Convergence Laws for Random Graphs

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Abstract. A convergence law in logic pertains to a language and class of finite structures on which a probability measure has been assigned to each set of structures of a given size. It states that, for every sentence in the language, the probability that it holds for a random structure approaches a limit as the size of the structure grows. A 0-1 law states that this limit must be 0 or 1. The original convergence law was a 0-1 law for first-order logic and relational structures with a uniform probability distribution. This expository article shows how it has been extended by numerous authors to more powerful logics and the class of random structures known as random graphs. In many cases, the 0-1 law no longer holds, but a convergence law can still be proven. Full proofs are not given, but a uniform framework is provided which emphasizes ideas common to all the proofs.

1 Background

Let us begin by introducing the conventions we will use. $\mathcal{L}$ will denote a logic of some type τ. $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots$ will stand for a sequence of sets of τ-structures where all structures in $\mathcal{C}_n$ have universe $\{0, 1, \ldots, n-1\}$, which we will abbreviate as n. For every $n \in \omega$, pr_n is a probability measure on $\mathcal{C}_n$. For a sentence $\sigma \in \mathcal{L}$, we put $\mathrm{pr}(\sigma, n)$ for

$$\mathrm{pr}_n(\{\mathfrak{A} \models \sigma : \mathfrak{A} \in \mathcal{C}_n\}) \ .$$

In this article, we will examine the behavior of $\mathrm{pr}(\sigma, n)$ for growing n (given $\mathcal{L}$, $(\mathcal{C}_n)_{n\in\omega}$, and $(\mathrm{pr}_n)_{n\in\omega}$). The first theorems on this topic considered a first-order logic $\mathcal{L}$ of a purely relational type τ, where each $\mathcal{C}_n$ is the set of all τ-structures on n, and pr_n is the uniform distribution. For example, if $\mathcal{L}$ has a single relational symbol of arity r, then

$$\mathrm{pr}(\sigma, n) = \frac{|\{\mathfrak{A} \models \sigma : \mathfrak{A} \in \mathcal{C}_n\}|}{2^{n^r}} \ .$$

Theorem 1 Fagin [11], Glebskiĭ et al. [13]. *For every $\sigma \in \mathcal{L}$,*

$$\lim_{n\to\infty} \mathrm{pr}(\sigma, n) = 0 \ \mathit{or}\ 1 \ .$$

* Research supported by NSF Grant CCR-94006809

When this theorem first appeared, it did not attract much attention from the logic community. It was mostly combinatorialists and computer scientists who appreciated it, and they suggested two orthogonal directions for future research. The combinatorialists pointed out that there are many other classes of random structures besides uniformly distributed relations, and the computer scientists felt that first-order logic was not powerful enough to describe interesting computational processes.

However, when one attempts to extend the 0-1 law in either direction, it often fails. Here are some examples where it fails on classes of structures which are not purely relational (but still with a uniform distribution).

Glebskiĭ et. al. [13]: Add constants $c_1, c_2, \ldots$ to $\mathcal{L}$. Then if R is a binary relation symbol,

$$\mathrm{pr}(R(c_1, c_1), n) = \frac{1}{2} .$$

In fact, every dyadic rational (i.e., of the form $u/2^v$) in $[0,1]$ is $\mathrm{pr}(\sigma, n)$ for some σ in this language and sufficiently large n.

Mycielski [23]: To each $\mathfrak{A} \in \mathcal{C}$ we add the successor relation

$$\{ (x, x+1) : x < n-1 \}.$$

Then

$$\mathrm{pr}(\exists x \forall y (x \neq y + 1 \wedge R(x,x)), n) = \frac{1}{2} .$$

Again, all possible dyadic rationals are realized.

Fagin [11]: Let f be a unary function symbol. Then

$$\begin{aligned} \mathrm{pr}(\forall x (f(x) \neq x), n) &= \left(\frac{n-1}{n}\right)^n \\ &\to \frac{1}{e} \text{ as } n \to \infty . \end{aligned}$$

In cases where the 0-1 law fails, it is sometimes possible to prove something almost as strong. A *convergence law* states that for every $\sigma \in \mathcal{L}$, $\lim_{n\to\infty} \mathrm{pr}(\sigma, n)$ exists. The three examples above have convergence laws:

Glebskiĭ et. al. [13]: Let $\mathcal{L}_1$ be a first-order language with relational and constant symbols only. Then for every $\sigma \in \mathcal{L}_1$,

$$\lim_{n\to\infty} \mathrm{pr}(\sigma, n) = \frac{u}{2^v} \text{ for some } u, v \in \omega .$$

Lynch [16]: Let $\mathcal{L}_2$ be a first-order language with relational symbols only, plus one binary relational symbol which is always interpreted as the successor relation. Then for every $\sigma \in \mathcal{L}_2$,

$$\lim_{n\to\infty} \mathrm{pr}(\sigma, n) = \frac{u}{2^v} \text{ for some } u, v \in \omega .$$

Lynch [17]: Let $\mathcal{L}_3$ be a first-order language with unary function symbols only. Then for every $\sigma \in \mathcal{L}_3$,

$$\lim_{n\to\infty} \mathrm{pr}(\sigma, n)$$

exists, and is denoted by an expression constructed from 1, +, −, ·, /, and the Poisson operators $e^{-x}x^k/k!$.

Note that these convergence laws are still subject to the criticisms of the computer scientists and combinatorialists, since they pertain only to first-order logic and uniform distributions. In the remainder of this article, we will explore two directions of research: more powerful logics, and an important class of random structures known as random graphs. Another problem we will look at is obtaining *rate estimates.* These are approximations of how fast the probabilities of sentences converge. As we will see, rate estimates have potential applications to optimization of database queries.

2 Recursive Logics and Infinitary Logics

Recursive logics and infinitary logics were developed by Barwise [3], Moschovakis [22], and others, in order to study the model theory of infinite structures. However, they are currently the focus of much of the research in finite model theory, because of motivations from computer science. For example, various complexity classes are *captured* by recursive logics. That is, every problem in such a class can be defined by a sentence in the appropriate logic. Also, recursive logics are needed to express the queries in modern relational database languages. Before defining some of these logics, we will make a digression into database theory. There are two reasons. First, it will provide some motivation for the study of recursive and infinitary logics in finite model theory, and second, it will introduce an important equivalence relation that has been used in proofs of the convergence laws (and nonconvergence laws) and rate estimates.

A *database machine* is an abstraction of a relational database system. It consists of a control unit that interacts with a finite model. The control unit is similar to a Turing machine, but restrictions are often imposed on its power. In addition, it cannot manipulate its data (the finite model) directly. In a single step, it can only perform a first-order query. A *query* is an automorphism-invariant mapping of finite models to finite models. That is, given a collection of relations, it computes a collection of relations. A 0-ary query computes the value TRUE or FALSE. A first-order query is one that is defined by a first-order formula. Thus, a 0-ary first-order query is defined by a sentence.

The earliest relational database systems could perform only first-order queries. It was quickly realized that these were inadequate for many applications; for instance, it is not possible to test whether a relation is connected or to compute the transitive closure of a relation using only first-order queries. Consequently, relational database systems that had the power of recursion were developed. The canonical example of a database machine for testing the connectedness of a graph is the following. Let

E be the symbol for the edge relation of the graph,
R be another binary relation symbol,
$\Phi(E, R)$ be the query defined by $\{ (x, y) : \exists z(E(x, z) \wedge R(z, y)) \}$.

INPUT: graph $\langle V, E\rangle$.
PROGRAM: $R := \{ (x, x) : x \in V \}$;
while $R \neq \Phi(E, R)$ **do** $R := \Phi(E, R)$.
OUTPUT: **if** $\forall x \forall y R(x, y)$ **then** "YES" **else** "NO."

Note that the final value of R computed by the program is the transitive closure of E. Because the ability to compute transitive closures of relations is important in many database applications, a language known as *transitive closure logic* (TCL) has been developed. It is first-order logic augmented with the transitive closure operator. Let $\sigma(\overline{x}, \overline{y}, \overline{z})$ be a formula of type τ whose free variables are

$$\overline{x} = x_1, \ldots, x_i,$$
$$\overline{y} = y_1, \ldots, y_j, \text{ and}$$
$$\overline{z} = z_1, \ldots, z_j,$$

where $i, j \geq 0$. Then $\mathrm{TC}(\sigma, \overline{x}, \overline{y}, \overline{z})$ is a formula with the same free variables as σ such that for any τ-structure $\mathfrak{A}$ with universe A and $\overline{a} = a_1, \ldots, a_i \in A$,

$$\{(\overline{b}, \overline{c}) \in A^{2j} : \mathfrak{A} \models \mathrm{TC}(\sigma, \overline{a}, \overline{b}, \overline{c})\}$$

is the transitive closure of

$$\{(\overline{b}, \overline{c}) \in A^{2j} : \mathfrak{A} \models \sigma(\overline{a}, \overline{b}, \overline{c})\}.$$

The least fixed point is another logical operator more powerful than the transitive closure. Let $\varphi(\overline{x}, S)$ be a formula of type τ extended with the new k-ary relational symbol S whose free variables are $\overline{x} = x_1, \ldots, x_k$. For any τ-model $\mathfrak{A}$ with universe A, φ induces an operator $\Phi : \mathcal{P}(A^k) \to \mathcal{P}(A^k)$. Given $S \subseteq A^k$,

$$\Phi(S) = \{ (a_1, \ldots, a_k) \in A^k : \mathfrak{A} \models \varphi(a_1, \ldots, a_k, S) \}.$$

Inductively, we can define the iterates of Φ:

$$\Phi^0 = \emptyset,$$
$$\Phi^{i+1} = \Phi(\Phi^i) \ .$$

We say the operator Φ is monotone if $S \subseteq T$ implies $\Phi(S) \subseteq \Phi(T)$. In that case, $\Phi^i \subseteq \Phi^{i+1}$ for all $i \in \omega$. If there is some i such that $\Phi^i = \Phi^{i+1}$, then Φ^i is said to be the *least fixed point* of Φ. The formula φ is positive in S if each occurrence of S is within an even number of $\neg$ signs. It is easily seen that if φ is positive, then Φ is monotone. The Knaster-Tarski theorem (see [22]) states that if Φ is monotone then Φ has a least fixed point. Of course, if $\mathfrak{A}$ is finite, this is obvious. In fact, it must occur on or before $|A|^k$ iterations. We put $\varphi^\infty(\overline{x})$ for the least fixed point of Φ. The least fixed point logic (LFP) is first-order logic closed under the least fixed point of positive formulas.

Other operators, such as the partial fixed point, can be added to first-order logic, resulting in the logic PFP. But all the logics mentioned here are contained in the infinitary logic $L^{\omega}_{\infty\omega}$.

Definition 2. For $k \in \omega$, $L^{k}_{\infty\omega}$ is the extension of first-order logic obtained by closure under conjunction and disjunction of arbitrary sets of formulas, provided only the variables $x_1, \ldots, x_k$ occur among them. Then

$$L^{\omega}_{\infty\omega} = \bigcup_{k\in\omega} L^{k}_{\infty\omega} \ .$$

It can be shown that

$$\mathrm{TC} \subset \mathrm{LFP} \subseteq \mathrm{PFP} \subset L^{\omega}_{\infty\omega} \ .$$

The first inclusion is known to be proper. The last inclusion is easily seen to be proper because $L^{\omega}_{\infty\omega}$ can express non-recursive properties, whereas all properties expressible in the other logics are recursive. It is not known whether $LFP \subset PFP$. As shown by Abiteboul and Vianu [1], this is equivalent to showing that P is properly contained in PSPACE, so it should not be expected that it is easy to answer.

Continuing with our example, let us show how transitive closure and connectedness can be expressed in $L^{\omega}_{\infty\omega}$. Let R^i be the value of R after the ith iteration of the above program. Then

$$R^0 = \{ (x,x) : x \in V \} \text{ and}$$
$$R^i = \{ (x,y) : \exists z (E(x,z) \wedge R^{i-1}(z,y)) \} \ .$$

Inductively, we can express $R^i(x,y)$ using only three variables:

$$\exists z (E(x,z) \wedge \exists x (x = z \wedge R^{i-1}(x,y))) \ .$$

Then we can express "$\langle V, E\rangle$ is connected" using an infinite disjunction and only three variables:

$$\forall x \forall y \Big(\bigvee_{i\in\omega} R^i(x,y) \Big) \ .$$

A fundamental problem in the model theory of $L^{\omega}_{\infty\omega}$ is, given $k \in \omega$, to characterize those relations (i.e., queries) that can be defined by a formula in $L^{k}_{\infty\omega}$. The following equivalence relation is central to this problem.

Definition 3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of the same type, $l \leq k$, $a_1, \ldots, a_l \in \mathfrak{A}$, and $b_1, \ldots, b_l \in \mathfrak{B}$. Then $\langle \mathfrak{A}, a_1, \ldots, a_l \rangle \equiv^k_\infty \langle \mathfrak{B}, b_1, \ldots, b_l \rangle$ if and only if for all $\sigma(x_1, \ldots, x_l) \in L^{k}_{\infty\omega}$,

$$\mathfrak{A} \models \sigma(a_1, \ldots, a_l) \Leftrightarrow \mathfrak{B} \models \sigma(b_1, \ldots, b_l) \ .$$

This equivalence relation plays a key role in many proofs of the expressibility of properties in $L^{\omega}_{\infty\omega}$, and as we will outline later, the convergence laws for $L^{\omega}_{\infty\omega}$. Since database queries are expressible in this infinitary logic, it is also useful in studying the power of database languages. For example, let M be a database machine. Given an input $\mathfrak{A}$, M computes the model $M(\mathfrak{A})$. As is often the case, we assume that $\mathfrak{A}$ and $M(\mathfrak{A})$ have the same universe. Let R be an l-ary relation in $M(\mathfrak{A})$. For inputs $\mathfrak{A}$ and $\mathfrak{B}$, $a_1, \ldots, a_l \in \mathfrak{A}$, and $b_1, \ldots, b_l \in \mathfrak{B}$, put $\langle \mathfrak{A}, a_1, \ldots, a_l \rangle \equiv_{M,R} \langle \mathfrak{B}, b_1, \ldots, b_l \rangle$ if and only if

$$M(\mathfrak{A}) \models R(a_1, \ldots, a_l) \Leftrightarrow M(\mathfrak{B}) \models R(b_1, \ldots, b_l) \ .$$

Then, for some k, $\equiv^k_\infty$ refines $\equiv_{M,R}$.

We conclude this section with a potential application of these ideas to database query optimization. When a model $\mathfrak{A}$ is the input to a database machine, it must be encoded as a string of bits. If the the universe of $\mathfrak{A}$ has n elements, then the length of this encoding is on the order of n^r for some fixed r. Let $l \leq k$. Suppose that there is a function $d\colon \omega \to \omega$ such that $d(n) \ll n^r$, and for every n, there is a collection $C_1, \ldots, C_{d(n)}$ of $\equiv^k_\infty$ classes of l-tuples on inputs $\mathfrak{A}$ of size n such that for almost all $\mathfrak{A}$, every l-tuple is in $\bigcup_{i=1}^{d(n)} C_i$. Then there is the possibility of optimizing $\sigma(x_1, \ldots, x_l) \in L^k_{\infty\omega}$. The case studied by Abiteboul, Compton, and Vianu [2] was for the partial fixed-point logic of a relational type, and the uniform probability distribution on models of that type. Using rate estimates, they proved that, for every $l \leq k \in \omega$, there is a *finite* set of $\equiv^k_\infty$ classes $C_1, \ldots, C_d$ and first-order formulas $\phi_1(x_1, \ldots, x_l), \ldots, \phi_d(x_1, \ldots, x_l)$ such that:

1. For all $\mathfrak{A}$, $a_1, \ldots, a_l \in \mathfrak{A}$, and all $i = 1, \ldots, d$,

$$\langle \mathfrak{A}, a_1, \ldots, a_l \rangle \in C_i \text{ if and only if } \mathfrak{A} \models \phi_i(a_1, \ldots, a_l) \ .$$

2. There is $c < 1$ such that

$$\mathrm{pr}\left(\exists x_1 \ldots \exists x_l \left((x_1, \ldots, x_l) \notin \bigcup_{i=1}^{d} C_i \right), n \right) < c^n \ .$$

On a parallel processor, the time complexity of a first-order query is bounded above by a constant, but the known methods for answering a fixed-point query require time n^t for some t, in the worst case. However, by exploiting facts 1. and 2. above, the average time complexity of the fixed-point query σ is

$$\begin{aligned} &\leq \mathrm{pr}\left(\forall x_1 \ldots x_l \left((x_1, \ldots, x_l) \in \textstyle\bigcup_{i=1}^{d} C_i \right), n \right) \\ &\qquad \times \max\nolimits_{1 \leq i \leq d}(\text{complexity}(\langle \mathfrak{A}, x_1, \ldots, x_l \rangle \in C_i)) \\ &\qquad + \mathrm{pr}\left(\exists x_1 \ldots \exists x_l \left((x_1, \ldots, x_l) \notin \textstyle\bigcup_{i=1}^{d} C_i \right), n \right) \times \text{complexity}(\sigma) \\ &\leq \max\nolimits_{1 \leq i \leq d}(\text{complexity}(\phi_i)) + c^n \times n^t \leq K \end{aligned}$$

for some constant K. Thus, on average, the cost of a fixed-point query is also constant.

3 Random Graphs

Random graph theory was initiated by Erdős and Rényi [10], and has since grown into a very active branch of combinatorics. A good introductory text is Palmer [24]; for those with some familiarity with the subject, a standard reference is Bollobás [5]. We will consider only one model of random graph, but it is by far the most widely studied. Unlike our previous examples, the probability distribution on random graphs of a given size is not in general uniform.

For each $n \in \omega$, $\mathcal{C}_n$ is the collection of undirected graphs whose vertex set is n. The probability distribution pr_n on $\mathcal{C}_n$ is defined in terms of the *edge probability*, a function $p(n)$ taking values in $[0,1]$. For any $\langle n, E\rangle \in \mathcal{C}_n$,

$$\mathrm{pr}_n(\{\langle n, E\rangle\}) = p(n)^{|E|}(1-p(n))^{\binom{n}{2}-|E|} \ .$$

Another, perhaps more intuitive, way of regarding this probability is to construct the random graph $\langle n, E\rangle$ by choosing independently for each pair $\{i,j\} \subseteq n$, that $\{i,j\} \in E$ with probability $p(n)$. It is evident that when $p(n) = 1/2$ for all n, this is identical to the uniform distribution.

There are many possible edge probabilities, but we will restrict our attention to those of the form $p(n) = \beta n^{-\alpha}$, where $\alpha, \beta \geq 0$. This is a crude characterization of edge probabilities, but it is important because it is simple, involving only two parameters α and β, and yet by varying these parameters, a wide range of monotonically decreasing edge probabilities is covered. (Increasing edge probabilities are a symmetric case because we can replace $p(n)$ by $1-p(n)$.) Constant edge probabilities occur when $\alpha = 0$. In this case, we assume $0 < \beta < 1$. When $\alpha > 0$, the graph is said to be "sparse." As α increases, the random graph becomes sparser, i.e., its edge density decreases, until $\alpha > 2$, when almost all random graphs have no edges. Thus the interesting values of α are in the range $[0,2]$. A slight generalization of the proofs of Fagin and Glebskiĭ et. al. extends their 0-1 law to all random graphs with constant edge probabilities.

Random graph theorists are particularly interested in threshold functions. These are parameterized functions that characterize the random graph such that as the parameter passes through a certain value (the threshold), the qualitative behavior of the random graph changes. In the logics we will be considering, α is the important parameter for the edge probability. For example, the asymptotic probability that the graph has a 4-clique is 0 if $\alpha > 2/3$, 1 if $\alpha < 2/3$, and $1 - e^{-\beta^6/24}$ if $\alpha = 2/3$. In the logics we will be studying, when $\alpha > 0$, the value of β does not affect the qualitative behavior of the random graph as long as it is greater than 0, so we will assume it is 1 in this case.

Table 1. summarizes the behavior of random graphs, categorized horizontally according to α, with the sparsest graphs to the left, and vertically according to the logic: FOL (first-order logic), TCL (transitive closure logic), LFP (least fixed point logic), and $L^{\omega}_{\infty\omega}$.

	$\alpha > 1$, $\alpha = \frac{l+1}{l}$	$\alpha > 1$, $\alpha \neq \frac{l+1}{l}$	$\alpha = 1$	$\alpha \in (0,1)$, rational	$\alpha \in (0,1)$, irrational	$\alpha = 0$
$L^{\omega}_{\infty\omega}$	Con[7]	0-1[7]	Noncon	Noncon	Noncon	0-1[5]
LFP	Con	0-1	Noncon	Noncon	Noncon[9]	0-1[3]
TCL	Con	0-1	Noncon[8]	Noncon	0-1[9]	0-1
FOL	Con[6]	0-1[4]	Con[6]	Noncon[4]	0-1[4]	0-1[1,2]

Table 1: Summary of convergence laws.

Three types of behavior are distinguished: 0-1, Con, and Noncon, standing for 0-1 law, convergence law, and nonconvergence law respectively. The superscripts are cross-references to the following list of citations. They are arranged in chronological order of journal publication dates, although in many cases, they were preceded by publications in conference proceedings. In the cases that are not cross-referenced, the behavior is a simple consequence of either a 0-1 or convergence law above, or a nonconvergence result below. Perhaps the most striking feature about this table is the threshold at $\alpha = 1$. For all other values of α, the behavior of FOL and TCL is the same. For all four logics, it demarks the onset of complex behavior, i.e., nonconvergence. This is not surprising because it has been long known that the probabilities of important graph properties change when α passes through the value 1. For example, when $\alpha > 1$, the random graph is almost surely planar and not connected, but when $\alpha < 1$, it is almost surely nonplanar and connected. In fact, a much more precise analysis of this threshold has been done (see Bollobás [5]).

1. Glebskiĭ et al. [13]. Also contains the convergence law for structures with relations and constant mentioned earlier.
2. Fagin [11]. Also contains a 0-1 law for unlabelled relational structures. That is, two structures are identified if they are isomorphic.
3. Blass, Gurevich, and Kozen [4]. Actually proven for the stronger partial fixed point logic.
4. Shelah and Spencer [27]. A rate estimate when α is irrational is given in Lynch [20].
5. Kolaitis and Vardi [15].
6. Lynch [18].
7. Lynch [19].
8. Tyszkiewicz [28]. Improved to deterministic transitive closure logic in Lynch and Tyszkiewicz [21].
9. Lynch, McArthur, Tyszkiewicz, and Spencer (in preparation). They also show that the convergence law fails for LFP, which immediately imples that TCL is weaker than LFP. It was incorrectly stated in [21] that convergence holds for $L^{\omega}_{\infty\omega}$

The proofs of 0-1 or convergence law in each category $\alpha > 1$, $\alpha = 1$, $\alpha < 1$ irrational, and $\alpha = 0$ are all quite different. There is, however, a unifying principle in all the proofs. It is the use of combinatorial pebble games to characterize definable properties of structures. The original pebble game was invented by Ehrenfeucht [8] to characterize the equivalence classes of structures indistinguishable by first-order sentences of bounded quantifier rank. The quantifier rank of a first-order sentence is the depth of nesting of its quantifiers. These classes had been studied earlier by Fraïssé [12], and the game is often called the Ehrenfeucht-Fraïssé game in their honor. We will describe the game as it is played on two graphs $G_i = \langle V_i, E_i \rangle$, for $i = 0, 1$. The game consists of k rounds numbered $1, \ldots, k$, for some fixed $k \in \omega$, and is played by two players, which we will refer to as I and II. There are two sets of pebbles numbered $1, \ldots, k$. In each round r, player I places one of the pebbles numbered r on a vertex of G_0 or G_1, and then player II places the other pebble numbered r on a vertex of the other graph (G_1 or G_0). For $i = 0, 1$ and $r = 1, \ldots, k$, let $a_{i,r}$ be the vertex in V_i that is pebbled (by either player) in round r. After round k, we say that player II has won if and only if the structures $\langle \{a_{i,1}, \ldots, a_{i,k}\}, E_i \restriction \{a_{i,1}, \ldots, a_{i,k}\}, a_{i,1}, \ldots, a_{i,k}\} \rangle$, $i = 0, 1$, are isomorphic. That is, the induced subgraphs on $\{a_{i,1}, \ldots, a_{i,k}\}$ are isomorphic via $a_{0,r} \mapsto a_{1,r}$.

Definition 4. Let G_0 and G_1 be two graphs and $k \in \omega$.

1. $G_0 \sim^k G_1$ if and only if player II has a winning strategy for the k-round Ehrenfeucht-Fraïssé game on G_0 and G_1. That is, no matter how player I moves, player II can always respond so that he wins after k rounds.
2. $G_0 \equiv^k G_1$ if and only if G_0 and G_1 satisfy the same sentences of quantifier rank $\leq k$.

It is easily seen that $\sim^k$ and $\equiv^k$ are equivalence relations. The key result on the game is the following.

Theorem 5 Ehrenfeucht [8]. *Let G_0, G_1, and k be fixed. Then the following are equivalent.*

1. $G_0 \sim^k G_1$.
2. $G_0 \equiv^k G_1$.

There is a similar game for the infinitary logic $L^{\omega}_{\infty\omega}$ which does not seem to have a widely accepted name, but it is sometimes called the eternal game. Again, it is played by two players on two graphs G_0 and G_1, and there are two sets of pebbles numbered $1, \ldots, k$. The first k rounds are played exactly like the k-round Ehrenfeucht-Fraïssé game, but after that, it may continue indefinitely. In succeeding rounds, player I moves a pebble to some (possibly the same) vertex in the same graph. Then player II responds by moving the other pebble with the same number to some vertex in the other graph. The game ends and player I wins if, after any round, the mapping $a_{0,j} \mapsto a_{1,j}$ for $j = 1, \ldots, k$ is not an isomorphism.

Definition 6. Let G_0 and G_1 be two graphs and $k \in \omega$.

1. $G_0 \sim^k_\infty G_1$ if and only if player II has a winning strategy for the k-pebble eternal game on G_0 and G_1. That is, no matter how player I moves, player II can always respond so that the game continues.
2. $G_0 \equiv^k_\infty G_1$ if and only if G_0 and G_1 satisfy the same sentences in $L^k_{\infty\omega}$.

Again, $\sim^k_\infty$ and $\equiv^k_\infty$ are the same equivalence relations.

Theorem 7 Barwise [3]. *Let G_0, G_1, and k be fixed. Then the following are equivalent.*

1. $G_0 \sim^k_\infty G_1$.
2. $G_0 \equiv^k_\infty G_1$.

The original proofs of the first-order 0-1 law for $\alpha = 0$ by Fagin and Glebskiĭ et al. did not rely on pebble games. But the Ehrenfeucht-Fraïssé game provides a simpler proof by showing that, for every k, almost all graphs belong to the same $\sim^k$ class. Therefore by Theorem 5, every sentence is almost always false or almost always true. We shall not present it here because the $L^\omega_{\infty\omega}$ 0-1 law for $\alpha = 0$ by Kolaitis and Vardi follows from essentially the same proof using the eternal game instead of the Ehrenfeucht-Fraïssé game. That is, we show that almost all graphs belong to the same $\sim^k_\infty$ class. This equivalence class has a very simple axiomatization. We say a graph is k-extendible if it has at least $k-1$ vertices and, for every set of $k-1$ vertices, every possible one-vertex extension is realized. In other words, it satisfies the following finite set of first-order axioms:

$$\left\{ \exists x_1 \ldots \exists x_{k-1} \Big(\bigwedge_{1 \le i < j < k} x_i \ne x_j \Big) \right\} \cup$$
$$\left\{ \forall x_1 \ldots \forall x_{k-1} \exists y \Big[\Big(\bigwedge_{1 \le i < j < k} x_i \ne x_j \Big) \to \right. \tag{1}$$
$$\left. \Big(\bigwedge_{1 \le i < k} x_i \ne y \wedge \bigwedge_{1 \le i \le j} E(x_j, y) \wedge \bigwedge_{j < i < k} \neg E(x_j, y) \Big) \Big] : j = 0, \ldots, k-1 \right\}$$

The proof of the $L^\omega_{\infty\omega}$ 0-1 law for $\alpha = 0$ is finished by showing:

1. $G_0 \sim^k_\infty G_1$ for any two k-extendible graphs.
2. Almost all graphs are k-extendible.

Both parts have elementary proofs. Part 1. follows because the extension axioms (1) state that player II can always match any move of player I. To prove Part 2., let θ_j, $j = 0, \ldots, k-1$, enumerate the extension axioms, where the j is as in (1). Then, letting $p(n) = \beta$ be the edge probability,

$$\mathrm{pr}\Big(\neg \Big(\bigwedge_{j=0}^{k-1} \theta_j \Big), n \Big) = \mathrm{pr}\Big(\bigvee_{j=0}^{k-1} \neg\theta_j, n \Big)$$
$$\le \sum_{j=0}^{k-1} n^{k-1} \times [\beta^j (1-\beta)^{k-j-1}]^{n-k+1} .$$

Since $0 < \beta < 1$, there is some $b < 1$ such that $\beta^j(1-\beta)^{k-j-1} \leq b$ for all $j = 0, \ldots, k-1$. Then

$$\mathrm{pr}\left(\neg\left(\bigwedge_{j=0}^{k-1} \theta_j\right), n\right) \leq kn^{k-1}b^{n-k+1}$$

$$\to 0 \text{ as } n \to \infty \ .$$

Note that this also yields a rate estimate. Letting $c = \sqrt{b}$, $kn^{k-1}b^{n-k+1} < c^n$ for sufficiently large n, so

$$\mathrm{pr}(\sigma, n) < c^n, \text{ or}$$
$$\mathrm{pr}(\sigma, n) > 1 - c^n \ .$$

A slight generalization of the proof shows that when $\alpha = 0$, $\mathcal{L}^{\omega}_{\infty\omega}$ almost surely collapses to first-order logic. That is, for every formula $\sigma(x_1, \ldots, x_l) \in L^k_{\infty\omega}$, there is a first-order formula $\sigma'(x_1, \ldots, x_l)$ such that

$$\lim_{n\to\infty} \mathrm{pr}(\forall x_1 \ldots \forall x_l(\sigma(x_1, \ldots, x_l) \leftrightarrow \sigma'(x_1, \ldots, x_l)), n) = 1 \ . \tag{2}$$

To construct σ', let $\gamma_1(x_1, \ldots, x_l), \ldots, \gamma_d(x_1, \ldots, x_l)$ be an enumeration of the quantifier-free formulas that describe the possible isomorphism types of l (not necessarily distinct) vertices. For $i = 1, \ldots, d$, let

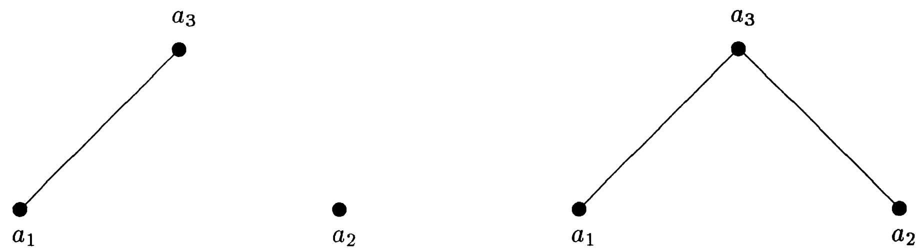
$$\phi_i(x_1, \ldots, x_l) = \gamma_i(x_1, \ldots, x_l) \wedge \bigwedge_{j=0}^{k-1} \theta_j \ .$$

Then, as in 1. above, each first-order formula ϕ_i defines an $\equiv^k_\infty$ class C_i of l-tuples, and, as in 2., for almost all graphs, every l-tuple belongs to some C_i. In fact, the probability of not belonging to some C_i is less than c^n for some $c < 1$. (Note that we have just proven the claims of Abiteboul, Compton, and Vianu mentioned earlier.) Since each C_i is an $\equiv^k_\infty$-class of l-tuples, either

1. for every graph G and vertices $a_1, \ldots, a_l$ in G, $(a_1, \ldots, a_l) \in C_i \Rightarrow G \models \sigma(a_1, \ldots, a_l)$, or
2. for every graph G and vertices $a_1, \ldots, a_l$ in G, $(a_1, \ldots, a_l) \in C_i \Rightarrow G \models \neg\sigma(a_1, \ldots, a_l)$.

Let I be the set of i such that a. holds, and $\sigma' = \bigvee_{i \in I} \phi_i$. Equation (2) now follows.

Player II's strategy in the case $\alpha = 0$ is the simplest possible. He just imitates player I's move without regard to any future moves. But this does not work for nonzero α because there are some moves that are "hard to imitate". This is true even for the first-order Ehrenfeucht-Fraïssé game. For example, take the case when $1 > \alpha > \frac{1}{2}$ and $k \geq 3$. Suppose, as illustrated below, player I places the third pebble on a_3 in one of the graphs, where a_3 is adjacent to a_1 but not to a_2. This is an easy move to imitate, because almost surely, for every two vertices in the random graph, there is a third vertex adjacent to exactly one of them. However, if player I had chosen a_3 adjacent to both a_1 and a_2, then player II will probably lose because almost surely, the random graph contains only $O(n^{3-2\alpha}) = o(n^2)$ pairs of vertices with a common neighbor.



Easy Hard

Thus player II must look ahead to possible future moves. But it is not sufficient to look ahead only one move. In the fourth round, if player I chooses a new vertex adjacent to a_1 and a_2, it is again unlikely that player II can match this move. There is, however, a probabilistic upper bound on the number of such vertices. Almost surely, there is no pair of vertices with more than $2/(2\alpha - 1)$ common neighbors. More generally, there is a probabilistic upper bound on the number of hard moves that player II must consider, and this makes a winning strategy possible.

The notions of hard moves and looking ahead were formalized by Shelah and Spencer in their proof of the first-order 0-1 law for $1 > \alpha > 0$ and irrational. The full proof is much longer than the previous proof of the 0-1 law for $L^{\omega}_{\infty\omega}$, and we will only sketch the highlights.

Definition 8. We will use the convention that if $\langle V, E\rangle$ is a graph, and $W \subseteq V$, then $\langle W, E\rangle$ is the graph $\langle W, E \restriction W\rangle$.

1. $\langle V, E, c_1, \ldots, c_i\rangle$ is a rooted graph if $\langle V, E\rangle$ is a graph, and $c_1, \ldots, c_i \in V$. We also say $\langle V, E, R\rangle$ is a rooted graph when $R \subseteq V$, and the order of the vertices in R is immaterial.
2. The weight of $\langle V, E, R\rangle$, $\zeta(\langle V, E, R\rangle)$, is $|V - R| - \alpha|E - E \restriction R|$.
3. $\langle V, E, R\rangle$ is dense if $\zeta(\langle V, E, R\rangle) < 0$.
4. $\langle V, E, R\rangle$ is rigid if $\langle V, E, S\rangle$ is dense for every set of vertices S such that $R \subseteq S \subset V$. By convention, we will say that $\langle R, E, R\rangle$ is rigid.

Note that, since α is irrational, the weight of a rooted graph cannot be 0 unless $V = R$.

Lemma 9. *If $\langle V, E, R\rangle$ and $\langle W, E, R\rangle$ are rigid, then so is $\langle V \cup W, E, R\rangle$.*

Definition 10. For a rooted graph $\langle V, E, R\rangle$, $Rig(\langle V, E, R\rangle)$ is the maximal set S such that $R \subseteq S \subseteq V$ and $\langle S, E, R\rangle$ is rigid. Note that by Lemma 9, S is unique.

The closure operator Cl in the next definition captures the notion of how far ahead player II must look when there are j moves left in the game.

Definition 11. Let $G = \langle V, E\rangle$. For $i, j \in \omega$ and $a_1, \dots, a_i \in V$, $Cl^j(G; a_1, \dots, a_i)$ is defined by induction on j.

$$\mathrm{Cl}^0(G; a_1, \dots, a_i) = \langle \{a_1, \dots, a_i\}, E, a_1, \dots, a_i\rangle \text{ for all } i \in \omega \ .$$

Assume Cl^j has been defined for all $i \in \omega$. Fixing i and $a_1, \dots, a_i \in V$, for all $a_{i+1} \in V$, let

$$\begin{aligned} \mathrm{Cl}^j(G; a_1, \dots, a_{i+1}) &= \langle V(a_{i+1}), E, a_1, \dots, a_{i+1}\rangle, \text{ and} \\ S(a_{i+1}) &= \mathrm{Rig}(\langle V(a_{i+1}), E, a_1, \dots, a_i\rangle) \ . \end{aligned}$$

Then

$$\mathrm{Cl}^{j+1} = \left\langle \bigcup_{a_{i+1} \in V} S(a_{i+1}), E, a_1, \dots, a_i \right\rangle \ .$$

Lemma 12. *For every $i, j \in \omega$, there is m such that*

$$\mathrm{pr}(\forall a_1 \dots \forall a_i \in G(|\mathrm{Cl}^j(G; a_1, \dots, a_i)| \leq m), n) \to 1$$

as $n \to \infty$.

Because of the preceding lemma, for every k, there is some m such that with probability asymptotic to 1, for all $i \leq k$,

$$\forall a_1 \dots \forall a_i \in G(|\mathrm{Cl}^{k-i}(G; a_1, \dots, a_i)| \leq m) \ .$$

Therefore, the following can be expressed as a finite set of first-order axioms, where $i = 0, \dots, k-1$:

$$\begin{aligned} &\forall x_1 \dots \forall x_i(\\ &\quad \text{for any graph } H = \langle W, F\rangle \text{ and } c_1, \dots, c_{i+1} \in W \text{ such that} \\ &\qquad |\mathrm{Cl}^{k-i}(H; c_1, \dots, c_i)|, |\mathrm{Cl}^{k-i-1}(H; c_1, \dots, c_{i+1})| \leq m, \\ &\qquad\quad \mathrm{Cl}^{k-i}(G; x_1, \dots, x_i) \cong \mathrm{Cl}^{k-i}(H; c_1, \dots, c_i) \\ &\qquad \to \\ &\qquad\quad \exists y(\mathrm{Cl}^{k-i-1}(G; x_1, \dots, x_i, y) \cong \mathrm{Cl}^{k-i-1}(H; c_1, \dots, c_{i+1}))) \ . \end{aligned}$$

We say that a graph is (α, k)-extendible if it satisfies all these axioms. The proof of the first-order 0-1 law for irrational α is completed by showing:

1. $G_0 \sim^k G_1$ for any two (α, k)-extendible graphs.
2. Almost all graphs are (α, k)-extendible.

The proof of the TCL 0-1 law for irrational α uses many of the concepts from the proof of the first-order 0-1 law. In essence, it shows that for almost all graphs, the transitive closure operation stops growing after a bounded number of steps. This happens in two ways: either its growth rate is so rapid that it absorbs all the tuples that it can very quickly, or its growth rate slows down until it stops absorbing new tuples. This is a generalization of the fact that, in a random graph with edge probability $n^{-\alpha}$, if $\alpha < 1$, then every pair of vertices is joined by a path of length $\lceil 1/(1-\alpha) \rceil$, but if $\alpha > 1$, then there is no path of length greater than $1/(\alpha - 1)$.

Definition 13. Let $\sigma(\overline{x}, \overline{y}, \overline{z})$ be a first-order formula in the language of graphs whose free variables are

$$\begin{aligned} \overline{x} &= x_1, \ldots, x_i, \\ \overline{y} &= y_1, \ldots, y_j, \text{ and} \\ \overline{z} &= z_1, \ldots, z_j, \end{aligned}$$

where $i, j \geq 0$. Let $G = \langle V, E \rangle$ be a graph and $\overline{a} = a_1, \ldots, a_i \in V$.

1. $\mathrm{TC}(G, \sigma, \overline{a})$ is the transitive closure of $\{(\overline{b}, \overline{c}) \in V^{2j} : G \models \sigma(\overline{a}, \overline{b}, \overline{c})\}$.
2. An $\overline{a}$-link is a rigid rooted graph $\langle W, F, \overline{a}, \overline{b}, \overline{c} \rangle$ where $\overline{b}, \overline{c} \in V^j$.
3. An $\overline{a}$-chain of length l is a sequence of $\overline{a}$-links $\langle \langle W_k, F_k, \overline{a}, \overline{b_k}, \overline{c_k} \rangle : 1 \leq k \leq l \rangle$ such that $\overline{b_{k+1}} = \overline{c_k}$ for $1 \leq k < l$ and for all $u, v \in W_h \cap W_k$ $(1 \leq h, k \leq l)$, $F_h(u, v)$ if and only if $F_k(u, v)$. We say the chain connects $\overline{b_1}$ to $\overline{c_l}$.

Lemma 14. *There is a finite set of $\overline{a}$-links $\mathcal{S}$ such that for almost all graphs $G = \langle V, E \rangle$, for all $\overline{a} \in V^i$ and $\overline{b}, \overline{c} \in V^j$, $G \models \sigma(\overline{a}, \overline{b}, \overline{c})$ if and only if there is $W \subseteq V$ such that $\langle W, E \restriction W, \overline{a}, \overline{b}, \overline{c} \rangle \in \mathcal{S}$.*

Lemma 15. *For almost all graphs $G = \langle V, E \rangle$, for all $\overline{a} \in V^i$ and $\overline{b}, \overline{c} \in V^j$, $(\overline{b}, \overline{c}) \in \mathrm{TC}(G, \sigma, \overline{a})$ if and only if there is an $\overline{a}$-chain connecting $\overline{b}$ to $\overline{c}$ with links in $\mathcal{S}$.*

Here is the main lemma.

Lemma 16. *There is a constant m such that for almost all graphs $G = \langle V, E \rangle$, for all $\overline{a} \in V^i$ and $\overline{b}, \overline{c} \in V^j$, if $(\overline{b}, \overline{c}) \in \mathrm{TC}(G, \sigma, \overline{a})$, then there is an $\overline{a}$-chain of length at most m connecting $\overline{b}$ to $\overline{c}$ with links in $\mathcal{S}$.*

A consequence of this lemma and the finiteness of $\mathcal{S}$ is that almost surely TCL collapses to first-order logic when $\alpha < 1$ is irrational. The 0-1 law for TCL in this case is an immediate consequence of the 0-1 law for first-order logic.

The proof of the first-order convergence law at the threshold $\alpha = 1$ will not be covered here, since it is quite involved. It has many similarities to the convergence law for unary functions [17] mentioned earlier. Some flavor of the proof can be obtained from a sketch of the $L^{\omega}_{\infty\omega}$ convergence law for $\alpha = (l+1)/l$. Thus we will finish our outline of 0-1 and convergence proofs by moving to the region $\alpha > 1$, i.e., the region of very sparse random graphs. In fact, we will outline a stronger result for all $p(n) = n^{-\alpha}$, $\alpha > 1$: for every $\sigma \in L^{\omega}_{\infty\omega}$, either

$$\mathrm{pr}(\sigma, n) < 2^{-n^d} \text{ for some } d > 0, \text{ or} \quad (3)$$
$$\mathrm{pr}(\sigma, n) \sim cn^{-d} \text{ for some } c > 0,\ d \geq 0\ . \quad (4)$$

If we take $f(n) \sim n^{-\infty}$ to mean $f(n) < n^{-d}$ for all $d > 0$, then the conclusion implies $\mathrm{pr}(\sigma, n) \sim cn^{-d}$ for some $c \in (0, \infty)$, $d \in [0, \infty]$. This kind of rate estimate is known as a *power law* in physics and engineering. When $\alpha > 1$ but not of the form $(l+1)/l$, the proof of the rate estimate shows that if $d = 0$, then $c = 1$. Thus the 0-1 law holds in this case.

When $\alpha > 1$, no version of extension axioms seems applicable. Instead, we use a theorem of Kolaitis which states that for each k, the $\equiv^k_\infty$ class of a graph is determined by counting how many components it has in each $\equiv^k_\infty$ class. In the following, $C \sqsubseteq G$ means that C is a component of G.

Theorem 17 Kolaitis [14]. *Let G_0 and G_1 be two graphs such that for every connected graph C, either*

$$|\{C_0 \sqsubseteq G_0 : C_0 \equiv^k_\infty C\}|, |\{C_1 \sqsubseteq G_1 : C_1 \equiv^k_\infty C\}| \geq k$$

or

$$|\{C_0 \sqsubseteq G_0 : C_0 \equiv^k_\infty C\}| = |\{C_1 \sqsubseteq G_1 : C_1 \equiv^k_\infty C\}|\ .$$

Then $G_0 \equiv^k_\infty G_1$.

We will actually use a weaker version of this result.

Corollary 18 *Let G_0 and G_1 be two graphs such that for every connected graph C, either*

$$|\{C_0 \sqsubseteq G_0 : C_0 \cong C\}|, |\{C_1 \sqsubseteq G_1 : C_1 \cong C\}| \geq k$$

or

$$|\{C_0 \sqsubseteq G_0 : C_0 \cong C\}| = |\{C_1 \sqsubseteq G_1 : C_1 \cong C\}|\ .$$

Then $G_0 \equiv^k_\infty G_1$.

We will use $G_0 \cong^k G_1$ to mean the condition in the Corollary is satisfied by G_0 and G_1. Thus $\cong^k$ is a refinement of $\equiv^k_\infty$.

We will categorize connected graphs in three ways:

(I) Trees with v vertices, $v < \alpha/(\alpha - 1)$.
(II) Trees with v vertices, $v = \alpha/(\alpha - 1)$.
(III) Connected graphs with v vertices and e edges, $v - \alpha e < 0$.

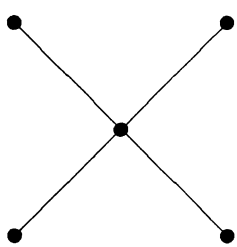
To see that these three classes partition the set of connected graphs, consider any connected graph that is not of type (I) or (II). If it is not a tree then $v \leq e$, which implies $v - \alpha e < 0$ since $\alpha > 1$. If it is a tree, then $v > \alpha/(\alpha - 1)$ and $e = v - 1$, again implying $v - \alpha e < 0$. Conversely, similar calculations show that any connected graph of type (III) is not of type (I) or (II).

The proof consists of three cases. Let $\sigma \in L^k_{\infty\omega}$.

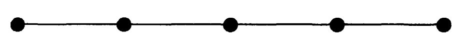
Case 1. For every graph G such that $G \models \sigma$, there is some tree T of type (I) such that G has less than k components isomorphic to T.
Case 2. Not Case 1, and $\alpha/(\alpha - 1)$ is not an integer.
Case 3. Not Case 1, and $\alpha/(\alpha - 1)$ is an integer.

Case 1. has probability $< 2^{-n^d}$ for some $d > 0$. Combined with Corollary 18, (3) follows. In Case 2., the existence of components of type (III) has probability $\sim cn^{-d}$ for some $c > 0$, $d \geq 0$, which implies (4). Further, if $d = 0$ it can be shown that $c = 1$, i.e., the 0-1 law holds. Lastly, in Case 3., the existence of components of type (II) has a Poisson distribution, which also implies (4), but here $d = 0$ and $c \neq 1$ is possible, as we show next.

We conclude with an example of the kinds of computations used to derive asymptotic probabilities in Case 3. They rely on sieve methods that are generalizations of Ch. Jordan's formula and Bonferroni's inequalities. These formulas may be found in Bollobás [5]; their generalizations are explained in detail in Lynch [17]. As previously mentioned, similar computations are needed when $\alpha = 1$, but they are considerably more complicated. Let $\alpha = -5/4$, and consider the property that the graph has exactly 3 components isomorphic to the tree (a) below and at least 7 components isomorphic to the 5-path (b).



(a)



(b)

The probability of this property is

$$\sim \sum_{s\geq 3}\sum_{t\geq 7}(-1)^{s+t-10}\binom{s}{3}\binom{t-1}{6}\frac{n^{5s+5t)}}{s!t!} \times \left(\frac{n^{-4\alpha}}{24}\right)^s \times \left(\frac{n^{-4\alpha}}{2}\right)^t$$

$$\sim \sum_{s\geq 3}(-1)^{s-3}\frac{(1/24)^s}{3!(s-3)!} \times \sum_{t\geq 7}(-1)^{t-7}\frac{(1/2)^t}{6!(t-7)!t}$$

$$= \frac{(1/24)^3 e^{-1/24}}{3!} \times \left(1 - \sum_{j\leq 6}\frac{(1/2)^j e^{-1/2}}{j!}\right) .$$

4 Future Directions

As Table 1. shows, there is a complete classification of convergence laws for the four logics according to the value of α. Future research is likely to be concerned with subtler questions. For example, in our analysis, we ignored the value of β in the edge probability $\beta n^{-\alpha}$. This was because it did not affect the existence of 0-1 or convergence laws as long as it was strictly between 0 and 1. It has long been known, however, that the asymptotic probability of particular properties can be strongly affected by β. For example, in an early paper Erdős and Rényi [9] showed that for $\alpha = 1$, if $\beta < 1$, then the random graph is almost surely planar, but if $\beta > 1$, then it is almost surely nonplanar. It is not known whether planarity is expressible in $L^{\omega}_{\infty\omega}$, but the property that all components have at most one cycle is expressible in TCL, and it is almost surely false when $\beta < 1$ but almost surely true when $\beta > 1$. A metatheorem characterizing properties that have thresholds involving β would be of great interest to random graph theorists.

Other forms of edge probability should be studied. An important one is $\beta n \log_2 n$. When $\beta < 1$, the random graph is almost surely not connected, but when $\beta > 1$, it is almost surely connected. As we saw earlier, connectedness is expressible in TCL.

The original motivation of Fagin and Glebskiĭ et al. in studying 0-1 laws was to find limitations on the expressive power of logics. The general idea would be to establish a 0-1 law or convergence law for some logic $\mathcal{L}$, and then show that some property violates the law, and is therefore not expressible in $\mathcal{L}$. Potential benefits could be lower bounds in computational complexity. That is, if $\mathcal{L}$ captures some complexity class, this would show that the property is not in this class. The fond hope that this approach will eventually produce significant inexpressibility results may not be unreasonable. A recent article by Rosen, Shelah, and Weinstein [25] uses the methods described in Section 3 to prove that certain graph properties are not definable as infinitary disjunctions of first-order existential sentences that involve only a finite number of variables.

On the pessimistic side, it seems that any logic that is powerful enough to capture a complexity class is also too strong to obey a convergence law. For example, many of these logics have a symbol $\leq$, which is interpreted on n in the standard way, and the presence of a linear ordering is enough to destroy the convergence law. We have seen that random graphs have 0-1 and convergence laws for many edge probabilities of the form $\beta n^{-\alpha}$. But Compton, Henson, and Shelah [6] showed that when $\leq$ is added as a new relation, the convergence law fails for constant edge probabilities, and Dolan and Lynch [7] showed that convergence fails for any edge probability $p(n)$ as long as $p(n)n^2$ and $(1-p(n))n^2$ are bounded below by some unbounded recursive function. A phenomenon that several researchers have started to investigate is *slow variation*. This happens when $|\operatorname{pr}(\sigma, n) - \operatorname{pr}(\sigma, n+1)|$ is asymptotic to 0 for every $\sigma \in \mathcal{L}$. In at least some of the cases where the convergence law fails, including random graphs with constant edge probability and the $\leq$ relation, the slow variation law holds (Shelah [26]). Perhaps this is the kind of behavior that will serve as a discriminator of complexity classes.

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