Modular forms and arithmetic geometry

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The aim of these notes is to describe some examples of modular forms whose Fourier coefficients involve quantities from arithmetical algebraic geometry. At the moment, no general theory of such forms exists, but the examples suggest that they should be viewed as a kind of arithmetic analogue of theta series and that there should be an arithmetic Siegel–Weil formula relating suitable averages of them to special values of derivatives of Eisenstein series. We will concentrate on the case for which the most complete picture is available, the case of generating series for cycles on the arithmetic surfaces associated to Shimura curves over $\mathbb{Q}$, expanding on the treatment in [**40**]. A more speculative overview can be found in [**41**].

In section 1, we review the basic facts about the arithmetic surface $\mathcal{M}$ associated to a Shimura curve over $\mathbb{Q}$. These arithmetic surfaces are moduli stacks over $\mathrm{Spec}(\mathbb{Z})$ of pairs (A, ι) over a base S, where A is an abelian scheme of relative dimension 2 and ι is an action on A of a maximal order O_B in an indefinite quaternion algebra B over $\mathbb{Q}$. In section 2, we recall the definition of the arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{M})$, following Bost, [**7**], and we discuss the metrized Hodge line bundle $\hat{\omega}$ and the conjectural value of $\langle \hat{\omega}, \hat{\omega} \rangle$, where $\langle\ ,\ \rangle$ is the height pairing on $\widehat{\mathrm{CH}}^1(\mathcal{M})$. In the next two sections, we describe divisors $\mathcal{Z}(t)$, $t \in \mathbb{Z}_{>0}$, on $\mathcal{M}$. These are defined as the locus of (A, ι, x)'s where x is

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a special endomorphism (Definition 3.1) of (A, ι) with $x^2 = -t$. Since such an x gives an action on (A, ι) of the order $\mathbb{Z}[\sqrt{-t}]$ in the imaginary quadratic field $\boldsymbol{k}_t = \mathbb{Q}(\sqrt{-t})$, the cycles $\mathcal{Z}(t)$ can be viewed as analogues of the familiar CM points on modular curves. In section 3, the complex points and hence the horizontal components of $\mathcal{Z}(t)$ are determined. In section 4, the vertical components of $\mathcal{Z}(t)$ are determined using the p-adic uniformization of the fibers $\mathcal{M}_p$ of bad reduction of $\mathcal{M}$. In section 5, we construct Green functions $\Xi(t, v)$ for the divisors $\mathcal{Z}(t)$, depending on a parameter $v \in \mathbb{R}^\times_{>0}$. When $t < 0$, the series defining $\Xi(t, v)$ becomes a smooth function on $\mathcal{M}(\mathbb{C})$. These Green functions are used in section 6 to define classes $\widehat{\mathcal{Z}}(t, v) \in \widehat{\mathrm{CH}}^1(\mathcal{M})$, for $t \in \mathbb{Z}$, $t \neq 0$, and an additional class $\widehat{\mathcal{Z}}(0, v)$ is defined using $\hat{\omega}$. The main result of section 6 (Theorem 6.3) says that generating series

$$\widehat{\theta}(\tau) = \sum_{t \in \mathbb{Z}} \widehat{\mathcal{Z}}(t, v)\, q^t, \qquad \tau = u + iv, \qquad q = e(\tau),$$

is the q–expansion of a (nonholomorphic) modular form of weight $\frac{3}{2}$, which we call an arithmetic theta function. The proof of this result is sketched in section 7. The main ingredients are (i) the fact that the height pairing of $\widehat{\theta}(\tau)$ with various classes in $\widehat{\mathrm{CH}}^1(\mathcal{M})$, e.g., $\hat{\omega}$, can be shown to be modular, and (ii) the result of Borcherds, [**5**], which says that a similar generating series with coefficients in the usual Chow group of the generic fiber $\mathrm{CH}^1(\mathcal{M}_{\mathbb{Q}})$ is a modular form of weight $\frac{3}{2}$. In section 8, we use the arithmetic theta function to define an arithmetic theta lift

$$\widehat{\theta} : S_{\frac{3}{2}} \longrightarrow \widehat{\mathrm{CH}}^1(\mathcal{M}), \qquad f \mapsto \widehat{\theta}(f) = \langle\, f, \widehat{\theta}\,\rangle_{\mathrm{Pet}},$$

from a certain space of modular forms of weight $\frac{3}{2}$ to the arithmetic Chow group. This lift is an arithmetic analogue of the classical theta lift from modular forms of weight $\frac{3}{2}$ to automorphic forms of weight 2 for $\Gamma = O_B^\times$. According to the results of Waldspurger, reviewed in section 9, the nonvanishing of this classical lift is controlled by a combination of local obstructions and, most importantly, the central value $L(1, F)$ of the standard Hecke L-function[2] of the cusp form F of weight 2 coming from f via the Shimura lift. In section 10, we describe a doubling integral representation (Theorem 10.1) of the Hecke L-function, involving f and an Eisenstein series $\mathcal{E}(\tau, s, B)$ of weight $\frac{3}{2}$ and genus

[2]Here we assume that F is a newform, so that $L(1, F) = L(\frac{1}{2}, \pi)$, where π is the corresponding cuspidal automorphic representation.

2. At the central point $s = 0$, $\mathcal{E}(\tau, 0, B) = 0$. In the case in which the root number of the L-function is -1, we obtain a formula (Corollary 10.2)

$$\langle\, \mathcal{E}_2'(\begin{pmatrix} \tau_1 & \\ & -\bar{\tau}_2 \end{pmatrix}, 0; B)\, , \overline{f(\tau_2)}\, \rangle_{\mathrm{Pet},\ \tau_2} = f(\tau_1) \cdot C(0) \cdot L'(\frac{1}{2}, \pi),$$

for an explicit constant $C(0)$, whose vanishing is controlled by local obstructions. Finally, in section 11, we state a conjectural identity (Conjecture 11.1)

$$\langle\, \widehat{\theta}(\tau_1), \widehat{\theta}(\tau_2)\, \rangle \overset{??}{=} \mathcal{E}_2'(\begin{pmatrix} \tau_1 & \\ & -\bar{\tau}_2 \end{pmatrix}, 0; B)$$

relating the height pairing of the arithmetic theta function and the restriction to the diagonal of the derivative at $s = 0$ of the weight $\frac{3}{2}$ Eisenstein series. This identity is equivalent to a series of identities of Fourier coefficients, (11.1),

$$\begin{aligned} &\langle\, \widehat{\mathcal{Z}}(t_1, v_1), \widehat{\mathcal{Z}}(t_2, v_2)\, \rangle \cdot q_1^{t_1} q_2^{t_2} \\ &\qquad = \sum_{\substack{T \in \mathrm{Sym}_2(\mathbb{Z})^\vee \\ \mathrm{diag}(T) = (t_1, t_2)}} \mathcal{E}_{2,T}'(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}, 0; B). \end{aligned}$$

Here

$$\mathrm{Sym}_2(\mathbb{Z})^\vee = \{\ T = \begin{pmatrix} t_1 & m \\ m & t_2 \end{pmatrix} \mid t_1, t_2 \in \mathbb{Z},\ m \in \frac{1}{2}\mathbb{Z}\ \}$$

is the dual lattice of $\mathrm{Sym}_2(\mathbb{Z})$ with respect to the trace pairing. We sketch the proof of these identities in the case where $t_1 t_2$ is not a square (Theorem 11.2). As a consequence, we prove Conjecture 11.1 up to a linear combination of theta series for quadratic forms in one variable (Corollary 11.3). Assuming that f is orthogonal to such theta series, we can substitute the height pairing $\langle\, \widehat{\mathcal{Z}}(t_1, v_1), \widehat{\mathcal{Z}}(t_2, v_2)\, \rangle$ for the derivative of the Eisenstein series in the doubling identity and obtain,

$$\langle\, \widehat{\theta}(\tau_1), \widehat{\theta}(f)\, \rangle = f(\tau_1) \cdot C(0) \cdot L'(\frac{1}{2}, \pi),$$

in the case of root number -1. This yields the arithmetic inner product formula

$$\langle\, \widehat{\theta}(f), \widehat{\theta}(f)\, \rangle = \langle\, f, f\, \rangle \cdot C(0) \cdot L'(\frac{1}{2}, \pi),$$

analogous to the Rallis inner product formula for the classical theta lift. Some discussion of the relation of this result to the Gross-Kohnen-Zagier formula, [**24**], is given at the end of section 11.

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Contents

1. Shimura curves and arithmetic surfaces

Let B be an indefinite quaternion algebra over $\mathbb{Q}$ and let $D(B)$ be the product of the primes p for which $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra. For the moment, we allow the case $B = M_2(\mathbb{Q})$, where $D(B) = 1$. The three dimensional $\mathbb{Q}$–vector space

$$V = \{x \in B \mid \mathrm{tr}(x) = 0\} \tag{1.1}$$

is equipped with the quadratic form $Q(x) = \nu(x) = -x^2$. Here ν (resp. tr) is the reduced norm (resp. trace) on B; in the case $B = M_2(\mathbb{Q})$, this is the usual determinant (resp. trace). The bilinear form associated to Q is given by $(x, y) = \mathrm{tr}(xy^\iota)$, where $x \mapsto x^\iota$ is the involution on B given by $x^\iota = \mathrm{tr}(x) - x$. The action of $H = B^\times$ on V by conjugation, $h : x \to hxh^{-1}$, preserves the quadratic form and induces an isomorphism

$$H \xrightarrow{\sim} \mathrm{GSpin}(V), \tag{1.2}$$

where $\mathrm{GSpin}(V)$ is the spinor similitude group of V. Since B is indefinite, i.e., since $B_{\mathbb{R}} = B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})$, V has signature $(1, 2)$. Let

$$D = \{\ w \in V(\mathbb{C}) \ \mid\ (w, w) = 0,\ (w, \bar{w}) < 0\ \}/\,\mathbb{C}^\times \ \subset\ \mathbb{P}(V(\mathbb{C})), \tag{1.3}$$

so that D is an open subset of a quadric in $\mathbb{P}(V(\mathbb{C}))$. Then the group $H(\mathbb{R})$ acts naturally on D, and, if we fix an isomorphism $B_{\mathbb{R}} \simeq M_2(\mathbb{R})$, then there is an identification

$$\mathbb{C} \setminus \mathbb{R} \xrightarrow{\sim} D, \qquad z \mapsto w(z) := \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \quad \bmod \mathbb{C}^\times, \tag{1.4}$$

which is equivariant for the action of $H(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{R})$ on $\mathbb{C} \setminus \mathbb{R}$ by fractional linear transformations.

Let O_B be a maximal order in B and let $\Gamma = O_B^\times$. In the case $B = M_2(\mathbb{Q})$, one may take $O_B = M_2(\mathbb{Z})$, so that $\Gamma = \mathrm{GL}_2(\mathbb{Z})$. Also let

$$K = (\hat{O}_B)^\times \subset H(\mathbb{A}_f), \tag{1.5}$$

where $\hat{O}_B = O_B \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, for $\hat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N\mathbb{Z}$. Then the quotient

$$M(\mathbb{C}) = H(\mathbb{Q}) \backslash \Big(D \times H(\mathbb{A}_f)/K \Big) \simeq \Gamma \backslash D, \tag{1.6}$$

which should be viewed as an orbifold, is the set of complex points of a Shimura curve M, if $D(B) > 1$, or of the modular curve (without its cusp), if $D(B) = 1$. From now on, we assume that $D(B) > 1$, although much of what follows can be carried over for $D(B) = 1$ with only slight modifications. The key point is to interpret M as a moduli space.

Let $\mathcal{M}$ be the moduli stack over $\mathrm{Spec}(\mathbb{Z})$ for pairs (A, ι) where A is an abelian scheme over a base S with an action $\iota : O_B \to \mathrm{End}_S(A)$ satisfying the determinant condition, [**49**], [**36**], [**9**],

$$\det(\iota(b); \mathrm{Lie}(A)) = \nu(b). \tag{1.7}$$

Over $\mathbb{C}$, such an (A, ι) is an abelian surface with O_B action. For example, for $z \in D \simeq \mathbb{C} \setminus \mathbb{R}$, the isomorphism

$$\lambda_z : B_{\mathbb{R}} \simeq M_2(\mathbb{R}) \xrightarrow{\sim} \mathbb{C}^2, \qquad b \mapsto b \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{1.8}$$

determines a lattice $L_z = \lambda_z(O_B) \subset \mathbb{C}^2$. The complex torus $A_z = \mathbb{C}^2/L_z$ is an abelian variety with a natural O_B action given by left multiplication, and hence defines an object $(A_z,\iota)\in\mathcal{M}(\mathbb{C})$. Two points in D give the same lattice if and only if they are in the same $O_B^\times$–orbit, and, up to isomorphism, every (A,ι) over $\mathbb{C}$ arises in this way. Thus, the construction just described gives an isomorphism

$$[\,\Gamma\backslash D\,]\xrightarrow{\sim}\mathcal{M}(\mathbb{C}) \tag{1.9}$$

of orbifolds, and $\mathcal{M}$ gives a model of $M(\mathbb{C})$ over $\mathrm{Spec}(\mathbb{Z})$ with generic fiber

$$M=\mathcal{M}\times_{\mathrm{Spec}(\mathbb{Z})}\mathrm{Spec}(\mathbb{Q}), \tag{1.10}$$

the Shimura curve over $\mathbb{Q}$.

Since we are assuming that $D(B)>1$, $\mathcal{M}$ is proper of relative dimension 1 over $\mathrm{Spec}(\mathbb{Z})$ and smooth over $\mathrm{Spec}(\mathbb{Z}[\,D(B)^{-1}])$. We will ignore the stack aspect from now on and simply view $\mathcal{M}$ as an arithmetic surface over $\mathrm{Spec}(\mathbb{Z})$.

The surface $\mathcal{M}$ has bad reduction at primes $p\mid D(B)$ and this reduction can be described via p-adic uniformization [**9**], [**46**]. Let $\hat{\Omega}_p$ be Drinfeld's p-adic upper half plane. It is a formal scheme over $\mathbb{Z}_p$ with a natural action of $\mathrm{PGL}_2(\mathbb{Q}_p)$. Let $W=W(\bar{\mathbb{F}}_p)$ be the Witt vectors of $\bar{\mathbb{F}}_p$ and let

$$\hat{\Omega}_W=\hat{\Omega}_p\times_{\mathrm{Spf}(\mathbb{Z}_p)}\mathrm{Spf}(W) \tag{1.11}$$

be the base change of $\hat{\Omega}_p$ to W. Also, let $\hat{\Omega}_W^\bullet=\hat{\Omega}_W\times\mathbb{Z}$, and let $g\in\mathrm{GL}_2(\mathbb{Q}_p)$ act on $\hat{\Omega}_W^\bullet$ by

$$g:(z,i)\to(\,g(z),i+\mathrm{ord}_p(\det(g))\,). \tag{1.12}$$

Let $B^{(p)}$ be the definite quaternion algebra over $\mathbb{Q}$ with invariants

$$\mathrm{inv}_\ell(B^{(p)})=\begin{cases}-\mathrm{inv}_\ell(B) & \text{if } \ell=p,\ \infty,\\ \ \ \mathrm{inv}_\ell(B) & \text{otherwise.}\end{cases} \tag{1.13}$$

Let $H^{(p)}=(B^{(p)})^\times$ and

$$V^{(p)}=\{\ x\in B^{(p)}\mid \mathrm{tr}(x)=0\ \}. \tag{1.14}$$

For convenience, we will often write $B'=B^{(p)}$, $H'=H^{(p)}$ and $V'=V^{(p)}$ when p has been fixed. Fix isomorphisms

$$H'(\mathbb{Q}_p)\simeq\mathrm{GL}_2(\mathbb{Q}_p),\qquad\text{and}\qquad H'(\mathbb{A}_f^p)\simeq H(\mathbb{A}_f^p). \tag{1.15}$$

Let $\widehat{\mathcal{M}}_p$ be the base change to W of the formal completion of $\mathcal{M}$ along its fiber at p. Then, the Drinfeld–Cherednik Theorem gives an isomorphism of formal schemes over W

$$\widehat{\mathcal{M}}_p \xrightarrow{\sim} H'(\mathbb{Q})\backslash\Big(\hat{\Omega}^\bullet_W \times H(\mathbb{A}_f^p)/K^p \Big) \simeq \Gamma'\backslash\hat{\Omega}^\bullet_W, \tag{1.16}$$

where $K = K_pK^p$ and $\Gamma' = H'(\mathbb{Q}) \cap H'(\mathbb{Q}_p)K^p$. The special fiber $\hat{\Omega}_p \times_{\mathbb{Z}_p} \mathbb{F}_p$ of $\hat{\Omega}_p$ is a union of projective lines $\mathbb{P}_{[\Lambda]}$ indexed by the vertices $[\Lambda]$ of the building $\mathcal{B}$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$. Here $[\Lambda]$ is the homothety class of the $\mathbb{Z}_p$–lattice Λ in $\mathbb{Q}_p^2$. The crossing points of these lines are double points indexed by the edges of $\mathcal{B}$, and the action of $\mathrm{PGL}_2(\mathbb{Q}_p)$ on components is compatible with its action on $\mathcal{B}$. Thus, the dual graph of the special fiber of $\widehat{\mathcal{M}}_p$ is isomorphic to $\Gamma'\backslash\mathcal{B}^\bullet$, where $\mathcal{B}^\bullet = \mathcal{B} \times \mathbb{Z}$.

2. Arithmetic Chow groups

The modular forms of interest in these notes will take values in the arithmetic Chow groups of $\mathcal{M}$. We will use the version of these groups with real coefficients defined by Bost, [**7**], section 5.5. Let $\widehat{Z}^1(\mathcal{M})$ be the real vector space spanned by pairs (Z, g), where Z is a real linear combination of Weil divisors on $\mathcal{M}$ and g is a Green function for Z. In particular, if Z is a Weil divisor, g is a C^∞ function on $\mathcal{M}(\mathbb{C}) \setminus Z(\mathbb{C})$, with a logarithmic singularity along the $Z(\mathbb{C})$, and satisfies the Green equation

$$dd^c g + \delta_Z = [\omega_Z], \tag{2.1}$$

where ω_Z is a smooth $(1,1)$-form on $\mathcal{M}(\mathbb{C})$, and $[\omega_Z]$ is the corresponding current. If $Z = \sum_i c_i\, Z_i$ is a real linear combination of Weil divisors, then $g = \sum_i c_i\, g_i$ is a real linear combination of such Green functions. By construction, $\alpha \cdot (Z, g) = (\alpha Z, \alpha g)$ for $\alpha \in \mathbb{R}$. The first arithmetic Chow group, with real coefficients, $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$, is then the quotient of $\widehat{Z}^1(\mathcal{M})$ by the subspace spanned by pairs $\widehat{\mathrm{div}}(f) = (\mathrm{div}(f), -\log|f|^2)$ where f is a rational function on $\mathcal{M}$, and $\mathrm{div}(f)$ is its divisor. Finally, we let

$$\widehat{\mathrm{CH}}^1(\mathcal{M}) = \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M}) \otimes_{\mathbb{R}} \mathbb{C}. \tag{2.2}$$

Note that restriction to the generic fiber yields a degree map

$$\deg_{\mathbb{Q}} : \widehat{\mathrm{CH}}^1(\mathcal{M}) \longrightarrow \mathrm{CH}^1(M) \otimes \mathbb{C} \xrightarrow{\deg} \mathbb{C}. \tag{2.3}$$

The group $\widehat{\mathrm{CH}}^2(\mathcal{M})$ is defined analogously, and the arithmetic degree map yields an isomorphism

$$\widehat{\deg} : \widehat{\mathrm{CH}}^2(\mathcal{M}) \xrightarrow{\sim} \mathbb{C}. \tag{2.4}$$

Moreover, there is a symmetric $\mathbb{R}$–bilinear height pairing[3]

$$\langle\ ,\ \rangle :\ \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M}) \times \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M}) \longrightarrow \mathbb{R}. \tag{2.5}$$

According to the index Theorem, cf. [**7**], Theorem 5.5, this pairing is nondegenerate and has signature $(+,-,-,\dots)$. We extend it to an Hermitian pairing on $\widehat{\mathrm{CH}}^1(\mathcal{M})$, conjugate linear in the second argument.

Let $\mathcal{A}$ be the universal abelian scheme over $\mathcal{M}$ with zero section ϵ, and let

$$\omega = \epsilon^* \Omega^2_{\mathcal{A}/\mathcal{M}} \tag{2.6}$$

be the Hodge line bundle on $\mathcal{M}$. We define the natural metric on ω by letting

$$||s_z||^2_{\mathrm{nat}} = \left| \left(\frac{i}{2\pi}\right)^2 \int_{A_z} s_z \wedge \bar{s}_z \right|. \tag{2.7}$$

for any section $s : z \mapsto s_z$, where, for $z \in \mathcal{M}(\mathbb{C})$, A_z is the associated abelian variety. As in section 3 of [**49**], we set

$$||\ ||^2 = e^{-2C}\, ||\ ||^2_{\mathrm{nat}} \tag{2.8}$$

where $2C = \log(4\pi) + \gamma$, where γ is Euler's constant. The reason for this choice of normalization is explained in the introduction to [**49**]. The pair $\hat{\omega} = (\omega, ||\ ||)$ defines an element of $\widehat{\mathrm{Pic}}(\mathcal{M})$, the group of metrized line bundles on $\mathcal{M}$. We write $\hat{\omega}$ for the image of this class in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$ under the natural map $\widehat{\mathrm{Pic}}(\mathcal{M}) \rightarrow \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$.

The pullback to D of the restriction of ω to $\mathcal{M}(\mathbb{C})$ is trivialized by the section α defined as follows. For $z \in D$, let α_z be the holomorphic 2–form on $A_z = \mathbb{C}^2/L_z$ given by

$$\alpha_z = D(B)^{-1} (2\pi i)^2\, dw_1 \wedge dw_2 \tag{2.9}$$

[3]Here the symmetry must still be checked in the case of a stack, cf. section 4 of [**49**].

where w_1 and w_2 are the coordinates on the right side of (1.6). Then

$$\omega_{\mathbb{C}} = \left[\ \Gamma\backslash(D \times \mathbb{C})\,\right], \tag{2.10}$$

where the action of $\gamma \in \Gamma$ is given by

$$\gamma : (z, \zeta) \longmapsto (\gamma(z), (cz+d)^2\,\zeta). \tag{2.11}$$

Thus, on $\mathcal{M}(\mathbb{C})$, ω is isomorphic to the canonical bundle $\Omega^1_{\mathcal{M}(\mathbb{C})}$, under the map which sends α_z to dz. The resulting metric on $\Omega^1_{\mathcal{M}(\mathbb{C})}$ is

$$\begin{aligned} ||dz||^2 = ||\alpha_z||^2 &= e^{-2C}\,\left|\,\left(\frac{i}{2\pi}\right)^2 \int_{A_z} \alpha_z \wedge \bar{\alpha}_z\,\right| \\ &= e^{-2C}\,(2\pi)^{-2}\,(2\pi)^4\,D(B)^{-2}\,\mathrm{vol}(M_2(\mathbb{R})/O_B\,)\cdot \mathrm{Im}(z)^2 \\ &= e^{-2C}\,(2\pi)^2\cdot \mathrm{Im}(z)^2. \end{aligned} \tag{2.12}$$

In [**49**], it was conjectured that

$$\langle\,\hat{\omega},\hat{\omega}\,\rangle \overset{??}{=} \zeta_{D(B)}(-1)\left[\,2\frac{\zeta'(-1)}{\zeta(-1)} + 1 - 2C - \sum_{p|D(B)} \frac{p\log(p)}{p-1}\,\right], \tag{2.13}$$

where

$$\zeta_{D(B)}(s) = \zeta(s)\cdot \prod_{p|D(B)} (1-p^{-s}), \tag{2.14}$$

and $2C = \log(4\pi) + \gamma$, as before. In the case $D(B) = 1$, i.e., for a modular curve, the analogous value for $\langle\,\hat{\omega},\hat{\omega}\,\rangle$ was established, independently, by Bost, [**8**], and Kühn, [**53**], cf. the introduction to [**49**] for a further discussion of normalizations. It will be convenient to define a constant $\mathbf{c}$ by

$$\frac{1}{2}\,\deg_{\mathbb{Q}}(\hat{\omega})\cdot \mathbf{c} := \langle\,\hat{\omega},\hat{\omega}\,\rangle - \text{RHS of (2.13)}. \tag{2.15}$$

In particular, $\langle\,\hat{\omega},\hat{\omega}\,\rangle$ has the conjectured value if and only if $\mathbf{c} = 0$. It seems likely that this conjecture can be proved using recent work of Bruinier, Burgos and Kühn, [**12**], on heights of curves on Hilbert modular surfaces. Their work uses an extended theory of arithmetic Chow groups, developed by Burgos, Kramer and Kühn, [**13**], which allows metrics with singularities of the type which arise on compactified Shimura varieties. In addition, they utilize results of Bruinier, [**10**], [**11**], concerning Borcherds forms. Very general conjectures about such arithmetic degrees and their connections with the equivariant arithmetic Riemann–Roch formula have been given by Maillot and Roessler, [**55**]. Some connections of arithmetic degrees with Fourier coefficients of derivatives of Eisenstein series are discussed in [**39**].

3. Special cycles, horizontal components

Just as the CM points on the modular curves are constructed as the points where the corresponding elliptic curves have additional endomorphisms, cycles on $\mathcal{M}$ can be defined by imposing additional endomorphisms as follows.

Definition 3.1. *For a given (A,ι), the space of* **special endomorphisms** *of (A,ι) is*

$$V(A,\iota) = \{\ x \in \mathrm{End}_S(A) \ \mid \ \mathrm{tr}(x) = 0 \ \textit{and}\ x\circ\iota(b) = \iota(b)\circ x,\ \forall b \in O_B\ \}. \tag{3.1}$$

Over a connected base, this free $\mathbb{Z}$–module of finite rank has a $\mathbb{Z}$–valued quadratic form Q given by

$$-x^2 = Q(x)\cdot \mathrm{id}_A. \tag{3.2}$$

Definition 3.2. *For a positive integer t, let $\mathcal{Z}(t)$ be the moduli stack over* $\mathrm{Spec}(\mathbb{Z})$ *of triples (A,ι,x), where (A,ι) is as before and $x \in V(A,\iota)$ is a special endomorphism with $Q(x) = t$.*

There is a natural morphism

$$\mathcal{Z}(t) \longrightarrow \mathcal{M}, \qquad (A,\iota,x) \mapsto (A,\iota), \tag{3.3}$$

which is unramified, and, by a slight abuse of notation, we write $\mathcal{Z}(t)$ for the divisor on $\mathcal{M}$ determined by this morphism.

Over $\mathbb{C}$, such a triple (A,ι,x) is an abelian surface with an action of $O_B\otimes_{\mathbb{Z}}\mathbb{Z}[\sqrt{-t}]$, i.e., with additional 'complex multiplication' by the order $\mathbb{Z}[\sqrt{-t}]$ in the imaginary quadratic field $\boldsymbol{k}_t = \mathbb{Q}(\sqrt{-t})$. Suppose that $A \simeq A_z = \mathbb{C}^2/L_z$, so that, by (1.8), the tangent space $T_e(A_z)$ is given as $B_\mathbb{R} \simeq \mathbb{C}^2 = T_e(A_z)$. Since the lift $\tilde{x}_\mathbb{R}$ of x in the diagram

$$\begin{array}{rcccc} & B_\mathbb{R} & \simeq & \mathbb{C}^2 & \longrightarrow & A_z \\ \tilde{x}_\mathbb{R} = r(j_x)\downarrow & & & \tilde{x}\downarrow & & x\downarrow \\ & B_\mathbb{R} & \simeq & \mathbb{C}^2 & \longrightarrow & A_z \end{array} \tag{3.4}$$

commutes with the left action of O_B and carries O_B into itself, it is given by right multiplication $r(j_x)$ by an element $j_x \in O_B \cap V$ with $\nu(j_x) = t$. Since the map $\tilde{x}$ is holomorphic, it follows that $z \in D_x$, the fixed point set of j_x on D. To simplify notation, we will write x in place of j_x. Then, we find that

$$\mathcal{Z}(t)(\mathbb{C}) = [\ \Gamma\backslash D_t\ ], \tag{3.5}$$

where

$$D_t = \coprod_{\substack{x\in O_B\cap V \\ Q(x)=t}} D_x. \tag{3.6}$$

In particular, $\mathcal{Z}(t)(\mathbb{C})$, which can be viewed as a set of CM points on the Shimura curve $M(\mathbb{C})$, is nonempty if and only if the imaginary quadratic field $\boldsymbol{k}_t$ embeds in B.

The horizontal part $\mathcal{Z}^{\mathrm{hor}}(t)$ of $\mathcal{Z}(t)$ is obtained by taking the closure in $\mathcal{M}$ of these CM–points, i.e.,

$$\mathcal{Z}^{\mathrm{hor}}(t) := \overline{\mathcal{Z}(t)_\mathbb{Q}}. \tag{3.7}$$

Finally, if we write $4t = n^2 d$ where $-d$ is the discriminant of the imaginary quadratic field $\boldsymbol{k}_t = \mathbb{Q}(\sqrt{-t})$, then the degree of the divisor $\mathcal{Z}(t)_\mathbb{Q}$ is given by

$$\deg_\mathbb{Q} \mathcal{Z}(t) = 2\delta(d, D(B))\, H_0(t, D(B)), \tag{3.8}$$

where

$$\delta(d, D(B)) = \prod_{p|D(B)} (1 - \chi_d(p)), \tag{3.9}$$

and

(3.10)
$$\begin{aligned} H_0(t, D(B)) &= \sum_{c|n} \frac{h(c^2d)}{w(c^2d)} \\ &= \frac{h(d)}{w(d)} \cdot \Bigg(\sum_{\substack{c|n \\ (c,D(B))=1}} c \prod_{\ell|c} (1-\chi_d(\ell)\ell^{-1}) \Bigg). \end{aligned}$$

Here $h(c^2d)$ is the class number of the order O_{c^2d} in $\boldsymbol{k}_t$ of conductor c, $w(c^2d)$ is the number of roots of unity in O_{c^2d}, and χ_d is the Dirichlet character for the field $\boldsymbol{k}_t$. Note that, in this formula, we are counting points on the orbifold $[\Gamma\backslash D]$, so that each point $\text{pr}(z)$, $z \in D$, is counted with multiplicity e_z^{-1} where $e_z = |\Gamma_z|$ is the order of the stabilizer of z in Γ. For example, suppose that $z \in D_x$ for $x \in V \cap O_B$ with $Q(x) = t$. Then, since $\mathbb{Z}[x] \simeq \mathbb{Z}[\sqrt{-t}]$ is an order of conductor n, $\mathbb{Q}[x] \cap O_B \supset \mathbb{Z}[x]$ is an order of conductor c for some $c \mid n$, and $e_z = w(c^2d) = |\ (\mathbb{Q}[x] \cap O_B)^\times|$.

4. Special cycles, vertical components

In this section, we describe the vertical components of the special cycles $\mathcal{Z}(t)$ in some detail, following [**46**]. In the end, we obtain a 'p-adic uniformization', (4.27), quite analogous to the expression (3.5) for $\mathcal{Z}(t)(\mathbb{C})$. We first review the construction of the p-adic uniformization isomorphism (1.16).

Let $\mathbb{B}$ be the division quaternion algebra over $\mathbb{Q}_p$ and let $\mathcal{O}_B$ be its maximal order. Let $\mathbb{Z}_{p^2}$ be the ring of integers in the unramified quadratic extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$. Fixing an embedding of $\mathbb{Q}_{p^2}$ into $\mathbb{B}$, we have $\mathbb{Z}_{p^2} \hookrightarrow \mathcal{O}_B$, and we can choose an element $\Pi \in \mathcal{O}_B$ with $\Pi^2 = p$ such that $\Pi a = a^\sigma \Pi$, for all $a \in \mathbb{Q}_{p^2}$, where σ is the generator of the Galois group of $\mathbb{Q}_{p^2}$ over $\mathbb{Q}_p$. Then $\mathcal{O}_B = \mathbb{Z}_{p^2}[\Pi]$.

Recall that $W = W(\bar{\mathbb{F}}_p)$. Let Nilp be the category of W–schemes S such that p is locally nilpotent in $\mathcal{O}_S$, and for $S \in$ Nilp, let $\bar{S} = S \times_W \bar{\mathbb{F}}_p$.

A *special formal* (s.f.) $\mathcal{O}_B$–module over a W scheme S is a p-divisible formal group X over S of dimension 2 and height 4 with an action $\iota : \mathcal{O}_B \hookrightarrow \text{End}_S(X)$.

The Lie algebra $\mathrm{Lie}(X)$, which is a $\mathbb{Z}_{p^2} \otimes \mathcal{O}_S$–module, is required to be free of rank 1 locally on S.

Fix a s.f. $\mathcal{O}_B$–module $\mathbb{X}$ over $\mathrm{Spec}(\bar{\mathbb{F}}_p)$. Such a module is unique up to $\mathcal{O}_B$–linear isogeny, and

$$\mathrm{End}^0_{\mathcal{O}_B}(\mathbb{X}) \simeq M_2(\mathbb{Q}_p). \tag{4.1}$$

We fix such an isomorphism. Consider the functor

$$\mathcal{D}^\bullet : \mathrm{Nilp} \to \mathrm{Sets} \tag{4.2}$$

which associates to each $S \in \mathrm{Nilp}$ the set of isomorphism classes of pairs (X, ρ), where X is a s.f. $\mathcal{O}_B$–module over S and

$$\rho : \mathbb{X} \times_{\mathrm{Spec}(\bar{\mathbb{F}}_p)} \bar{S} \longrightarrow X \times_S \bar{S} \tag{4.3}$$

is a quasi-isogeny[4]. The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\mathcal{D}^\bullet$ by

$$g : (X, \rho) \mapsto (X, \rho \circ g^{-1}). \tag{4.4}$$

There is a decomposition

$$\mathcal{D}^\bullet = \coprod_i \mathcal{D}^i, \tag{4.5}$$

where the isomorphism class of (X, ρ) lies in $\mathcal{D}^i(S)$ if ρ has height i. The action of $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ carries $\mathcal{D}^i$ to $\mathcal{D}^{i-\mathrm{ord}\det(g)}$. Drinfeld showed that $\mathcal{D}^\bullet$ is representable by a formal scheme, which we also denote by $\mathcal{D}^\bullet$, and that there is an isomorphism of formal schemes

$$\mathcal{D}^\bullet \xrightarrow{\sim} \hat{\Omega}^\bullet_W \tag{4.6}$$

which is equivariant for the action of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Similarly, using the notation, B, O_B, H, etc., of section 1, the formal scheme $\widehat{\mathcal{M}}_p$ represents the functor on Nilp which associates to S the set of isomorphism classes of triples $(A, \iota, \bar{\eta})$ where A is an abelian scheme of relative dimension 2 over S, up to prime to p–isogeny, with an action $O_B \hookrightarrow \mathrm{End}_S(A)$ satisfying the

[4] This means that, locally on S, there is an integer r such that $p^r\rho$ is an isogeny.

determinant condition (1.7), and $\bar{\eta}$ is K^p–equivalence class of O_B–equivariant isomorphisms

$$\eta : \hat{V}^p(A) \xrightarrow{\sim} B(\mathbb{A}_f^p), \tag{4.7}$$

where

$$\hat{V}^p(A) \xrightarrow{\sim} \prod_{\ell \neq p} T_\ell(A) \otimes \mathbb{Q} \tag{4.8}$$

is the rational Tate module of A. Two isomorphisms η and η' are equivalent iff there exists an element $k \in K^p$ such that $\eta' = r(k) \circ \eta$.

Fix a base point $(A_0, \iota_0, \bar{\eta}_0)$ in $\widehat{\mathcal{M}}_p(\bar{\mathbb{F}}_p)$, and let

$$\widehat{\mathcal{M}}_p^\sim : \mathrm{Nilp} \longrightarrow \mathrm{Sets} \tag{4.9}$$

be the functor which associates to S the set of isomorphism classes of tuples $(A, \iota, \bar{\eta}, \psi)$, where $(A, \iota, \bar{\eta})$ is as before, and

$$\psi : A_0 \times_{\mathrm{Spec}(\bar{\mathbb{F}}_p)} \bar{S} \longrightarrow A \times_S \bar{S} \tag{4.10}$$

is an O_B–equivariant p-primary isogeny.

To relate the functors just defined, we let

$$B' = \mathrm{End}^0(A_0, \iota_0), \qquad \text{and} \qquad H' = (B')^\times, \tag{4.11}$$

and fix $\eta_0 \in \bar{\eta}_0$. Since the endomorphisms of $\hat{V}^p(A_0)$ coming from $B'(\mathbb{A}_f^p) = B' \otimes_{\mathbb{Q}} \hat{\mathbb{Z}}^p$ commute with O_B, the corresponding endomorphisms of $B(\mathbb{A}_f^p)$, obtained via η_0, are given by right multiplications by elements of $B(\mathbb{A}_f^p)$. Thus we obtain identifications, as in (1.15),

$$B'(\mathbb{A}_f^p) \xrightarrow{\sim} B(\mathbb{A}_f^p)^{\mathrm{op}}, \qquad \text{and} \qquad H'(\mathbb{A}_f^p) \xrightarrow{\sim} H(\mathbb{A}_f^p)^{\mathrm{op}}, \tag{4.12}$$

where the order of multiplication is reversed in B^{op} and H^{op}. We also identify $\mathbb{B}$ with B_p and take $\mathbb{X} = A_0(p)$, the p-divisible group of A_0, with the action of $\mathcal{O}_B = O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ coming from ι_0. Note that we also obtain an identification $B'_p \simeq \mathrm{GL}_2(\mathbb{Q}_p)$, via (4.1).

Once these identifications have been made, there is a natural isomorphism

$$\widehat{\mathcal{M}}_p^{\sim} \xrightarrow{\sim} \mathcal{D}^\bullet \times H(\mathbb{A}_f^p)/K^p \tag{4.13}$$

defined as follows. To a given $(A, \iota, \bar\eta, \psi)$ over S, we associate:

$$\begin{aligned} X = A(p) = &\ \text{the p-divisible group of } A,\\ \iota = &\ \text{the action of } \mathcal{O}_B = O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p \text{ on } A(p),\\ \rho = \rho(\psi) = &\ \text{the quasi-isogeny}\\ &\quad \rho(\psi) : \mathbb{X} \times_{\mathrm{Spec}(\bar{\mathbb{F}}_p)} \bar S \longrightarrow A(p) \times_S \bar S\\ &\ \text{determined by } \psi, \end{aligned} \tag{4.14}$$

so that (X, ρ) defines an element of $\mathcal{D}^\bullet$. For $\eta \in \bar\eta$, there is also a diagram

$$\begin{array}{ccc} \hat V^p(A_0) & \overset{\eta_0}{\underset{\sim}{\longrightarrow}} & B(\mathbb{A}_f^p) \\ \psi_* \downarrow & & \downarrow r(g) \\ \hat V^p(A) & \overset{\eta}{\underset{\sim}{\longrightarrow}} & B(\mathbb{A}_f^p) \end{array} \tag{4.15}$$

where $r(g)$ denotes right multiplication[5] by an element $g \in H(\mathbb{A}_f^p)$. The coset gK^p is then determined by the equivalence class $\bar\eta$, and the isomorphism (4.13) sends $(A, \iota, \bar\eta, \psi)$ to $((X, \rho), gK^p)$.

The Drinfeld-Cherednik Theorem says that, by passing to the quotient under the action of $H'(\mathbb{Q})$, we have

$$\begin{array}{ccc} \widehat{\mathcal{M}}_p^{\sim} & \xrightarrow{\sim} & \mathcal{D}^\bullet \times H(\mathbb{A}_f^p)/K^p \\ \downarrow & & \downarrow \\ \widehat{\mathcal{M}}_p & \xrightarrow{\sim} & H'(\mathbb{Q})\backslash\Big(\mathcal{D}^\bullet \times H(\mathbb{A}_f^p)/K^p\Big). \end{array} \tag{4.16}$$

Via the isomorphism $\mathcal{D}^\bullet \xrightarrow{\sim} \hat\Omega_W^\bullet$ of (4.6), this yields (1.16).

We can now describe the formal scheme determined by the cycle $\mathcal{Z}(t)$, following section 8 of [**46**]. Let[6] $\widehat{\mathcal{C}}_p(t)$ be the base change to W of the formal completion

[5]This is the reason for the "op" in the isomorphism (4.12), since we ultimately want a left action of $H'(\mathbb{A}_f^p)$.

[6]We change to $\widehat{\mathcal{C}}$ to avoid confusion with the notation $\widehat{\mathcal{Z}}(t, v)$ used for classes in the arithmetic Chow group.

of $\mathcal{Z}(t)$ along its fiber at p, and let $\widehat{\mathcal{C}}_p^{\sim}(t)$ be the fiber product:

$$\begin{array}{ccc} \widehat{\mathcal{C}}_p^{\sim}(t) & \longrightarrow & \widehat{\mathcal{M}}_p^{\sim} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{C}}_p(t) & \longrightarrow & \widehat{\mathcal{M}}_p. \end{array} \tag{4.17}$$

A point of $\widehat{\mathcal{C}}_p(t)$ corresponds to a collection $(A, \iota, \bar{\eta}, x)$, where $x \in V(A, \iota)$ is a special endomorphism with $Q(x) = t$. In addition, since we are now working with A's up to prime to p isogeny, we also require that the endomorphism $\eta_*(x)$ of $B(\mathbb{A}_f^p)$, obtained by transfering, via η, the endomorphism of $\hat{V}^p(A)$ induced by x, is given by right multiplication by an element $j_f^p(x) \in V(\mathbb{A}_f^p) \cap \hat{O}_B$. This condition does not depend on the choice of η in the K^p–equivalence class $\bar{\eta}$.

Next, we would like to determine the image of $\widehat{\mathcal{C}}_p^{\sim}(t)$ in $\mathcal{D}^\bullet \times H(\mathbb{A}_f^p)/K^p$ under the isomorphism in the top line of (4.16). Let

$$V' = \{\ x \in B' \ \mid\ \mathrm{tr}(x) = 0\ \} = V(A_0, \iota_0) \otimes_{\mathbb{Z}} \mathbb{Q}, \tag{4.18}$$

where $V(A_0, \iota_0)$ is the space of special endomorphisms of (A_0, ι_0). We again write Q for the quadratic form on this space. For a point of $\widehat{\mathcal{C}}_p^{\sim}(t)$ associated to a collection $(A, \iota, \bar{\eta}, x, \psi)$, let $((X, \rho), gK^p)$ be the corresponding point in $\mathcal{D}^\bullet \times H(\mathbb{A}_f^p)/K^p$. The special endomorphism $x \in V(A, \iota)$ induces an endomorphism of $A \times_S \bar{S}$ and, thus, via the isogeny ψ, there is an associated endomorphism $\psi^*(x) \in V'(\mathbb{Q})$. This element satisfies two compatibility conditions with the other data:

(i) $\psi^*(x)$ determines an element $j = j(x) \in V'(\mathbb{Q}_p) = \mathrm{End}^0_{\mathcal{O}_B}(\mathbb{X})$. By construction, this element has the property that the corresponding element

$$\rho \circ j \circ \rho^{-1} \in \mathrm{End}^0_{\mathcal{O}_B}(X \times_S \bar{S}) \tag{4.19}$$

is, in fact, the restriction of an element of $\mathrm{End}_{\mathcal{O}_B}(X)$, viz. the endomorphism induced by original x. Said another way, j defines an endomorphism of the reduction $X \times_S \bar{S}$ which lifts to an endomorphism of X.

(ii) Via the diagram (4.15),

$$g^{-1}\, \psi^*(x)\, g \in V(\mathbb{A}_f^p) \cap \hat{O}_B^p. \tag{4.20}$$

Here we are slightly abusing notation and, in effect, are identifying $\psi^*(x) \in V'(\mathbb{Q})$ with an element of $V(\mathbb{A}_f^p)$ obtained via the identification

$$V'(\mathbb{A}_f^p) \xrightarrow{\sim} V(\mathbb{A}_f^p) \tag{4.21}$$

coming from (4.12).

Condition (i) motivates the following basic definition, [**46**], Definition 2.1.

Definition 4.1. *For a special endomorphism $j \in V'(\mathbb{Q}_p)$ of $\mathbb{X}$, let $Z^\bullet(j)$ be the closed formal subscheme of $\mathcal{D}^\bullet$ consisting of the points (X, ρ) such that $\rho \circ j \circ \rho^{-1}$ lifts to an endomorphism of X.*

We will also write $Z(j) \subset \mathcal{D}$ for the subschemes where the height of the quasi-isogeny ρ is 0. We will give a more detailed description of $Z(j)$ in a moment.

As explained above and in more detail in [**46**], section 8, there is a map

$$\widehat{\mathcal{C}}_p^{\sim}(t) \hookrightarrow V'(\mathbb{Q}) \times \mathcal{D}^\bullet \times H(\mathbb{A}_f^p)/K^p \tag{4.22}$$

whose image is the set:

$$(\star) := \left\{ (y, (X, \rho), gK^p) \;\middle|\; \begin{array}{c} (i) \quad Q(y) = t \\ (ii) \quad (X, \rho) \in Z^\bullet(j(y)) \\ (iii) \quad y \in g\,(V(\mathbb{A}_f^p) \cap \hat{O}_B^p)\,g^{-1} \end{array} \right\}. \tag{4.23}$$

Taking the quotient by the group $H'(\mathbb{Q})$, we obtain the following p-adic uniformization of the special cycle:

Proposition 4.2. *The construction aboves yields isomorphisms:*

$$\begin{array}{ccc} \widehat{\mathcal{C}}_p(t) & \xrightarrow{\sim} & H'(\mathbb{Q})\backslash(\star) \\ \downarrow & & \downarrow \\ \widehat{\mathcal{M}}_p & \xrightarrow{\sim} & H'(\mathbb{Q})\backslash\Big(\ \mathcal{D}^\bullet \times H(\mathbb{A}_f^p)/K^p\ \Big) \end{array}$$

of formal schemes over W.

Remark 4.3. In fact, in this discussion, the compact open subgroup K^p giving the level structure away from p can be arbitrary. In the case of interest, where

$K^p = (\hat{O}_B^p)^\times$, the last diagram can be simplified as follows. Let

$$L' = V'(\mathbb{Q}) \cap \big(B'(\mathbb{Q}_p) \times \hat{O}_B^p \big), \tag{4.24}$$

so that L' is a $\mathbb{Z}[p^{-1}]$–lattice in $V'(\mathbb{Q})$. Also, as in section 1, let

$$\Gamma' = H'(\mathbb{Q}) \cap \big(H'(\mathbb{Q}_p) \times K^p \big), \tag{4.25}$$

so that (the projection of) Γ' is an arithmetic subgroup of $H'(\mathbb{Q}_p) \simeq \mathrm{GL}_2(\mathbb{Q}_p)$. Finally, let

$$\mathcal{D}_t^\bullet = \coprod_{\substack{y \in L' \\ Q(y)=t}} Z^\bullet(j(y)). \tag{4.26}$$

Then,

$$\begin{array}{ccc} \widehat{\mathcal{C}}_p(t) & \overset{\sim}{\longrightarrow} & \big[\,\Gamma' \backslash \mathcal{D}_t^\bullet\,\big] \\ \downarrow & & \downarrow \\ \widehat{\mathcal{M}}_p & \overset{\sim}{\longrightarrow} & \big[\,\Gamma' \backslash \mathcal{D}^\bullet\,\big]. \end{array} \tag{4.27}$$

Of course, we should now view these quotients as orbifolds, and, in fact, should formulate the discussion above in terms of stacks.

To complete the picture of the vertical components of our cycle $\mathcal{Z}(t)$, we need a more precise description of the formal schemes $Z^\bullet(j)$ for $j \in V'(\mathbb{Q}_p)$, as obtained in the first four sections of [**46**]. Note that we are using the quadratic form $Q(j) = \det(j)$, whereas, in [**46**], the quadratic form $q(j) = j^2 = -Q(j)$ was used. It is most convenient to give this description in the space $\hat{\Omega}_W^\bullet = \hat{\Omega}_W \times \mathbb{Z}$. In fact, we will just consider $Z(j) \subset \hat{\Omega}_W$, and we will assume that $p \neq 2$. The results for $p = 2$, which are very similar, are described in section 11 of [**49**]. Recall that the special fiber of $\hat{\Omega}_W$ is a union of projective lines $\mathbb{P}_{[\Lambda]}$, indexed by the vertices $[\Lambda]$ of the building $\mathcal{B}$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$.

The first result, proved in section 2 of [**46**], describes the support of $Z(j)$ in terms of the building.

Proposition 4.4.

(i)

$$\mathbb{P}_{[\Lambda]} \cap Z(j) \neq \emptyset \iff j(\Lambda) \subset \Lambda.$$

In particular, if $Z(j) \neq \emptyset$*, then* $\mathrm{ord}_p(Q(j)) \geq 0$.

(ii)

$$j(\Lambda) \subset \Lambda \quad \iff \quad d([\Lambda], \mathcal{B}^j) \ \leq\ \frac{1}{2} \cdot \mathrm{ord}_p(Q(j)).$$

Here $\mathcal{B}^j$ *is the fixed point set of* j *on* $\mathcal{B}$*, and* $d(x,y)$ *is the distance between the points* x *and* $y \in \mathcal{B}$*.*

Recall that the distance function on the building $\mathcal{B}$ is $\mathrm{PGL}_2(\mathbb{Q}_p)$–invariant and gives each edge length 1.

In effect, if we write $Q(j) = \epsilon\, p^\alpha$, for $\epsilon \in \mathbb{Z}_p^\times$, then the support of $Z(j)$ lies in the set of $\mathbb{P}_{[\Lambda]}$'s indexed by vertices of $\mathcal{B}$ in the 'tube' $\mathcal{T}(j)$ of radius $\frac{\alpha}{2}$ around the fixed point set $\mathcal{B}^j$ of j.

Next, the following observation of Genestier is essential, [**46**], Theorem 3.1: Write $Q(j) = \epsilon\, p^\alpha$. Then

$$Z(j) = \begin{cases} (\hat{\Omega}_W)^j & \text{if } \alpha = 0, \\ (\hat{\Omega}_W)^{1+j} & \text{if } \alpha > 0. \end{cases} \tag{4.28}$$

where $(\hat{\Omega}_W)^x$ denotes the fixed point set of the element $x \in \mathrm{GL}_2(\mathbb{Q}_p)$ acting on $\hat{\Omega}_W$, via its projection to $\mathrm{PGL}_2(\mathbb{Q}_p)$. Using this fact, one can obtain local equations for $Z(j)$ in terms of the local coordinates on $\hat{\Omega}_W$, cf. [**46**], section 3. Recall that there are standard coordinate neighborhoods associated to each vertex $[\Lambda]$ and each edge $[\Lambda_0, \Lambda_1]$ of $\mathcal{B}$, cf. [**46**], section 1. By Proposition 4.4, it suffices to compute in the neighborhoods of those $\mathbb{P}_{[\Lambda]}$'s which meet the support of $Z(j)$. For the full result, see Propositions 3.2 and 3.3 in [**46**]. In particular, embedded components can occur. But since these turn out to be negligible, e.g., for intersection theory, cf. section 4 of [**46**], we can omit them from $Z(j)$ to obtain the divisor $Z(j)^{\mathrm{pure}}$ which has the following description.

Proposition 4.5. *Write* $Q(j) = \epsilon\, p^\alpha$*, and let*

$$\mu_{[\Lambda]}(j) = \max\{\, 0,\, \frac{\alpha}{2} - d([\Lambda], \mathcal{B}^j)\,\}.$$

(i) If α is even and $-\epsilon \in \mathbb{Z}_p^{\times,2}$, then

$$Z(j)^{\mathrm{pure}} = \sum_{[\Lambda]} \mu_{[\Lambda]}(j) \cdot \mathbb{P}_{[\Lambda]}.$$

(ii) If α is even and $-\epsilon \notin \mathbb{Z}_p^{\times,2}$, then

$$Z(j)^{\mathrm{pure}} = Z(j)^h + \sum_{[\Lambda]} \mu_{[\Lambda]}(j) \cdot \mathbb{P}_{[\Lambda]},$$

where the horizontal part $Z(j)^h$ is the disjoint union of two divisors projecting isomorphically to $\mathrm{Spf}(W)$ *and meeting the special fiber in 'ordinary special' points of $\mathbb{P}_{[\Lambda(j)]}$, where $[\Lambda(j)]$ is the unique vertex $\mathcal{B}^j$ of $\mathcal{B}$ fixed by j.*
(iii) If α is odd, then

$$Z(j)^{\mathrm{pure}} = Z(j)^h + \sum_{[\Lambda]} \mu_{[\Lambda]}(j) \cdot \mathbb{P}_{[\Lambda]},$$

where the horizontal divisor $Z(j)^h$ is $\mathrm{Spf}(W')$ *where W' is the ring of integers in a ramified quadratic extension of $W \otimes_{\mathbb{Z}} \mathbb{Q}$ and $Z(j)^h$ meets the special fiber in the double point* $\mathrm{pt}_{\Delta(j)}$ *where $\Delta(j)$ is the edge of $\mathcal{B}$ containing the unique fixed point $\mathcal{B}^j$ of j.*
Thus $Z(j)^{\mathrm{pure}}$ is a sum with multiplicities of regular one dimensional formal schemes.

In case (i), the split case, $\mathbb{Q}_p(j) \simeq \mathbb{Q}_p \oplus \mathbb{Q}_p$, the element j lies in a split torus A in $\mathrm{GL}_2(\mathbb{Q}_p)$, and $\mathcal{B}^j$ is the corresponding apartment in $\mathcal{B}$. More concretely, if e_0 and e_1 are eigenvectors of j giving a basis of $\mathbb{Q}_p^2$, then $\mathcal{B}^j$ is the geodesic arc connecting the vertices $[\mathbb{Z}_p e_0 \oplus p^r \mathbb{Z}_p e_1]$, for $r \in \mathbb{Z}$. The $\mathbb{P}_{[\Lambda]}$'s for $[\Lambda] \in \mathcal{B}^j$ have multiplicity $\frac{\alpha}{2}$ in $Z(j)$, and the multiplicity decreases linearly with the distance from $\mathcal{B}^j$. The cycle $Z(j)$ is infinite and there are no horizontal components.

In case (ii), the inert case, $\mathbb{Q}_p(j)$ is an unramified quadratic extension of $\mathbb{Q}_p$, the element j lies in the corresponding nonsplit Cartan subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathcal{B}^j = [\Lambda(j)]$ is a single vertex. The corresponding $\mathbb{P}_{[\Lambda(j)]}$ occurs with multiplicity $\frac{\alpha}{2}$ in $Z(j)$, and the multiplicity of the vertical components $\mathbb{P}_{[\Lambda]}$ decreases linearly with the distance $d([\Lambda], [\Lambda(j)])$.

Finally, in case (iii), the ramified case, $\mathbb{Q}_p(j)$ is a ramified quadratic extension of $\mathbb{Q}_p$, the element j lies in the corresponding nonsplit Cartan subgroup of

$\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathcal{B}^j$ is the midpoint of a unique edge $\Delta(j) = [\Lambda_0, \Lambda_1]$. The vertical components $\mathbb{P}_{[\Lambda_0]}$ and $\mathbb{P}_{[\Lambda_1]}$ occur with multiplicity $\frac{\alpha-1}{2}$ in $Z(j)$, and, again, the multiplicity of the vertical components $\mathbb{P}_{[\Lambda]}$ decreases linearly with the distance $d([\Lambda], \mathcal{B}^j)$, which is now a half-integer.

This description of the $Z(j)$'s, together with the p-adic unformization of Proposition 4.2, gives a fairly complete picture of the vertical components of the cycles $\mathcal{Z}(t)$ in the fibers $\mathcal{M}_p$, $p \mid D(B)$, of bad reduction. Several interesting features are evident.

For example, the following result gives a criterion for the occurrence of such components.

Proposition 4.6. *For $p \mid D(B)$, the cycle $\mathcal{Z}(t)$ contains components of the fiber $\mathcal{M}_p$ of bad reduction if and only if $\mathrm{ord}_p(t) \geq 2$, and no prime $\ell \neq p$ with $\ell \mid D(B)$ is split in $\boldsymbol{k}_t$.*

Note that the condition amounts to (i) $\mathrm{ord}_p(t) \geq 2$, and (ii) the field $\boldsymbol{k}_t$ embeds into $B^{(p)}$

For example, if more than one prime $p \mid D(B)$ splits in the quadratic field $\boldsymbol{k}_t = \mathbb{Q}(\sqrt{-t})$, then $\mathcal{Z}(t)$ is empty. If $p \mid D(B)$ splits in $\boldsymbol{k}_t$, and all other primes $\ell \mid D(B)$ are not split in $\boldsymbol{k}_t$, then the generic fiber $\mathcal{Z}(t)_{\mathbb{Q}}$ is empty, and $\mathcal{Z}(t)$ is a vertical cycle in the fiber at p, provided $\mathrm{ord}_p(t) \geq 2$. In general, if $p \mid D(B)$, and $\boldsymbol{k}_t$ embeds in $B^{(p)}$, then the vertical component in $\mathcal{M}_p$ of the cycle $\mathcal{Z}(p^{2r}t)$ grows as r goes to infinity, while the horizontal part does not change. Indeed, if we change t to $p^{2r}t$, then, by (3.8), $\deg_{\mathbb{Q}} \mathcal{Z}(p^{2r}t) = \deg_{\mathbb{Q}} \mathcal{Z}(t)$, while both the radius of the tube $\mathcal{T}(p^r j)$ and the multiplicity function $\mu_{[\Lambda]}(p^r j)$ increase.

Analogues of these results about cycles defined by special endomorphisms are obtained in [**45**] and [**47**] for Hilbert–Blumenthal varieties and Siegel modular varieties of genus 2 respectively.

5. Green functions

To obtain classes in the arithmetic Chow group $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$ from the $\mathcal{Z}(t)$'s, it is necessary to equip them with Green functions. These are defined as follows; see [**37**] for more details. For $x \in V(\mathbb{R})$, with $Q(x) \neq 0$, let

$$D_x = \{\ z \in D \ \mid\ (x, w(z)) = 0\ \}. \tag{5.1}$$

Here $w(z) \in V(\mathbb{C})$ is any vector with image z in $\mathbb{P}(V(\mathbb{C}))$. The set D_x consists of two points if $Q(x) > 0$, and is empty if $Q(x) < 0$. By (3.5) and (3.6), we have

$$\mathcal{Z}(t)(\mathbb{C}) = \sum_{\substack{x \in O_B \cap V \\ Q(x)=t \\ \mathrm{mod}\ \Gamma}} \mathrm{pr}(D_x) \tag{5.2}$$

where $\mathrm{pr} : D \to \Gamma\backslash D$. For $x \in V(\mathbb{R})$, with $Q(x) \neq 0$, and $z \in D$, let

$$R(x, z) = |(x, w(z))|^2 |(w(z), \overline{w(z)})|^{-1}. \tag{5.3}$$

This function on D vanishes precisely on D_x. Let

$$\beta_1(r) = \int_1^\infty e^{-ru}\, u^{-1}\, du = -\mathrm{Ei}(-r) \tag{5.4}$$

be the exponential integral. Note that

$$\beta_1(r) = \begin{cases} -\log(r) - \gamma + O(r), & \text{as } r \to 0, \\ O(e^{-r}), & \text{as } r \to \infty. \end{cases} \tag{5.5}$$

Thus, the function

$$\xi(x, z) = \beta_1(2\pi R(x, z)) \tag{5.6}$$

has a logarithmic singularity on D_x and decays exponentially as z goes to the boundary of D. A straightforward calculation, [**37**], section 11, shows that $\xi(x, \cdot)$ is a Green function for D_x.

Proposition 5.1. *As currents on D,*

$$dd^c\xi(x) + \delta_{D_x} = [\varphi^0_\infty(x)\,\mu],$$

where, for $z \in D \simeq \mathbb{C} \setminus \mathbb{R}$ with $y = \mathrm{Im}(z)$,

$$\mu = \frac{1}{2\pi}\,\frac{i}{2}\,\frac{dz \wedge d\bar{z}}{y^2}$$

is the hyperbolic volume form and

$$\varphi^0_\infty(x,z) = \big[\; 4\pi(\, R(x,z) + 2Q(x)\,) - 1 \;\big] \cdot e^{-2\pi R(x,z)}.$$

Recall that $dd^c = -\frac{1}{2\pi i}\partial\bar{\partial}$.

Remark. For fixed $z \in D$,

$$(x,x)_z = (x,x) + 2\,R(x,z) \tag{5.7}$$

is the majorant attached to z, [**63**]. Thus, the function

$$\varphi_\infty(x,z)\cdot\mu = \varphi^0_\infty(x,z)\cdot e^{-2\pi Q(x)}\cdot\mu \tag{5.8}$$

is (a very special case of) the Schwartz function valued in smooth $(1,1)$–forms on D defined in [**43**]. In fact, the function $\xi(x,z)$ was first obtained by solving the Green equation of Proposition 5.1 with this right hand side.

Because of the rapid decay of $\xi(x,\cdot)$, we can average over lattice points.

Corollary 5.2. *For $v \in \mathbb{R}^\times_{>0}$, let*

$$\Xi(t,v)(z) := \sum_{\substack{x\in O_B\cap V\\ Q(x)=t}} \xi(v^{\frac{1}{2}}x, z).$$

(i) For $t > 0$, $\Xi(t,v)$ defines a Green function for $\mathcal{Z}(t)$.
(ii) For $t < 0$, $\Xi(t,v)$ defines a smooth function on $M(\mathbb{C})$.

Note, for example, that for $t > 0$,

(5.9)
$$\begin{matrix}\mathcal{Z}(t)(\mathbb{C}) = \emptyset \\ \text{and } \Xi(t,v) = 0\end{matrix} \iff \{x \in O_B \cap V,\ Q(x) = t\} = \emptyset \iff \begin{matrix}\boldsymbol{k}_t \text{ does not} \\ \text{embed in } B.\end{matrix}$$

An explicit construction of Green functions for divisors in general locally symmetric varieties is given by Oda and Tsuzuki, [**58**], by a different method.

6. The arithmetic theta series

At this point, we can define a family of classes in $\widehat{\mathrm{CH}}^1(\mathcal{M})$. These can be viewed as an analogue for the arithmetic surface $\mathcal{M}$ of the Hirzebruch-Zagier classes T_N in the middle cohomology of a Hilbert modular surface, [**29**].

Definition 6.1. *For $t \in \mathbb{Z}$, with $t \neq 0$, and for a parameter $v \in \mathbb{R}_+^\times$, define classes in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$ by*

$$\widehat{\mathcal{Z}}(t,v) = \begin{cases} (\,\mathcal{Z}(t), \Xi(t,v)\,) & \text{if } t > 0, \\ (\,0, \Xi(t,v)\,) & \text{if } t < 0. \end{cases}$$

For $t = 0$, define

$$\widehat{\mathcal{Z}}(0,v) = -\widehat{\omega} - (0, \log(v)) + (0, \mathbf{c}),$$

where $\mathbf{c}$ is the constant defined by (2.15).

We next construct a generating series for these classes; again, this can be viewed as an arithmetic analogue of the Hirzebruch–Zagier generating series for the T_N's. For $\tau = u + iv \in \mathfrak{H}$, the upper half plane, let $q = e(\tau) = e^{2\pi i \tau}$.

Definition 6.2. *The arithmetic theta series is the generating series*

$$\widehat{\theta}(\tau) = \sum_{t \in \mathbb{Z}} \widehat{\mathcal{Z}}(t,v)\, q^t \ \in \widehat{\mathrm{CH}}^1(\mathcal{M})[[q]].$$

Note that, since the imaginary part v of τ appears as a parameter in the coefficient $\widehat{\mathcal{Z}}(t,v)$, this series is not a holomorphic function of τ.

The arithmetic theta function $\widehat{\theta}(\tau)$ is closely connected with the generating series for quadratic divisors considered by Borcherds, [**5**], and the one for Heegner points introduced by Zagier, [**78**]. The following result, [**42**], justifies the terminology. Its proof, which will be sketched in section 7, depends on the results of [**49**] and of Borcherds, [**5**].

Theorem 6.3. *The arithmetic theta series $\widehat{\theta}(\tau)$ is a (nonholomorphic) modular form of weight $\frac{3}{2}$ valued in $\widehat{\mathrm{CH}}^1(\mathcal{M})$.*

As explained in Proposition 7.1 below, the arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{M})$ can be written as direct sum

$$\widehat{\mathrm{CH}}^1(\mathcal{M}) = \widehat{\mathrm{CH}}^1(\mathcal{M},\mu)_{\mathbb{C}} \oplus C_0^\infty(\mathcal{M}(\mathbb{C}))$$

where $\widehat{\mathrm{CH}}^1(\mathcal{M},\mu)_{\mathbb{C}}$ a finite dimensional complex vector space, the Arakelov Chow group of $\mathcal{M}$ for the hyperbolic metric μ, (7.7), and $C_0^\infty(\mathcal{M}(\mathbb{C}))$ is the space of smooth functions on $\mathcal{M}(\mathbb{C})$ with integral 0 with respect to μ. Theorem 6.3 then means that there is a smooth function of ϕ_{Ar} of τ valued in $\widehat{\mathrm{CH}}^1(\mathcal{M},\mu)_{\mathbb{C}}$ and a smooth function $\phi(\tau,z)$ on $\mathfrak{H}\times\mathcal{M}(\mathbb{C})$, with

$$\int_{\mathcal{M}(\mathbb{C})} \phi(\tau,z)\,d\mu(z) = 0,$$

and such that the sum $\phi(\tau) = \phi_{\mathrm{Ar}}(\tau) + \phi(\tau,z)$ satisfies the usual transformation law for a modular form of weight $\frac{3}{2}$, for $\Gamma_0(4D(B))$, and such that the q–expansion of $\phi(\tau)$ is the formal generating series $\widehat{\theta}(\tau)$ of Definition 6.2. By abuse of notation, we write $\widehat{\theta}(\tau)$ both for $\phi(\tau)$ and for its q–expansion.

7. Modularity of the arithmetic theta series

Although the arithmetic theta series $\widehat{\theta}(\tau)$ can be viewed as a kind of generating series for lattice vectors in the spaces $V(A,\iota)$ of special endomorphisms, there is no evident analogue of the Poisson summation formula, which is the key ingredient of the proof of modularity of classical theta series. Instead, the modularity of $\widehat{\theta}(\tau)$ is proved by computing its height pairing with generators of the group $\widehat{\mathrm{CH}}^1(\mathcal{M})$ and identifying the resulting functions of τ with known modular forms.

First we recall the structure of the arithmetic Chow group $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$, [**20**], [**66**], [**7**]. There is a map

$$a : C^\infty(M(\mathbb{C})) \longrightarrow \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M}), \qquad \phi \longmapsto (0,\phi) \tag{7.1}$$

from the space of smooth functions on the curve $M(\mathbb{C})$. Let $\mathbb{1} = a(1)$ be the image of the constant function. Let $\mathrm{Vert} \subset \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$ be the subspace generated by classes of the form $(Y_p, 0)$, where Y_p is a component of a fiber $\mathcal{M}_p$. The relation $\widehat{\mathrm{div}}(p) = (\mathcal{M}_p, -\log(p)^2) \equiv 0$ implies that $(\mathcal{M}_p, 0) \equiv 2\log(p) \cdot \mathbb{1}$, so that $\mathbb{1} \in \mathrm{Vert}$. The spaces $a(C^\infty(M(\mathbb{C}))$ and Vert span the kernel of the restriction map

$$\mathrm{res}_{\mathbb{Q}} : \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M}) \longrightarrow \mathrm{CH}^1(\mathcal{M}_{\mathbb{Q}})_{\mathbb{R}} \tag{7.2}$$

to the generic fiber. Here $\mathrm{CH}^1(\mathcal{M}_{\mathbb{Q}})_{\mathbb{R}} = \mathrm{CH}^1(\mathcal{M}_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{R}$.

Let

$$\mathrm{MW}_{\mathbb{R}} = \mathrm{MW}(M)_{\mathbb{R}} = \mathrm{Jac}(M)(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R} \tag{7.3}$$

be the Mordell–Weil space of the Shimura curve $M = \mathcal{M}_{\mathbb{Q}}$, and recall that this space is the kernel of the degree map

$$\mathrm{MW}_{\mathbb{R}} \longrightarrow \mathrm{CH}^1(M)_{\mathbb{R}} \xrightarrow{\deg} \mathbb{R}. \tag{7.4}$$

Note that the degree of the restriction of the class $\widehat{\omega}$ to the generic fiber is positive, since it is given by the integral of the hyperbolic volume form over $[\Gamma\backslash D]$.

Finally, let $C_0^\infty = C^\infty(M(\mathbb{C}))_0$ be the space of smooth functions which are orthogonal to the constants with respect to the hyperbolic volume form.

The following is a standard result in the Arakelov theory of arithmetic surfaces, [**32**], [**15**], [**7**].

Proposition 7.1. *Let*

$$\widetilde{\mathrm{MW}} := \Big(\mathbb{R}\widehat{\omega} \oplus \mathrm{Vert} \oplus a(C_0^\infty) \Big)^{\perp}$$

be the orthogonal complement of $\mathbb{R}\widehat{\omega} \oplus \mathrm{Vert} \oplus a(C_0^\infty)$ *with respect to the height pairing. Then*

$$\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M}) = \widetilde{\mathrm{MW}} \oplus \Big(\mathbb{R}\widehat{\omega} \oplus \mathrm{Vert} \Big) \oplus a(C_0^\infty),$$

where the three summands are orthogonal with respect to the height pairing. Moreover, the restriction map $\mathrm{res}_{\mathbb{Q}}$ *induces an isometry*

$$\mathrm{res}_{\mathbb{Q}}: \ \widetilde{\mathrm{MW}} \xrightarrow{\sim} \mathrm{MW}$$

with respect to the Gillet–Soulé–Arakelov height pairing on $\widetilde{\mathrm{MW}}$ *and the negative of the Neron–Tate height pairing on* MW*.*

In addition, there are some useful formulas for the height pairings of certain classes. For example, for any $\widehat{\mathcal{Z}} \in \widehat{\mathrm{CH}}^1(\mathcal{M})$,

$$\langle\, \widehat{\mathcal{Z}}, \mathbb{1}\,\rangle = \frac{1}{2}\,\deg_{\mathbb{Q}}(\widehat{\mathcal{Z}}). \tag{7.5}$$

In particular, $\mathbb{1}$ is in the radical of the restiction of the height pairing to Vert, and $\langle\, \widehat{\omega}, \mathbb{1}\,\rangle = \frac{1}{2}\,\deg_{\mathbb{Q}}(\widehat{\omega}) > 0$. Also, for ϕ_1 and $\phi_2 \in C_0^\infty$,

$$\langle\, a(\phi_1), a(\phi_2)\,\rangle = \frac{1}{2}\int_{M(\mathbb{C})} dd^c\phi_1\cdot\phi_2. \tag{7.6}$$

Note that the subgroup

$$\widehat{\mathrm{CH}}^1(\mathcal{M},\mu)_{\mathbb{R}} = \widetilde{\mathrm{MW}} \oplus \mathbb{R}\widehat{\omega} \oplus \mathrm{Vert} \tag{7.7}$$

is the Arakelov Chow group (with real coefficients) for the hyperbolic metric μ.

Returning to the arithmetic theta series $\widehat{\theta}(\tau)$, we consider its pairing with various classes in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$. To describe these, we first introduce an Eisenstein series of weight $\frac{3}{2}$ associated to the quaternion algebra B, [**49**]. Let $\Gamma' = \mathrm{SL}_2(\mathbb{Z})$ and let Γ'_∞ be the stabilizer of the cusp at infinity. For $s \in \mathbb{C}$, let

$$\mathcal{E}(\tau,s;B) = v^{\frac{1}{2}(s-\frac{1}{2})}\sum_{\gamma\in\Gamma'_\infty\backslash\Gamma'} (c\tau+d)^{-\frac{3}{2}}\,|c\tau+d|^{-(s-\frac{1}{2})}\,\Phi^B(\gamma,s), \tag{7.8}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ and $\Phi^B(\gamma,s)$ is a function of γ and s depending on B. This series converges absolutely for $\mathrm{Re}(s) > 1$ and has an analytic continuation to the whole s–plane. It satisfies the functional equation, [**49**], section 16,

$$\mathcal{E}(\tau,s;B) = \mathcal{E}(\tau,-s;B), \tag{7.9}$$

normalized as in Langlands theory.

The following result is proved in [**49**].

Theorem 7.2.
(i)

$$2\cdot\langle\,\widehat{\theta}(\tau),1\!\!1\,\rangle = -\mathrm{vol}(M(\mathbb{C})) + \sum_{t>0}\deg(\mathcal{Z}(t)_{\mathbb{Q}})\,q^t$$
$$= \mathcal{E}(\tau,\frac{1}{2};B).$$

(ii)

$$\langle\,\widehat{\theta}(\tau),\widehat{\omega}\,\rangle = \sum_t \langle\,\widehat{\mathcal{Z}}(t,v),\widehat{\omega}\,\rangle\,q^t$$
$$= \mathcal{E}'(\tau,\frac{1}{2};B).$$

This result is proved by a direct computation of the Fourier coefficients on the two sides of (i) and (ii). For (i), the identity amounts to the formula

$$\deg(\mathcal{Z}(t)_{\mathbb{Q}})\,q^t = 2\,\delta(d;D(B))\,H_0(t;D(B))\,q^t = \mathcal{E}'_t(\tau,\frac{1}{2};D(B)), \tag{7.10}$$

with the notation as in (3.8), (3.9) and (3.10) above. Recall that we write $4t=n^2d$ where $-d$ is the discriminant of the field $\boldsymbol{k}_t$.

For example, if $D(B)=1$, i.e., in the case of $B=M_2(\mathbb{Q})$,

$$H_0(t;1) = \sum_{c|n}\frac{h(c^2d)}{w(c^2d)}. \tag{7.11}$$

Thus, $2\,H_0(t;1) = H(4t)$, where $H(t)$ is the 'class number' which appears in the Fourier expansion of Zagier's nonholomorphic Eisenstein series of weight $\frac{3}{2}$, [**14**], [**78**]:

$$\mathcal{F}(\tau) = -\frac{1}{12} + \sum_{t>0}H(t)\,q^t + \sum_{m\in\mathbb{Z}}\frac{1}{16\pi}\,v^{-\frac{1}{2}}\int_1^\infty e^{-4\pi m^2 vr}\,r^{-\frac{3}{2}}\,dr\,q^{-m^2}. \tag{7.12}$$

In fact, when $D(B)=1$, the value of our Eisenstein series is

$$\mathcal{E}(\tau,\frac{1}{2};1) = -\frac{1}{12} + \sum_{t>0}2\,H_0(t;1)\,q^t + \sum_{m\in\mathbb{Z}}\frac{1}{8\pi}\,v^{-\frac{1}{2}}\int_1^\infty e^{-4\pi m^2 vr}\,r^{-\frac{3}{2}}\,dr\cdot q^{-m^2}. \tag{7.13}$$

As in (i) of Theorem 7.2, both of these series have an interpretation in term of degrees of 0–cycles of CM points on the modular curve, [**76**]. Their relation to a regularized integral of a theta series is proved by J. Funke in [**16**], [**17**].

The computations involved in (ii) are significantly more difficult. For example, if $t > 0$ and $\mathcal{Z}(t)_{\mathbb{Q}}(\mathbb{C})$ is nonempty, then, [**49**], Theorem 8.8, the t-th Fourier coefficient of $\mathcal{E}'(\tau, \frac{1}{2}; B)$ is

(7.14)
$$\begin{aligned}
&\mathcal{E}'_t(\tau, \frac{1}{2}; B) \\
&= 2\,\delta(d;D)\,H_0(t;D)\cdot q^t \cdot \left[\frac{1}{2}\log(d) + \frac{L'(1,\chi_d)}{L(1,\chi_d)} - \frac{1}{2}\log(\pi) - \frac{1}{2}\gamma\right. \\
&\qquad\qquad \left. + \frac{1}{2}J(4\pi t v) + \sum_{\substack{p \\ p\nmid D}} \left(\log|n|_p - \frac{b'_p(n,0;D)}{b_p(n,0;D)} \right) + \sum_{\substack{p \\ p\mid D}} K_p \log(p) \right].
\end{aligned}$$

Here, we write D for $D(B)$, $4t = n^2 d$, as before, and

$$K_p = \begin{cases} -k + \frac{(p+1)(p^k-1)}{2(p-1)} & \text{if } \chi_d(p) = -1, \text{ and} \\ -1 - k + \frac{p^{k+1}-1}{p-1} & \text{if } \chi_d(p) = 0, \end{cases} \tag{7.15}$$

with $k = k_p = \mathrm{ord}_p(n)$. Also,

$$J(x) = \int_0^\infty e^{-xr}\left[\,(1+r)^{\frac{1}{2}} - 1\,\right] r^{-1}\,dr. \tag{7.16}$$

Finally, for a prime $p \nmid D$,

$$\frac{1}{\log p}\cdot\frac{b'_p(n,0;D)}{b_p(n,0;D)} = \frac{\chi_d(p) - \chi_d(p)\,(2k+1)p^k + (2k+2)p^{k+1})}{1 - \chi_d(p) + \chi_d(p)\,p^k - p^{k+1}} - \frac{2p}{1-p} \tag{7.17}$$

The connection of this rather complicated quantity with arithmetic geometry is not at all evident. Nonetheless, the identity of part (ii) of Theorem 7.2 asserts that

$$\mathcal{E}'_t(\tau, \frac{1}{2}; B) = \langle\, \widehat{\mathcal{Z}}(t,v), \widehat{\omega}\,\rangle\, q^t. \tag{7.18}$$

Recall that the points of the generic fiber $\mathcal{Z}(t)_{\mathbb{Q}}$ correspond to abelian surfaces A with an action of $O_B \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{-t}]$, an order in $M_2(\boldsymbol{k}_t)$. The contribution to $\langle\, \widehat{\mathcal{Z}}(t,v), \widehat{\omega}\,\rangle$ of the associated horizontal component of $\mathcal{Z}(t)$ is the Faltings height of A. Due to the action of $O_B \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{-t}\,]$, A is isogenous to a product $E \times E$ where E is an elliptic curve with CM by the order O_d, and so the Faltings heights are related by

$$h_{\mathrm{Fal}}(A) = 2\, h_{\mathrm{Fal}}(E) + \text{an isogeny correction.} \tag{7.19}$$

Up to some combinatorics involving counting of the points $\mathcal{Z}(t)_{\mathbb{Q}}$, the terms on the first line of the right side of (7.14) come from $h_{\mathrm{Fal}}(E)$, while sum on $p \nmid D(B)$ on the second line arises from the isogeny correction. The $J(4\pi tv)$ term comes from the contribution of the Green function $\Xi(t,v)$ with parameter v. Finally, the cycle $\mathcal{Z}(t)$ can have vertical components, and the sum on $p \mid D(B)$ comes from their pairing with $\widehat{\omega}$.

The analogue of (ii) also holds for modular curves, i.e., when $D(B) = 1$. In this case, there are some additional terms, as in the nonholomorphic parts of (7.12) and (7.13). These terms are contributions from a class in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$ supported in the cusp, [**76**], [**51**].

To continue the proof of modularity of $\widehat{\theta}(\tau)$, we next consider the height pairing of $\widehat{\theta}(\tau)$ with classes of the form $(Y_p, 0)$ in Vert, and with classes of the form $(0, \phi)$, for $\phi \in C_0^\infty$, which might be thought of as 'vertical at infinity'.

First we consider $\langle\, \widehat{\theta}(\tau), (Y_p, 0)\,\rangle$. For $p \neq 2$, the intersection number of a component Y_p indexed by $[\Lambda]$ with a cycle $\mathcal{Z}(t)$ is calculated explicitly in [**46**]. For $p = 2$, the computation is very similar. Using this result, we obtain, [**42**]:

Theorem 7.3. *Assume that $p \mid D(B)$ and that $p \neq 2$. Then, for a component Y_p of $\mathcal{M}_p$ associated to a homothety class of lattices $[\Lambda]$, there is a Schwartz function $\varphi_{[\Lambda]} \in S(V^{(p)}(\mathbb{A}_f))$ and associated theta function of weight $\frac{3}{2}$*

$$\theta(\tau, \varphi_{[\Lambda]}) = \sum_{x \in V^{(p)}(\mathbb{Q})} \varphi_{[\Lambda]}(x)\, q^{Q(x)},$$

such that

$$\langle\, \widehat{\theta}(\tau), (Y_p, 0)\,\rangle = \theta(\tau, \varphi_{[\Lambda]}).$$

Next consider the height pairings with classes of the form $a(\phi) = (0,\phi)$ for $\phi \in C_0^\infty$. Here we note that, for a class $(Z, g_Z) \in \widehat{\mathrm{CH}}^1_R(\mathcal{M})$,

$$\langle\, (Z, g_Z), a(\phi)\,\rangle = \frac{1}{2} \int_{\mathcal{M}(\mathbb{C})} \omega_Z\, \phi, \tag{7.20}$$

where $\omega_Z = dd^c g_Z + \delta_Z$ is the smooth $(1,1)$–form on the right side of the Green equation. The map $(Z, g_Z) \mapsto \omega_Z$ defines a map, [**20**], [**66**]

$$\omega : \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M}) \longrightarrow A^{(1,1)}(\mathcal{M}(\mathbb{C})). \tag{7.21}$$

By the basic construction of the Green function $\Xi(t, v)$ and Proposition 5.1, we have

$$\begin{aligned} \omega(\,\widehat{\theta}(\tau)\,) &= \sum_{x \in O_B \cap V} \varphi^0_\infty(x v^{\frac{1}{2}}, z)\, q^{Q(x)} \cdot \mu \\ &:= \theta(\tau, \varphi^0_\infty)(z) \cdot \mu, \end{aligned} \tag{7.22}$$

where $\theta(\tau, \varphi^0_\infty)(z)$ is the theta series of weight $\frac{3}{2}$ for the rational quadratic space V of signature $(1, 2)$. Thus,

$$\langle\, \widehat{\theta}(\tau), a(\phi)\,\rangle = \frac{1}{2} \int_{\mathcal{M}(\mathbb{C})} \theta(\tau, \varphi^0_\infty)(z)\, \phi(z)\, d\mu(z), \tag{7.23}$$

is the classical theta lift of ϕ. To describe this more precisely, recall that

$$dd^c \phi = \frac{1}{2}\, \Delta\{\phi\} \cdot \mu \tag{7.24}$$

where Δ is the hyperbolic Laplacian, and consider functions $\phi_\lambda \in C_0^\infty(\mathcal{M}(\mathbb{C}))$ satisfying $\Delta\phi_\lambda + \lambda\phi_\lambda = 0$ for $\lambda > 0$. Let ϕ_{λ_i}, for $0 < \lambda_1 \le \lambda_2 \le \dots$ be a basis of such eigenfunctions, orthonormal with respect to μ. These are just the Maass forms of weight 0 for the cocompact Fuchsian group $\Gamma = O_B^\times$. Recalling (7.6), we see that the classes $a(\phi_\lambda) \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$ are orthogonal with respect to the height pairing and span the subspace $a(C_0^\infty)$. By (7.23), we have the following, [**42**].

Theorem 7.4. *For a Maass form ϕ_λ of weight 0 for Γ, let*

$$\theta(\tau; \phi_\lambda) := \int_{\mathcal{M}(\mathbb{C})} \theta(\tau, \varphi^0_\infty)(z)\, \phi_\lambda(z)\, d\mu(z)$$

be its classical theta lift, a Maass form of weight $\frac{3}{2}$ *and level* $4D(B)$. *Then*

$$\langle\, \widehat{\theta}(\tau), a(\phi_\lambda)\,\rangle = \frac{1}{2}\,\theta(\tau;\phi_\lambda).$$

Finally, we must consider the component $\widehat{\theta}_{\mathrm{MW}}(\tau)$ of $\widehat{\theta}(\tau)$ in the space $\widetilde{\mathrm{MW}}$. Recall the isomorphism

$$\mathrm{res}_{\mathbb{Q}}:\ \widetilde{\mathrm{MW}} \xrightarrow{\sim} \mathrm{MW} \subset \mathrm{CH}^1(M). \tag{7.25}$$

Write

$$\theta_B(\tau) := \mathrm{res}_{\mathbb{Q}}(\widehat{\theta}(\tau)) = -\omega + \sum_{t>0} Z(t)\, q^t \quad \in \mathrm{CH}^1(M), \tag{7.26}$$

where $Z(t)$ (resp. ω) is the class of the 0–cycle $\mathcal{Z}(t)_{\mathbb{Q}}$ (resp. the line bundle $\omega = \mathrm{res}_{\mathbb{Q}}(\widehat{\omega})$) in $\mathrm{CH}^1(M)$. This series is essentially a special case of the generating function for divisors considered by Borcherds, [**5**], [**6**]. Assuming that certain spaces of vector valued modular forms have bases with rational Fourier coefficients, Borcherds proved that his generating series are modular forms, and McGraw, [**56**], verified Borcherds assumption. The main point in Borcherds' proof is the existence of enough relations among the divisors in question, and such relations can be explicitly given via Borcherds construction of meromorphic modular forms with product expansions, [**3**], [**4**]. Thus, the series $\theta_B(\tau)$ is a modular form of weight $\frac{3}{2}$ valued in $\mathrm{CH}^1(M)$.

By (i) of Theorem 7.2, we have

$$\deg(\theta_B(\tau)) = \mathcal{E}(\tau, \frac{1}{2}; B), \tag{7.27}$$

and so, the function

$$\theta_{\mathrm{MW}}(\tau) := \theta_B(\tau) - \mathcal{E}(\tau, \frac{1}{2}; B)\cdot \frac{\omega}{\deg(\omega)} \quad \in \mathrm{MW} \tag{7.28}$$

is also modular of weight $\frac{3}{2}$. Thus we obtain, [**42**],

Theorem 7.5. *The image of* $\widehat{\theta}_{\mathrm{MW}}(\tau)$ *under the isomorphism (7.25), is given by*

$$\begin{aligned}\mathrm{res}_{\mathbb{Q}}(\ \widehat{\theta}_{\mathrm{MW}}(\tau)\ ) &= \mathrm{res}_{\mathbb{Q}}(\ \widehat{\theta}(\tau) - \mathcal{E}(\tau, \frac{1}{2}; B)\, \deg_{\mathbb{Q}}(\widehat{\omega})^{-1}\cdot \widehat{\omega}\ ) \\ &= \theta_{\mathrm{MW}}(\tau) \in \mathrm{MW}.\end{aligned}$$

Thus, $\widehat{\theta}_{\mathrm{MW}}(\tau)$ is a modular form of weight $\frac{3}{2}$.

This completes the proof of the modularity of the arithmetic theta function $\widehat{\theta}(\tau)$.

8. The arithmetic theta lift

The arithmetic theta function $\widehat{\theta}(\tau)$ can be used to define an arithmetic theta lift

$$\widehat{\theta}: \ S_{\frac{3}{2}} \longrightarrow \widehat{\mathrm{CH}}^1(\mathcal{M}), \qquad f \mapsto \widehat{\theta}(f),$$

where $S_{\frac{3}{2}}$ is the space of cusp forms of weight $\frac{3}{2}$ for $\Gamma' = \Gamma_0(4D(B))$, as follows. Given a cusp form f of weight $\frac{3}{2}$ for $\Gamma' = \Gamma_0(4D(B))$, let

$$\widehat{\theta}(f) = \langle\, f, \widehat{\theta}\,\rangle_{\mathrm{Pet}} = \int_{\Gamma'\backslash\mathfrak{H}} f(\tau)\,\overline{\widehat{\theta}(\tau)}\,v^{\frac{3}{2}}\,\frac{du\,dv}{v^2} \quad \in \widehat{\mathrm{CH}}^1(\mathcal{M}). \tag{8.1}$$

This construction is analogous to the construction of Niwa, [**57**], of the classical Shimura lift, [**64**], from $S_{\frac{3}{2}}$ to modular forms of weight 2 for $O_B^\times$. Recall that we extend the height pairing $\langle\ ,\ \rangle$ on $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$ to an Hermitian pairing on $\widehat{\mathrm{CH}}^1(\mathcal{M})$, conjugate linear in the second argument. By adjointness, and Theorem 7.2,

$$\langle\widehat{\theta}(f), 1\!\!1\rangle = \langle f, \langle\widehat{\theta}, 1\!\!1\rangle\,\rangle_{\mathrm{Pet}} = \frac{1}{2}\,\langle f, \mathcal{E}(\frac{1}{2}; B)\rangle_{\mathrm{Pet}} = 0, \tag{8.2}$$

and

$$\langle\widehat{\theta}(f), \hat{\omega}\rangle = \langle f, \langle\widehat{\theta}, \widehat{\omega}\rangle\,\rangle_{\mathrm{Pet}} = \langle f, \mathcal{E}'(\frac{1}{2}; B)\rangle_{\mathrm{Pet}} = 0. \tag{8.3}$$

If f is holomorphic, then, by Theorem 7.4, for any $\phi \in C_0^\infty(\mathcal{M}(\mathbb{C}))$,

$$\langle\widehat{\theta}(f), a(\phi)\rangle = \langle\, f, \langle\,\widehat{\theta}, a(\phi)\,\rangle\,\rangle = \langle\, f, \theta(\phi)\,\rangle_{\mathrm{Pet}} = 0, \tag{8.4}$$

since $\theta(\tau;\phi)$ is a combination of Maass forms of weight $\frac{3}{2}$. Thus, for f holomorphic

$$\widehat{\theta}(f) \in \widetilde{\mathrm{MW}} \oplus \mathrm{Vert}^0 \ \subset \widehat{\mathrm{CH}}^1(\mathcal{M}) \tag{8.5}$$

where

$$\text{Vert}^0 = \text{Vert} \cap \ker\langle \cdot, \widehat{\omega} \rangle. \tag{8.6}$$

It remains to describe the components of $\hat{\theta}(f)$ in the spaces Vert and $\widetilde{\text{MW}}$. This is best expressed in terms of automorphic representations.

9. Theta dichotomy: Waldspurger's theory

We begin with a brief review of Waldspurger's theory of the correspondence between cuspidal automorphic representations of the metaplectic cover of SL_2 and automorphic representations of PGL_2 and its inner forms. For a more detailed survey, the reader can consult [**59**], [**70**], [**31**], as well as the original papers [**68**], [**69**], and especially, [**73**].

We fix the additive character ψ of $\mathbb{A}/\mathbb{Q}$ which has trivial conductor, i.e., is trivial on $\widehat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p$ and has archimedean component $\psi_\infty(x) = e(x) = e^{2\pi i x}$. Since ψ is fixed, we suppress it from the notation.

Let $G = \text{SL}_2$ and let $G'_{\mathbb{A}}$ be the 2–fold cover of $G(\mathbb{A})$ which splits over $G(\mathbb{Q})$. Let

$$\begin{aligned} \mathcal{A}_0(G') &= \text{the space of genuine cusp forms for } G'_{\mathbb{A}}. \\ \mathcal{A}_{00}(G') &= \text{the space of genuine cusp forms for } G'_{\mathbb{A}}. \\ &\qquad \text{orthogonal to all } O(1) \text{ theta series} \end{aligned} \tag{9.1}$$

For an irreducible cuspidal automorphic representation $\sigma \simeq \otimes_{p\leq\infty}\sigma_p$ in $\mathcal{A}_{00}(G')$, Waldspurger constructs an irreducible cuspidal automorphic representation

$$\pi = \pi(\sigma) = \text{Wald}(\sigma) = \text{Wald}(\sigma, \psi) \tag{9.2}$$

of $\text{PGL}_2(\mathbb{A})$, which serves as a kind of reference point for the description of the global theta lifts for various ternary quadratic spaces. For each $p \leq \infty$, there is a corresponding local construction $\sigma_p \mapsto \text{Wald}(\sigma_p, \psi_p)$, and the local and global constructions are compatible, i.e.,

$$\text{Wald}(\sigma, \psi) \ \simeq \ \otimes_{p\leq\infty}\text{Wald}(\sigma_p, \psi_p).$$

Since we have fixed the additive character $\psi = \otimes_p \psi_p$, we will often omit it from the notation.

For a quaternion algebra B over $\mathbb{Q}$, let

$$V^B = \{\ x \in B \mid \mathrm{tr}(x) = 0\ \}, \tag{9.3}$$

with quadratic form[7] $Q(x) = -x^2 = \nu(x)$, and let

$$H^B = B^\times \simeq \mathrm{GSpin}(V). \tag{9.4}$$

For a Schwartz function $\varphi \in S(V^B(\mathbb{A}))$, $g' \in G'_{\mathbb{A}}$ and $h \in H^B(\mathbb{A})$, define the theta kernel by

$$\theta(g', h; \varphi) = \sum_{x \in V^B(\mathbb{Q})} \omega(g')\varphi(h^{-1}x). \tag{9.5}$$

Here $\omega = \omega_\psi$ is the Weil representation of $G'_{\mathbb{A}}$ on $S(V(\mathbb{A}))$. For $f \in \sigma$ and $\varphi \in S(V^B(\mathbb{A}))$, the classical theta lift is

$$\theta(f; \varphi) = \langle\, f, \theta(\varphi)\,\rangle_{\mathrm{Pet}} = \int_{G'_{\mathbb{Q}}\backslash G'_{\mathbb{A}}} f(g')\,\overline{\theta(g', h; \varphi)}\, dg' \quad \in \mathcal{A}_0(H^B). \tag{9.6}$$

The global theta lift of σ to H^B is the space

$$\theta(\sigma; V^B)\ \subset\ \mathcal{A}_0(H^B), \tag{9.7}$$

spanned by the $\theta(f;\varphi)$'s for $f \in \sigma$ and $\varphi \in S(V(\mathbb{A}))$. Here $\mathcal{A}_0(H^B)$ is the space of cusp forms on $H^B(\mathbb{Q})\backslash H^B(\mathbb{A})$. Then $\theta(\sigma; V^B)$ is either zero or is an irreducible cuspidal representation of $H^B(\mathbb{A})$, [**73**], Prop. 20, p. 290.

For any irreducible admissible genuine representation σ_p of G'_p, the metaplectic cover of $G(\mathbb{Q}_p)$, there are analogous local theta lifts $\theta(\sigma_p, V_p^B)$. Each of them is either zero or is an irreducible admissible representation of H_p^B. The local and global theta lifts are compatible in the sense that

$$\theta(\sigma; V^B) \simeq \begin{cases} \otimes_p \theta(\sigma_p; V_p^B) & \text{or} \\ 0. \end{cases} \tag{9.8}$$

[7]Note that Waldspurger uses the opposite sign.

In particular, the global theta lift is zero if any local $\theta(\sigma_p; V_p^B) = 0$, but the global theta lift can also vanish even when there is no such local obstruction, i.e., even if $\otimes_{p\le\infty}\theta(\sigma_p; V_p^B) \ne 0$.

For each p, let $B_p^{\pm}$ be the quaternion algebra over $\mathbb{Q}_p$ with invariant

$$\mathrm{inv}_p(B_p^{\pm}) = \pm 1. \tag{9.9}$$

Then the two ternary quadratic spaces

$$V_p^{\pm} = \{\ x \in B_p^{\pm} \mid \mathrm{tr}(x) = 0\ \} \tag{9.10}$$

have the same discriminant and opposite Hasse invariants. A key local fact established by Waldspurger is:

Theorem 9.1. (Local theta dichotomy) *For an irreducible admissible genuine representaton σ_p of G'_p, precisely one of the spaces $\theta(\sigma_p; V_p^+)$ and $\theta(\sigma_p; V_p^-)$ is nonzero.*

Definition 9.2. Let $\epsilon_p(\sigma_p) = \pm 1$ be the unique sign such that with

$$\theta(\sigma_p, V_p^{\epsilon_p(\sigma_p)}) \ne 0.$$

Examples: (i) If p is a finite prime and σ_p is an unramified principal series representation, then $\epsilon_p(\sigma_p) = +1$ and $\theta(\sigma_p; V_p^+)$ is a principal series representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. It is unramified if σ_p is unramified.
(ii) For a finite prime p and a character μ_p of $\mathbb{Q}_p^\times$, with $\mu_p^2 = |\ |$, there is a special representation $\sigma_p(\mu_p)$ of G'_p. For $\sigma_p = \sigma_p(\mu_p)$, with $\mu_p = |\ |^{\frac{1}{2}}$,

$$\theta(\sigma_p, V_p^-) = 1\!\!1 \ne 0, \qquad B_p^- = \mathbb{B}_p$$
$$\theta(\sigma_p, V_p^+) = 0, \qquad B_p^+ = M_2(\mathbb{Q}_p)$$
$$\mathrm{Wald}(\sigma_p, \psi_p) = \text{ unramified special } \sigma(|\ |^{\frac{1}{2}}, |\ |^{-\frac{1}{2}}) \text{ of } GL_2(\mathbb{Q}_p).$$

If $\mu_p \ne |\ |^{\frac{1}{2}}$, then

$$\theta(\sigma_p, V^+) = \mathrm{Wald}(\sigma_p, \psi_p) = \sigma(\mu, \mu^{-1})$$

is a special representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\theta(\sigma_p, V_p^-) = 0$.
(iii) If $\sigma_\infty = \mathrm{HDS}_{\frac{3}{2}}$, the holomorphic discrete series representation of $G'_{\mathbb{R}}$ of weight $\frac{3}{2}$, then

$$\begin{aligned} \theta(\sigma_\infty, V_\infty^-) &= 1\!\!1 \neq 0, & B_\infty^- &= \mathbb{H} \\ \theta(\sigma_\infty, V_\infty^+) &= 0, & B_\infty^+ &= M_2(\mathbb{R}) \end{aligned}$$
$$\mathrm{Wald}(\sigma_\infty, \psi_\infty) = \mathrm{DS}_2 = \text{weight 2 disc. series of } GL_2(\mathbb{R}).$$

Since we will be considering only holomorphic cusp forms of weight $\frac{3}{2}$ and level $4N$ with N odd and square free, these examples give all of the relevant local information.

The local root number $\epsilon_p(\frac{1}{2}, \mathrm{Wald}(\sigma_p)) = \pm 1$ and the invariant $\epsilon_p(\sigma_p)$ are related as follows. There is an element $-\mathbf{1}$ in the center of G'_p which maps to $-1 \in G_p$, and Waldspurger defines a sign $\epsilon(\sigma_p, \psi_p)$, the central sign of σ_p, by

$$\sigma_p(-\mathbf{1}) = \epsilon(\sigma_p, \psi_p)\, \chi_\psi(-1) \cdot I_{\sigma_p}, \tag{9.11}$$

[**73**], p.225. Then,

$$\epsilon_p(\sigma_p) = \epsilon_p(\frac{1}{2}, \mathrm{Wald}(\sigma_p, \psi_p))\, \epsilon(\sigma_p, \psi_p). \tag{9.12}$$

Note that

$$\prod_p \epsilon(\sigma_p, \psi_p) = 1. \tag{9.13}$$

Thus, for a given global $\sigma \simeq \otimes_p \sigma_p$,

(9.14)
$$\begin{aligned} \epsilon(\frac{1}{2}, \mathrm{Wald}(\sigma, \psi)) = +1 &\iff \text{there is a } B/\mathbb{Q} \text{ with } \mathrm{inv}_p(B) = \epsilon_p(\sigma_p) \text{ for all } p \\ &\iff \text{there is a } B/\mathbb{Q} \text{ with } \otimes_p \theta(\sigma_p, V_p^B) \neq 0. \end{aligned}$$

The algebra B is then unique, and, if $B' \not\simeq B$, the global theta lift $\theta(\sigma; V^{B'}) = 0$ for local reasons. On the other hand, the global theta lift for V^B is

$$\theta(\sigma, V^B) \simeq \begin{cases} \otimes_p \theta(\sigma_p, V_p^B) & \text{or} \\ 0, \end{cases} \tag{9.15}$$

and Waldspurger's beautiful result, [**73**], is that

$$\theta(\sigma, V^B) \neq 0 \iff L(\frac{1}{2}, \mathrm{Wald}(\sigma)) \neq 0. \tag{9.16}$$

10. The doubling integral

The key to linking Waldspurger's theory to the arithmetic theta lift is the doubling integral representation of the Hecke L-function, [**60**], [**54**], [**52**], and in classical language, [**18**], [**2**]. Let $\widetilde{G} = \mathrm{Sp}_2$ be the symplectic group of rank 2 over $\mathbb{Q}$, and let $\widetilde{G}'_{\mathbb{A}}$ be the 2–fold metaplectic cover of $\widetilde{G}(\mathbb{A})$. Recall that $G = \mathrm{SL}_2 = \mathrm{Sp}_1$. Let $i_0 : G \times G \to \widetilde{G}$ be the standard embedding:

$$i_0 : \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \longmapsto \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}. \tag{10.1}$$

For $g \in G$, let

$$g^\vee = \mathrm{Ad}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \cdot g, \tag{10.2}$$

and let

$$i(g_1, g_2) = i_0(g_1, g_2^\vee). \tag{10.3}$$

Let $\widetilde{P} \subset \widetilde{G}$ be the standard Siegel parabolic, and, for $s \in \mathbb{C}$, let $\widetilde{I}(s)$ be the induced representation

$$\widetilde{I}(s) = \mathrm{Ind}_{\widetilde{P}'_{\mathbb{A}}}^{\widetilde{G}'_{\mathbb{A}}}(\delta^{s+\frac{3}{2}}) \simeq \otimes_p \widetilde{I}_p(s), \tag{10.4}$$

where δ is a certain lift to the cover $\widetilde{P}'_{\mathbb{A}}$ of the modulus character of $\widetilde{P}(\mathbb{A})$. Here we are using unnormalized induction. For a section $\Phi(s) \in \widetilde{I}(s)$, there is a Siegel–Eisenstein series

$$E(g', s, \Phi) = \sum_{\gamma \in \widetilde{P}'_{\mathbb{Q}} \backslash \widetilde{G}'_{\mathbb{Q}}} \Phi(\gamma g', s), \tag{10.5}$$

convergent for $\mathrm{Re}(s) > \frac{3}{2}$, and with an analytic continuation in s satisfying a functional equation relating s and $-s$. For $\sigma \simeq \otimes_p \sigma_p$ a cuspidal representation

in $\mathcal{A}_0(G')$, as above, the doubling integral is defined as follows:
For vectors f_1, $f_2 \in \sigma$ and $\Phi(s) \in \widetilde{I}(s)$,

$$(10.6)\quad Z(s, f_1, f_2, \Phi) = \int_{G'_{\mathbb{Q}}\backslash G'_{\mathbb{A}} \times G'_{\mathbb{Q}}\backslash G'_{\mathbb{A}}} \overline{f_1(g'_1)}\, f_2(g'_2)\, E(i(g'_1, g'_2), s, \Phi)\, dg'_1\, dg'_2.$$

If f_1, f_2 and $\Phi(s)$ are factorizable and unramified outside of a set of places S, including ∞ and 2, then, [**60**], [**54**], [**44**][8],

$$(10.7)\quad Z(s, f_1, f_2, \Phi) = \frac{1}{\zeta^S(2s+2)}\, L^S(s+\frac{1}{2}, \mathrm{Wald}(\sigma)) \cdot \prod_{p \in S} Z_p(s, f_{1,p}, f_{2,p}, \Phi_p),$$

where $Z_p(s, f_{1,p}, f_{2,p}, \Phi_p)$ is a local zeta integral depending on the local components at p.

Now suppose that $\sigma_\infty = \mathrm{HDS}_{\frac{3}{2}}$, and take the archimedean local components $f_{1,\infty} = f_{2,\infty}$ to be the weight $\frac{3}{2}$ vectors. We can identify f_1 and f_2 with classical cusp forms of weight $\frac{3}{2}$. Taking $\Phi_\infty(s) \in \widetilde{I}_\infty(s)$ to be the standard weight $\frac{3}{2}$ section $\Phi_\infty^{\frac{3}{2}}(s)$, we can write the Eisenstein series as a classical Siegel Eisenstein series $E(\tau, s, \Phi_f)$ of weight $\frac{3}{2}$, where $\tau \in \mathfrak{H}_2$, the Siegel space of genus 2, and $\Phi_f(s)$ is the finite component of $\Phi(s)$.

For a given indefinite quaternion algebra B, there is a section $\Phi_f(s)$ defined as follows. For a finite prime p, the group $\widetilde{G}'_p$ acts on the Schwartz space $S((V_p^{\pm})^2)$ via the Weil representation ω determined by ψ_p, and there is a map

$$(10.8)\qquad \lambda_p : S((V_p^{\pm})^2) \longrightarrow \widetilde{I}_p(0), \qquad \varphi_p \mapsto \lambda_p(\varphi_p)(g') = \omega(g')\varphi_p(0).$$

Here $V_p^{\pm}$ is the ternary quadratic space defined in (5.10). Note that a section $\Phi_p(s)$ is determined by its restriction to the compact open subgroup K'_p, the inverse image of $\widetilde{G}(\mathbb{Z}_p)$ in $\widetilde{G}'_p$. A section is said to be standard if this restriction is independent of s. The function $\lambda_p(\varphi_p) \in \widetilde{I}_p(0)$ has a unique extension to a standard section of $\widetilde{I}_p(s)$. Fix a maximal order $R_p^{\pm}$ in $B_p^{\pm}$, and let $\varphi_p^{\pm} \in S((V_p^{\pm})^2)$ be the characteristic function of $(R_p^{\pm} \cap V_p^{\pm})^2$. Also let $R_p^e \subset R_p^+$ be the Eichler order[9] of index p, and let φ_p^e be the characteristic function of $(R_p^e \cap V_p^+)^2$. We then have standard sections $\Phi_p^0(s)$, $\Phi_p^-(s)$ and $\Phi_p^e(s)$ whose

[8]Note that [**60**] and [**44**] deal with the symplectic case, while the metaplectic case needed here is covered in [**54**].
[9]So $R_p^+ = M_2(\mathbb{Z}_p)$ and $x \in R_p^e$ iff $c \equiv 0 \mod p$.

restrictions to K'_p are $\lambda_p(\varphi_p^+)$, $\lambda_p(\varphi_p^-)$, and $\lambda_p(\varphi_p^e)$ respectively. Following [**46**], let

$$\widetilde{\Phi}_p(s) = \Phi_p^-(s) + A_p(s)\Phi_p^0(s) + B_p(s)\,\Phi_p^e(s), \tag{10.9}$$

where $A_p(s)$ and $B_p(s)$ are entire functions of s such that

$$A_p(0) = B_p(0) = 0, \qquad \text{and} \qquad A'_p(0) = -\frac{2}{p^2-1}\,\log(p), \quad B'_p(0) = \frac{1}{2}\frac{p+1}{p-1}\,\log(p). \tag{10.10}$$

Then let

$$\widetilde{\Phi}^B(s) = \Big(\otimes_{p|D(B)}\,\widetilde{\Phi}_p(s)\Big) \otimes \Big(\otimes_{p\nmid D(B)}\,\Phi_p^0(s)\Big). \tag{10.11}$$

Using this section, we can define a normalized Siegel Eisenstein series of weight $\frac{3}{2}$ and genus 2 attached to B by

$$\mathcal{E}_2(\tau, s; B) = \eta(s,B)\,\zeta_{D(B)}(2s+2)\,E(\tau, s; \widetilde{\Phi}^B), \tag{10.12}$$

where $\eta(s,B)$ is a certain normalizing factor and the partial zeta function is as in (2.14). Then we have a precise version of the doubling identity, [**52**]:

Theorem 10.1. [10] *For every indefinite quaternion algebra B over $\mathbb{Q}$, the associated Siegel–Eisenstein series $\mathcal{E}_2(\tau, s; B)$ of genus 2 and weight $\frac{3}{2}$ has the following property. For each holomorphic 'newform' f of weight $\frac{3}{2}$ and level $4D(B)$ associated to an irreducible cuspidal representation $\sigma = \otimes_p\sigma_p$:*

$$\begin{aligned}\langle\, \mathcal{E}_2(\begin{pmatrix}\tau_1 & \\ & -\bar{\tau}_2\end{pmatrix}, s; B)\,,\, \overline{f(\tau_2)}\,\rangle_{\mathrm{Pet},\ \tau_2} \\ = C(s)\,C(s;\sigma;B)\,L(s+\frac{1}{2}, \mathrm{Wald}(\sigma))\cdot f(\tau_1),\end{aligned}$$

where[11] $C(0) \neq 0$, *and*

$$C(s;\sigma;B) = \prod_{p|D(B)} C_p(s;\sigma_p;B_p),$$

[10] At the time of this writing, we must assume that $2\nmid D(B)$ and that $\epsilon_2(\sigma_2) = +1$. It should not be difficult to remove these restrictions.

[11] These are explicit elementary factors.

with

$$C_p(0;\sigma_p;B_p) = \begin{cases} 1 & \textit{if } \epsilon_p(\sigma_p) = -1, \\ 0 & \textit{if } \epsilon_p(\sigma_p) = +1, \end{cases}$$

and, if $\epsilon_p(\sigma_p) = +1$,

$$C_p'(0;\sigma_p;B_p) = \log(p).$$

Note that $\epsilon_p(\sigma_p) = 1$ for $p \nmid 4D(B)$, and that

$$\epsilon(\frac{1}{2}, \mathrm{Wald}(\sigma)) = -\prod_{p|D(B)} \epsilon_p(\sigma_p). \tag{10.13}$$

Thus, for example, for $\epsilon(\frac{1}{2}, \mathrm{Wald}(\sigma)) = +1$, an odd number of $p \mid D(B)$ have $\epsilon_p(\sigma_p) = +1$.

Corollary 10.2. *With the notation and assumptions of Theorem 10.1,*

$$\begin{aligned} &\langle\, \mathcal{E}_2'(\begin{pmatrix} \tau_1 & \\ & -\bar{\tau}_2 \end{pmatrix}, 0; B)\,,\, \overline{f(\tau_2)}\,\rangle_{\mathrm{Pet},\ \tau_2} \\ &= f(\tau_1)\cdot C(0)\cdot \begin{cases} L'(\frac{1}{2}, \mathrm{Wald}(\sigma)), & \textit{if } \epsilon_p(\sigma_p) = -1 \\ & \textit{for all } p \mid D(B), \\ L(\frac{1}{2}, \mathrm{Wald}(\sigma)) & \textit{if } \epsilon_p(\sigma_p) = +1 \\ \quad \times \log(p), & \textit{for a unique } p \mid D(B), \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

11. The arithmetic inner product formula

Returning to the arithmetic theta lift and arithmetic theta function $\widehat{\theta}(\tau)$, there should be a second relation between derivatives of Eisenstein series and arithmetic geometry, [**37**]:

Conjecture 11.1.

$$\langle\, \widehat{\theta}(\tau_1), \widehat{\theta}(\tau_2)\,\rangle = \mathcal{E}_2'(\begin{pmatrix} \tau_1 & \\ & -\bar{\tau}_2 \end{pmatrix}, 0; B).$$

Here recall that $\langle\ ,\ \rangle$ has been extended to be conjugate linear in the second factor. Additional discussion can be found in [**38**], [**40**], [**41**], [**51**]. This conjecture amounts to identities on Fourier coefficients:

$$\begin{aligned}(11.1)\qquad & \langle\, \widehat{\mathcal{Z}}(t_1, v_1), \widehat{\mathcal{Z}}(t_2, v_2)\,\rangle \cdot q_1^{t_1} q_2^{t_2} \\ & = \sum_{\substack{T\in \mathrm{Sym}_2(\mathbb{Z})^\vee \\ \mathrm{diag}(T)=(t_1,t_2)}} \mathcal{E}'_{2,T}(\begin{pmatrix} \tau_1 & \\ & \tau_2\end{pmatrix}, 0; B).\end{aligned}$$

If t_1t_2 is not a square, then any $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$ with $\mathrm{diag}(T) = (t_1, t_2)$ has $\det(T) \neq 0$. Thus, only the nonsingular Fourier coefficients of $\mathcal{E}_2(\tau, s, B)$ contribute to the right hand side of (11.1) in this case. Under the same condition, the cycles $\mathcal{Z}(t_1)$ and $\mathcal{Z}(t_2)$ do not meet on the generic fiber, although they may have common vertical components in the fibers of bad reduction.

On the other hand, if $t_1t_2 = m^2$, then the singular matrices $T = \begin{pmatrix} t_1 & \pm m \\ \pm m & t_2\end{pmatrix}$ occur on the right side of (11.1), and the cycles $\mathcal{Z}(t_1)$ and $\mathcal{Z}(t_2)$ meet in the generic fiber and have common horizontal components.

The results of [**37**] and [**46**] yield the following.

Theorem 11.2. *Suppose that t_1t_2 is not a square. In addition, assume that* (11.11) (*resp.* (11.26)) *below holds for $p = 2$ if $2 \nmid D(B)$ (resp. $2 \mid D(B)$). Then the Fourier coefficient identity* (11.1) *holds.*

We now briefly sketch the proof of Theorem 11.2. The basic idea is that there is a decomposition of the height pairing on the left hand side of (11.1) into terms indexed by $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$ with $\mathrm{diag}(T) = (t_1, t_2)$. One can prove identities between terms on the two sides corresponding to a given T.

Recalling the modular definition of the cycles given in section 3, the intersection $\mathcal{Z}(t_1) \cap \mathcal{Z}(t_2)$ can be viewed as the locus of triples $(A, \iota, \mathbf{x})$, where $\mathbf{x} = [x_1, x_2]$ is a pair of special endomorphisms $x_i \in V(A, \iota)$ with $Q(x_i) = t_i$. Associated to $\mathbf{x}$ is the 'fundamental matrix' $Q(\mathbf{x}) = \frac{1}{2}\,((x_i, x_j)) \in \mathrm{Sym}_2(\mathbb{Z})^\vee$, where $(\ ,\ )$ is the bilinear form on the quadratic lattice $V(A,\iota)$. Thus we may write

$$\mathcal{Z}(t_1)\cap\mathcal{Z}(t_2) = \coprod_{\substack{T\in \mathrm{Sym}_2(\mathbb{Z})^\vee \\ \mathrm{diag}(T)=(t_1,t_2)}} \mathcal{Z}(T), \tag{11.2}$$

where $\mathcal{Z}(T)$ is the locus of triples with $Q(\mathbf{x}) = T$. Note that the fundamental matrix is always positive semidefinite, since the quadratic form on $V(A,\iota)$ is positive definite. On the other hand, if $\det(T)\neq 0$, then $\mathcal{Z}(T)_{\mathbb{Q}}$ is empty, since the space of special endomorphisms $V(A,\iota)$ has rank 0 or 1 in characteristic 0.

For a nonsingular $T\in \mathrm{Sym}_2(\mathbb{Q})$, there is a unique global ternary quadratic space V_T with discriminant -1 which represents T; the matrix of the quadratic form on this space is

$$Q_T = \begin{pmatrix} T & \\ & \det(T)^{-1}\end{pmatrix}. \tag{11.3}$$

The space V_T is isometric to the space of trace zero elements for some quaternion algebra B_T over $\mathbb{Q}$, and the local invariants of this algebra must differ from those of the given indefinite B at a finite set of places

$$\mathrm{Diff}(T,B) := \{\ p\le\infty \ \mid\ \mathrm{inv}_p(B_T) = -\mathrm{inv}_p(B)\ \}, \tag{11.4}$$

with $|\mathrm{Diff}(T,B)|$ even[12].

The nonsingular Fourier coefficients of $\mathcal{E}_2(\tau,s;B)$ have a product formula

$$\mathcal{E}_{2,T}(\tau,s;B) = \eta(s;B)\,\zeta_{D(B)}(2s+2)\cdot W_{T,\infty}(\tau,s;\frac{3}{2})\cdot\prod_p W_{T,p}(s;\widetilde{\Phi}^B_p), \tag{11.5}$$

where $\widetilde{\Phi}^B(s) = \otimes_p\widetilde{\Phi}^B_p(s)$ is given by (10.11). Moreover, for a finite prime p,

$$p\in \mathrm{Diff}(T,B) \iff W_{T,p}(0,\widetilde{\Phi}^B_p) = 0, \tag{11.6}$$

while

$$\mathrm{ord}_{s=0}\, W_{T,\infty}(\tau,s;\frac{3}{2}) = \begin{cases} 0 & \text{if } T>0, \text{ and}\\ 1 & \text{if } \mathrm{sig}(T) = (1,1) \text{ or } (0,2).\end{cases} \tag{11.7}$$

[12] This is a slightly different definition than that used in [**37**], where ∞ is taken to be in the Diff set for T indefinite. Hence the difference in parity.

First suppose that $T > 0$, so that $\infty \in \mathrm{Diff}(T, B)$, since V^B has signature $(1, 2)$ and hence cannot represent T. If $|\mathrm{Diff}(T, B)| \geq 4$, then $\mathcal{Z}(T)$ is empty and $\mathcal{E}'_{2,T}(\tau, 0; B) = 0$, so there is no contribution of such T's on either side of (11.1). If $T > 0$ and $|\mathrm{Diff}(T, B)| = 2$, then $\mathrm{Diff}(T, B) = \{\infty, p\}$ for a unique finite prime p. In this situation, it turns out that $\mathcal{Z}(T)$ is supported in the fiber $\mathcal{M}_p$ at p. There are two distinct cases:

(i) If $p \nmid D(B)$, then $\mathcal{Z}(T)$ is a finite set of points in $\mathcal{M}_p$.

(ii) If $p \mid D(B)$, then $\mathcal{Z}(T)$ can be a union of components Y_p of the fiber at p, with multiplicities.

In case (i), the contribution to the height pairing on the left side of (11.1) is $\log(p)$ times the sum of the local multiplicities of points in $\mathcal{Z}(T)$. It turns out that all points have the same multiplicity, $e_p(T)$, so that the contribution to (11.1) has the form

$$e_p(T) \cdot |\mathcal{Z}(T)(\bar{\mathbb{F}}_p)|. \tag{11.8}$$

The computation of $e_p(T)$ can be reduced to a special case of a problem in the deformation theory of p–divisible groups which was solved by Gross and Keating, [**23**]. As explained in section 14 of [**37**], their result yields the formula

$$e_p(T) = \begin{cases} \sum_{j=0}^{\frac{\alpha-1}{2}} (\alpha + \beta - 4j)\, p^j & \text{if } \alpha \text{ is odd,} \\ \sum_{j=0}^{\frac{\alpha}{2}-1} (\alpha + \beta - 4j)\, p^j + \frac{1}{2}(\beta - \alpha + 1)\, p^{\frac{\alpha}{2}} & \text{if } \alpha \text{ is even,} \end{cases} \tag{11.9}$$

where, for $p \neq 2$, T is equivalent, via the action of $\mathrm{GL}_2(\mathbb{Z}_p)$, to $\mathrm{diag}(\epsilon_1\, p^\alpha, \epsilon_2\, p^\beta)$, with ϵ_1, $\epsilon_2 \in \mathbb{Z}_p^\times$ and $0 \leq \alpha \leq \beta$. The same result holds for $p = 2$, but with a slightly different definition, [**23**], of the invariants α and β of T.

On the other hand, if $\mathrm{Diff}(T, B) = \{\infty, p\}$ with $p \nmid D(B)$, then

(11.10)

$$\begin{aligned} \mathcal{E}'_{2,T}(\tau, 0; B) \;&=\; \eta(0; B)\, \zeta_{D(B)}(2) \cdot W_{T,\infty}(\tau, 0; \frac{3}{2}) \\ &\qquad \times \frac{W'_{T,p}(0, \Phi_p^0)}{W_{T,p}(0, \Phi_p^-)} \cdot \bigg(W_{T,p}(0, \Phi_p^-) \cdot \prod_{\ell \neq p} W_{T,\ell}(0, \widetilde{\Phi}_\ell^B) \bigg). \end{aligned}$$

Here recall that the local sections $\widetilde{\Phi}_\ell^B(s)$ are as defined in (10.9) and (10.11). For $p \neq 2$, formulas of Kitaoka, [**35**], for representation densities $\alpha_p(S, T)$ of binary forms T by unimodular quadratic forms S can be used to compute the derivatives of the local Whittaker functions, and yield, [**37**], Proposition 8.1 and Proposition 14.6,

$$\frac{W'_{T,p}(0,\Phi^0_p)}{W_{T,p}(0,\Phi^-_p)} = \frac{1}{2}(p-1)\,\log(p)\cdot e_p(T), \tag{11.11}$$

where $e_p(T)$ is precisely the multiplicity (11.9)! On the other hand, up to simple constants[13],

$$W_{T,\infty}(\begin{pmatrix}\tau_1 & \\ & \tau_2\end{pmatrix},0;\frac{3}{2}) \;\mathrel{\overset{\circ}{=}}\; q_1^{t_1}q_2^{t_2}, \tag{11.12}$$

and

$$W_{T,p}(0,\Phi^-_p)\cdot\prod_{\ell\neq p} W_{T,\ell}(0,\widetilde{\Phi}^B_\ell) \;\mathrel{\overset{\circ}{=}}\; |\mathcal{Z}(T)(\bar{\mathbb{F}}_p)|. \tag{11.13}$$

Note that to obtain (11.11) in the case $p=2$, one needs to extend Kitaoka's representation density formula to this case; work on this is in progress.

Next, we turn to case (ii), where the component $\mathcal{Z}(T)$ is attached to T with $\mathrm{diag}(T)=(t_1,t_2)$ and $\mathrm{Diff}(T,B)=\{\infty,p\}$ with $p\mid D(B)$. This case is studied in detail in [**46**] under the assumption that $p\neq 2$, using the p–adic uniformization described in section 4 above. First, we can base change to $\mathbb{Z}_{(p)}$ and use the intersection theory explained in section 4 of [**46**]. The contribution of $\mathcal{Z}(T)$ to the height pairing is then

$$\chi(\,\mathcal{Z}(T),\mathcal{O}_{\mathcal{Z}(t_1)}\overset{\mathbb{L}}{\otimes}\mathcal{O}_{\mathcal{Z}(t_2)}\,)\cdot\log(p). \tag{11.14}$$

Here χ is the Euler–Poincaré characteristic and, for quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on $\mathcal{M}\times_{\mathrm{Spec}(\mathbb{Z})}\mathrm{Spec}(\mathbb{Z}_{(p)})$, with $\mathrm{supp}(\mathcal{F})\cap\mathrm{supp}(\mathcal{G})$ contained in the special fiber and proper over $\mathrm{Spec}(\mathbb{Z}_{(p)})$,

$$\chi(\mathcal{F}\overset{\mathbb{L}}{\otimes}\mathcal{G}) \;=\; \chi(\mathcal{F}\otimes\mathcal{G})-\chi(\mathcal{T}or_1(\mathcal{F},\mathcal{G}))+\chi(\mathcal{T}or_2(\mathcal{F},\mathcal{G})). \tag{11.15}$$

Recall that, for $i=1,\,2$, $\widehat{\mathcal{C}}_p(t_i)$ is the base change to W of the formal completion of $\mathcal{Z}(t_i)$ along its fiber at p. Similarly, we write $\widehat{\mathcal{C}}_p(T)$ for the analogous formal scheme over W determined by $\mathcal{Z}(T)$. By Lemma 8.4 of [**46**],

$$\chi(\,\mathcal{Z}(T),\mathcal{O}_{\mathcal{Z}(t_1)}\overset{\mathbb{L}}{\otimes}\mathcal{O}_{\mathcal{Z}(t_2)}\,) \;=\; \chi(\,\widehat{\mathcal{C}}_p(T),\mathcal{O}_{\widehat{\mathcal{C}}_p(t_1)}\overset{\mathbb{L}}{\otimes}\mathcal{O}_{\widehat{\mathcal{C}}_p(t_2)}\,), \tag{11.16}$$

[13]Hence the notation $\mathrel{\overset{\circ}{=}}$.

so that we can calculate after passing to the formal situation. The same arguments which yield the p–adic uniformization, Proposition 4.2, of $\widehat{\mathcal{C}}_p(t)$ yields a diagram

$$\begin{array}{ccc} \widehat{\mathcal{C}}_p(T) & \xrightarrow{\sim} & H'(\mathbb{Q})\backslash(\star\star) \\ \downarrow & & \downarrow \\ \widehat{\mathcal{M}}_p & \xrightarrow{\sim} & H'(\mathbb{Q})\backslash\Big(\,\mathcal{D}^\bullet \times H(\mathbb{A}_f^p)/K^p\,\Big) \end{array} \tag{11.17}$$

where

$$(\star\star) := \left\{ (\mathbf{y},(X,\rho),gK^p) \;\middle|\; \begin{array}{c} (i)\quad Q(\mathbf{y}) = T \\ (ii)\quad (X,\rho)\in Z^\bullet(j(\mathbf{y})) \\ (iii)\quad \mathbf{y}\in \big(\,g\,(V(\mathbb{A}_f^p)\cap \hat{O}_B^p)\,g^{-1}\big)^2 \end{array} \right\}. \tag{11.18}$$

Proceeding as in Remark 4.3, we obtain

$$\widehat{\mathcal{C}}_p(T) \xrightarrow{\sim} \big[\,\Gamma'\backslash\mathcal{D}_T^\bullet\,\big], \tag{11.19}$$

where

$$\mathcal{D}_T^\bullet = \coprod_{\substack{\mathbf{y}\in(L')^2\\ Q(\mathbf{y})=T}} Z^\bullet(j(\mathbf{y})). \tag{11.20}$$

Here, recall from (4.25) that

$$\Gamma' = H'(\mathbb{Q}) \cap \big(\,H'(\mathbb{Q}_p)\times K^p\,\big) = \big(\,O'_B[\frac{1}{p}]\,\big)^\times. \tag{11.21}$$

Since any $\mathbf{y}\in L'$ with $Q(\mathbf{y}) = T$ spans a nondegenerate 2–plane in V',

$$\Gamma'_{\mathbf{y}} = \Gamma' \cap Z(\mathbb{Q}) \simeq \big(\,\mathbb{Z}[\frac{1}{p}]\,\big)^\times = \{\pm1\}\times p^{\mathbb{Z}}, \tag{11.22}$$

and the central element p acts on $\mathcal{D}^\bullet$ by translation by 2, i.e. carries $\mathcal{D}^i$ to $\mathcal{D}^{i+2}$. Thus, unfolding, as in [**46**], p.216, we have

$$\chi(\,\widehat{\mathcal{C}}_p(T), \mathcal{O}_{\widehat{\mathcal{C}}_p(t_1)} \overset{\mathbb{L}}{\otimes} \mathcal{O}_{\widehat{\mathcal{C}}_p(t_2)}\,) = \sum_{\substack{\mathbf{y}\in(L')^2\\ Q(\mathbf{y})=T\\ \mathrm{mod}\ \Gamma'}} \chi(\,Z(\mathbf{j}), \mathcal{O}_{Z(j_1)} \overset{\mathbb{L}}{\otimes} \mathcal{O}_{Z(j_2)}\,). \tag{11.23}$$

Here we have used the orbifold convention, which introduces a factor of $\frac{1}{2}$ from the ±1 in $\Gamma'_{\mathbf{y}}$ which acts trivially, and the fact that the two 'sheets' $Z(\mathbf{j}) = Z^0(\mathbf{j})$ and $Z^1(\mathbf{j})$ make the same contribution.

Two of the main results of [**46**], Theorem 5.1 and 6.1, give the following:

Theorem 11.3. *(i) The quantity*

$$e_p(T) := \chi(\,Z(\mathbf{j}), \mathcal{O}_{Z(j_1)} \overset{\mathbb{L}}{\otimes} \mathcal{O}_{Z(j_2)}\,)$$

is the intersection number, [**46**], *section 4, of the cycles $Z(j_1)$ and $Z(j_2)$ in the formal scheme $\mathcal{D}_p$. It depends only on the $\mathrm{GL}_2(\mathbb{Z}_p)$–equivalence class of T.*
(ii) For $p \neq 2$, and for $T \in \mathrm{Sym}_2(\mathbb{Z}_p)$ which is $\mathrm{GL}_2(\mathbb{Z}_p)$–equivalent to $\mathrm{diag}(\epsilon_1 p^\alpha, \epsilon_2 p^\beta)$, with $0 \le \alpha \le \beta$ and $\epsilon_1,\ \epsilon_2 \in \mathbb{Z}_p^\times$,

$$e_p(T) = \alpha+\beta+1-\begin{cases} p^{\alpha/2} + 2\frac{p^{\alpha/2}-1}{p-1} & \text{if } \alpha \text{ is even and } (-\epsilon_1,p)_p = -1, \\ (\beta-\alpha+1)\,p^{\alpha/2} + 2\frac{p^{\alpha/2}-1}{p-1} & \text{if } \alpha \text{ is even and } (-\epsilon_1,p)_p = 1, \\ 2\frac{p^{(\alpha+1)/2}-1}{p-1} & \text{if } \alpha \text{ is odd.} \end{cases}$$

Part (i) of Theorem 11.3, together with (11.16) and (11.23), yields

$$\chi(\,\mathcal{Z}(T), \mathcal{O}_{\mathcal{Z}(t_1)} \overset{\mathbb{L}}{\otimes} \mathcal{O}_{\mathcal{Z}(t_2)}\,) \;=\; e_p(T)\cdot\Big(\sum_{\substack{\mathbf{y}\in(L')^2\\ Q(\mathbf{y})=T\\ \bmod \Gamma'}} 1\Big), \tag{11.24}$$

which is the analogue of (11.8) in the present case.

On the other hand, if $\mathrm{Diff}(T,B) = \{\infty, p\}$ with $p \mid D(B)$, the term on the right side of (11.1) is

$$\mathcal{E}'_{2,T}(\tau,0;B) = c\cdot W_{T,\infty}(\tau,0;\frac{3}{2})\cdot W'_{T,p}(0,\widetilde{\Phi}_p)\cdot\prod_{\ell\neq p} W_{T,\ell}(0,\widetilde{\Phi}_\ell^B). \tag{11.25}$$

where $c = \eta(0;B)\,\zeta_{D(B)}(2)$. By [**46**], Corollary 11.4, the section $\widetilde{\Phi}_p(s)$ in (10.9) satisfies,

$$W'_{T,p}(0,\widetilde{\Phi}_p) = p^{-2}(p+1)\,\log(p)\cdot e_p(T) \tag{11.26}$$

while

$$\prod_{\ell\neq p} W_{T,\ell}(0,\Phi_\ell) \overset{\circ}{=} \Big(\sum_{\substack{\mathbf{y}\in(L')^2\\ Q(\mathbf{y})=T\\ \bmod \Gamma'}} 1\Big). \tag{11.27}$$

can be thought of as the number of 'connected components' of $\mathcal{Z}(T)$. Note that, the particular choice (10.10) of the coefficients $A_p(s)$ and $B_p(s)$ in the definition of $\widetilde{\Phi}_p(s)$ was dictated by the identity (11.16), the proof of which is based on results of Tonghai Yang, [**74**], on representation densities $\alpha_p(S,T)$ of binary forms T by nonunimodular forms S. To obtain (11.16) in the case $p=2$, one needs to extend both the intersection calculations of [**46**] and the density formulas of [**74**] to this case. The first of these tasks, begun in the appendix to section 11 of [**50**], is now complete. The second is in progress.

Finally, the contribution to the right side of (11.1) of the terms for T of signature $(1,1)$ or $(0,2)$, which can be calculated using the formulas of [**65**], coincides with the contribution of the star product of the Green functions $\Xi(t_1,v_1)$ and $\Xi(t_2,v_2)$ to the height pairing. This is a main result of [**37**]; for a sketch of the ideas involved, cf. [**38**].

This completes the sketch of the proof of Theorem 11.2. $\square$ $\square$

Before turning to consequences, we briefly sketch how part (ii) of Theorem 11.3 is obtained. By part (i) of that Theorem, it will suffice to compute the intersection number $(Z(j_1),Z(j_2))$ for j_1 and $j_2\in V'(\mathbb{Q}_p)$ with $j_1^2=-Q(j_1)=-\epsilon_1p^\alpha$, $j_2^2=-Q(j_2)=-\epsilon_2p^\beta$ and $(j_1,j_2)=j_1j_2+j_2j_1=0$. By Theorem 5.1 of [**46**], $(Z(j_1),Z(j_2))=(Z(j_1)^{\rm pure},Z(j_2)^{\rm pure})$, so that we may use the description of $Z(j)^{\rm pure}$ given in Proposition 4.5, above.

The following result, which combines Lemmas 4.7, 4.8, and 4.9 of [**46**], describes the intersection numbers of individual components. Recall that the vertical components $\mathbb{P}_{[\Lambda]}$ of $\mathcal{D}$ are indexed by vertices $[\Lambda]$ in the building $\mathcal{B}$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$.

Lemma 11.4. *(i) For a pair of vertices* $[\Lambda]$ *and* $[\Lambda']$*,*

$$(\mathbb{P}_{[\Lambda]},\mathbb{P}_{[\Lambda']})=\begin{cases}1 & \textit{if } ([\Lambda],[\Lambda']) \textit{ is an edge,}\\ -(p+1) & \textit{if } [\Lambda]=[\Lambda'], \textit{ and}\\ 0 & \textit{otherwise.}\end{cases}$$

(ii)

$$(Z(j_1)^h, Z(j_2)^h) = \begin{cases} 1 & \textit{if } \alpha \textit{ and } \beta \textit{ are odd,} \\ 0 & \textit{otherwise.} \end{cases}$$

(iii)

$$(Z(j_1)^h, \mathbb{P}_{[\Lambda]}) = \begin{cases} 2 & \textit{if } \alpha \textit{ is even, } (-\epsilon_1, p)_p = -1, \textit{ and } \mathcal{B}^{j_1} = [\Lambda], \\ 1 & \textit{if } \alpha \textit{ is odd and } d([\Lambda], \mathcal{B}^{j_1}) = \frac{1}{2}, \textit{ and} \\ 0 & \textit{otherwise.} \end{cases}$$

and similarly for $Z(j_2)^h$.

The computation of the intersection number $(Z(j_1)^{\text{pure}}, Z(j_2)^{\text{pure}})$ is thus reduced to a combinatorial problem. Recall from Proposition 4.5 above that the multiplicity in $Z(j_1)^{\text{pure}}$ of a vertical component $\mathbb{P}_{[\Lambda]}$ indexed by a vertex $[\Lambda]$ is determined by the distance of $[\Lambda]$ from the fixed point set $\mathcal{B}^{j_1}$ of j_1 on $\mathcal{B}$ by the formula

$$\mu_{[\Lambda]}(j_1) = \max\{\, 0, \, \frac{\alpha}{2} - d([\Lambda], \mathcal{B}^{j_1}) \,\}.$$

In particular, this multiplicity is zero outside of the tube $\mathcal{T}(j_1)$ of radius $\frac{\alpha}{2}$ around $\mathcal{B}^{j_1}$. Of course, the analogous description holds for $Z(j_2)^{\text{pure}}$. Our assumption that the matrix of inner products of j_1 and j_2 is diagonal implies that j_1 and j_2 anticommute, and hence the relative position of the fixed point sets and tubes $\mathcal{T}(j_1)$ and $\mathcal{T}(j_2)$ is particularly convenient.

For example, consider the case in which α and β are both even, with $(-\epsilon_1, p)_p = 1$ and $(-\epsilon_2, p)_p = -1$. These conditions mean that $\mathbb{Q}_p(j_1)^\times$ is a split Cartan in $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathbb{Q}_p(j_2)^\times$ is a nonsplit, unramified, Cartan. The fixed point set $\mathcal{A} = \mathcal{B}^{j_1}$ is an apartment, and $\mathcal{T}(j_1)$ is a tube of radius $\frac{\alpha}{2}$ around it. Moreover, $Z(j_1)^h = \emptyset$, so that $Z(j_1)$ consists entirely of vertical components. The fixed point set $\mathcal{B}^{j_2}$ is a vertex $[\Lambda_0]$ and $\mathcal{T}(j_2)$ is a ball of radius $\frac{\beta}{2}$ around it. Since j_1 and j_2 anticommute, the vertex $[\Lambda_0]$ lies in the apartment $\mathcal{A} = \mathcal{B}^{j_1}$. For any vertex $[\Lambda]$, the geodesic from $[\Lambda_0]$ to $[\Lambda]$ runs a distance ℓ inside the apartment $\mathcal{A}$ and then a distance r outside of it. By (i) of Lemma 11.4, the contribution to the intersection number of the vertices with $\ell = 0$ is

$$-(p+1)\,\frac{\alpha}{2} + (1-p)\sum_{r=1}^{\alpha/2-1}(\frac{\alpha}{2} - r)(p-1)p^{r-1} \;=\; 1 - \alpha - p^{\alpha/2}. \tag{11.28}$$

Here the first term is the contribution of $[\Lambda_0]$, since

$$(\mathbb{P}_{[\Lambda_0]}, Z(j_2)^v) = -(p+1)\frac{\beta}{2} + (p+1)(\frac{\beta}{2} - 1) = -(p+1), \tag{11.29}$$

where $Z(j_2)^v$ is the vertical part of $Z(j_2)^{\rm pure}$. Similarly, for $r>0$, there are $(p-1)p^{r-1}$ vertices $[\Lambda]$ at distance r (with $\ell=0$), and each contributes

$$(11.30)\quad (\mathbb{P}_{[\Lambda]}, Z(j_2)^v) = (\frac{\beta}{2}-r+1)-(p+1)(\frac{\beta}{2}-r)+p(\frac{\beta}{2}-r-1) \ = \ 1-p.$$

Here we have used the fact that all such vertices lie inside the ball $\mathcal{T}(j_2)$. Next, consider vertices with $1\le \ell\le (\beta-\alpha)/2$. Note that, if such a vertex $[\Lambda]$ has $r=\frac{\alpha}{2}$, then $[\Lambda]\in\mathcal{T}(j_2)$. The contribution of vertices with ℓ values in this range is

$$(11.31)\quad 2(1-p)\sum_{\ell=1}^{(\beta-\alpha)/2}\bigg(\frac{\alpha}{2}+\sum_{r=1}^{\alpha/2-1}(\frac{\alpha}{2}-r)(p-1)p^{r-1}\bigg) = (\alpha-\beta)(p^{\alpha/2-1}-1).$$

Next, in the range $(\beta-\alpha)/2<\ell<\beta/2$ and with $r<\beta/2-\ell-1$, $[\Lambda]\in\mathcal{T}(j_2)$, and such $[\Lambda]$'s contribute

(11.32)
$$2(1-p)\sum_{\ell=(\beta-\alpha)/2+1}^{\beta/2-1}\bigg(\frac{\alpha}{2}+\sum_{r=1}^{\beta/2-\ell-1}(\frac{\alpha}{2}-r)(p-1)p^{r-1}\bigg) = 2\alpha-4\,\frac{p^{\alpha/2-1}-1}{p-1}.$$

Finally, the vertices on the boundary of the ball $\mathcal{T}(j_2)$ contribute

$$(11.33)\qquad \alpha+2\sum_{\ell=(\beta-\alpha)/2+1}^{\beta/2-1}(\frac{\alpha}{2}-(\frac{\beta}{2}-\ell))(p-1)p^{\beta/2-\ell-1} \ = \ 2\,\frac{p^{\alpha/2-1}-1}{p-1}.$$

Summing these contributions and adding $(\mathbb{P}_{[\Lambda_0]}, Z(j_2)^h)\cdot\alpha/2=\alpha$, we obtain the quantity claimed in (ii) of Theorem 11.3 in this case. The other cases are similar. $\square$

We now describe the consequences of Theorem 11.2.

Corollary 11.5. *Assume that the p–adic density identity (11.11) (resp. (11.26)) holds for $p=2$ when $2\mid D(B)$ (resp. $2\nmid D(B)$). Then*

$$\begin{aligned}\langle\,\widehat{\theta}(\tau_1),\widehat{\theta}(\tau_2)\,\rangle = \ &\mathcal{E}_2'(\begin{pmatrix}\tau_1&\\&-\bar\tau_2\end{pmatrix},0;B)\\ &+\sum_{t_1}\bigg(\sum_{\substack{t_2\\ t_1t_2=\text{square}}} c(t_1,t_2,v_1,v_2)\cdot\bar q_2^{t_2}\bigg)\,q_1^{t_1}.\end{aligned}$$

for some coefficients $c(t_1, t_2, v_1, v_2)$.

The extra term on the right hand side should vanish, according to Conjecture 11.1, but, in any case, the coefficient of each $q_1^{t_1}$ is a modular form of weight $\frac{3}{2}$ in τ_2 with only one square class of Fourier coefficients, i.e., is a distinguished form. But such forms all come from $O(1)$'s, and so are orthogonal to cusp forms in $\mathcal{A}_{00}(G')$, [**19**].

Corollary 11.6. *With the notation and assumptions of Theorem 10.1 and assuming that the p–adic density identity (11.11) holds for* $p = 2$,

$$\langle\, \widehat{\theta}(\tau_1), \widehat{\theta}(f)\,\rangle = f(\tau_1)\cdot C(0)\cdot \begin{cases} L'(\frac{1}{2}, \mathrm{Wald}(\sigma)), \\ L(\frac{1}{2}, \mathrm{Wald}(\sigma))\cdot \log(p), \\ 0 \end{cases}.$$

Proof.

$$\begin{aligned}
\langle\, \widehat{\theta}(\tau_1), \widehat{\theta}(f)\,\rangle &= \langle\, \widehat{\theta}(\tau_1), \langle\, \widehat{\theta}, f\,\rangle_{\mathrm{Pet}}\,\rangle \\
&= \big\langle\, \langle\, \widehat{\theta}(\tau_1), \widehat{\theta}(\tau_2)\,\rangle, \overline{f(\tau_2)}\,\big\rangle_{\mathrm{Pet}} \qquad (11.34)\\
&= \big\langle\, \mathcal{E}_2'(\begin{pmatrix}\tau_1 & \\ & -\bar{\tau}_2\end{pmatrix}, 0; B), \overline{f(\tau_2)}\,\big\rangle_{\mathrm{Pet}} \\
&= f(\tau_1)\cdot C(0)\cdot \begin{cases} L'(\frac{1}{2}, \mathrm{Wald}(\sigma)), \\ L(\frac{1}{2}, \mathrm{Wald}(\sigma))\cdot \log(p), \\ 0 \end{cases}
\end{aligned}$$

□

Corollary 11.7. *With the notation and assumptions of Corollary 11.6,*
(i) If $\epsilon_p(\sigma_p) = +1$ *for more than one* $p \mid D(B)$, *then* $\widehat{\theta}(f) = 0$.
Otherwise
(ii) If $\epsilon(\frac{1}{2}, \mathrm{Wald}(\sigma)) = +1$, *then there is a unique prime* $p \mid D(B)$ *such that* $\epsilon_p(\sigma_p) = +1$, $\widehat{\theta}(f) \in \mathrm{Vert}_p \subset \mathrm{Vert}$, *and*

$$\langle\, \widehat{\theta}(f), \widehat{\theta}(f)\,\rangle = C(0)\cdot\langle\, f, f\,\rangle\cdot L(\frac{1}{2}, \mathrm{Wald}(\sigma))\cdot\log(p).$$

(iii) If $\epsilon(\frac{1}{2}, \mathrm{Wald}(\sigma)) = -1$, *then* $\widehat{\theta}(f) \in \widetilde{\mathrm{MW}}$ *and*

$$\langle\, \widehat{\theta}(f), \widehat{\theta}(f)\,\rangle = C(0)\cdot \langle\, f, f\,\rangle \cdot L'(\frac{1}{2}, \mathrm{Wald}(\sigma)).$$

This result gives an analogue of Waldspurger's theory, described in section 9 above, and of the Rallis inner product formula, [**61**], for the arithmetic theta lift (8.1). Of course, the result only applies to forms of weight $\frac{3}{2}$, and we have made quite strong restrictions on the level of f. The restrictions on the level can most likely be removed with some additional work, but the restriction on the weight is essential to our setup.

In the case $\epsilon(\frac{1}{2}, \mathrm{Wald}(\sigma)) = +1$, part (ii) gives a geometric interpretation of the value $L(\frac{1}{2}, \mathrm{Wald}(\sigma))$ in terms of vertical components of $\mathcal{M}$, analogous to the geometric interpretation of the value of the base change L-function given by Gross in his Montreal paper, [**21**].

Finally, if $\widehat{\theta}(f) \neq 0$, let

$$\widehat{\mathcal{Z}}(t)(f) = \widehat{\mathcal{Z}}(t,v)(f) = \frac{\langle\, \widehat{\mathcal{Z}}(t,v), \widehat{\theta}(f)\,\rangle}{\langle\, \widehat{\theta}(f), \widehat{\theta}(f)\,\rangle}\cdot \widehat{\theta}(f) \tag{11.35}$$

be the component of the cycle $\widehat{\mathcal{Z}}(t,v)$ along the line spanned by $\widehat{\theta}(f)$. Note that, by Corollary 11.6, this projection does not depend on v. The following is an analogue of the Gross-Kohnen-Zagier relation, [**24**], [**78**]. Note that it holds in both cases $\epsilon(\frac{1}{2}, \mathrm{Wald}(\sigma)) = +1$ or $\epsilon(\frac{1}{2}, \mathrm{Wald}(\sigma)) = -1$.

Corollary 11.8.

$$\sum_t \widehat{\mathcal{Z}}(t)(f)\cdot q^t = \frac{f(\tau)\cdot \widehat{\theta}(f)}{\langle\, f, f\,\rangle}.$$

Thus, the Fourier coefficients of f encode the position of the cycles $\widehat{\mathcal{Z}}(t)(f)$ on the $\widehat{\theta}(f)$ line. Both sides are invariant under scaling of f by a constant factor.

For the related results of Gross–Kohnen–Zagier in the case $\epsilon(\frac{1}{2}, \mathrm{Wald}(\sigma)) = -1$, cf. [**24**], especially Theorem C, p. 503, the discussion on pp. 556–561, and the examples in [**78**], where the analogies with the work of Hirzebruch–Zagier are also explained and used in the proof! Our class $\widehat{\theta}(f)/ < f, f >$, which arises as the image of f under the arithmetic theta lift, is the analogue of the class y_f in [**24**]. To proved the main part of Theorem C of [**24**] in our case, namely that the π_f–isotypic components, where $\pi = \mathrm{Wald}(\sigma)$, of the classes $\widehat{\mathcal{Z}}(t, v)$ either vanish or lie on a line, we can use the Howe duality Theorem [**30**], [**72**] for the local theta correspondence, cf. [**42**].

In [**22**], Gross gives a beautiful representation theoretic framework in which the Gross–Zagier formula, [**26**], and a result of Waldspurger, [**71**], can be viewed together. He works with unitary similitude groups $G = G^B = GU(2)$, constructed from the choice of a quaternion algebra B over a global field k and a quadratic extension E/k which splits B. The torus T over k with $T(k) = E^\times$ embeds in

$$G^B(k) = (B^\times \times E^\times)/\Delta k^\times.$$

For a place v of k, the results of Tunnel, [**67**], and Saito, [**62**], show that the existence of local $T(k_v)$–invariant functionals on an irreducible admissible representation Π_v of G_v is controlled by the local root number $\epsilon_v(\Pi_v)$ of the Langlands L-function defined by a 4 dimensional symplectic representation of ${}^LG$, [**22**], section 10. There is a dichotomy phenomenon. If v is a place which is not split in E/k, then the local quaternion algebras B_v^+ and B_v^- are both split by E_v, so that the torus T_v embeds in both similitude groups G_v^+ and G_v^-. If Π_v is a discrete series representation of G_v^+, then, by the local Jacquet–Langlands correspondence, there is an associated representation Π'_v of G_v^-, and

$$\dim \mathrm{Hom}_{T_v}(\Pi_v, \mathbb{C}) + \dim \mathrm{Hom}_{T_v}(\Pi'_v, \mathbb{C}) = 1.$$

For a global cuspidal automorphic representation Π of $G(\mathbb{A}) = (\mathrm{GL}_2(\mathbb{A}) \times E_{\mathbb{A}}^\times)/\mathbb{A}^\times$, there is a finite collection of quaternion algebras B over k, split by E, with automorphic cuspidal representations Π^B associated to Π by the global Jacquet–Langlands correspondence. When the global root number $\epsilon(\Pi)$ for the degree 4 Langlands L-function $L(s, \Pi)$ is $+1$, Waldspurger's theorem, [**71**], says that the nonvanishing of the automorphic $T(\mathbb{A})$–invariant linear functional, defined on Π^B by integration over $\mathbb{A}^\times T(k)\backslash T(\mathbb{A})$, is equivalent to (i) the nonvanishing of the local linear functionals[14] and (ii) the nonvanishing of the central value $L(\frac{1}{2}, \Pi)$ of the L-function. Suppose that k is a totally real number field, E

[14]By local root number conditions, this uniquely determines the quaternion algebra B for the given E and Π.

is a totally imaginary quadratic extension, and Π_v is a discrete series of weight 2 at every archimedean place of k. Then, in the case $\epsilon(\Pi) = -1$, the local root number conditions determine a Shimura curve M^B over k, and Gross defines a $T(\mathbb{A}_f)$–invariant linear functional on the associated Mordell–Weil space MW^B using the 0–cycle attached to T. Gross conjectures that the nonvanishing of this functional on the Π_f^B component of MW^B is equivalent to the nonvanishing of the central derivative $L'(\frac{1}{2}, \Pi)$, and that there is an explicit expression for this quantity in terms of the height pairing of a suitable 'test vector'. In the case $k = \mathbb{Q}$, this is the classical Gross–Zagier formula, [**26**], and in there is work of Zhang in the general case [**79**].

Thus, there is a close parallel between Gross's 'arithmetic' version of Waldspurger's central value result [**71**] and our 'arithmetic' version of Waldspurger's results on the Shimura lift. It should be possible to formulate our conjectures above for Shimura curves over an arbitrary totally real field k. It would be interesting to find a direct connection between our constructions and the methods used by Shou-Wu Zhang, [**79**].

Finally, it is possible to formulate a similar theory for central value/derivative of the triple product L-function. This is discussed in [**25**], cf. also [**27**], [**28**]. It may be that all of these examples can be covered by some sort of arithmetic version of Jacquet's relative trace formula, [**33**], [**34**], [**1**]. Further speculations about 'arithmetic theta functions' and derivatives of Eisenstein series can be found in [**40**] and [**41**].

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