PREFACE

THIS introduction to the classical theory of invariants of algebraic forms is divided into three parts of approximately equal length.

Part I treats of linear transformations both from the standpoint of a change of the two points of reference or the triangle of reference used in the definition of the homogeneous coördinates of points in a line or plane, and also from the standpoint of projective geometry. Examples are given of invariants of forms f of low degrees in two or three variables, and the vanishing of an invariant of f is shown to give a geometrical property of the locus $f=0$, which, on the one hand, is independent of the points of reference or triangle of reference, and, on the other hand, is unchanged by projection. Certain covariants such as Jacobians and Hessians are discussed and their algebraic and geometrical interpretations given; in particular, the use of the Hessian in the solution of a cubic equation and in the discussion of the points of inflexion of a plane cubic curve. In brief, beginning with ample illustrations from plane analytics, the reader is led by easy stages to the standpoint of linear transformations, their invariants and interpretations, employed in analytic projective geometry and modern algebra.

Part II treats of the algebraic properties of invariants and covariants, chiefly of binary forms: homogeneity, weight, annihilators, seminvariant leaders of covariants, law of reciprocity, fundamental systems, properties as functions of the roots, and production by means of differential operators. Any quartic equation is solved by reducing it to a canonical form by means of the Hessian (§ 33). Irrational invariants are illustrated by a carefully selected set of exercises (§ 35).

Part III gives an introduction to the symbolic notation of Aronhold and Clebsch. The notation is first explained at length for a simple case; likewise the fundamental theorem on the types of symbolic factors of a term of a covariant of binary forms is first proved for a simple example by the method later used for the general theorem. In view of these and similar attentions to the needs of those making their first acquaintance with the symbolic notation, the difficulties usually encountered will, it is believed, be largely avoided. This notation must be mastered by those who would go deeply into the theory of invariants and its applications.

Hilbert's theorem on the expression of the forms of a set linearly in terms of a finite number of forms of the set is proved and applied to establish the finiteness of a fundamental set of covariants of a system of binary forms. The theory of transvectants is developed as far as needed in the discussion of apolarity of binary forms and its application to rational curves (§§ 53–57), and in the determination by induction of a fundamental system of covariants of a binary form without the aid of the more technical supplementary concepts employed by Gordan. Finally, there is a discussion of the types of symbolic factors in any term of a concomitant of a system of forms in three or four variables, with remarks on line and plane coördinates.

For further developments reference is made at appropriate places to the texts in English by Salmon, Elliott, and Grace and Young, as well as to Gordan's *Invariantentheorie.* The standard work on the geometrical side of invariants is Clebsch-Lindemann, *Vorlesungen über Geometrie.* Reference may be made to books by W. F. Meyer, *Apolarität und Rationale Curve*, *Bericht über den gegenwartigen Stand der Invariantentheorie*, and *Formentheorie.* Concerning invariant-factors, elementary divisors, and pairs of quadratic or bilinear forms, not treated here, see Muth, *Elementartheiler*, Bromwich, *Quadratic Forms and their Classification by Means of Invariant Factors*, and Bôcher's *Introduction to Higher Algebra.* Lack of space prevents also the discussion of the invariants and covariants arising in the theory of numbers; but an elementary exposition is available in the author's recent book, *On Invariants and the Theory of Numbers*, published, together with Osgood's lectures on functions of several complex variables, by the American Mathematical Society, as *The Madison Colloquium*.

In addition to numerous illustrative examples, there are fourteen sets of exercises which were carefully selected on the basis of experience with classes in this subject.

The author is indebted to Professor H. S. White for suggesting certain additions to the initial list of topics and for reading the proofs of Part I.

CHICAGO, May, 1914.