lemma 7.13. *Let $j : V \to \mathit{Ult}(V, U)$, where U is a normal ultrafilter on μ; then there are almost normal iteration trees $\mathcal{T}$ on K and $\mathcal{U}$ on $j(K)$,* *having common last model Q and associated embeddings $k : K \to Q$ and $\ell : j(K) \to Q$, such that $k = \ell \circ j$.*

Proof. Let W be a weasel such that Ω is thick in W, and W has the hull property at all $\alpha < \Omega$. Lemma 4.5 shows that such a weasel exists. By Lemma 8.2, there is an iteration tree $\mathcal{T}_0$ on K having last model W whose associated embedding $t_0 : K \to W$ satisfies BDef(W) $= t_0''K$. In fact, $\mathcal{T}_0$ is a linear iteration by normal measures. Notice that $j(\mathcal{T}_0)$ is an iteration tree on $j(K)$ with last model $j(W)$ and associated embedding $j(t_0)$. Since the class of fixed points of j is thick in W, Ω is thick in $j(W)$ and Def($j(W)$) $= j''$ Def(W).

Now let $(\mathcal{T}_1, \mathcal{U}_1)$ be the successful coiteration of W with $j(W)$, using their unique $\Omega + 1$ iteration strategies, and let Q be the common last model of $\mathcal{T}_1$ and $\mathcal{U}_1$. BLet $t_1 : W \to Q$ and $u : j(W) \to Q$ be the associated iteration maps.

We have the diagram:

$$\begin{tikzcd}
 & Q \\
W \arrow[ur, "t_1"] \arrow[r, "j"'] & j(W) \arrow[u, "u"'] \\
K \arrow[u, "t_0"] \arrow[r, "j"'] & j(K) \arrow[u, "j(t_0)"']
\end{tikzcd}$$

The bottom rectangle commutes: $j \circ t_0 = j(t_0) \circ j$ because j is elementary on V. The upper "triangle" may not commute, but it commutes on ran(t_0), since

$$\begin{aligned}
t_1''(t_0''K) &= t_1'' \text{ Def}(W) = \text{Def}(Q) \\
&= u'' \text{ Def}(j(W)) = u''(j'' \text{ Def}(W)) \\
&= u''(j''(t_0''K)) .
\end{aligned}$$

It follows that, setting $\mathcal{T} = \mathcal{T}_0^\frown \mathcal{T}_1$, $k = t_1 \circ t_0$, $\mathcal{U} = j(\mathcal{T}_0)^\frown \mathcal{U}_1$, and $\ell = u \circ j(t_0)$, the conclusion of 7.13 holds. □

We can now complete the proof of Theorem 7.9. By 7.10 and 7.11 it is enough to show $\delta = u_2$, so assume $\delta < u_2$. Let M, N, i, and $\bar{\mu}$ be as in the proof of 7.12. So $M \in V_\mu$, and $i : M \to N$ is the iteration map coming from hitting a normal meaAsure of M on $\bar{\mu}$ repeatedly, μ times in all. LeAt $i_{\alpha\beta} : M_\alpha \to M_\beta$ be the natural map, where M_α and M_β are the αth and βth iterates of M. So $i = i_{0\mu}$.

We define premice Q_α, for $\alpha \leq \mu$, by induction on α. We shall have that $Q_{\alpha+1}$ is the last model on an almost normal iteration tree $\mathcal{T}_\alpha$ on Q_α, with an associated iteration map $k_{\alpha,\alpha+1} : Q_\alpha \to Q_{\alpha+1}$. We shall simultaneously define embeddings $\ell_\alpha : K^{M_\alpha} \to Q_\alpha$ so that for $\alpha \leq \beta \leq \mu$

$$\begin{tikzcd}
Q_\alpha \arrow[r, "k_{\alpha,\beta}"] & Q_\beta \\
K^{M_\alpha} \arrow[u, "\ell_\alpha"] \arrow[r, "i_{\alpha\beta}"'] & K^{M_\beta} \arrow[u, "\ell_\beta"']
\end{tikzcd}$$

commutes. (Here we are setting $k_{\alpha,\gamma+1} = k_{\gamma,\gamma+1} \circ k_{\alpha\gamma}$, and $k_{\alpha\lambda} : Q_\alpha \to Q_\lambda$ to be the canonical embedding into $Q_\lambda = \text{dir lim}_{\alpha<\lambda} Q_\alpha$ for λ limit.)

Set $Q_0 = K^{M_0}$ and $\ell =$ identity. Now, given Q_α and ℓ_α, we apply 7.13 inside the model M_α to the ultrapower which produces $M_{\alpha+1}$. This gives an almost normal iteration tree $\mathcal{T}$ on K^{M_α} with last model Q and iteration map $k : K^{M_\alpha} \to Q$, and an embedding $\ell : K^{M_{\alpha+1}} \to Q$ such that $k = \ell \circ i_{\alpha,\alpha+1}$. Note $\mathcal{T} \in M_\alpha$. Let $\mathcal{T}_\alpha = \ell_\alpha \mathcal{T}$ be the result of copying $\mathcal{T}$ to a tree on Q_α, and let $Q_{\alpha+1}$ be the last model of $\mathcal{T}_\alpha$. (K^{M_α} is a model of ZFC, $\mathcal{T}$ doesn't drop on its main branch, and k and ℓ are fully elementary. So, by induction, all Q_γ are ZFC models, no $\mathcal{T}_\gamma$ drops on its main branch, and all $k_{\eta\gamma}$ and ℓ_γ are fully elementary. So we can copy.) Let $u : Q \to Q_{\alpha+1}$ be given by the copy construction, and $\ell_{\alpha+1} = u \circ \ell$. The commutative diagram below summarizes the construction of $Q_{\alpha+1}$ and $\ell_{\alpha+1}$:

$$\begin{tikzcd}
Q_\alpha \arrow[r, "\ell_\alpha\mathcal{T}"] & Q_{\alpha+1} \\
 & {} \arrow[u, "u"] \\
K^{M_\alpha} \arrow[uu, "\ell_\alpha"] \arrow[ur, "\mathcal{T}"] \arrow[r, "i_{\alpha,\alpha+1}"'] & K^{M_{\alpha+1}} \arrow[u, "\ell"] \arrow[uu, bend right, "\ell_{\alpha+1}"']
\end{tikzcd}$$

For λ a limit $\leq \mu$, let $\ell_\lambda(i_{\alpha\lambda}(x)) = k_{\alpha\lambda}(\ell_\alpha(x))$ whenever $\alpha < \lambda$ and $x \in K^{M_\alpha}$. This completes the inductive definition of Q_α and ℓ_α.

The Q_α's are not properly small, but we can easily finesse this problem. Let $\psi : V \to \text{Ult}(V, U)$ be the canonical embedding, where U is a normal ultrafilter on Ω. Let $\mathcal{P} = \mathcal{J}_\alpha^{\psi(K)}$, where $\alpha = \Omega^+ = (\Omega^+)^{\psi(K)}$. Clearly, $\mathcal{P}$ is properly small, its largest cardinal being Ω, and $K = \mathcal{J}_\Omega^{\mathcal{P}}$. Also, $\mathcal{P}$ is $\Omega+1$ iterable in V, since $\mathcal{P}$ is $\psi(\Omega+1)$ iterable in $\mathrm{Ult}(V,U)$ and $\mathrm{Ult}(V,U)$ is closed under Ω-sequences. It follows that any iteration tree on K of length $\leq \Omega+1$ which is built according to the unique $\Omega+1$ iteration strategy for K can be regarded as an iteration tree on $\mathcal{P}$. Now we can assume that the hull $Z \prec V_\Omega$ collapsing to M is such that $Z = Y \cap V_\Omega$ for some $Y \prec V_\theta$ (for $\theta > \Omega$ large) with $\Omega, \mathcal{P} \in Y$. Let M' be the transitive collapse of Y and Q_0' be the image of $\mathcal{P}$ under this collapse. Thus $Q_0' \in \mathcal{F}$ and $Q_0 = \mathcal{J}_\alpha^{Q_0'}$, where α is the collapse of Ω. We can interpret $\mathcal{T}_0$ as a tree on Q_0' according to its unique $\Omega+1$ iteration strategy, and let Q_1' be the last model of $\mathcal{T}_0$, so interpreted. Then $Q_1 = \mathcal{J}_\alpha^{Q_1'}$, where α is the largest cardinal of Q_1', and $Q_1' \in \mathcal{F}$. Proceeding similarly by induction, we define Q_α' for $\alpha \leq \mu$ so that $Q_\alpha = \mathcal{J}_\beta^{Q_\alpha'}$ for β the largest cardinal of Q_α'.

Now let $\mathcal{R} \in \mathcal{F}^N$. Working in N, we see that $\mathcal{R} \leq^* \mathcal{J}_\beta^{K^N}$ for some $\beta < \mu$. Since K^N is elementarily embedded into Q_μ by ℓ_μ, and Q_μ' is an almost normal iterate of Q_0' by its unique $\Omega+1$ iteration strategy, $\mathcal{R} \leq^* Q_0'$. Thus $\mathcal{F}^N$ is not $\leq^*$-cofinal in $\mathcal{F}$; Q_0' is an upper bound. The argument in the proof of 7.12 now yields a contradiction. □

We can use our Σ_3^1 correctness theorem to show that certain apparently weak consequences of $\boldsymbol{\Delta}_2^1$ determinacy actually imply Δ_2^1 determinacy. The ideas here are due to A. S. Kechris; what we have contributed is just Theorem 7.9.

Corollary 7.14. *Suppose $\forall x \in {}^\omega\omega$ ($x^\sharp$ exists), and $\forall x \in {}^\omega\omega$ (the class of $\Sigma_3^1(x)$ subsets of ω has the separation property). Then $\boldsymbol{\Delta}_2^1$ determinacy holds.*

Proof. We show that Δ_2^1 determinacy holds; the proof relativizes routinely to an arbitrary real. By a theorem of Woodin, it is enough to show that there is a transitive proper class model M and an ordinal δ such that $M \models \delta$ is Woodin, and $V_{\delta+1}^M$ is countable.

Let x be a real which codes up witnesses to all true Σ_3^1 sentences; that is, let x be such that whenever P is a nonempty Σ_3^1 set of reals, then $\exists y \in P$ $(y \leq_T x)$. Using the Jensen-Mitchell Σ_3^1 correctness theorem, we get a proper class model N such that $x \in N$ and $N \models$ "There is are two measurable cardinals". For if there is no such N, then $K_{DJ}(x)$ is Σ_3^1 correct, where $K_{DJ}(x)$ is the Dodd-Jensen-Mitchell core model for two measurable cardinals, relativised to x. Now $K_{DJ}(x) \models$" There is a $\Delta_3^1(x)$-good wellorder of $\mathbb{R}$", and thus $K_{DJ}(x) \models$ "There are $\Sigma_3^1(x)$ sets A, $B \subseteq \omega$ such that $A \cap B = \emptyset$ and for all $\Delta_3^1(x)$ sets C, $A \subseteq C \Rightarrow B \cap C \neq \emptyset$". Since $K_{DJ}(x)$ is Σ_3^1 correct, there really are such sets A and B, and thus $\Sigma_3^1(x)$ separation fails.

Now let N be as described in the previous paragraph, and let $N \models$ "μ and Ω are measurable", where $\mu < \Omega$. If $(K^c)^N \models$" there is a Woodin cardinal", then we get the desired proper class model M with one Woodin cardinal δ such that $V^M_{\delta+1}$ is countable. (Let P be the transitive collapse of a countable elementary submodel of $V_{\Omega+\omega}$, and $i : P \to P_\infty$ the result of iterating a normal measure on the image under collapse of Ω through OR, and let $M = i((K^c)^P)$.) But if $(K^c)^N$ satisfies that there are no Woodin cardinals, then K^N is Σ^1_3 correct in N by 7.9. The choice of x guarantees that, since $\Sigma^1_3 \cap P(\omega)$ has the separation property in V, it has the separation property in N. The correctness of K^N implies that $\Sigma^1_3 \cap P(\omega)$ has the separation property in K^N. But $K^N \models$ "$\mathbb{R}$ has a Δ^1_3-good wellorder", so $K^N \models$ "$\Sigma^1_3 \cap P(\omega)$ does not have the separation property". □

If $\Pi^1_3 \cap P({}^\omega\omega)$ has the reduction property, then for all $x \in {}^\omega\omega$ of sufficiently large Turing degree, $\Sigma^1_3(x) \cap P(\omega)$ has the separation property. [Let (A, B) reduce a universal pair of Π^1_3 subsets of ${}^\omega\omega \times {}^\omega\omega$. Then whenever A and B are $\Pi^1_3(x)$, $\Sigma^1_3(x) \cap P(\omega)$ has the separation property.] Thus the proof of 7.14 shows that $\forall x \in {}^\omega\omega$ ($x^\sharp$ exists) + "$\Pi^1_3 \cap P({}^\omega\omega)$ has the reduction property" implies Δ^1_2 determinacy. We do not know whether $\forall x \in {}^\omega\omega$ ($x^\sharp$ exists) + "$\Sigma^1_3 \cap P({}^\omega\omega)$ has the separation property" implies Δ^1_2 determinacy.

We conjecture that $\forall x \in {}^\omega\omega$ ($x^\sharp$ exists) plus "$\Sigma^1_3 \cap P(\omega)$ has the separation property" implies Δ^1_2 determinacy. If one tries to prove this lightface refinement of 7.14 by the method of 7.14, then the fact that our Σ^1_3 correctness theorem required two measurable cardinals, (rather than none) becomes a problem.

Another application of our Σ^1_3 correctness theorem in "reverse descriptive set theory" can be found in [Hj], where Hjorth uses it to show that $\boldsymbol{\Pi}^1_2$ Wadge determinacy implies $\boldsymbol{\Pi}^1_2$ determinacy.

A problem which is closely related to the Σ^1_3 correctness problem is: what is the consistency strength of ZFC + $\forall x \in {}^\omega\omega(x^\sharp$exists) + $\boldsymbol{\delta}^1_2 = \omega_2$? Woodin has shown that the strength of ZFC+ "there is a Woodin cardinal with a measurable cardinal above it" is an upper bound. It is shown in [SW] that ZFC + "There is a strong cardinal" is a lower bound. We conjecture that the lower bound can be improved to ZFC + "There is a Woodin cardinal". Unfortunately, our proof of 7.9 does not seem to help with this conjecture, because of our use of the measurable cardinals μ and Ω. One wants to replace μ with ω_1 (and V_μ with HC), and avoid Ω altogether, and we don't see how to do this. However, our proof of 7.9 does give the consistency strength lower bound ZFC + "There is a Woodin cardinal" for a certain variant of ZFC + "$\forall x \in {}^\omega\omega(x^\sharp$ exists) + $\boldsymbol{\delta}^1_2 = \omega_2$" which we now explain.

Let $\mu < \Omega$ be measurable, and let u_α be the αth uniform indiscernible relative to elements of V_μ, as in the proof of 7.9. Notice that in $V^{\mathrm{Col}(\omega,<\mu)}$, μ_α is the αth uniform indiscernible relative to reals, and so $u_2 = (\boldsymbol{\delta}^1_2)^{V^{\mathrm{Col}(\omega,<\mu)}}$. One can ask whether $V^{\mathrm{Col}(\omega,<\mu)} \models \boldsymbol{\delta}^1_2 = \omega_2$; we do not know whether it is consistent relative to any large cardinal hypothesis that this be true. But if we replace $V^{\mathrm{Col}(\omega,<\mu)}$ by its $L(\mathbb{R})$, then the resulting proposition follows from $AD^{L(\mathbb{R})}$ in $V^{\mathrm{Col}(\omega,<\mu)}$, which of course holds if there are enough Woodin cardinals in V. We now show that "$V^{\mathrm{Col}(\omega,<\mu)} \models (L(\mathbb{R}) \models \boldsymbol{\delta}^1_2 = \omega_2)$" is at least as strong as the existence of one Woodin cardinal.

Theorem 7.15. *Let $\mu < \Omega$ be measurable, and suppose $V^{\mathrm{Col}(\omega,<\mu)} \models \boldsymbol{\delta}^1_2 = \omega_2^{L(\mathbb{R})}$; then $K^c \models$ There is a Woodin cardinal.*

Proof. Suppose $K^c \models$ There are no Woodin cardinals. Letting u_α be the αth uniform indiscernible relative to parameters in V_μ, we have $u_2^K = u_2^V$ from the proof of 7.9. Now let G be V-generic/$\mathrm{Col}(\omega, < \mu)$. It is easy to see that u_2^V is the second uniform indiscernible relative to reals in $V[G]$, so that $u_2^V = (\boldsymbol{\delta}^1_2)^{V[G]}$. Thus $u_2^K = \omega_2^{L(\mathbb{R}^*)}$, where $\mathbb{R}^* = \mathbb{R}^{V[G]}$. On the other hand, $\mathcal{J}^K_\mu$ is $\Sigma_\omega(L_\mu(\mathbb{R}^*))$ definable, by §6 and the fact that $K = K^{V[G]}$. Since $\mathcal{J}^K_\mu$ is essentially a subset of μ, we get $u_2^K < \omega_2^{L(\mathbb{R}^*)}$, a contradiction. □