7. DEGREES OF INTERPRETABILITY

Suppose PA$\dashv$ T. We shall use A, B, etc. for extensions of T. (Thus, T, A, B, etc. are essentially reflexive.) The relation $\leq$ of interpretability is reflexive and transitive. Thus, the relation $\equiv$ of mutual interpretability (restricted to extensions of T) is an equivalence relation; its equivalence classes will be called *degrees* (*of interpretability*) and will be written a, b, c, etc. D_T is the set of degrees of extensions of T. A is of degree a if $A \in a$ and d(A) is the degree of A. The relation $\leq$ among degrees is the relation induced by the relation $\leq$ among theories: $d(A) \leq d(B)$ iff $A \leq B$. $\mathbf{D}_T = (D_T, \leq)$, the partially ordered set of degrees defined in this way, will be studied in some detail in this chapter.

§1. Algebraic properties. In this § we restrict ourselves to purely algebraic properties of $\mathbf{D}_T$. First we define the theory A^T and the operations $\downarrow$ and $\uparrow$ on theories as follows.

$A^T = T + \{Con_{A\restriction k}: k \in N\}$,

$A \downarrow B = T + \{Con_{A\restriction k} \vee Con_{B\restriction k}: k \in N\}$,

$A \uparrow B = T + \{Con_{A\restriction k} \wedge Con_{B\restriction k}: k \in N\}$.

From Lemma 6.2 and Theorem 6.6, we get the following:

Lemma 1. (a) $A \leq B$ iff $A^T \dashv B$. Thus, $A^T \equiv A$ and $A \leq B$ iff $A^T \dashv B^T$.
(b) $A \leq B, C$ iff $A \leq B \downarrow C$,
(c) $A, B \leq C$ iff $A \uparrow B \leq C$.

The following lemma is little more than a restatement of Lemma 4.4.

Lemma 2. If θ is Π_1 and $A \vdash \theta$, there is a k such that $PA \vdash Con_{A\restriction k} \rightarrow \theta$.

Instead of $A \downarrow B$ it is sometimes convenient to use the theory $A \vee B$ defined by

$A \vee B = \{\varphi \vee \psi: \varphi \in A \ \& \ \psi \in B\}$.

$Th(A \vee B) = Th(A) \cap Th(B)$. Evidently, $A \downarrow B \dashv A \vee B$ and, by Lemma 2, $A \vee B \dashv_{\Pi_1} A \downarrow B$. But then, by Theorem 6.6, that $A \vee B \leq A \downarrow B$ and so $A \vee B \equiv A \downarrow B$. It follows that for every sentence φ, $(A + \varphi) \downarrow (A + \neg\varphi) \leq A$.

From Lemma 2 and Lemma 6.1 we get:

Lemma 3. For every Π_1 sentence π, $T + \pi \leq A \uparrow B$ iff $A \uparrow B \vdash \pi$ iff there are Π_1 sentences φ, ψ such that $A \vdash \varphi$, $B \vdash \psi$, and $T + \varphi \wedge \psi \vdash \pi$.

For $A \in a$ and $B \in b$, let $a \cap b = d(A \downarrow B)$ and $a \cup b = d(A \uparrow B)$. By Lemma 1, $\cap$ and $\cup$

are well–defined, $a \cap b$ is the g.l.b. of a and b and $a \cup b$ is l.u.b. of a and b. Thus, we have proved part of the following:

Theorem 1. $\mathbf{D}_T$ is a distributive lattice.

To prove distributivity we need the following lemma whose proof is left to the reader.

Lemma 4. (a) For every k, there is an m such that

$$PA \vdash Con_{(A \vee B)\restriction m} \rightarrow Con_{A\restriction k} \vee Con_{B\restriction k}.$$

(b) For every k, there is an m such that

$$PA \vdash Con_{A\restriction m} \vee Con_{B\restriction m} \rightarrow Con_{(A\vee B)\restriction k}.$$

Proof of Theorem 1. Let $D = A^T \vee (B \uparrow C)$ and $E = (A \vee B) \uparrow (A \vee C)$. To prove that $\mathbf{D}_T$ is distributive, it is, by Lemma 1, sufficient to show that $D \dashv\vdash E$.

Let k be arbitrary. By Lemma 4 (a), there is an m such that

$$PA \vdash Con_{(A\vee B)\restriction m} \rightarrow Con_{A\restriction k} \vee Con_{B\restriction k},$$
$$PA \vdash Con_{(A\vee C)\restriction m} \rightarrow Con_{A\restriction k} \vee Con_{C\restriction k}.$$

But then

$$E \vdash Con_{A\restriction k} \vee (Con_{B\restriction k} \wedge Con_{C\restriction k}).$$

It follows that $D \dashv E$. The proof that $D \vdash E$ is similar. ■

$\mathbf{D}_T$ has a minimal element $0_T = d(T)$ and a maximal element 1_T, the common degree of all inconsistent theories.

In our next result we answer a number of standard questions concerning $\mathbf{D}_T$; in particular, it follows that $\mathbf{D}_T$ is dense.

Theorem 2. Suppose $a < b < 1_T$, $d_0 \not\leq a$, and $b \not\leq d_1$. There are then degrees c_0, c_1 such that $a < c_i < b$, $d_0 \not\leq c_i \not\leq d_1$, $i = 0, 1$, $c_0 \cap c_1 = a$, and $c_0 \cup c_1 = b$.

We derive this from:

Lemma 5. Suppose X is r.e. and monoconsistent with PA. Let θ be any true Π_1 sentence. There are then Π_1 sentences θ_0, θ_1 such that

(i) $PA \vdash \theta_0 \vee \theta_1$,

(ii) $PA \vdash \theta_0 \wedge \theta_1 \rightarrow \theta$,

(iii) $\theta_i^j \notin X$, $i, j = 0, 1$.

Proof. We may assume that if $\varphi \in X$ and $PA \vdash \varphi \rightarrow \psi$, then $\psi \in X$. Let $\theta := \forall x \gamma(x)$, where $\gamma(x)$ is PR. Let R(k,m) be a primitive recursive relation such that $X = \{k: \exists m R(k,m)\}$ and let $\rho(x,y)$ be a PR binumeration of R(k,m). Finally, let θ_0 and θ_1 be such that

(1) $PA \vdash \theta_0 \leftrightarrow \forall y((\rho(\theta_0, y) \vee \neg\gamma(y)) \rightarrow \exists z \leq y \rho(\theta_1, z))$,

(2) $PA \vdash \theta_1 \leftrightarrow \forall z(\rho(\theta_1,z) \rightarrow \exists y<z(\rho(\theta_0,y) \vee \neg\gamma(y)))$.

Then (i) and (ii) follow directly (cf. Lemma 1.3).

Suppose $\theta_0 \in X$ or $\theta_1 \in X$ and let m be the least number such that $R(\theta_0,m)$ or $R(\theta_1,m)$. If $R(\theta_1,m)$, then $\theta_1 \in X$. Also, by (2), and since θ is true, $PA \vdash \neg\theta_1$, a contradiction. It follows that not $R(\theta_1,m)$ and, therefore, $R(\theta_0,m)$. But then $\theta_0 \in X$ and, by (1), $PA \vdash \neg\theta_0$, again a contradiction. Thus, $\theta_0 \notin X$ and $\theta_1 \notin X$.

Finally, if $\neg\theta_i \in X$, then, by (i), $\theta_{1-i} \in X$. It follows that $\neg\theta_0 \notin X$ and $\neg\theta_1 \notin X$. ■

Proof of Theorem 2. Let $A \in a$, $B \in b$, $D_i \in d_i$. By Orey's compactness theorem (Theorem 6.5) there are sentences ψ, χ such that $B \vdash \psi$, $\psi \not\leq A$, $D_0 \vdash \chi$, $\chi \not\leq A$. By Theorem 6.6, there is a Π_1 sentence π such that $B \vdash \pi$, $A \nvdash \pi$, and $D_1 \nvdash \pi$. Let

$X_0 = \{\varphi: A \vdash \varphi \vee \pi\}$, $X_1 = \{\varphi: \psi \leq A + \neg\varphi\}$,

$X_2 = \{\varphi: \chi \leq A + \neg\varphi\}$, $X_3 = \{\varphi: D_1 \vdash \varphi \vee \pi\}$.

Let $X = X_0 \cup X_1 \cup X_2 \cup X_3$. Then X is r.e. (cf. Lemma 6.5). It is also easy to verify that X is monoconsistent with PA. By Lemma 5, there are then Π_1 sentences θ_0, θ_1 such that

(1) $PA \vdash \theta_0 \vee \theta_1$,

(2) $PA \vdash \theta_0 \wedge \theta_1 \rightarrow Con_B$,

(3) $\theta_i^j \notin X$, $i, j = 0, 1$.

Let $e_i = d(A+\theta_i)$, $i = 0, 1$. Then $a \leq e_i$. $b \not\leq e_i$, since $\neg\theta_i \notin X_1$. $d_0 \not\leq e_i$, since $\neg\theta_i \notin X_2$. Let $c_i = e_i \cap b$. Then $c_i < b$ and $d_0 \not\leq c_i$. If $c_i \leq a$, then, since θ_i is Π_1, $A \vdash \theta_i \vee \pi$, contradicting the fact that $\theta_i \notin X_0$. Thus, $c_i \not\leq a$ and so $a < c_i$. Similarly, $c_i \not\leq d_1$, since $\theta_i \notin X_3$. By (1), $c_0 \cap c_1 = a$. By (2), Theorem 6.4, and Lemma 3, $e_0 \cup e_1 \geq b$, whence, by distributivity, $c_0 \cup c_1 = b \cap (e_0 \cup e_1) = b$. ■

From Lemma 5 we can also derive the following:

Corollary 1. T is not Σ_1–sound iff there are degrees $a_0, a_1 < 1_T$ such that $a_0 \cup a_1 = 1_T$ (and $a_0 \cap a_1 = 0_T$).

Proof. Suppose T is Σ_1–sound. Let $a, b < 1_T$, $A \in a$, $B \in b$. Then $A \uparrow B$ is consistent and so $a \cup b < 1_T$. Next, suppose T is not Σ_1–sound. There is then a true Π_1 sentence θ such that $T \vdash \neg\theta$. Let θ_i be as in Lemma 5 with $X = Th(T)$. Let $a_i = d(T+\theta_i)$. Then $a_i < 1_T$, by Lemma 5 (iii), and $a_0 \cap a_1 = 0_T$, by Lemma 5 (i). Finally, by Lemma 3, $(T + \theta_0)\uparrow(T + \theta_1) \vdash \theta$. Since $T \vdash \neg\theta$, it follows that $(T + \theta_0)\uparrow(T + \theta_1)$ is inconsistent and so $a_0 \cup a_1 = 1_T$. ■

By Corollary 1, if $PA \dashv S$ and S is Σ_1–sound but T is not, then $\mathbf{D}_S$ and $\mathbf{D}_T$ are not isomorphic. But suppose S and T are both Σ_1–sound. It is an open problem if this implies that $\mathbf{D}_S$ and $\mathbf{D}_T$ are isomorphic.

Given that there are $c_0, c_1 > a$ such that $c_0 \cap c_1 = a$, we may ask if any b such that $a < b < 1_T$ *caps to* a in the sense that there is a $c > a$ such that $b \cap c = a$. (Dually, b *cups to* a if there is a $c < a$ such that $b \cup c = a$.) In our next result this question and its dual are answered in the negative. We write $a \ll_\cap b$ to mean that $a < b$ and b does not cap to a. Dually, $a \ll_\cup b$ means that $a < b$ and a does not cup to b.

Theorem 3. (a) Suppose $0_T < a \not\leq c$. There is a b such that $0_T < b \ll_{\cup} a$ and $b \not\leq c$.
(b) Suppose $c \not\leq$) $a < 1_T$. There is a b such that $a \ll_{\cap} b < 1_T$ and $c \not\leq b$.

Proof. (a) Let $A \in a$ and $C \in c$. There is a Π_1 sentence θ such that $A \vdash \theta$ and $C \dashv \theta$. Let $X = \mathrm{Th}(C + \neg\theta)$. X is r.e. and (mono)consistent with $T + \neg\theta$. By Theorem 5.2, there is a Π_1 sentence $\psi \notin X$ such that ψ is Σ_1–conservative over $T + \neg\theta$. Let $B = T + \psi \vee \theta$ and $b = d(B)$. Then $0_T < b \not\leq c$ and $b \leq a$. Suppose $b \cup d = a$. Let $D \in d$. Then, by Lemma 6.2, there is an m such that $T + \psi + \mathrm{Con}_{D\restriction m} \vdash \theta$ and so $T + \neg\theta + \psi \vdash \neg\mathrm{Con}_{D\restriction m}$. Since ψ is Σ_1–conservative over $T + \neg\theta$, it follows that $T + \neg\theta \vdash \neg\mathrm{Con}_{D\restriction m}$ and so $D \vdash \theta$. Thus, $d \geq b$ and so $d = b \cup d = a$. ♦

The proof of the following lemma from Lemma 6.2, Theorem 6.6, and Lemma 2 is straightforward.

Lemma 6. The following conditions are equivalent:
(i) $A \downarrow B \leq C$.
(ii) $A \leq C + \neg\mathrm{Con}_{B\restriction k}$ for every k.
(iii) $A \leq C + \neg\theta$ for every Π_1 sentence θ such that $B \vdash \theta$.

Let σ be any Σ_1 sentence. By Corollary 6.3, the degree $d(A+\sigma)$ is uniquely determined by σ and $d(A)$. Thus, we may denote the former by $d(A) + \sigma$. A degree of the form $a + \sigma$ will be called a Σ_1*–extension* of a. If X is an r.e. set of Σ_1 sentences, then, by Theorem 6.11 (b), $d(A+X)$ is a Σ_1–extension of $d(A)$.

Lemma 7. The following conditions are equivalent:
(i) $a \ll_{\cap} b$.
(ii) $a < b$ and for every Σ_1–extension c of a, if $b \leq c$, then $c = 1_T$.

Proof. Suppose (i) holds. Let $A \in a$ and $B \in b$. Let σ be Σ_1 and such that $b \leq a + \sigma$. Then $B\downarrow(A + \neg\sigma) \leq (A + \sigma)\downarrow(A + \neg\sigma) \leq A$. Hence, by assumption, $A + \neg\sigma \leq A$, whence $A \vdash \neg\sigma$ and so $a + \sigma = 1_T$. Thus, (ii) holds.

Next suppose (ii) holds. Let c be such that $b \cap c = a$. Let $A \in a$, $B \in b$, $C \in c$. Let θ be any Π_1 sentence provable in C. It suffices to show that $A \vdash \theta$. By Lemma 6, $B \leq A + \neg\theta$. But then, by assumption, $A \vdash \theta$, as desired. ■

Lemma 8. If π is Π_1, $A \leq B + \pi$, and $\neg\pi$ is Π_1–conservative over A, then $d(A) \ll_{\cap} d(B + \pi)$.

Proof. Suppose $B + \pi \leq A + \sigma$. Then, by Lemma 6.1, $A + \sigma \vdash \pi$, whence $A + \neg\pi \vdash \neg\sigma$ and so $A \vdash \neg\sigma$, in other words, $A + \sigma$ is inconsistent. Now use Lemma 7. ■

Proof of Theorem 3 (b). Let $A \in a$, $C \in c$. By Theorem 6.5, there is a sentence ψ such that $C \vdash \psi \not\leq A$. Let $X = \{\varphi\colon \psi \leq A + \neg\varphi\}$. Then, by Theorem 5.2, there is a Σ_1 sentence $\chi \notin X$ such that χ is Π_1–conservative over A. Let $B = A + \neg\chi$ and $b = d(B)$. Then $c \not\leq$

$b < 1_T$. Finally, by Lemma 8, $a \ll_\cap b$. ■

From Theorem 6.4 and Lemma 8, and Theorem 5.1, we get the following (compare Theorem 6.2):

Corollary 2. $d(A) \ll_\cap d(T+Con_A)$.

Theorem 3 (a) leads to the problem if for any $a < 1_T$, there is a b such that $a \ll_\cup b < 1_T$. (The dual of this is false: if $0_T < b < a$ and not $0_T \ll_\cap a$, then not $b \ll_\cap a$.) We now show that the answer is negative.

a is a *cupping* degree if $a < 1_T$ and a cups to every b such that $a \leq b < 1_T$. Let

$$CON_T = \{a < 1_T: a = d(T+Con_\tau) \text{ for some PR binumeration } \tau(x) \text{ of } T\}.$$

By Corollary 2.4, $CON_T \neq \emptyset$.

Theorem 4. Every member of CON_T is a cupping degree.

Proof. Suppose $a = d(T+Con_\tau) < 1_T$, where $\tau(x)$ is a PR binumeration of T. Let b be any degree such that $a \leq b < 1_T$. Let $B \in b$. We want to define a degree d such that $d \not\geq a$ and $a \cup d \geq b$. The obvious way to try is to let $d = d(T+\theta)$, where

$$\theta := \forall u(Prf_B(\bot,u) \to \exists z{<}u Prf_\tau(\bot,z)).$$

But it seems difficult to prove, and may not even be true, that $d \not\geq a$ so we have to proceed in a somewhat different way.

Let φ be such that

$$PA \vdash \varphi \leftrightarrow \forall z(Prf_\tau(\varphi,z) \to \exists u{\leq}z Prf_B(\bot,u)),$$

and let

$$\psi := \forall u(Prf_B(\bot,u) \to \exists z{<}u Prf_\tau(\varphi,z)).$$

Then

(1) $T \nvdash \varphi$,

(2) $PA \vdash \varphi \vee \psi$,

(3) $PA \vdash \varphi \wedge \psi \to Con_B$.

Clearly, $PA \vdash \neg\varphi \to Pr_\tau(\varphi)$. Since $\neg\varphi$ is Σ_1, we also have, by provable Σ_1–completeness, $PA \vdash \neg\varphi \to Pr_\tau(\neg\varphi)$. Thus,

(4) $PA \vdash Con_\tau \to \varphi$.

Let $d = d(T+\psi)$. Then, since ψ and Con_τ are Π_1, it follows from (3), (4), Lemma 3, and Theorem 6.4 that $a \cup d \geq b$. Suppose $a \leq d$. Then $T + \psi \vdash Con_\tau$. But then, by (2) and (4), $T \vdash \varphi$, contradicting (1). Thus, $a \nleq d$. Let $c = d \cap b$. Then $c < b$. Finally, $a \cup c = (a \cup d) \cap (a \cup b) = b$. Thus, a is cupping. ■

Theorem 14′, below, is an improvement of Theorem 4.

A set G of degrees is *cofinal* in $\mathbf{D}_T$ if for every degree $a < 1_T$, there is a degree $b \in G$ such that $a \leq b < 1_T$.

Lemma 9. CON_T is cofinal in $\mathbf{D}_T$.

Proof. Suppose $b < 1_T$. By Corollary 2.4, even if T is not Σ_1–sound, there is a PR binumeration $\beta(x)$ of a theory of degree b such that T + Con_β is consistent. By Theorem 6.4, $b \le d(T+Con_\beta)$. By Theorem 2.8 (b), there is a PR binumeration $\tau(x)$ of T such that $T \vdash Con_\tau \leftrightarrow Con_\beta$. Let $a = d(T+Con_\tau)$. Then $b \le a \in CON_T$. ■

Let P be a property of degrees. We shall say that there are *arbitrarily large* degrees having property P if the set of degrees having P is cofinal in $\mathbf{D}_T$. Every *sufficiently large* degree has P if for every degree $a < 1_T$, there is a b such that $a \le b < 1_T$ and every degree c such that $b \le c < 1_T$ has P.

If a is cupping and $a \le b$, b is cupping. Thus, from Theorem 4 and Lemma 9 we get:

Corollary 3. Every sufficiently large degree is a cupping degree.

By Corollary 1, if T is Σ_1–sound, no degree, except 0_T and 1_T, has a complement whereas if T is not Σ_1–sound, some do. Also, of course, if $0_T \ll_\cap a < 1_T$, then a has no complement. But, even if a has no complement, it may still have a pseudocomplement (p.c.). For example, if $0_T \ll_\cap a$, 0_T is the p.c. of a. By Lemma 6, if π is Π_1, then $d(T+\neg\pi)$ is the p.c. of $d(T+\pi)$. On the other hand we have the following:

Theorem 5. There is a degree which has no p.c.

The proof of this (and more) will be given in § 3 (Theorem 17).

In addition to the usual (finitary) distributive laws, $\mathbf{D}_T$ also satisfies the following infinitary distributive laws. Let G be a set of degrees. $\bigcup G$ ($\bigcap G$) is then the l.u.b. (g.l.b.) of G, if it exists.

Theorem 6. (a) If $\bigcup G$ exists, then $\bigcup G \cap b = \bigcup\{a \cap b: a \in G\}$.
(b) If $\bigcap G$ exists, then $\bigcap G \cup b = \bigcap\{a \cup b: a \in G\}$.

By Theorem 6 (a), if a has no p.c., then $\{b: b \cap a = 0_T\}$ has no l.u.b. In Lemma 23, below, we give a nontrivial example of a set G which has no g.l.b.

To prove Theorem 6 (b) we need the following:

Lemma 10. The following conditions are equivalent:
(i) $A{\uparrow}B \ge C$.
(ii) For all (Σ_1) sentences χ and all m, if $A^T + \neg Con_{C\restriction m} \dashv_{\Sigma_1} T + \chi$, then $B \vdash \neg\chi$.

Proof. Suppose (i) holds. Let χ and m be such that $A^T + \neg Con_{C\restriction m} \dashv_{\Sigma_1} T + \chi$. There is a k such that $A^T + Con_{B\restriction k} \vdash Con_{C\restriction m}$. It follows that $T + \chi \vdash \neg Con_{B\restriction k}$, whence $B \vdash \neg\chi$. Thus, (ii) holds.

To prove that (ii) implies (i), suppose (i) fails, i.e. $A{\uparrow}B \not\ge C$. There is then an m such that for every k, $A^T + Con_{B\restriction k} \nvdash Con_{C\restriction m}$. But then, by Theorem 4.3, there is a

Σ_1 sentence χ such that $A^T + \neg Con_{C|m} \dashv_{\Sigma_1} T + \chi$ and $T + \chi \nvdash \neg Con_{B|k}$ for every k. Since $\neg\chi$ is Π_1, it follows, by Lemma 2, that $B \nvdash \neg\chi$. Thus, (ii) is false, as desired. ■

Proof of Theorem 6. (a) Let $c = \bigcup G$. Clearly $c \cap b$ is an upper bound of $\{a \cap b: a \in G\}$. Suppose d is any upper bound of $\{a \cap b: a \in G\}$. It is then sufficient to show that $c \cap b \leq d$. Let $B \in b$, $C \in c$, $D \in d$. Then $A{\downarrow}B \leq D$ for every A such that $d(A) \in G$. But then, by Lemma 6, $A \leq D + \neg Con_{B|k}$ for every such A and every k. It follows that for every k, $C \leq D + \neg Con_{B|k}$ for every k, whence, by Lemma 6, $C{\downarrow}B \leq D$ and so $c \cap b \leq d$. ♦

(b) Let $c = \bigcap G$. Clearly $c \cup b$ is a lower bound of $\{a \cup b: a \in G\}$. Suppose d is any lower bound of $\{a \cup b: a \in G\}$. It is then sufficient to show that $d \leq c \cup b$. Again let $B \in b$ etc. Then $D \leq A{\uparrow}B$ for every A such that $d(A) \in G$. But then, by Lemma 10, for every such A, every m and every Σ_1 sentence χ, if $B^T + \neg Con_{D|m} \dashv_{\Sigma_1} T + \chi$, then $A \vdash \neg\chi$. It follows that for every m and every Σ_1 sentence χ, if $B^T + \neg Con_{D|m} \dashv_{\Sigma_1} T + \chi$, then $C \vdash \neg\chi$. Hence, again by Lemma 10, $D \leq C{\uparrow}B$ and so $d \leq c \cup b$. ■

Suppose $a \leq b$. Let [a,b] be the *interval* $\{c: a \leq c \leq b\}$. (We also write [a,b) for $\{c: a \leq c < b\}$ etc.) A natural (global) question concerning $\mathbf{D}_T$ is if all intervals [a,b], where $a < b < 1_T$, are isomorphic (in the obvious sense). The answer is negative.

If $c < d$, let $[d,c] = ([c,d], \geq)$. Another natural question is, under what conditions [a,b] is isomorphic to [d,c], where $a < b$ and $c < d$.

Theorem 7. (a) There are degrees $a, b \in (0_T, 1_T)$ such that the intervals $[0_T,a]$ and $[0_T,b]$ are not isomorphic.

(b) Suppose $a < b$ and $c < d$. Then [a,b] is not isomorphic to [d,c].

Theorem 7 (a) follows at once from our next two lemmas.

The interval $[a_0,a_1]$, where $a_0 \leq a_1$, is said to satisfy the *reduction principle* if for any $b_0, b_1 \in [a_0,a_1]$, if $b_0 \cup b_1 = a_1$, there are $c_i \leq b_i$, i = 0, 1, such that $c_0 \cap c_1 = a_0$ and $c_0 \cup c_1 = a_1$. A degree a is *r.p.* if $[0_T,a]$ satisfies the reduction principle.

Lemma 11. If $a = d(T+\theta)$, where θ is Π_1, then a is r.p.

Proof. Let b_0, b_1 be such that $b_0 \cup b_1 = a$. There are then Π_1 sentences ψ_0, ψ_1 such that $d(T+\psi_i) \leq b_i$ and $T + \psi_0 \wedge \psi_1 \vdash \theta$. By Lemma 5.5, there are Π_1 sentences θ_0, θ_1 such that $T \vdash \theta_0 \vee \theta_1$, $T \vdash \psi_i \rightarrow \theta_i$, i = 0, 1, $T \vdash \theta_0 \wedge \theta_1 \rightarrow \psi_0 \wedge \psi_1$. Let $c_i = d(T+\theta_i)$, i = 0, 1. Then $c_i \leq b_i$, $c_0 \cap c_1 = 0_T$, and, by Lemma 3, $c_0 \cup c_1 = b_0 \cup b_1 = a$. ■

Lemma 12. There is a degree $a < 1_T$ which is not r.p.

Proof. Let π be a Π_1 sentence undecidable in T. In case T is not Σ_1-sound we also need to assume that π is Σ_1-conservative over T (cf. Theorem 5.2). We now effectively define r.e. sets X_k of Π_1 sentences such that

(1) $T + X_k + \pi^i$ is consistent, i = 0, 1,

(2) $X_k \subseteq X_{k+1}$,

(3) $T + X_k + \pi \nvdash X_{k+1}$,

(4) $T + X_k + \neg\pi \leq T + X_{k+1}$.

Let $X_0 = \emptyset$. Then (1) holds for k = 0. Now suppose (1) holds for k = n. By (the proof of) Lemma 2.1, we can effectively find a Π_1 sentence ψ_n such that

(5) $T + X_n + \pi^i + \neg\psi_n^i$ is consistent, i = 0, 1.

Let $T_n =_{df} T + X_n + \neg\pi + \psi_n$. It follows that

(6) there is no Π_1 sentence θ such that $T_n \vdash \theta$ and $T + \theta \vdash \neg\pi$.

For suppose $T + \theta \vdash \neg\pi$. Then $T + \pi \vdash \neg\theta$ and so $T \vdash \neg\theta$, whence, by (5), $T_n \nvdash \theta$.

Let $X_{n+1} = Th(T_n) \cap \Pi_1$. Let k = n+1. Then (1) is satisfied for i = 1 and, by (6), (1) is satisfied for i = 0. Moreover (2) and (4) hold for k = n. Finally, $T + X_{n+1} \vdash \psi_n$ and so, by (5), (3) holds for k = n.

Let $a_0 = d(T+\bigcup\{X_k:k\in N\})$, $a_1 = d(T+\pi)$, and $a = a_0 \cup a_1$. Since $a_0 < 1_T$ and π is Σ_1–conservative over T, we have $a < 1_T$. We now show that a is not r.p. Let b_0 and b_1 be such that $b_0 \leq a_0$, $b_1 \leq a_1$, $b_0 \cap b_1 = 0_T$, and $b_0 \cup b_1 \geq a_1$. It is then sufficient to show that $b_0 \cup b_1 \ngeq a_0$.

Let $\theta_{i,k}$ be Π_1 sentences such that $b_i = d(T+\{\theta_{i,k}:k\in N\})$, i = 0, 1. We may assume that $T + \theta_{i,k+1} \vdash \theta_{i,k}$ for i = 0, 1 and all k. By Lemma 3, there is then an m such that $T + \theta_{0,m} \wedge \theta_{1,m} \vdash \pi$. $d(T+\theta_{0,m}) \leq b_0 \leq a_0$. Thus, by (2), there is an n such that $T + \theta_{0,m} \leq T + X_n$. Since $b_0 \cap b_1 = 0_T$, for every k, $T + \theta_{0,k} \vee \pi \leq T + \theta_{0,k} \vee (\theta_{0,m} \wedge \theta_{1,m}) \leq T + \theta_{0,m}$. It follows that $T + \theta_{0,k} \vee \pi \leq T + X_n$, whence $T + \theta_{0,k} \leq T + X_n + \neg\pi$ (cf. Corollary 6.3) and so, by (4), $T + \theta_{0,k} \leq T + X_{n+1}$. But this holds for all k, whence $b_0 \leq d(T+X_{n+1})$. Next, by (3), $b_0 \cup b_1 \leq b_0 \cup a_1 \leq d(T+X_{n+1}+\pi) \ngeq a_0$. It follows that $b_0 \cup b_1 \ngeq a_0$ and so the proof is complete. ■

Proof of Theorem 7 (b). Let $A \in a$ and let π be a Π_1 sentence such that $A \nvdash \pi$. Then $[a,d(A^T+\pi)]$ satisfies the reduction principle (see the proof of Lemma 11). It follows that in [a,b] there is a degree e > a such that [a,e] satisfies the reduction principle. Thus, it is sufficient to show that the dual of the reduction principle is false in [c,d] whenever c < d.

Let $C \in c$ and $D \in d$, and let π be such that $C \nvdash \pi$ and $D \vdash \pi$. Then, by Theorem 5.5 (b) with $X = Th(C^T + \neg\pi)$, there are Σ_1 sentences σ_i such that $C^T + \neg\sigma_i \equiv C^T + \sigma_{1-i}$, i = 0, 1, and $C^T + \neg\sigma_0 \wedge \neg\sigma_1 \nvdash \pi$. Let $c_i = d(C^T+\sigma_i) = d(C^T+\neg\sigma_{1-i})$. Then $c_0 \cap c_1 = c$ and $c_0 \cup c_1 \ngeq d$. Let $d_i = c_i \cap d$. Then $d_0 \cap d_1 = c$ and $d_0 \cup d_1 < d$. Suppose now $d_i \leq e_i \leq d$, i = 0, 1, and $e_0 \cap e_1 = c$. We have to show that $e_0 \cup e_1 < d$. Let $E_0 \in e_0$. $c_1 \cap e_0 = c_1 \cap d \cap e_0 = d_1 \cap e_0 \leq e_1 \cap e_0 = c$. It follows that $(C^T + \neg\sigma_0){\downarrow}E_0 \leq C^T$. But then, by Lemma 6, for every Π_1 sentence θ, if $E_0 \vdash \theta$, then $C^T + \neg\sigma_0 \leq C^T + \neg\theta$, whence $C^T+\sigma_0 \vdash \theta$. It follows that $e_0 \leq c_0$ and so $e_0 = d_0$. Similarly, $e_1 = d_1$. Hence $e_0 \cup e_1 = d_0 \cup d_1 < d$ and the proof is complete. ■

Theorem 7 (a) leads to the problem of determining the exact number of nonisomorphic intervals of D_T. This problem remains open.

We have actually proved more than is stated in Theorem 7. Let $L = \{\leq, \cap, \cup, 0, 1\}$ be the language of the theory of lattices with a bottom and a top element.

Formulated in L, the reduction principle is an $\forall\exists$ sentence. Hence, by the proof of Theorem 7 (a), there are degrees a, b$\in(0_T,1_T)$ and an $\forall\exists$ sentence of L which holds in $[0_T,a]$ but not in $[0_T,b]$. (This is, so far, the only known way of proving that two intervals of $\mathbf{D}_T$ are nontrivially nonisomorphic.) Similarly, the proof of Theorem 7 (b) shows that if a < b and c < d, there is an $\exists\forall\exists$ sentence which is true in [a,b] and false in [d,c].

§2. A classification of degrees. When there is no risk of confusion we shall use φ and X in place of T + φ and T + X. Thus, d(φ) is d(T+φ), X < φ means that T + X < T + φ, $\varphi \equiv \psi$ that T + φ $\equiv$ T + ψ, etc. We also write a $\ll$b to mean that a $\ll_{\Pi}$ b. A $\ll$B means that d(A) $\ll$ d(B). σ, σ_0, etc. will be used to denote Σ_1 sentences and π, π_0, etc. to denote Π_1 sentences.

A degree a is Φ if there is a Φ sentence φ such that a = d(φ). By the proof of Theorem 6.11 (a), it is clear that every degree is Π_2 and Σ_2. This can be somewhat improved:

Theorem 8. Every degree is Δ_2.

Proof. Let a be any degree. There is a primitive recursive set X of Π_1 sentences such that a = d(X). Let $\xi(x)$ be a PR binumeration of X and let φ be such that

$$\text{PA}\vdash \varphi \leftrightarrow \forall z([\Pi_1](\varphi,z) \rightarrow (\xi(z) \rightarrow \text{Tr}_{\Pi_1}(z))).$$

Then φ is Π_2 and T + φ is a Π_1–conservative extension of T + X (cf. the proof of Theorem 5.4 (a)). It follows that a = d(φ). Using Lemma 5.1 (i) and Lemma 1.3 (v) (applied to $\neg\varphi$), we get:

$$\text{PA}\vdash \varphi \leftrightarrow \forall z(\xi(z) \rightarrow \text{Tr}_{\Pi_1}(z)) \vee \exists z(\neg[\Pi_1](\varphi,z) \wedge \forall u{<}z(\xi(u) \rightarrow \text{Tr}_{\Pi_1}(u))).$$

Thus, φ is Δ_2. ■

By Theorem 8, in terms of the arithmetical hierarchy, the only interesting (proper) subsets of D_T are the sets of B_1 degrees, Σ_1 degrees, Π_1 degrees, and degrees which are both Σ_1 and Π_1. (If T is not Σ_1–sound, there are also Δ_1^T degrees other than 0_T and 1_T; see e.g. the proof of Corollary 1.) The object of the rest of this § is simply to show that these sets are different and that there is a non–B_1 degree. More detailed information about the Σ_1 and the Π_1 degrees will be given in the next §.

Our next lemma is a restatement of Theorem 6.11 (b).

Lemma 13. If X is an r.e. set of Σ_1 sentences, then d(X) is Σ_1.

The following lemma is occasionally useful.

Lemma 14. There exist a (primitive) recursive sequence $\langle\sigma_k\rangle_{k<\omega}$ and a sentence σ such that (i) T + $\sigma_{k+1}\vdash \sigma_k$, for all k, (ii) $\sigma_k < \sigma_{k+1}$, for all k, (iii) $\sigma \equiv \{\sigma_k\text{: } k\in N\}$.

This follows at once from (the proof of) Lemma 2.1 (applied to the sets $\{\varphi: Q + \varphi \leq T + \sigma_k\}$; $\sigma_0 := 0 = 0$) and Lemma 13.

Theorem 9. (a) There is a Π_1 degree which isn't Σ_1.
(b) There is a Σ_1 degree which isn't Π_1.
(c) There is a degree other than 0_T and 1_T which is both Σ_1 and Π_1.
(d) There is a B_1 degree which is neither Σ_1 nor Π_1.
(e) There is a degree which isn't B_1.

Proof. (a) Let π be such that $\neg\pi$ is Π_1–conservative over T and $T \nvdash \neg\pi$. Then, by Lemma 8, $0_T << d(\pi)$ and so, by Lemma 7, $d(\pi)$ is not Σ_1. ♦

(b) Let $<\sigma_k>_{k<\omega}$ and σ be as in Lemma 14. Suppose $d(\sigma)$ is Π_1 and let π be such that $\sigma \equiv \pi$. Then $\pi \equiv \{\sigma_k: k \in N\}$ and so, by Lemma 14 (i), there is an m such that $T + \sigma_m \vdash \pi$. But then $\{\sigma_k: k \in N\} \leq \sigma_m$, contradicting Lemma 14 (ii). Thus, $d(\sigma)$ isn't Π_1. ♦

(d) The easiest way to prove this is to define π as in the proof of (a) and then σ as in the proof of (b), except that T is replaced by $T + \pi$. Then $d(\pi \wedge \sigma)$ is neither Σ_1 nor Π_1. Details are left to the reader. ♦

Theorem 9 (c) will be derived from the following lemma, which will also be used later.

Lemma 15. There are Π_1 sentences θ_i, i = 0, 1, such that
(i) $T \nvdash \theta_i$,
(ii) $T \vdash \theta_0 \vee \theta_1$,
(iii) $T \vdash \theta_0 \wedge \theta_1 \rightarrow Con_T$,
(iv) $T + Con_T \vdash \neg Pr_T(\theta_i)$,
(v) $T + Con_T \vdash \theta_i$,
(vi) $\theta_i \equiv \neg\theta_{1-i}$.

Proof. Let θ_i, i = 0, 1, be such that

$$PA \vdash \theta_i \leftrightarrow \forall z(Prf_T(\theta_i,z) \rightarrow \exists u<z+i Prf_T(\theta_{1-i},u)).$$

A standard argument proves (i). Formalizing this argument we get (iv). (ii) and (iii) are immediate. (v) follows from (iv). By (ii),

(1) $PA \vdash Pr_T(\neg\theta_i) \rightarrow Pr_T(\theta_{1-i})$.

Also,

(2) $PA \vdash \neg\theta_{1-i} \leftrightarrow Pr_T(\theta_{1-i}) \wedge \theta_i$.

By Theorem 6.8, $\theta_i \wedge Pr_T(\neg\theta_i) \leq \theta_i$. By (1), it follows that $\theta_i \wedge Pr_T(\theta_{1-i}) \leq \theta_i$ and so, by (2), $\neg\theta_{1-i} \leq \theta_i$. But then, by (ii), $\theta_i \equiv \neg\theta_{1-i}$, i.e. (v) holds. ■

Proof of Theorem 9 (c). Let θ_i be as in Lemma 15. Let $a = d(\theta_0)$. Then a is Π_1 and, by Lemma 15 (vi), a is Σ_1. By Lemma 15 (i), $a > 0_T$. Finally, by Lemma 15 (i) and (ii), $a < 1_T$. ♦

To prove Theorem 9 (e) we need the following:

Lemma 16. Suppose X is r.e. and for every k, $X \upharpoonright k \ll X$. Then if φ is B_1 and $X \leq \varphi$, then $X \ll \varphi$. Thus, *a fortiori* d(X) is not B_1.

Proof. φ can be written in the form $(\pi_0 \wedge \sigma_0) \vee ... \vee (\pi_n \wedge \sigma_n)$. It is easily checked that for any degrees a, b, c, if $a \ll b$ and $a \ll c$, then $a \ll b \cap c$. Thus, it is sufficient to show that if $X \leq \pi \wedge \sigma$, then $X \ll \pi \wedge \sigma$. Let χ be a Σ_1 sentence such that $\pi \wedge \sigma \leq X + \chi$. Then, by Lemma 7, it suffices to show that $T + X \vdash \neg\chi$. By assumption, there is a k such that $T + X \upharpoonright k + \chi \vdash \pi$. Hence $T + \pi \wedge \sigma \dashv T + X \upharpoonright k + (\chi \wedge \sigma)$ and so $X \leq X \upharpoonright k + (\chi \wedge \sigma)$. But then, since $X \upharpoonright k \ll X$, by Lemma 7, $T + X \vdash \neg(\chi \wedge \sigma)$. But $X \leq \pi \wedge \sigma$. It follows that $T + \pi \wedge \sigma \vdash \neg\chi$, whence $T + X + \chi \vdash \neg\chi$ and so $T + X \vdash \neg\chi$, as was to be shown. ■

Proof of Theorem 9 (e). By (the proof of) Theorem 5.2, we can effectively construct sentences π_n such that $\neg\pi_n$ is Π_1-conservative over but not provable in $T + \{\pi_k: k < n\}$. Let $X = \{\pi_k: k \in N\}$. Then, by Lemma 8, $X \upharpoonright k \ll X$ for all k. So, by Lemma 16, d(X) is not B_1. ■

§3. Σ_1 and Π_1 degrees. This § is devoted to a discussion of the Σ_1 and Π_1 degrees and the relations between them.

The l.u.b. of two Π_1 degrees is Π_1 and the g.l.b. of two Π_1 (Σ_1) degrees is Π_1 (Σ_1).

Let us say that a is *high* if $a \gg 0_T$, *low* otherwise. Thus, by Lemma 7, a is low iff there is a Σ_1 degree b such that $a \leq b < 1_T$. By Lemma 8, if $\neg\pi$ is Π_1–conservative over T, d(π) is high. By Corollary 2, every member of CON_T is high.

The following lemma is sometimes useful.

Lemma 17. Suppose a is high. Then for any b, $[a \cap b, b)$ contains no Σ_1 degree; in fact, if c is Σ_1 and $a \cap b \leq c$, then $b \leq c$.

Proof. Let $A \in a$, $B \in b$, and $c = d(\sigma)$. Suppose $A \downarrow B \leq T + \sigma$. Then, by Lemma 6, $A \leq T + \sigma \wedge \neg Con_{B \upharpoonright k}$ for every k. Since a is high, it follows that $T + \sigma \vdash Con_{B \upharpoonright k}$, for every k, and so $B \leq T + \sigma$. ■

Theorem 10. (a) The set of Π_1 degrees is cofinal in $\mathbf{D}_T$.
(b) The set of Σ_1 degrees is not cofinal in $\mathbf{D}_T$; in fact, for every degree $a > 0_T$, there is a degree $b < a$ such that $[b,a)$ contains no Σ_1 degree.
(c) There is a low Π_1 degree which is not Σ_1.

Proof. (a) Since all members of CON_T are Π_1, this follows from Lemma 9. ♦

(b) By Theorem 3 (b), there is a high degree c such $a \nleq c$. Let $b = c \cap a$. Then, by Lemma 17, b is as desired. ♦

(c) Let a be any low Π_1 degree $> 0_T$. By Theorem 3 (b), there is a high Π_1 degree

$c \not\geq a$. Let $b = a \cap c$. Then b is low and and Π_1. Finally, by Lemma 17, b is not Σ_1. ■

Using Theorem 2 we can now prove the following corollary.

Corollary 4. (a) Suppose a is not Π_1 and $a \in (b,c)$. There are then degrees b′, c′ such that $a \in (b',c') \subseteq (b,c)$ and $[b',c']$ contains no Π_1 degree.

(b) Suppose a is not Σ_1 and $a \in (b,c)$. There are then degrees b′, c′ such that $a \in (b',c') \subseteq (b,c)$ and $[b',c']$ contains no Σ_1 degree.

Proof. (a) By Theorem 2, there are degrees b_0, b_1 such that $b \leq b_i < a$, $i = 0, 1$, and $b_0 \cup b_1 = a$. Either $[b_0,a]$ or $[b_1,a]$ contains no Π_1 degree. If not, then a would be the l.u.b. of two Π_1 degrees and therefore Π_1. Suppose $[b_i,a]$ contains no Π_1 degree and let $b' = b_i$. By Theorem 2, there are degrees c_0, c_1 such that $a < c_i \leq c$, $i = 0, 1$, and $c_0 \cap c_1 = a$. Either $[b',c_0]$ or $[b',c_1]$ contains no Π_1 degree. For suppose $d_i \in [b',c_i]$ and d_i is Π_1, $i = 0, 1$. Then $d_0 \cap d_1 \in [b',a]$ and $d_0 \cap d_1$ is Π_1, a contradiction. Suppose $[b',c_j]$ contains no Π_1 degree and let $c' = c_j$. Then b′ and c′ are as desired. ♦

(b) By a slight modification of the proof of Theorem 10 (b), which we leave to the reader, there is a degree b′ such that $b \leq b' < a$ and $[b',a]$ contains no Σ_1 degree. The rest of the proof is the same as the proof of (a). ■

Theorem 10 (b) leads to the question if there are arbitrarily small Σ_1 degrees. By our next result, the answer is affirmative; later we shall prove a stronger result (Theorem 15).

Theorem 11. If $0_T < a$, then there is a Σ_1 and Π_1 degree $b \in (0_T,a)$.

To prove this we need a lemma on partial conservativity.

Lemma 18. Let X be an r.e. set. There is then a PR formula $\eta(y,x,z)$ such that for all k and θ,

(i) if $k \in X$, then $T + \theta \vdash \neg\exists z\eta(\theta,k,z)$,

(ii) if $k \notin X$, then $\exists z\eta(\theta,k,z)$ is Π_1-conservative over $T + \theta$.

The proof of Lemma 18 is similar to the proof of Lemma 5.3 (for $\Gamma = \Pi_1$) and is left to the reader.

Proof of Theorem 11. Let $\forall u\delta(u)$, where $\delta(u)$ is PR, be a Π_1 sentence such that $0_T < d(\forall u\delta(u)) < a$. By Lemma 18, there is a PR formula $\gamma(x,z)$ such that

(1) if $T \vdash \varphi$, then $T \vdash \neg\exists z\gamma(\varphi,z)$,

(2) if $T \nvdash \varphi$, then $\exists z\gamma(\varphi,z)$ is Π_1–conservative over $T + \varphi$.

Let θ be such that

(3) $PA \vdash \theta \leftrightarrow \forall u(\neg\delta(u) \to \exists z{<}u\gamma(\theta,z))$,

and let

$$\sigma := \exists z(\gamma(\theta,z) \wedge \forall u{\leq}z\delta(u)).$$

Then

(4) $PA\vdash \sigma \leftrightarrow \exists z\gamma(\theta,z) \wedge \theta$,
(5) $PA + \theta + \neg\exists z\gamma(\theta,z)\vdash \forall u\delta(u)$.
It follows that
(6) $T\nvdash \theta$.
For suppose not. Then, by (1), $T\vdash \neg\exists z\gamma(\theta,z)$ and so, by (5), $T\vdash \forall u\delta(u)$, contrary to the choice of $\delta(u)$.

By (3), $\theta \le \forall u\delta(u)$ and so $d(\theta) < a$. By (6), $0_T < d(\theta)$. Finally, by (4), (6), (2), $\sigma \equiv \theta$. Thus, $b = d(\sigma)$ is as claimed. ■

It is natural to ask if D_T is "generated" by some "small" set of degrees, for example, the set of Σ_1 degrees. We prove two negative results, Theorems 12 and 13, and one partial positive result, Theorem 14 (and 14′).

Let E_T be the set of l.u.b.s of (finite) sets of Σ_1 degrees. Note that E_T is closed under $\cap$. By Lemma 15, $CON_T \subseteq E_T$.

Theorem 12. There is a Π_1 degree not in E_T.

This is an immediate consequence of the following two lemmas.

Lemma 19. If $a\in E_T$, there is a smallest Σ_1 degree $\ge a$.

Proof. Suppose $a = d(\sigma_0) \cup ... \cup d(\sigma_n)$. Then $d(\sigma_0\wedge...\wedge\sigma_n)$ is the smallest Σ_1 degree $\ge d(\sigma_0) \cup...\cup d(\sigma_n)$. This can be seen as follows. Suppose $d(\sigma_0) \cup...\cup d(\sigma_n) \le d(\sigma)$. Let π be such that $T + \sigma_0 \wedge...\wedge \sigma_n\vdash \pi$. Then $T + \sigma_0\vdash \sigma_1 \wedge...\wedge \sigma_n \rightarrow \pi$. Now, $\sigma_1 \wedge...\wedge \sigma_n \rightarrow \pi$ is a Π_1 sentence. It follows that $T + \sigma\vdash \sigma_1 \wedge...\wedge \sigma_n \rightarrow \pi$. But then $T + \sigma_1\vdash \sigma \wedge \sigma_2 \wedge...\wedge \sigma_n \rightarrow \pi$ and so $T + \sigma\vdash \sigma_2 \wedge...\wedge \sigma_n \rightarrow \pi$. Continuing in this way we eventually get $T + \sigma\vdash \pi$, as desired. ■

Lemma 20. There is a Π_1 degree a for which there is no smallest Σ_1 degree $\ge a$.

Proof. Let $<\sigma_k>_{k<\omega}$ and σ be as in Lemma 14. Let $a = d(\neg\sigma)$. Then a is Π_1. Now let χ be any Σ_1 sentence such that $a \le d(\chi)$. Then $T + \chi\vdash\neg\sigma$ and so $T + \sigma\vdash\neg\chi$. It follows that there is a k such that $T + \sigma_k\vdash\neg\chi$ and so
(1) $T + \chi\vdash\neg\sigma_k$.
Since $\sigma_k < \sigma$, there is a sentence π such that $T + \sigma\vdash \pi$ and $T + \sigma_k\nvdash \pi$. It follows that
(2) $T + \neg\pi\vdash\neg\sigma$,
(3) $T + \neg\pi\nvdash\neg\sigma_k$.
But then, by (2), $a \le d(\neg\pi)$ and, by (1) and (3), $\chi \not\le \neg\pi$. Thus, $d(\chi)$ is not the smallest Σ_1 degree $\ge a$. ■

A strengthening of Lemma 20 will be proved later (Lemma 23).

Let F_T be the set of l.u.b.s of (finite) sets of Σ_1 and Π_1 degrees. By Theorem 12, $F_T \supsetneq E_T$.

Theorem 13. $F_T \neq D_T$.

We need the following definition: A $\lll$ B iff A < B and for every set X of Σ_1 sentences, if $B \dashv_{\Pi_1} A + X$, then A + X is inconsistent. (Here X need not be r.e.) We write a $\lll$ b to mean that A $\lll$ B where A$\in$a and B$\in$b. (If A $\equiv$ A′ and B $\equiv$ B′, then A $\lll$ B iff A′ $\lll$ B′.) By Lemma 7, A $\lll$ B implies A $\ll$ B. As will become clear, the converse of this is not true. But if a is Π_1 and high, then $0_T \lll$ a.

Lemma 21. Suppose a$\in F_T$ and for all π, if d(π) $\leq$ a, then d(π) $\ll$ a. Then $0_T \lll$ a.

Proof. By assumption there are $\pi, \sigma_0,...,\sigma_n$ such that a = d(π) $\cup$ d(σ_0) $\cup ... \cup$ d(σ_n). Also d(π) $\ll$ a. Let A$\in$a. Then

(1) $T + \sigma_i \leq A$ for $i \leq n$.

Moreover, $\pi \ll \pi \wedge \sigma_0 \wedge ... \wedge \sigma_n$ and so, by Lemma 7, $T + \pi \vdash \neg\sigma_0 \vee ... \vee \neg\sigma_n$. But $A \vdash \pi$ and so

(2) $A \vdash \neg\sigma_0 \vee ... \vee \neg\sigma_n$.

Let X be any set of Σ_1 sentences such that

(3) $A \dashv_{\Pi_1} T + X$.

Then, by (2), $T + X \vdash \neg\sigma_0 \vee ... \vee \neg\sigma_n$, whence there is a k_0 such that $T + \sigma_0 \vdash \neg\bigwedge X \restriction k_0 \vee \neg\sigma_1 \vee ... \vee \neg\sigma_n$, and so, by (1) and (3), $T + X \vdash \neg\sigma_1 \vee ... \vee \neg\sigma_n$. Continuing in this way we eventually obtain the conclusion that T + X is inconsistent. ∎

Proof of Theorem 13. We effectively construct sentences $\psi_0, \psi_1,...$ such that if $A_n = T + \{\psi_k: k < n\}$ and $A = T + \{\psi_k: k \in N\}$, then

(1) $A_n \ll A_{n+1}$,

(2) not $T \lll A$.

Let a = d(A). Then for all π, if d(π) $\leq$ a, there is an n such that d(π) $\leq$ d(A_n). Also d(A_n) $\ll$ d(A_{n+1}) $\leq$ a and so d(π) $\ll$ a. Thus, by (2) and Lemma 21, a$\notin F_T$.

There is an r.e. relation S(n,k,p,q) such that

(not $T + \psi \ll T + \psi + \varphi$) iff $\exists p \forall q S(\psi,\varphi,p,q)$.

By Lemma 3.2 (b), there are a Π_1 formula σ(x,y,z,u) and a Σ_1 formula σ'(x,y,z,u) such that

(3) if S(n,k,p,q), then $T \vdash \sigma'(n,k,p,q)$,

(4) $T \vdash \sigma'(n,k,p,q) \to \sigma(n,k,p,q)$,

(5) T + Y is consistent where Y = $\{\neg\sigma(n,k,p,q)$: not S(n,k,p,q)$\}$.

Let A_0 = T. Suppose A_n has been defined and set $\theta_n := \bigwedge\{\psi_k: k < n\}$. Then

(6) not $A_n \ll A_n + \varphi$ iff $\exists p \forall q S(\theta_n,\varphi,p,q)$.

By (3) and Lemma 5.2, there is a Σ_1 formula ρ_n(x,y) such that

(7) $A_n \vdash \rho_n(\varphi,p) \to \sigma'(\theta_n,\varphi,p,q)$,

(8) if $\forall q S(\theta_n,\varphi,p,q)$, then $\rho_n(\varphi,p)$ is Π_1-conservative over A_n.

By Theorem 5.4 (b), there is a formula η_n(x) such that

(9) $A_n + \eta_n(\varphi)$ is a Π_1-conservative extension of $A_n + \{\neg\rho_n(\varphi,p): p \in N\}$.

Finally, let ψ_n be such that

(10) $T \vdash \psi_n \leftrightarrow \eta_n(\psi_n)$.

The formulas $\rho_n(x,y)$, $\eta_n(x)$ and the sentences ψ_n can be found effectively in n.

To prove (1) assume it is false. Then, by (6), there is a p such that $\forall q S(\theta_n,\psi_n,p,q)$. But then, by (8), $\rho_n(\psi_n,p)$ is Π_1-conservative over A_n. By (9) and (10), $A_{n+1} \vdash \neg\rho_n(\psi_n,p)$. But, by Lemma 8, this implies that $A_n \ll A_{n+1}$, a contradiction. This proves (1).

Next we prove (2). Let Y be as in (5). Then T + Y is consistent. To prove that $A \dashv_{\Pi_1} T + Y$ we first show that

(11) $A_{n+1} + Y \dashv_{\Pi_1} A_n + Y$.

Suppose $A_{n+1} + Y \vdash \pi$. Then there is a k such that

(12) $A_{n+1} \vdash \neg\bigwedge Y \mid k \vee \pi$.

By (1) and (6), for each p there is a q_p such that

(13) not $S(\theta_n,\psi_n,p,q_p)$.

By (12), (9), and (10),

$$A_n + \{\neg\rho_n(\psi_n,p): p \in N\} \vdash \neg\bigwedge Y \mid k \vee \pi.$$

By (13), (4), (7), $A_n + Y \vdash \neg\rho_n(\psi_n,p)$ for every p. It follows that $A_n + Y \vdash \pi$. This proves (11).

Since (11) holds for all n, it follows that $A \dashv_{\Pi_1} T + Y$. This proves (2) and so the proof is complete. ■

Let G_T be the set of degrees obtained from the Σ_1 and the Π_1 degrees by closing under $\cap$ and $\cup$. It is an open problem if $G_T \neq D_T$.

The degree mentioned in Lemma 20 cannot be arbitrarily large: if a is high, there is a smallest Σ_1 degree $\geq$ a, namely 1_T. Similarly, the degree a defined in the proof of Theorem 13 cannot be arbitrarily large; it is not $>>> 0_T$. This is explained, at least partially, by the following surprising:

Theorem 14. (a) Every sufficiently large degree is the l.u.b. of a Σ_1 degree and a Π_1 degree.

(b) Every sufficiently large degree is the l.u.b. of two Σ_1 degrees.

Proof. We may assume that $d(Con_T) < 1_T$. By Lemma 9, it is sufficient to consider degrees a such that $d(Con_T) \leq a < 1_T$. Let $\pi_n := \forall u \delta_n(u)$, where $\delta_n(u)$ is PR, be Π_1 sentences such that $a = d(\{\pi_n : n \in N\})$. We may assume that for all n,

(1) $T \vdash \pi_0 \rightarrow Con_T$,

(2) $T \vdash \pi_{n+1} \rightarrow \pi_n$.

(a) We define Π_1 sentences φ_n and ψ_n in the following way:

(3) $T \vdash \varphi_n \leftrightarrow \forall z(Prf_T(\bigvee\{\varphi_k : k \leq n\}, z) \rightarrow \exists u \leq z \neg\delta_{n+1}(u))$,

$\psi_n := \forall u(\neg\delta_{n+1}(u) \rightarrow \exists z < u Prf_T(\bigvee\{\varphi_k : k \leq n\}, z))$.

It follows that

(4) $T \vdash \varphi_n \vee \psi_n$,

(5) $T + \varphi_n \wedge \psi_n \vdash \neg Pr_T(\bigvee\{\varphi_k : k \leq n\}) \wedge \pi_{n+1}$,

(6) $T + \pi_{n+1} \vdash \psi_n$,

(7) $T + \neg\varphi_n \vdash Pr_T(\bigvee\{\varphi_k{:}k<n\})$,

(8) $\neg\bigvee\{\varphi_k{:}\ k < n\} \leq \pi_n$.

($\bigvee\{\varphi_k{:}\ k < 0\} := \bot$.) (4), (5), (6) are standard.

Since $\neg\varphi_n$ is Σ_1, we have $T + \neg\varphi_n \vdash Pr_T(\neg\varphi_n)$. Also, by (3),

$$T + \neg\varphi_n \vdash Pr_T(\bigvee\{\varphi_k{:}k\leq n\}).$$

But then (7) follows.

By Theorem 6.4, (8) follows from

(9) $T + \pi_n \vdash \neg Pr_T(\bigvee\{\varphi_k{:}k<n\})$.

By (1), (9) holds for n = 0. Suppose (9) holds for n = m. To show that it holds for n = m+1, we argue in T as follows: "Suppose π_{m+1}. Then, by (6), ψ_m. Also, by (2) and the inductive assumption, $\neg Pr_T(\bigvee\{\varphi_k{:}k<m\})$ and so, by (7), φ_m. Finally, by (5), $\neg Pr_T(\bigvee\{\varphi_k{:}k<m+1\})$, as desired." Thus, (9) holds for n = m+1. This proves (9) and so we have proved (8).

Next we show that for all n,

(10) $T + \bigwedge\{\psi_k{:}\ k < n\} + Con_T \vdash \varphi_n$.

We first show that

(11) $T + Con_T \vdash \varphi_0$,

(12) $T + \psi_n + \varphi_n \vdash \varphi_{n+1}$.

(11) follows from (7) with n = 0. (12) follows from (5) and (7).

Now (10) follows from (11) and (12).

Let $a_0 = d(\{\neg\varphi_k{:}k\in N\})$, $a_1 = d(Con_T)$, $a_2 = d(\{\varphi_k{:}k\in N\})$. Then, by Lemma 13, a_0 is Σ_1. a_1 is Π_1. By (8) and Orey's compactness theorem, $a_0 \leq a$ and, by hypothesis, $a_1 \leq a$. But then $a_0 \cup a_1 \leq a$. By (4) and (5), $a_0 \cup a_2 \geq a$. By (4) and (10), $a_0 \cup a_1 \geq a_2$. It follows that $a_0 \cup a_1 \geq a$ and so $a_0 \cup a_1 = a$, as desired. ♦

(b) Let θ_i, i = 0, 1, be as in Lemma 15. We define Π_1 sentences φ_n and ψ_n in the following way:

(13) $T \vdash \varphi_n \leftrightarrow \forall z(Prf_T(\theta_0 \vee \bigvee\{\varphi_k{:}k\leq n\},z) \rightarrow \exists u{\leq}z\neg\delta_{n+1}(u))$,

$\psi_n := \forall u(\neg\delta_{n+1}(u) \rightarrow \exists z{<}u Prf_T(\theta_0 \vee \bigvee\{\varphi_k{:}k\leq n\},z))$.

It follows that

(14) $T \vdash \varphi_n \vee \psi_n$,

(15) $T + \varphi_n \wedge \psi_n \vdash \neg Pr_T(\theta_0 \vee \bigvee\{\varphi_k{:}k\leq n\}) \wedge \pi_{n+1}$,

(16) $T + \pi_{n+1} \vdash \psi_n$,

(17) $T + \neg\varphi_n \vdash Pr_T(\theta_0 \vee \bigvee\{\varphi_k{:}k<n\})$,

(18) $\neg\theta_0 \wedge \neg\bigvee\{\varphi_k{:}\ k < n\} \leq \pi_n$.

The proofs of (14) – (18) are almost the same as the proofs of (4) – (8).

Next we show that for all n,

(19) $T + \bigwedge\{\psi_k{:}\ k < n\} + \theta_0 \wedge \theta_1 \vdash \varphi_n$.

We first show that

(20) $T + \theta_0 \wedge \theta_1 \vdash \varphi_0$,

(21) $T + \psi_n + \varphi_n \vdash \varphi_{n+1}$.

(20) follows from Lemma 15 (iii) and (iv) and (17) with n = 0. (21) follows from (15) and (17).

Now (19) follows from (20) and (21).

Let $a_0 = d(\neg\theta_0+\{\neg\varphi_k : k\in N\})$, $a_1 = d(\theta_0)$. Then a_0 is Σ_1. By Lemma 15 (vi), a_1 is Σ_1. By (18), $a_0 \leq a$ and, by Lemma 15 (v), $a_1 \leq a$. But then $a_0 \cup a_1 \leq a$. By Lemma 15 (ii), (14), (19), and (15), $a_0 \cup a_1 \geq a$. Thus, $a_0 \cup a_1 = a$, as desired. ■

The proof of Theorem 14 actually yields the following stronger result; Theorem 14′ is also an improvement of Theorem 4.

Theorem 14′. (a) Suppose $a\in CON_T$ and $a \leq b < 1_T$. There is then a Σ_1 degree c such that $a \cup c = b$.

(b) Suppose $a\in CON_T$. There are then degrees a_0, a_1 such that (i) a_0 and a_1 are both Σ_1 and Π_1, (ii) $a_0 \cap a_1 = 0_T$, (iii) $a_0 \cup a_1 = a$, (iv) for every degree $b \geq a$, there is a Σ_1 degree b_i such that $a_i \cup b_i = b$, $i = 0, 1$.

One way to strengthen Theorem 12 would be to show that there is a Π_1 degree $a > 0_T$ such that no Σ_1 degree cups to a. This, however, is not the case:

Theorem 15. For every Π_1 degree $a > 0_T$, there is a Σ_1 (and Π_1) degree which cups to a.

Proof. The following proof is similar to that of Theorem 11. Let π be such that $a = d(\pi)$ and let $\delta(u)$ be a PR formula such that $\pi := \forall u\delta(u)$. By Lemma 18, there is a PR formula $\eta(x,y,z)$ such that for all φ, ψ,

(1) if $T + \varphi \vdash \pi$, then $T + \psi \vdash \neg\exists z\eta(\varphi,\psi,z)$,

(2) if $T + \varphi \nvdash \pi$, then $\exists z\eta(\varphi,\psi,z)$ is Π_1-conservative over $T + \psi$.

Next let θ and χ be such that

(3) $T \vdash \theta \leftrightarrow \forall u(\neg\delta(u) \rightarrow \exists z{<}u\,\eta(\chi,\theta,z))$,

$T \vdash \chi \leftrightarrow \forall z(\eta(\chi,\theta,z) \rightarrow \exists u{\leq}z\neg\delta(u))$.

Then

(4) $T \vdash \theta \vee \chi$,

(5) $T \vdash (\theta \wedge \chi) \rightarrow \pi$.

We now show that

(6) $T + \chi \nvdash \pi$.

Suppose not. Then, by (1) and (3), $T + \theta \vdash \pi$. But then, by (4), $T \vdash \pi$, contrary to assumption. This proves (6).

Now let

$$\sigma := \exists z(\eta(\chi,\theta,z) \wedge \forall u{\leq}z\delta(u)).$$

Then

$$T \vdash \sigma \leftrightarrow \exists z\eta(\chi,\theta,z) \wedge \theta.$$

By (3), $d(\theta) \leq a$. By (2) and (6), $\sigma \equiv \theta$. Thus, $d(\sigma)$ is Σ_1 and Π_1. Let $b = a \cap d(\chi)$. By (6), $a \nleq d(\chi)$ and so $b < a$. By (5), $d(\sigma) \cup b = a$. Thus, $d(\sigma)$ cups to a. Also note that b (is Π_1 and) $d(\sigma) \cap b = 0_T$. ■

The problem if for every degree $a > 0_T$, there is a Σ_1 degree which cups to a remains open. By Theorem 14, this is true of every sufficiently large degree.

Our next task is to show that the result of interchanging Σ_1 and Π_1 in Theorem 15 is false.

Theorem 16. There is a Σ_1 degree $a > 0_T$ such that no Π_1 degree cups to a.

Let $\xi(x)$ be as in Lemma 5.8 with $n = 1$ and let $a = d(\{\xi(k):k\in N\})$. Then $a > 0_T$ and no Π_1 degree cups to a (see the proof of Theorem 3 (a)). To obtain a Σ_1 degree satisfying these conditions we first prove the following refinement of Lemma 5.8 (for $n = 1$).

Lemma 22. There are Π_1 formulas $\xi(x)$, $\eta(x)$ and Σ_1 sentences χ_k such that

(i) $T \nvdash \xi(k)$

(ii) $T \vdash \eta(k) \rightarrow \xi(k)$,

(iii) $T \vdash \xi(k+1) \rightarrow \eta(k)$,

(iv) $\xi(k)$ is Σ_1–conservative over $T + \neg\eta(k)$,

(v) $\{\xi(k): k\in N\} \equiv \{\chi_k: k\in N\}$.

Proof. We combine the ideas of the proofs of Lemma 5.8 and Theorem 11. By Lemma 18, there is a PR formula $\gamma(x,z)$ such that for all φ,

(1) if $T \vdash \varphi$, then $T \vdash \neg\exists z\gamma(\varphi,z)$,

(2) if $T \nvdash \varphi$, then $\exists z\gamma(\varphi,z)$ Π_1–conservative over $T + \varphi$.

Let $\delta(u)$ be an arbitrary PR formula. Let $\kappa(z,u,x,y)$ and $\nu(z,u,x,y)$ be Π_1 formulas and $\mu(z,u,x,y,v)$ a PR formula such that

(3) $T \vdash \kappa(z,u,x,y) \leftrightarrow \forall v\mu(z,u,x,y,v)$,

(4) $T \vdash \neg\nu(z,u,x,0)$,

(5) $T \vdash \kappa(\delta,u,k,y) \leftrightarrow \nu(\delta,u,k,y) \vee \forall v([\Sigma_1](\neg\eta_\delta(k)\wedge\xi_\delta(k),v) \rightarrow \neg Prf_T(\xi_\delta(k),v))$,

(6) $T \vdash \nu(\delta,u,k,y+1) \leftrightarrow \forall v(\neg\mu(\delta,u,k+1,y,v) \rightarrow \exists z<\max\{u,v\}\gamma(\eta_\delta(k),z))$,

where

$$\xi_\delta(x) := \forall u(\delta(u) \rightarrow \kappa(\delta,u,x,u \dot{-} x)),$$

$$\eta_\delta(x) := \forall u(\delta(u) \rightarrow \nu(\delta,u,x,u \dot{-} x)).$$

As in the proof of Lemma 5.8, (5) implies that

(7) $T \vdash \xi_\delta(k) \leftrightarrow \eta_\delta(k) \vee \forall v([\Sigma_1](\neg\eta_\delta(k)\wedge\xi_\delta(k),v) \rightarrow \neg Prf_T(\xi_\delta(k),v))$.

Let

$$\eta'_\delta(x) := \forall u(\delta(u) \rightarrow \nu(\delta,u,x,(u \dot{-} (x+1)) + 1)).$$

Then, by (6),

(8) $T \vdash \eta'_\delta(k) \leftrightarrow \forall uv\big(\delta(u) \wedge \neg\mu(\delta,u,k+1,u \dot{-} (k+1),v) \rightarrow \exists z<\max\{u,v\}\gamma(\eta_\delta(k),z)\big)$.

Let

$$\chi_{\delta,k} := \exists z\big(\gamma(\eta_\delta(k),z) \wedge \forall uv\leq z\neg(\delta(u) \wedge \neg\mu(\delta,u,k+1,u \dot{-} (k+1),v))\big).$$

Then $\chi_{\delta,k}$ is Σ_1 and (cf. Lemma 1.3)

(9) $T \vdash \chi_{\delta,k} \leftrightarrow \exists z\gamma(\eta_\delta(k),z) \wedge \forall uv\big(\delta(u) \wedge \neg\mu(\delta,u,k+1,u \dot{-} (k+1),v) \rightarrow$
$\exists z<\max\{u,v\}\gamma(\eta_\delta(k),z)\big)$

and so, by (8),

(10) $T\vdash \chi_{\delta,k} \leftrightarrow \exists z\gamma(\eta_\delta(k),z) \wedge \eta'_\delta(k)$.

We now show that

(11) $T\vdash \eta_\delta(k) \to \xi_\delta(k)$,

(12) if $T\vdash \xi_\delta(k)$, then $T\vdash \eta_\delta(k)$,

(13) $T\vdash \xi_\delta(k+1) \to \eta'_\delta(k)$,

(14) if $T\vdash \delta(u) \to u > k$, then $T\vdash \eta'_\delta(k) \leftrightarrow \eta_\delta(k)$,

(15) if $T\vdash \delta(u) \to u > k$ and $T\vdash \eta_\delta(k)$, then $T\vdash \xi_\delta(k+1)$.

(11) follows from (7). (12) follows from (7) by the same argument as in the proof of Lemma 5.8. (13) follows by predicate logic from (3) and (8). (14) is obvious.

To prove (15), assume $T\vdash \delta(u) \to u > k$ and $T\vdash \eta_\delta(k)$. Then, by (14), $T\vdash \eta'_\delta(k)$. Also, by (1), $T\vdash \neg\exists z\gamma(\eta_\delta(k),z)$. By (8), it follows that

$$T\vdash \forall uv(\delta(u) \to \mu(\delta,u,k+1,u \dot{-} (k+1),v))$$

and so, by (3), $T\vdash \xi_\delta(k+1)$. This proves (15).

It can now be shown that

if $\exists u\delta(u)$ is true, then $T\nvdash \xi_\delta(0)$.

The proof of this from (4), (12), (15) is the same as that of (6) in the proof of Lemma 5.8.

As in the proof of Lemma 5.8 we can now find a PR formula $\delta'(x)$ such that $\exists u\delta'(u)$ is false and $T\nvdash \xi_{\delta'}(0)$. Let $\xi(x) := \xi_{\delta'}(x)$, $\eta(x) := \eta_{\delta'}(x)$, $\chi_k := \chi_{\delta',k}$.

The verification of (i) – (iv) is now straightforward or much the same as in the proof of Lemma 5.8; this is where (13) is needed.

To prove (v), we first note that $\{\xi(k)\colon k\in N\} \le \{\chi_k\colon k\in N\}$ follows from (10), (14), (11). Next suppose $T + \{\chi_k\colon k\in N\}\vdash \pi$. There is then an m such that $T + \chi_0\vdash \chi_1 \wedge\ldots\wedge \chi_m \to \pi$. By (i) and (ii), $T\nvdash \eta(0)$. Hence, by (2), $\exists z\gamma(\eta(0),z)$ is Π_1–conservative over $T + \eta(0)$. But then, by (10) and (14), $T + \eta(0)\vdash \chi_1 \wedge\ldots\wedge \chi_m \to \pi$. By (10), (14), (ii), (iii), $T + \chi_1\vdash \eta(0)$. It follows that $T + \chi_1 \wedge\ldots\wedge \chi_m\vdash \pi$. Continuing in this way we eventually get $T + \eta(m)\vdash \pi$ and so, by (iii), $T + \xi(m+1)\vdash \pi$. This shows that $\{\chi_k\colon k\in N\} \le \{\xi(k)\colon k\in N\}$ and so (v) is proved. ■

Proof of Theorem 16. Let $\xi(x)$ and $\eta(x)$ be as in Lemma 22. Let $a = d(\{\xi(k){:}k\in N\})$. Then, by Lemma 22 (i), $a > 0_T$. Also, by Lemma 22 (v) and Lemma 13, a is Σ_1. By Lemma 22 (ii), $d(\xi(k)) \le d(\eta(k))$ for every k. That $d(\xi(k))$ doesn't cup to $d(\eta(k))$ now follows from Lemma 22 (iv).

Suppose b is Π_1 and $b \le a$. Then, by Lemma 22 (ii) and (iii), $b \le d(\xi(k))$, for some k, and $d(\eta(k)) \le a$. Since $d(\xi(k))$ doesn't cup to $d(\eta(k))$, it follows that b doesn't cup to a. ■

Note that if a is as in Theorem 16, then a does not cup to any Π_1 degree. Indeed, let b be Π_1 and $\ge$ a. If a cups to b, there is a Π_1 degree $c \le a$ which cups to b. But then c cups to a, contrary to assumption.

Finally, we prove Theorem 5 (and a bit more). We have already observed that $d(\neg\pi)$ is the p.c. of $d(\pi)$. Thus, every Π_1 degree has a p.c. It follows that, in terms of our classification of degrees, the following result is the best we can do.

Theorem 17. There is a Σ_1 degree which has no p.c.

This is a consequence of the following strengthening of Lemma 20.

Lemma 23. There is a sentence σ such that $\{b \geq d(\neg\sigma): b \text{ is } \Sigma_1\}$ has no g.l.b.

To prove this, we need another:

Lemma 24. Suppose $\{\pi_k: k \in N\}$ is r.e. and let $G = \{d(\pi_k): k \in N\}$. Suppose there is no finite subset H of G such that $\bigcap H$ is a lower bound of G. Then G has no g.l.b.

Proof. Let $X = \{\pi: T + \pi_k \vdash \pi \text{ for every } k\}$. X is not r.e. This can be seen as follows. Let R(k,m) be a primitive recursive relation such that $Y = \{k: \forall m R(k,m)\}$ is not r.e. and let $\rho(x,y)$ be a PR binumeration of R(k,m). We may assume that $Z = \{\pi_k: k \in N\}$ is primitive recursive; let $\zeta(x)$ be a PR binumeration of Z. Finally, let $\eta(x) :=$

$\forall z(\neg\rho(x,z) \rightarrow \exists u \leq z(\zeta(u) \wedge Tr_{\Pi_1}(u)).$

It is sufficient to show that

(1) $Y = \{k: \eta(k) \in X\}$.

If $k \in Y$, then, clearly, $\eta(k) \in X$. Suppose $k \notin Y$. Let m be such that not R(k,m). Then $T + \eta(k) \vdash \bigvee Z \restriction m$. By assumption, there is an n such that $T + \pi_n \nvdash \bigvee Z \restriction m$ and so $\eta(k) \notin X$. Thus, (1) holds and so X is not r.e.

Suppose $d(A) \leq d(\pi_k)$ for every k. Then $Th(A) \cap \Pi_1 \subseteq X$. Since X is not r.e., it follows that there is a $\pi \in X$ such that $A \nvdash \pi$. But then $\pi \leq \pi_k$ for every k and $d(\pi) \nleq d(A)$. Thus, d(A) is not the g.l.b. of G. ■

Proof of Lemma 23. From the proof of Theorem 11 it is clear that there are (primitive) recursive functions f(n) and g(n) such that if π is any Π_1 sentence, then $f(\pi)$ is a Π_1 sentence, $g(\pi)$ is a Σ_1 sentence, and if $T \nvdash \pi$, then $T < T + f(\pi) \equiv T + g(\pi) \leq T + \pi$.

We now define π_k and σ_k as follows. Let π_0 be any Π_1 sentence not provable in T. Next suppose π_k has been defined and $T \nvdash \pi_k$. Let ψ be a Π_1 sentence undecidable in $T + \neg\pi_k$. Then $T < T + \pi_k \vee \psi < T + \pi_k$. Let $\sigma_k := g(\pi_k \vee \psi)$ and $\pi_{k+1} := f(\pi_k \vee \psi)$. Then $T \nvdash \pi_{k+1}$.

For every k,

(1) $\pi_{k+1} \leq \sigma_k < \pi_k$.

By Theorem 5.4 (a), there is a sentence σ such that

(2) $T + \sigma$ is a Π_1-conservative extension of $T + \{\neg\pi_k: k \in N\}$.

By (1) and (2),

(3) $\neg\sigma \leq \sigma_k$.

Moreover

(4) if b is Σ_1 and $b \geq d(\neg\sigma)$, there is a k such that $b \geq d(\pi_k)$.

For suppose $b = d(\chi) \geq d(\neg\sigma)$, where χ is Σ_1. Then $T + \chi \vdash \neg\sigma$, whence $T + \sigma \vdash \neg\chi$. But then, by (2), there is a k such that $T + \neg\pi_k \vdash \neg\chi$, whence $T + \chi \vdash \pi_k$ and so $b \geq d(\pi_k)$.

Let $G = \{d(\pi_k): k \in N\}$. If $\{b \geq d(\neg\sigma): b \text{ is } \Sigma_1\}$ has a g.l.b. c, then, by (1), (3), (4), c is the g.l.b. of G. But from (1) it follows that no $d(\pi_k)$ is a lower bound of G. Hence, by Lemma 24, G has no g.l.b. and so $\{b \geq d(\neg\sigma): b \text{ is } \Sigma_1\}$ has no g.l.b. ■

Proof of Theorem 17. Let σ be as in Lemma 23. By Lemma 6, for all B, $(T + \sigma){\downarrow}B \leq T$ iff $B \leq T + \chi$ for all Σ_1 sentences χ such that $T + \chi \vdash \neg\sigma$. But then the p.c. of $d(\sigma)$, if it had one, would also be the g.l.b. of $\{b \geq d(\neg\sigma): b \text{ is } \Sigma_1\}$. Thus, by Lemma 23, $d(\sigma)$ has no p.c. ■

Every Σ_1 degree is the p.c. of some degree. It is an open problem if the converse of this is true. If it is, the Σ_1 degrees can be characterized in a purely algebraic way as those degrees that are p.c.s.

Exercises for Chapter 7.

In the following exercises we assume that $PA \dashv T$ and that A, B, etc. are extensions of T.

1. Suppose $G \subseteq D_T$. G is *independent* if for any disjoint finite subsets G_0 and G_1 of G, $\bigcap G_0 \not\leq \bigcup G_1$. ($\bigcap\emptyset = 1_T$, $\bigcup\emptyset = 0_T$.) (Thus, for example, $\emptyset$ is independent and $\{a\}$ is independent iff $0_T < a < 1_T$.) Show that for every finite independent set G, there are degrees b_0, b_1 such that $G \cup \{b_i\}$ is independent, $i = 0, 1$, and $b_0 \cap b_1 = 0_T$. Conclude that every finite independent set is included in $2^{\aleph_0}$ many maximal independent sets.

2. Suppose $a < b$.

(a) c *cups to* b *above* a if there is a d such that $a \leq d < b$ and $c \cup d = b$. Show that there is a $c \in (a,b]$ which doesn't cup to b above a.

(b) c *caps to* a *below* b if there is a d such that $a < d \leq b$ and $c \cap d = a$. Show that there is a $c \in [a,b)$ which doesn't cap to a below b.

3. Suppose $a < b$ and $b < 1_T$ if T is Σ_1–sound. For $c \in [a,b]$, let c^* be the *complement of* c *in* [a,b] if it exists, i.e. $c \cap c^* = a$ and $c \cup c^* = b$. (Complements are unique.) Let $Cpl_{a,b}$ be the set of degrees in [a,b] having complements in [a,b].

(a) Show that $Cpl_{a,b}$ is closed under $\cap$, $\cup$, and $*$.

Let $\mathbf{Cpl}_{a,b} = (Cpl_{a,b}, \cap, \cup, {}^*, a, b)$. Then $\mathbf{Cpl}_{a,b}$ is a Boolean algebra.

(b) Show that if $c, d \in Cpl_{a,b}$ and $c < d$, there is an $e \in Cpl_{a,b}$ such that $c < e < d$. (It follows that the Boolean algebras $\mathbf{Cpl}_{a,b}$ are (denumerable and) atomless and therefore isomorphic.)

(c) Show that if $a \leq c < d \leq b$, there is an $e \in [c,d)$ such that $Cpl_{a,b} \cap [e,d) = \emptyset$. [Hint: $Cpl_{a,b} \cap [c,d] \subseteq Cpl_{c,d}$.]

(d) Show that if $a \leq c < e < d \leq b$ and $e \notin Cpl_{a,b}$, there are c′, d′ such that $c \leq c' < e < d' \leq d$ and $Cpl_{a,b} \cap [c',d'] = \emptyset$.

4. Suppose a is Σ_1.

(a) Show that if $a < b < 1_T$, then a caps to 0_T below b.

(b) Show that if $a < b$ and b is high, then $a \ll b$. Conclude that if $b_i > a$, $i = 0, 1$, and $b_0 \cap b_1 = a$, then b_0 and b_1 are low. ((a) and (b) are true of every a which is the p.c. of some degree,)

5. Show that for every low degree a, there is a low Π_1 degree $\geq a$. [Hint: Let $B = T + \sigma$ and $\sigma := \exists x\delta(x)$, where $\delta(x)$ is PR, be such that $a \leq d(B) < 1_T$. We may assume that $B \nvdash \neg Con_B$. Let

$$\theta := \forall y(Prf_B(\bot,y) \rightarrow \exists x{\leq}y\delta(x)),$$

$$\chi := \exists x(\delta(x) \wedge \forall y{\leq}x\neg Prf_B(\bot,y)).$$

Then $\sigma \leq \theta \leq \chi$ and $T + \chi$ is consistent.]

6. Referring to the proof of Theorem 4, show that there is a primitive recursive function g such that ψ can be replaced by the sentence

$$\chi := \forall u(Prf_B(\bot,u) \rightarrow \exists z{<}g(u)Prf_T(\bot,z)),$$

similar to θ. [Hint: Define g in such a way that $PA \vdash \neg\varphi \rightarrow \chi$.]

7. (a) Show that there is an r.p. degree a which is not Π_1 (compare Lemma 11). [Hint: Let $\kappa(x)$ be as in Exercise 2.11 and let $a = d(\{\kappa(k):k\in N\})$].

(b) Improve (a) by showing that there is a non-Π_1 Σ_1 degree a which is r.p. (compare Exercise 16 (c)). [Hint: Define π_k and σ_k so that $T \nvdash \pi_k$, $T \vdash \kappa(k) \rightarrow \pi_k$, where $\kappa(x)$ is as in (a), and $\pi_0 \wedge ... \wedge \pi_{k-1} \wedge \sigma_k \equiv \pi_0 \wedge ... \wedge \pi_k$. Let $a = d(\{\sigma_k:k\in N\})$.]

8. Suppose $A \dashv B$. Show that there is a Δ_2 sentence φ such that $A + \varphi \simeq B$ (compare Corollary 6.10 and Theorem 8).

9. Show that there is a Σ_1 sentence σ such that $0_T < d(\sigma) < 1_T$ and for every Σ_1 sentence χ, if $\sigma \leq \chi$, then $T + \chi \vdash \sigma$. [Hint: Let σ be such that $d(\neg\sigma)$ is Σ_1.]

10. Let $<\sigma_k>_{k<\omega}$ and σ be as in Lemma 14. Show that every Π_1 degree $\leq d(\sigma)$ caps to 0_T below $d(\sigma)$ (compare Exercise 26 (c)).

11. (a) Show that for every Π_1 sentence π, $d(\pi) \cup d(\neg\pi)$ is high; in fact, if b is the p.c. of a, then $a \cup b$ is high.

(b) Let $a = d(\sigma) \cup d(\neg\sigma)$. a is high. Let $<\sigma_k>_{k<\omega}$ and σ be as in Lemma 14. Show that if b is Π_1 and $b \leq a$, then b is low. [Hint: Use Exercise 4 (a).]

12. Show that there is a degree of the form $d(\sigma\vee\pi)$ which is neither Σ_1 nor Π_1.

13. Show that there is a Π_1 degree a such that for every Π_1 degree b, if $a \cap b = 0_T$, then $a \cup b$ is low (compare Exercise 18 (c)).

14. Suppose $a < b < 1_T$. Show that
(a) there is a degree $c < 1_T$ such that for every d, if $b \cap d = a$, then $d \leq c$,
(b) there is a degree $c > 0_T$ such that for every d, if $a \cup d = b$, then $d \geq c$.

15. (a) Verify that in any distributive lattice, for any a, b, the intervals $[a \cap b,a]$ and $[b,a \cup b]$ are isomorphic.
(b) Show that there are degrees a, b, c, d such that $a \ll b$, $c < d$, not $c \ll d$, and [a,b] and [c,d] are isomorphic. [Hint: Use Exercises 4 (b) and 11 (a).]

16. (a) Verify that in any distributive lattice, if $a < b < c$ and [a,c] satisfies the reduction principle, so does [b,c].
(b) Show that for each degree $a < 1_T$, there is a b such that $a < b < 1_T$ and [a,b] does not satisfy the reduction principle.
(c) The non–r.p. degree a defined in the proof of Lemma 12 is high (cf. Exercise 11 (a)). Show that there is a Σ_1 degree which is not r.p. Conclude from Exercise 7 (b) that there are non–Π_1 Σ_1 degrees such that $[0_T,a]$ and $[0_T,b]$ are not isomorphic. [Hint: Use Theorem 14′ (a).]

17. (a) Suppose φ and X are as in Lemma 16. Show that if $\varphi \leq X$, then $\varphi \ll X$.
(b) Suppose $a < b$. Show that there are c, d such that $a \leq c < d \leq b$ and [c,d] contains no B_1 degree.

18. (a) Show, by combining the proofs of Theorem 4 and Lemma 15, that there are cupping degrees a_0 and a_1 which are Σ_1 and Π_1 and such that $a_0 \cap a_1 = 0_T$. Conclude that there are low cupping degrees. (This also follows from Theorem 14′ (b).)
(b) Show that there is a high (Π_1) degree a which is not cupping. [Hint: Suppose $d(Con_T) < 1_T$. Let $a = d(\pi)$ where π is Σ_1–conservative over $T + \neg Con_T$ and $\neg\pi$ is Π_1–conservative over $T + \neg Con_T$.]
(c) Show that there is a low (Π_1) degree a such that for every degree b, if $a \cap b = 0_T$, then $a \cup b$ is not cupping (compare Exercise 13). [Hint: Let $d(\pi)$ be as in (b). Define a sentence σ such that $d(\sigma) > 0_T$ and $d(\sigma) \cup d(\neg\sigma) \leq d(\pi)$; use Theorem 11. Let $a = d(\neg\sigma)$.]

19. Show that there are degrees a, b such that a is Σ_1, b is both Σ_1 and Π_1, and $a \cup b$ is not B_1.

20. Prove Lemma 15 by letting θ_0 be a Π_1 Rosser sentence for T and $\theta_1 :=$
$$\forall u(Prf_T(\neg\theta_0,u) \rightarrow \exists z{\leq}u Prf_T(\theta_0,z)).$$
Conclude that $d(\theta_0)$ is Σ_1 (compare Exercise 6.9).

21. (a) Suppose $a \in E_T$ and $a > 0_T$. Show that there is a degree $b < a$ such that $[b,a] \subseteq$

E_T. [Hint: We may assume that a $\nleq$ d(Con_T). Let b = a $\cap$ d(Con_T) and use Theorem 14′ (b). (By Lemma 17, no member of [b,a) is Σ_1).]

(b) Suppose there is a Σ_1 degree which cups to a. Show that there is a b < a such that for every c$\in$[b,a], there is a Σ_1 degree which cups to c.

22. (a) Let E'_T be the set of degrees obtained from 0_T by taking l.u.b.s, g.l.b.s, and Σ_1–extensions. Show that if a$\in E'_T$, there is a least Σ_1 degree $\geq$ a. Conclude that there is a Π_1 degree not in E'_T. (This improves Theorem 12.)

(b) Let F'_T be the set of degrees obtained from E'_T and the Π_1 degrees by taking l.u.b.s and Σ_1–extensions. Show that the degree defined in the proof of Theorem 13 is not in F'_T. Conclude that there is a degree which is not the l.u.b. of a finite set of degrees of the form d($\pi\wedge\sigma$). (This improves Theorem 13.)

23. Show that for any a, if there is a member of G_T which cups to a, then there is a Σ_1 degree which cups to a. (This improves Theorem 15.)

24. (a) Show that not all non–Π_1 Σ_1 degrees are as stated in Theorem 16.

(b) Improve Theorem 16 by showing that for every degree b > 0_T, there is a Σ_1 degree a such that 0_T < a < b and no Π_1 degree cups to a. [Hint: By Theorem 11, there are sentences π and σ such that 0_T < d(π) = d(σ) < b. Let C = T + $\neg\pi$. By the proof of Lemma 22, with T replaced by C, there are Π_1 formulas $\xi(x)$, $\eta(x)$ and Σ_1 sentences χ_k such that (i) – (iv) hold with T replaced by C and C + {$\xi(k)$: k$\in$N} $\equiv$ C + {χ_k: k$\in$N}. Let a = d({$\xi(k)\vee\pi$:k$\in$N}).]

25. Show that in contrast to Lemma 24 we have the following: There is a set G = {d(σ_k): k$\in$N} of Σ_1 degrees, where {σ_k: k$\in$N} is (primitive) recursive, such that $\bigcap$H > 0_T for every finite subset H of G and $\bigcap$G = 0_T. [Hint: Let a be high and such that there is no high Π_1 degree $\leq$ a (cf. Exercise 11 (b)). Let A$\in$a and let $\sigma_k := \neg Con_{A\restriction k}$.]

26. (a) Show that there is a PR formula $\delta(u)$ such that if θ is defined as in the proof of Theorem 11, then d($\neg\theta$) isn't Π_1.

(b) Let θ be as in (a). Show that d($\neg\theta$) has a p.c. Conclude that there is a non–Π_1 Σ_1 degree which has a p.c.

(c) Let θ be as in (a). Show that there is a Π_1 degree < d($\neg\theta$) which does not cap to 0_T below d($\neg\theta$) (compare Exercise 10).

Notes for Chapter 7.

The lattice $\mathbf{D}_T$ was introduced by Lindström (1979), (1984b); a related lattice $\mathbf{V}_T$ (degrees of finite extensions of T) has been defined by Švejdar (1978) (see also Jeroslow (1971a)). (By Theorem 6.11 (a), $\mathbf{V}_T$ and $\mathbf{D}_T$ are isomorphic.) Theorem 1 is due to Lindström (1979), (1984b) and (for $\mathbf{V}_T$) to Švejdar (1978). Corollary 1 is, modulo Theorem 6.6, a restatement of the equivalence of Exercise 2.22 (i) and (ii). The proof of Theorem 4 was suggested by the proof of a related result in Hájková II (1971). Theorem 7 is new; the term "reduction principle" is borrowed from descriptive set theory and recursion theory (cf. Soare (1987)). (The only way of showing that intervals are isomorphic known so far is given in Exercise 15 (a) and works in all distributive lattices.) The remaining results of § 1 are due to Lindström (1979), (1984b). In connection with the proof of Theorem 4, see Exercise 6. Lemmas 11 and 12 lead to the question if there is a non–Π_1 r.p. degree; this question is answered in Exercise 7.

Theorem 8 (with a slightly different proof; see Exercise 6.12 (a)) is due to Montagna (cf. Lindström (1993)). Theorem 9 is due to Lindström (1979), (1984b), (1993); (a) and (c) were also proved by Švejdar (1978); for a different proof of Theorem 9 (d), see Exercise 12.

Theorem 10 is due to Lindström (1979), (1984b); (a) and the first half of (b) were also proved by Švejdar (1978). Theorems 14 and 16 are new, they were announced in Lindström (1993), where a weaker form of Theorem 16 is proved; Theorem 16 leads to the question if there is a Σ_1 degree a such that no Π_1 degree caps to a; this is answered negatively in Exercise 5; in connection with Theorem 16, see also Exercise 24. The remaining results of § 3 are due to Lindström (1984b), (1993). The definition of the sentences φ_n and ψ_n in the proof of Theorem 14 (a) and the observations concerning these sentences, except (8), were first used by Misercque (1982) in a different context. For improvements of Theorems 12, 13, 15, and 16, see Exercises 22 (a), 22 (b), 23, 24 (b). Theorem 17 leads to the question if no non–Π_1 Σ_1 degree has a p.c.; this question is answered in Exercise 26 (b).

For a proof of Exercise 26 (a), see Lindström (1993).