4. AXIOMATIZATIONS

S is an *axiomatization* of T if $S \dashv\vdash T$. Suppose $S \dashv T$. $S + X$ is an *axiomatization of* T *over* S if X is r.e. and $T \dashv\vdash S + X$. In this chapter we discuss some important properties of axiomatizations: finiteness, boundedness, and irredundance.

§1. Finite and bounded axiomatizability; reflection principles. We shall say that T is a *finite extension of* S if there is a sentence φ such that $T \dashv\vdash S + \varphi$. T is *essentially infinite over* S if no consistent extension of T is finite over S. T is *essentially infinite* if T is essentially infinite over the empty theory (logic). We already know that PA is essentially infinite (Corollary 2.1).

By the *local reflection principle for* S we understand the set

$Rfn_S = \{Pr_S(\varphi) \rightarrow \varphi \text{: } \varphi \text{ any sentence of } L_A\}$.

Thus, Rfn_S is a piecemeal (local) way of saying that every sentence provable in S is true. (The latter statement, the full (global) reflection principle for S, cannot be expressed in T, since, by the Gödel–Tarski theorem, truth is not definable.)

Clearly $PA + Rfn_T \vdash Con_T$ (let $\varphi := \bot$). Also note that T is essentially reflexive iff $T \vdash Rfn_{T \restriction k}$ for every k (cf. Corollary 1.9 (b)).

We now use the local reflection principle to construct an essentially infinite extension of a given theory S. Note that $Rfn_S \dashv T$ implies $S \dashv T$.

Theorem 1. If $Rfn_S \dashv T$, then T is essentially infinite over S.

Proof. Suppose $T \dashv S + \theta$. We are going to show that $S + \theta$ is inconsistent. Let ψ be such that

(1) $Q \vdash \psi \leftrightarrow \neg Pr_{S+\theta}(\psi)$.

By hypothesis,

$T \vdash Pr_S(\theta \rightarrow \psi) \rightarrow (\theta \rightarrow \psi)$.

From this and (1) it follows that $T \vdash \theta \rightarrow \psi$. But then

(2) $S + \theta \vdash \psi$.

It follows that $Q \vdash Pr_{S+\theta}(\psi)$ and so, by (1), $Q \vdash \neg\psi$. But $Q \dashv S + \theta$ and so, by (2), $S + \theta$ is inconsistent. ■

If $PA \dashv T$, the conclusion of Theorem 1 can be strengthened; see Corollary 2, below.

There is a stronger principle, the *uniform reflection principle*, which is a better approximation than Rfn_S of the full reflection principle for S, namely,

$RFN_S = \{\forall x(\Gamma(x) \wedge Pr_S(x) \rightarrow Tr_\Gamma(x))\text{: } \Gamma \text{ arbitrary}\}$.

Clearly $T + RFN_S \vdash Rfn_S$ provided that $PA \dashv T$. Applying the uniform reflection principle we can derive a stronger conclusion than in Theorem 1.

A set X of sentences is *bounded* if $X \subseteq \Gamma$ for some Γ. Let $Prf_{S,\Gamma}(x,y) :=$

$\exists z(\Gamma(z) \wedge Tr_\Gamma(z) \wedge Prf_{S+z}(x,y))$
and let $Pr_{S,\Gamma}(x) := \exists y Prf_{S,\Gamma}(x,y)$.

Lemma 1. For every φ,
$PA + RFN_S \vdash Pr_{S,\Gamma}(\varphi) \to \varphi$.

Proof. Suppose φ is Γ^d. Argue in $PA + RFN_S$: "Suppose $Pr_{S,\Gamma}(\varphi)$. There is then a Γ sentence ψ such that $Tr_\Gamma(\psi)$ and $Pr_S(\psi \to \varphi)$. By RFN_S, $\forall z(\Gamma^d(z) \wedge Pr_S(z) \to Tr_{\Gamma^d}(z))$. Since $\psi \to \varphi$ is Γ^d, it follows that $Tr_{\Gamma^d}(\psi \to \varphi)$. But $Tr_\Gamma(\psi)$. Consequently, by Fact 10 (a) $Tr_{\Gamma^d}(\varphi)$ and so φ, as desired." ■

Theorem 2. Suppose $PA \dashv T$ and $T \vdash RFN_S$. If X is any bounded (not necessarily r.e.) set of sentences such that $T \dashv S + X$, then $S + X$ is inconsistent.

Proof. Let Γ be such that $X \subseteq \Gamma$. Suppose $T \dashv S + X$. We are going to show that $S + X$ is inconsistent. Let ψ be such that
(1) $PA \vdash \psi \leftrightarrow \neg Pr_{S,\Gamma}(\psi)$.
By Lemma 1,
$T \vdash Pr_{S,\Gamma}(\psi) \to \psi$.
From this and (1) it follows that $T \vdash \psi$ and so
(2) $S + X \vdash \psi$.
But then there is a conjunction θ of members of X such that $S + \theta \vdash \psi$. It follows that $T + \theta \vdash Tr_\Gamma(\theta) \wedge Pr_{S+\theta}(\psi)$ and so $T + \theta \vdash Pr_{S,\Gamma}(\psi)$, whence, by (1), $T + \theta \vdash \neg\psi$ and so $S + X \vdash \neg\psi$. Thus, by (2), $S + X$ is inconsistent. ■

Note the obvious analogy between the proofs of Theorems 1 and 2, on the one hand, and the proof of Gödel's theorem (Theorem 2.1), on the other. Note also that if T is Σ_1–sound, then $X = \{\neg Pr_T(\varphi): T \nvdash \varphi\}$ is a (non–r.e.) set of Π_1 sentences such that $T + Rfn_T \dashv T + X$ and $T + X$ is consistent.

Since $PA \vdash RFN_\varnothing$ (Fact 11), we have (a) of the following corollary, improving Corollary 2.1.

Corollary 1. (a) There is no consistent bounded set X such that $PA \dashv X$.
(b) If $PA \dashv T$, there is no bounded set X such that $T + RFN_T \dashv T + X$ and $T + X$ is consistent.

If $PA \dashv S$, the above proof of Theorem 2 can be replaced by the following simple argument; the proof of Theorem 1 can be simplified in a similar way. Let
$\chi := \forall x(\Gamma(x) \wedge Pr_S(x) \to Tr_\Gamma(x))$.
Now let θ be any Γ^d sentence such that $S + \theta \vdash \chi$. Then $S + \theta \vdash \neg Pr_S(\neg\theta)$, whence $S + \theta \vdash Con_{S+\theta}$ and so $S + \theta$ is inconsistent, by Theorem 2.4.

This argument and (a somewhat more detailed version of) the above proof of Theorem 2 can be looked at from a different point of view which will be further elaborated in Chapter 5: Let φ be any Γ sentence such that $S + \neg\chi \vdash \varphi$. Then $S + \neg\varphi \vdash \chi$ and so $S \vdash \varphi$. Thus, $\neg\chi$ is Γ–conservative over S in the sense that if φ is any Γ sentence and $S + \neg\chi \vdash \varphi$, then $S \vdash \varphi$.

Next we show that if $PA \dashv T$, no bounded extension of T is essentially infinite over T (and a bit more).

Theorem 3. Suppose $PA \dashv T$, let X be an r.e. set of Γ sentences, and let Y be any r.e. set of sentences such that $T + X \nvdash \psi$ for every $\psi \in Y$. There is then a Γ sentence θ such that $T + \theta \vdash X$ and $T + \theta \nvdash \psi$ for every $\psi \in Y$.

Proof. By Craig's theorem, we may assume that X and Y are primitive recursive. Let $\xi(x)$ and $\eta(x)$ be PR binumerations of X and Y, respectively.

Case 1. $\Gamma = \Pi_n$. Let θ be such that

$$PA \vdash \theta \leftrightarrow \forall y\big(\xi(y) \wedge \forall zu{\le}y(\eta(z) \to \neg Prf_{T+\theta}(z,u)) \to Tr_{\Pi_n}(y)\big).$$

Suppose $\psi \in Y$ and $T + \theta \vdash \psi$. Let p be a proof of ψ in $T + \theta$ and let $q = \max\{p,\psi\}$. Then

(1) $PA \vdash \forall zu{\le}y(\eta(z) \to \neg Prf_{T+\theta}(z,u)) \to y < q.$

Let $\varphi_0,\ldots,\varphi_k$ be those members of X which are $< q$. Then, by (1) and Fact 10 (a) (ii),

$$PA + \varphi_0 + \ldots + \varphi_k \vdash \theta,$$

whence $T + X \vdash \theta$ and so $T + X \vdash \psi$, contrary to hypothesis. Thus, $T + \theta \nvdash \psi$ for all $\psi \in Y$. But then

$$PA \vdash \forall zu{\le}r(\eta(z) \to \neg Prf_{T+\theta}(z,u))$$

for all r. It follows that $T + \theta \vdash X$, as desired.

Case 2. $\Gamma = \Sigma_n$. Let θ be such that

$$PA \vdash \theta \leftrightarrow \exists y\big(\exists zu{\le}y(\eta(z) \wedge Prf_{T+\theta}(z,u)) \wedge \forall z{\le}y(\xi(z) \to Tr_{\Sigma_n}(z))\big).$$

The verification that θ is as desired is left to the reader. ■

From Theorem 1 and Theorem 3 with $Y = \{\bot\}$ we get the following:

Corollary 2. Suppose $PA \dashv T$. If X is any bounded r.e. set of sentences such that $Rfn_T \dashv T + X$, then $T + X$ is inconsistent.

Suppose T is Σ_1–sound. We have already mentioned that $PA + Rfn_T \vdash Con_T$. By Theorem 1, $T + Con_T \nvdash Rfn_T$. Clearly $PA + RFN_T \vdash Rfn_T$. It has been pointed out that $T + \{\neg Pr_T(\varphi)\colon T \nvdash \varphi\}$ is a consistent, bounded extension of $T + Rfn_T$. Thus, by Theorem 2, if $PA \dashv T$, then $T + Rfn_T \nvdash RFN_T$. These observations can be strengthened as follows.

We define the sentences Con(n,S), for $n > 0$, by: $Con(1,S) := Con_S$, $Con(n{+}1,S) := Con(1,S + Con(n,S))$. Let

$$Con_S^{\omega} = \{Con(n,S)\colon n > 0\}.$$

The proof of the following lemma is straightforward and left to the reader.

Lemma 2. (a) If $k > m > 0$, then

$PA \vdash Con(k,S) \rightarrow Con(m,S)$.

(b) For all k, m > 0,

$PA \vdash Con(k,S + Con(m,S)) \leftrightarrow Con(k+m,S)$.

The sets Rfn(n,S) are defined as follows: $Rfn(0,S) = \emptyset$, $Rfn(1,S) := RfnS$, $Rfn(n+1,S) := Rfn(1,S + Rfn(n,S))$. Next let

$$Rfn^{\omega}_{S} = \bigcup\{Rfn(n,S): n \in N\}.$$

We write $S \dashv_p S'$ to mean that S is a proper subtheory of S′.

Theorem 4. Suppose $PA \dashv T$ and T is Σ_1–sound.

(a) $T + Con^{\omega}_{T} \dashv_p T + Rfn_T$.

(b) $T + Rfn^{\omega}_{T} \dashv_p T + RFN_T$.

Lemma 3. (a) $PA + Rfn_T \vdash Rfn_{T+Con_T}$.

(b) $PA + RFN_T \vdash RFN_{T+Rfn_T}$.

Proof. (a) Let φ be any sentence.

$$PA + Rfn_T \vdash Pr_T(Con_T \rightarrow \varphi) \rightarrow (Con_T \rightarrow \varphi).$$

But, as we have already observed, $PA + Rfn_T \vdash Con_T$. It follows that $PA + Rfn_T \vdash Pr_{T+Con_T}(\varphi) \rightarrow \varphi$, as desired. ♦

(b) We give an informal proof using the fact that Fact 10 (a) is provable in PA. We assume, as we may, that the PR binumeration $\rho(x)$ of Rfn_T implicit in the notation RFN_{T+Rfn_T} is such that PA proves that every sentence satisfying $\rho(x)$ is of the form $Pr_T(\theta) \rightarrow \theta$. Suppose $\Sigma_1 \subseteq \Gamma$. Now argue in $PA + RFN_T$: "Let ψ be any Γ sentence provable in $T + Rfn_T$ and let $Pr_T(\varphi_i) \rightarrow \varphi_i$, for $i \leq n$, be the members of Rfn_T occurring in the proof. We may assume that $\neg Pr_T(\varphi_i)$, for $i \leq n$, since those $Pr_T(\varphi) \rightarrow \varphi$ for which $Pr_T(\varphi)$ are provable in T and we may add the proofs of them to the original proof. Since $\neg Pr_T(\varphi_i) \rightarrow (Pr_T(\varphi_i) \rightarrow \varphi_i)$ is (trivially) provable in T, it follows that $\theta :=$

$$\neg Pr_T(\varphi_0) \wedge \ldots \wedge \neg Pr_T(\varphi_n) \rightarrow \psi$$

is provable in T. By RFN_T, $Tr_\Gamma(\theta)$. But, by Fact 10 (a) (ii), $Tr_{\Gamma^d}(\neg Pr_T(\varphi_i))$, for $i \leq n$. Hence, by Fact 10 (a) (iii), $Tr_\Gamma(\psi)$, as desired." ■

Proof of Theorem 4. (a) In view of Lemma 3 (a), it follows, by induction, that $T + Rfn_T \vdash Con^{\omega}_{T}$. $T + Con^{\omega}_{T}$ is consistent, since T is Σ_1–sound, and Con^{ω}_{T} is an r.e. set of Π_1 sentences. Thus, by Corollary 2, $T + Con^{\omega}_{T} \nvdash Rfn_T$. ♦

(b) By Lemma 3 (b), $T + RFN_T \vdash Rfn^{\omega}_{T}$. Let $X_k = \{\neg Pr_{T+Rfn(k,T)}(\varphi): T + Rfn(k,T) \nvdash \varphi\}$. Then, by induction, $T + \bigcup\{X_k: k \leq n\} \vdash Rfn(n+1,T)$. Let $X = \bigcup\{X_k: k \in N\}$. Then X is a (non–r.e.) set of true Π_1 sentences, whence T + X is consistent, and $T + X \vdash Rfn^{\omega}_{T}$. Thus, by Theorem 2, $T + Rfn^{\omega}_{T} \nvdash RFN_T$. ■

If T is Σ_1–sound then, by Theorem 4 (a), $T + Con^{\omega}_{T}$ is a proper subtheory of $T + Rfn_T$. In our next result we show that if we restrict ourselves to Π_1 sentences, this is no longer true.

We write $S \dashv_{\Pi_1} S'$ to mean that S is a Π_1–*subtheory* of S′, i.e. every Π_1 sentence provable in S is provable in S′.

Theorem 5. If $PA \dashv T$, then $T + Rfn_T \dashv_{\Pi_1} PA + Con_T^{\omega}$.

In the proof we use the following observation.

Lemma 4. If $Q \dashv S$, then $S \dashv_{\Pi_1} PA + Con_S$.

Proof. Let π be a Π_1 sentence such that $S \vdash \pi$. Then $PA \vdash Pr_S(\pi)$. Since $\neg\pi$ is Σ_1, we also have, $PA \vdash \neg\pi \rightarrow Pr_S(\neg\pi)$. It follows that $PA \vdash \neg\pi \rightarrow \neg Con_S$ and so $PA + Con_S \vdash \pi$. ■

Proof of Theorem 5. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be all sentences of L_A. For every theory S, let

$$S_n = S + \bigwedge\{Pr_S(\varphi_i) \rightarrow \varphi_i : i \leq n\}.$$

It is sufficient to show that for every n, there is a k such that $T_n \dashv_{\Pi_1} PA + Con(k,T)$. By Lemma 4, $T_n \dashv_{\Pi_1} PA + Con_{T_n}$ and so we need only prove that $PA + Con(k,T) \vdash Con_{T_n}$.

First we note that

(1) for any sentence φ, $PA + Con(2,S) \vdash Con_{S+Pr_S(\varphi)\rightarrow\varphi}$.

Argue in PA: "Suppose $\neg Con_{S+Pr_S(\varphi)\rightarrow\varphi}$, in other words, $S + Pr_S(\varphi) \rightarrow \varphi \vdash \bot$. Then $S \vdash Pr_S(\varphi)$ and $S \vdash \neg\varphi$. But then $S \vdash Pr_S(\neg\varphi)$ and so $S \vdash \neg Con_S$, whence $\neg Con(2,S)$." This proves (1).

We now show that for every n,

(2) for every extension S of PA,
$PA + Con(2^{n+1},S) \vdash Con_{S_n}$.

For n = 0 this holds, by (1). Suppose (2) holds for n = k. Let S be any extension of PA. Then

(3) PA proves: $PA + Con(2^{k+1},S) \vdash Con_{S_k}$,

(4) PA proves: if $Con(2^{k+1}, S + Con_{S_k})$, then $(S + Con_{S_k})_k$ is consistent.

Now argue in PA: "Suppose $\neg Con_{S_{k+1}}$, in other words,

$$S_k + Pr_S(\varphi_{k+1}) \rightarrow \varphi_{k+1} \vdash \bot.$$

Then, since $S \dashv S_k$,

$$S_k + Pr_{S_k}(\varphi_{k+1}) \rightarrow \varphi_{k+1} \vdash \bot.$$

But then, by (1), $S_k + Con_{S_k} \vdash \bot$ and so

$$S + Con_{S_k} + \bigwedge\{Pr_S(\varphi_i) \rightarrow \varphi_i : i \leq k\} \vdash \bot.$$

It follows that

$$S + Con_{S_k} + \bigwedge\{Pr_{S+Con_{S_k}}(\varphi_i) \rightarrow \varphi_i : i \leq k\} \vdash \bot,$$

in other words, $(S + Con_{S_k})_k \vdash \bot$. But then, by (4),

(5) $\neg Con(2^{k+1}, S + Con_{S_k})$.

By (3), we also have

$$PA + Con(2^{k+1},S) \vdash Con_{S_k}.$$

From this and (5) we get

$\neg Con(2^{k+1}, S + Con(2^{k+1}, S))$,
and so, by Lemma 2 (b), $\neg Con(2^{k+2}, S)$."

Thus, we have shown that (2) holds for n = k+1. It follows that (2) holds for all n. For S = T, this yields the desired conclusion. ■

For completeness we mention, but do not prove, that PA + RFN_T is not a Π_1–subtheory of T + Rfn_T^{ω}; for example, PA + $RFN_T \vdash Con_{T+Rfn_T^{\omega}}$.

§2. Irredundant axiomatizability. A set X of sentences is *irredundant over* T if for every $\phi \in X$, $T + (X - \{\phi\}) \nvdash \phi$. An extension S of T is *irredundantly axiomatizable (i.a.) over* T if there is an axiomatization T + X of S such that X is irredundant over T. In this case we shall also say that T + X is *irredundant over* T. If S is a finite extension of T, then S is i.a. over T. A theory is *irredundantly axiomatizable (i.a.)* if it is i.a. over the empty theory (logic). If T is i.a. over a finite theory, then T is i.a.

Theorem 6. If PA$\dashv$ T, then T is i.a.

Lemma 5. Suppose X is recursive and S + X is not a finite extension of S. Then S + X is i.a. over S iff there is a recursive function f(n) such that for every conjunction χ of members of X, $S + X \vdash f(\chi)$ and $S \nvdash \chi \rightarrow f(\chi)$.

Proof. "If". Let f(n) be as assumed. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an effective enumeration of X. Let $\chi_n := \varphi_0 \wedge \ldots \wedge \varphi_n$. We may assume that $S \nvdash \varphi_0$. We effectively define sentences ψ_n in the following way. Let $\psi_0 := \varphi_0$. Suppose ψ_n has been defined and $S + X \vdash \psi_n$. We can then effectively find an m such that $S + \chi_m \vdash \psi_n$. Let $\psi_{n+1} := \chi_m \wedge f(\chi_m)$. Then $S + X \dashv\vdash S + \{\psi_n : n \in N\}$, $\vdash \psi_{n+1} \rightarrow \psi_n$, and $S \nvdash \psi_n \rightarrow \psi_{n+1}$ for every n. Next let $\theta_0 := \psi_0$ and $\theta_{n+1} := \psi_n \rightarrow \psi_{n+1}$. Again we have $S + X \dashv\vdash S + \{\theta_n : n \in N\}$. For every n, $S + \neg\theta_n$ is consistent. Also $\vdash \neg\theta_n \rightarrow \theta_k$ for every $k \neq n$. It follows that $S + \{\theta_k : k \neq n\} \nvdash \theta_n$. Thus, $S + \{\theta_n : n \in N\}$ is an axiomatization of S + X which is irredundant over S.

"Only if". Let S + Y be an axiomatization of S + X which is irredundant over S. Let χ be a conjunction of members of X. Given χ, we can effectively find a conjunction ψ of members $\psi_0, \ldots, \psi_k$ of Y such that $S + \psi \vdash \chi$. Since S + X is not finite over S, we can now effectively find a sentence $\theta \in Y - \{\psi_0, \ldots, \psi_k\}$. Let $f(\chi) = \theta$; if n is not a conjunction of members of X, let f(n) = 0. Since S + Y is irredundant over S, it follows that f(n) is as desired. ■

Proof of Theorem 6. Let φ be as in Theorem 2.1 with $T = Q + \chi$. If $T \vdash \chi$, then $Q + \chi$ is consistent and so $Q + \chi \nvdash \varphi$. By Theorem 2.4, PA + $Con_{Q+\chi} \vdash \varphi$. Set $f(\chi) = \varphi$. Then $\nvdash \chi \rightarrow f(\chi)$. Also, by Corollary 1.8, $T \vdash Con_{Q+\chi}$ and so $T \vdash f(\chi)$. The desired conclusion now follows from Lemma 5 with $S = \emptyset$ and X = T. ■

To prove the existence of non–i.a. theories we borrow the following lemma from recursion theory.

Lemma 6. There is a coinfinite r.e. set H such that for every recursive function h(n) (such that $h(n) < h(n+1)$ for every n), there is a number m such that $\{k: h(m) < k \leq h(m+1)\} \subseteq H$. (It follows that H is not recursive.)

Theorem 7. There is a Π_1 (Σ_1) formula $\eta(x)$ such that $T + \{\eta(k): k \in N\}$ is not i.a. over T.

Proof. Let H be as in Lemma 6. By Theorem 3.3, there is a Π_1 (Σ_1) formula $\eta(x)$ numerating H in T and such that if $k \notin H$, then
(1) $T + \{\eta(m): m \neq k\} \nvdash \eta(k)$.
Let $S = T + \{\eta(k): k \in N\}$. By (1) and since H is coinfinite, S is not finite over T. Suppose S is i.a. over T. Let $\varphi_n := \eta(0) \wedge ... \wedge \eta(n)$. By Lemma 5, there is then a recursive function f(n) such that for every n, $S \vdash f(\varphi_n)$ and $T \nvdash \varphi_n \to f(\varphi_n)$. There is a recursive function g(n) such that for every n, $T \vdash \varphi_{g(n)} \to f(\varphi_n)$. It follows that $T \nvdash \varphi_n \to \varphi_{g(n)}$. Let $h(0) = 0$ and $h(n+1) = g(h(n))$. Then for every n, $T \nvdash \varphi_{h(n)} \to \varphi_{h(n+1)}$. But $T \vdash \eta(k)$ for $k \in H$. It follows that $\{k: h(n) < k \leq h(n+1)\} \not\subseteq H$ for every n, contradicting Lemma 6. ■

Corollary 4. If T is finite, there is a Π_1 (Σ_1) formula $\eta(x)$ such that $T + \{\eta(k): k \in N\}$ is not i.a.

Let $S = T + \{\varphi_k: k \in N\}$. Suppose S is i.a. over T. By the proof of Lemma 5, there are conjunctions ψ_m of the sentences φ_k such that if $\theta_0 := \psi_0$, $\theta_{m+1} := \psi_m \to \psi_{m+1}$, then $T + \{\theta_k: k \in N\}$ is an axiomatization of S which is irredundant over T. However, irredundance has been obtained at the price of a slight increase in complexity: supposing that the sentences φ_k are Γ, it does not follow that this is true of the sentences θ_k. Thus, we may ask if irredundance can always be achieved without raising complexity. By our next result, the answer is negative.

Let us say that S is *irredundantly Γ-axiomatizable* (*i. Γ-a.*) *over* T, if there is an r.e. set $Z \subseteq \Gamma$ such that $T + Z$ is an axiomatization of S which is irredundant over T.

Theorem 8. If $PA \dashv T$, there is a Π_n formula $\xi(x)$ such that $T + \{\xi(k): k \in N\}$ is i.a. over T but not i. Π_n–a. over T.

The proof of Theorem 8 uses methods which will be developed in Chapter 5; it is given at the end of that chapter.

Exercises for Chapter 4.

1. (a) Show that

$$PA + \varphi + Rfn_S \vdash Rfn_{S+\varphi},$$

$PA + \varphi + RFN_S \vdash RFN_{S+\varphi}$.

(b) Let

$Rfn_S(\Gamma) = \{Pr_S(\varphi) \to \varphi : \varphi \text{ is } \Gamma\}$,

$RFN_S(\Gamma) := \forall x(\Gamma(x) \wedge Pr_S(x) \to Tr_\Gamma(x))$.

(i) Improve (a) by showing that

if φ is Γ, then $PA + \varphi + Rfn_S(\Gamma^d) \vdash Rfn_{S+\varphi}(\Gamma^d)$,

if φ is Γ, then $PA + \varphi + RFN_S(\Gamma^d) \vdash RFN_{S+\varphi}(\Gamma^d)$.

(ii) Show that

if $Q \dashv S$, then $PA + Con_S \vdash Rfn_S(\Pi_1)$,

$PA + RFN_S(\Sigma_n) \vdash RFN_S(\Pi_{n+1})$.

(iii) Suppose $PA \dashv T$. Show that

if $X \subseteq \Gamma$ is r.e. and $T + X \vdash Rfn_T(\Gamma^d)$, then $T + X$ is inconsistent,

if $X \subseteq \Gamma$ and $T + X \vdash RFN_T(\Gamma^d)$, then $T + X$ is inconsistent.

Define the sets $Rfn_S^\omega(\Gamma)$ and $RFN_S^\omega(\Gamma)$ in the natural way. Suppose S and T are true. Conclude that

$T + Rfn_S^\omega \nvdash RFN_T(\Sigma_1)$,

$T + Rfn_S^\omega(\Sigma_n) \nvdash Rfn_T(\Pi_n)$ for $n \geq 2$,

$T + RFN_S^\omega(\Pi_n) \nvdash Rfn_T(\Sigma_n)$.

2. Suppose $PA \dashv T$. Let

$RFN_\tau = \{\forall x(\Gamma(x) \wedge Pr_\tau(x) \to Tr_\Gamma(x)) : \Gamma \text{ arbitrary}\}$.

Let φ be any sentence such that $T \nvdash \varphi$. Show that there is a PR binumeration $\tau(x)$ of T such that $T + RFN_\tau \nvdash \varphi$.

3. Suppose $PA \dashv T$ and T is Σ_1–sound.

(a) Show that $T + Rfn_T(\Gamma)$ is not essentially infinite over T.

(b) Let S be such that $T + Rfn_T(\Sigma_1) \dashv S \dashv T + Rfn_T$. Show that S is infinite over T. [Hint: Use (the proof of) Theorem 5 and Theorem 2.4.]

4. (a) Suppose the formula $\alpha(x)$ is such that for every φ,

if $T \vdash \varphi$, then $T \vdash \alpha(\varphi)$.

Show that there is a sentence ψ such that $T \nvdash \alpha(\psi) \to \psi$. [Hint: Use Exercise 1.4.]

(b) Suppose there is a formula $\alpha(x)$ such that for every φ,

if $\vdash \varphi$, then $T \vdash \alpha(\varphi)$,

$T \vdash \alpha(\varphi) \to \varphi$.

Show that T is not finitely axiomatizable. (This also follows by the proof of Theorem 1 with $S = \varnothing$.)

5. T is *reducible to* S if there is a recursive function g(n) such that for all sentences φ, (i) $T \vdash g(\varphi)$ and (ii) if $T \vdash \varphi$, then $S \vdash g(\varphi) \to \varphi$. If T is a finite extension of S, $T = S + \theta$, then T is reducible to S: let $g(\varphi) = \theta$ for every φ. Prove the following result, a strengthening of Theorem 1: if $Rfn_S \dashv T$, T is not reducible to S. [Hint: Suppose T is

reducible to S and let g(n) be the relevant recursive function. Let $\delta(x,y)$ be such that for every sentence φ,

$$Q\vdash \delta(\varphi,y) \leftrightarrow y = (g(\varphi)\rightarrow\varphi)$$

(cf. Fact 3). Let ψ be such that

$$Q\vdash \psi \leftrightarrow \exists y(\delta(\psi,y) \wedge \neg Pr_S(y)).$$

Show that $T\vdash \psi$ and $Q\vdash \neg\psi$.]

6. (a) Suppose $S\nvdash \varphi$ and $S + \neg\varphi + Z$ is non–i.a. over $S + \neg\varphi$. Show that $S + \{\varphi \vee \psi: \psi\in Z\}$ is non–i.a. over S.

(b) Suppose $T\dashv_p T'$. Show that

(i) there is a theory S such that $T\dashv S\dashv T'$ and S is not i.a. over T,

(ii) if T is finitely axiomatizable, there is a theory S such that $T\dashv S\dashv T'$ and S is not i.a.

7. Suppose $PA\dashv T$. Let X and Y be any r.e. sets of Γ sentences such that if $\varphi\in X$ and $\psi\in Y$, then $T\vdash \varphi \rightarrow \psi$. Show that there is a Γ sentence θ such that if $\varphi\in X$ and $\psi\in Y$, then $T\vdash \varphi \rightarrow \theta$ and $T\vdash \theta \rightarrow \psi$. [Hint: Suppose $\Gamma = \Pi_n$. Suppose X and Y are primitive recursive. Let $\xi(x)$ and $\eta(x)$ be PR binumerations of X and Y. Let $\theta :=$

$$\forall x\big(\eta(x) \wedge \forall y{\le}x(\xi(y) \rightarrow \neg Tr_{\Pi_n}(y)) \rightarrow Tr_{\Pi_n}(x)\big).]$$

8. Suppose $PA\dashv T$ and T is Σ_1–sound.

(a) Show that there is a Π_1 formula $\beta(x)$ such that for every m, $T + \beta(m)$ is consistent and

$$T + \beta(m)\vdash Con_{T+\beta(m+1)}.$$

(Note that if T is true, so are the theories $T + \beta(m)$.) [Hint: Let the primitive recursive function f be defined (in T and in the real world) as follows; we assume that $\delta(x)$ is a PR formula:

$$f(\delta,\xi,0) = 0,$$

$f(\delta,\xi,n+1) = m$ if $m > f(\delta,\xi,n)$,
$\forall z{\le}m\neg\delta(z)$,
n is a proof in T of $\neg\xi(\delta,m)$,
if there is such a number m,
$= f(\delta,\xi,n)$ otherwise.

If the value of f(k,m,n) is not determined by these conditions, it is irrelevant and we may set $f(k,m,n) = 0$.

Next let $\gamma(z,x)$ be such that

$$PA\vdash \gamma(z,x) \leftrightarrow \forall y(f(z,\gamma,y) \le x).$$

Let $g(k,s) = f(k,\gamma,s)$.

Claim. If $\exists x\delta(x)$ is true, then for every n, $g(\delta,n) = 0$.

Proof. Let k be the least number such that $\delta(k)$ is true. Then for every n, $g(\delta,n) \le k$. Thus, if the claim is false, there is a largest n such that $g(\delta,n) \neq g(\delta,n+1)$. Let $m = g(\delta,n+1)$. Then n is a proof of $\neg\gamma(\delta,m)$. It follows that $\neg\forall y(g(\delta,y) \le m)$ is provable and so is true, a contradiction.

Let $\delta'(x)$ be a PR formula such that

$$PA \vdash \exists x\delta'(x) \leftrightarrow Pr_T(\neg\forall y(g(\delta',y) = 0)).$$

Let $\beta(x) := \forall y(g(\delta',y) \leq x)$.]

(b) Show that with each rational number $a \geq 0$, we can effectively associate a Π_1 sentence θ_a such that $T + \theta_a$ is consistent and if $a < b$, then $T + \theta_a \vdash Con_{T+\theta_b}$. [Hint: Define a function g in much the same way as in case (a) except that g may, in a sense, take rational numbers ≥ 0 as values.]

Notes for Chapter 4.

Theorems 1 and 2 are due to Kreisel and Lévy (1968). The formula $Pr_{S,\Gamma}(x)$ and the present formulation of the proof of Theorem 2 are due to Smoryński (1981b). Corollary 1 (a) is due to Montague (1961) and Rabin (1961). What we have called the uniform reflection principle RFN_S is not quite what is usually referred to by that term, but for theories containing PA the difference is negligible. Theorem 3 is due to Lindström (1984a). Corollary 2 is due to Kreisel and Lévy (1968). Theorem 4 (b) is a weak form of a result of Feferman (1962). For (partial) improvements of Theorems 1, 2, 4 and Corollaries 1, 2, see Exercise 1. Theorem 5 is due to Goryachev (1986) (with a different proof); the bound 2^{n+1} obtained in the proof is far from optimal; using methods not explained here, it can be shown that n+2 will do (cf. also Beklemishev (1995)).

More information on (transfinite) iterations of consistency statements and reflection principles, a rather technical subject which falls outside the scope of this book, can be found in Feferman (1962) and Beklemishev (1995).

What we have called an irredundant axiomatization is usually called an independent axiomatization. Theorem 6 is due to Montague and Tarski (1957). Lemma 5 is due to Tarski (cf. Montague and Tarski (1957)). For a proof of the existence of an r.e. set as described in Lemma 6, a so called *hypersimple* set, see Soare (1987). The idea of using hypersimple sets to construct non–i.a. theories is due to Kreisel (1957). Theorem 7 is related to a result of Pour–El (1968) and Corollary 4 is Pour–El's result restricted to theories in L_A. Theorem 8 is new; Theorem 8 with Π_n replaced by Σ_n and restricted to Σ_n–sound theories is also true but seems to require a quite different proof.

Exercise 3 (b) is due to Beklemishev (199?). Exercise 4 is due to Montague (1963). Exercise 5 is due to Kreisel and Lévy (1968). Exercise 8 (a) was proved by Harvey Friedman, Smoryński, and Solovay, independently, answering a question of Haim Gaifman; for a different proof, due to Friedman, see Smoryński (1985), p. 179. Exercise 8 (b) is due to Alex Wilkie (with a different proof); see Simmons (1988). The present proof can be modified to yield much stronger conclusions.