DIFFERENTIAL GAMES

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1. Introduction

In this paper, we shall give a brief account of the main ideas in the theory of differential games. Our presentation will be limited to two player, zero sum, differential games, except for a few words regarding the situations where there are more than two players or where the game is not zero sum.

To begin with, recall the game formulation when there are two players, I and II. The *actions* (*strategies*) available to I are represented by a set $A = \{\alpha\}$, whereas those available to II are described by the set $B = \{\beta\}$. There is specified a *payoff* function $P : A \times B \twoheadrightarrow R$, and I chooses α in A to maximize P while II chooses β in B to minimize P. In general, we know that the order in which the choices are made is essential, and we can only assert that

$$\sup_A \inf_B P(\alpha, \beta) \leqq \inf_B \sup_A P(\alpha, \beta). \tag{1.1}$$

When equality holds in (1.1), we say that the game (P, A, B) has a *saddle value*, and this common number is called the (saddle) value of the game. If there exist α^* in A and β^* in B such that,

$$P(\alpha, \beta^*) \leqq P(\alpha^*, \beta^*) \leqq P(\alpha^*, \beta), \qquad \alpha \in A, \beta \in B \tag{1.2}$$

then we say that (α^*, β^*) constitute a *saddle point* for the game. It is easy to see that in this case the game has a saddle value, and it is equal to $P(\alpha^*, \beta^*)$. Let us call games of this kind *matrix games*, since P has an obvious matrix representation when A and B are finite sets. The theory of matrix games started with the result of von Neumann, and since then many generalizations have appeared. A typical result is the following.

THEOREM 1.1. *Suppose A and B are convex, compact topological spaces. Suppose for fixed β in B, $P(\cdot, \beta)$ is concave, and upper semicontinuous, over A, and for fixed α in A, $P(\alpha, \cdot)$ is convex, and lower semicontinuous, over B. Then, the game (P, A, B) has a saddle point.*

Although the concave-convex assumption on P can be weakened slightly [14], it appears that in general this assumption (or an equivalent hypothesis) is essential [12]. This is in apparent striking contrast with the situation in differential games.

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We can think of a differential game as matrix games played continuously over some time interval, say $[0, 1]$. At each time t, the *states* (positions) of I and II are represented by n dimensional vectors $x(t)$ and $y(t)$, respectively. Having observed the past history $\{(\tau, y(\tau)) | 0 \leqq \tau \leqq t\}$ of his opponent, player I chooses an action (*control*) $u(t)$ from a specified set U, to guide his state according to the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)), x(0) = x_0. \tag{1.3}$$

Similarly, II chooses a control $v(t)$ from a set V, based upon the past history of I, to steer his own state according to the differential equation

$$\dot{y}(t) = g(t, y(t), v(t)), y(0) = y_0. \tag{1.4}$$

At the end of the time interval $[0, 1]$, we obtain the continuous *trajectories* $x : [0, 1] \to R^n$, $y : [0, 1] \to R^n$ of the two players, respectively, and II gives to I the amount $P(x, y)$ where $P : C \times C \to R$ is a specified *payoff* function. Here, C is the space of all continuous functions from $[0, 1]$ into R^n.

Difficult technical problems arise when we try to specify precisely the set of strategies available to each player. Since the controls $u(t)$ and $v(t)$ will in general be functionals of the past histories of y and x, respectively,

$$u(t) = F(t, y_{[0,t]}), \qquad v(t) = G(t, x_{[0,t]}), \tag{1.5}$$

when we insert these back into the differential equations, we are likely to lose the results of existence and uniqueness of solutions even if f and g are "nice" functions. Furthermore, elementary examples [10] show that there is no natural way to limit the arbitrariness of the functionals F and G. In the next section, we shall give a natural extension of the notion of solution which avoids these difficulties.

After defining strategies accurately, we shall discuss the questions of existence of saddle values and saddle points. We will indicate why saddle values and saddle points exist for a large class of differential games. In Section 3, we shall consider the synthesis problem: how do we find the saddle value and a saddle point when we know they exist? We shall also present a stochastic version of a differential game which sheds some light on the synthesis problem, and also exhibits an intriguing connection with Theorem 1.1. In the final section, we shall discuss generalizations.

2. Game formulation and existence results

Consider the differential equations:

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = x_0, \tag{2.1}$$

$$\dot{y}(t) = g(t, y(t), v(t)), \qquad y(0), y_0. \tag{2.2}$$

We assume that U and V are compact subsets of R^m and $f: [0, 1] \times R^n \times U \to R^n$ is measurable in the first, Lipschitz in the second, and continuous in the third variable. Furthermore, $|f(t, x, u)|$ grows at most linearly in $|x|$, uniformly in t, u. Similar assumptions are placed on g. Finally, we assume that the sets $f(t, x, U)$ and $g(t, y, V)$ are convex.

By an admissible control for player I (II), we mean any measurable function $u : [0, 1] \to U(v : [0, 1] \to V)$. For any admissible control u, there is defined a unique continuous *trajectory* $x : [0, 1] \to R^n$ satisfying (2.1) for almost all t in $[0, 1]$. Let X be the set of trajectories obtained from all the admissible controls of I. Similarly, let Y be the set of trajectories obtained from all the admissible controls v of player II. We consider X and Y as subsets of the Banach space C of all continuous functions $z : [0, 1] \to R^n$ under the norm, $\|z\| = \max \{|z(t)| \,|\, t \in [0, 1]\}$. The following result is well known [13].

THEOREM 2.1. *The sets X and Y are compact subsets of C.*

At each time t, player I chooses $u(t) \in U$ based on the past history of the trajectory y of II. Since the result of his actions determines a trajectory x in X, we can define a strategy for I as a map from Y into X, taking care to insure that only the past history is used, that is, the maps should be causal. This motivates the following definition.

DEFINITION 2.1. *An (admissible)* strategy *for I is any map $\alpha : Y \to X$ such that if y, y' in Y, satisfy $y(\tau) = y'(\tau)$, $0 \leqq \tau \leqq t$, then $\alpha(y)(\tau) = \alpha(y')(\tau)$, $0 \leqq \tau \leqq t$.*

Let A be the set of all strategies of I. Similarly, we define the set $B = \{\beta\}$ of strategies of II.

The notion of solution of an ordinary differential equation is generalized as follows.

DEFINITION 2.2. *Let $\alpha \in A$* and *$\beta \in B$. A pair of trajectories (x, y) of $X \times Y$ is said to be an outcome of (α, β) if there exist sequences $\{x_n\} \subset X$, and $\{y_n\} \subset Y$ such that*

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \alpha(y_n) = x \tag{2.3}$$

and

$$\lim_{n \to \infty} y_n = \lim_{n \to \infty} \beta(x_n) = y. \tag{2.4}$$

Let $0(\alpha, \beta)$ be the set of all outcomes of (α, β).

PROPOSITION 2.1 (Varaiya and Lin [16]). *For all (α, β), $0(\alpha, \beta)$ is a nonempty, closed subset of $X \times Y$.*

We remark that the most natural definition of an outcome of (α, β) should be a pair (x, y) such that $\alpha(y) = x$, $\beta(x) = y$. Indeed, if α, β are continuous, we get the same definition. The generalization consists in considering the closures of the graphs of α and β instead of α and β.

The payoff function is any map $P : C \times C \to R$.

DEFINITION 2.3. *Let π be the set valued map defined on $A \times B$ by*

$$\pi(\alpha, \beta) = \{P(x, y) \mid (x, y) \in 0(\alpha, \beta)\}. \tag{2.5}$$

DEFINITION 2.4. *The game (A, B, P) has a saddle value if*

$$\sup_{\alpha \in A} \inf_{\beta \in B} [\inf \pi(\alpha, \beta)] = \inf_{\beta \in B} \sup_{\alpha \in A} [\sup \pi(\alpha, \beta)] \tag{2.6}$$

and this number is called the value of the game. The game has a saddlepoint (α^*, β^*) if

$$\pi(\alpha, \beta^*) \leqq \pi(\alpha^*, \beta^*) \leqq \pi(\alpha^*, \beta) \tag{2.7}$$

for all $\alpha \in A$, $\beta \in B$.

Here we adopt the convention that $\pi(\alpha_1, \beta_1) \leqq \pi(\alpha_2, \beta_2)$ if $p_1 \leqq p_2$ for $p_i \in \pi(\alpha_i, \beta_i)$, $i = 1, 2$.

A basic result of the theory of differential games is the following theorem [16].

THEOREM 2.2. *If the differential equations* (2.1) *and* (2.2) *satisfy the assumptions stated earlier, and if $P : C \times C \twoheadrightarrow R$ is continuous, then the game (A, B, P) has a saddle point.*

We shall sketch the basic idea involved in showing existence of the saddle value for the game (A, B, P). The idea is due to Fleming [3]. We approximate the continuous time game $G = (A, B, P)$ by a sequence of discrete time games $G^\delta = (A^\delta, B_\delta, P)$ and $G^\delta = (A_\delta, B^\delta, P)$ in such a way that in the game G^δ, the information pattern is biased in favor of player I whereas the situation is reversed in the game G_δ. The bias in information vanishes as δ approaches 0. More precisely, consider the following definitions.

DEFINITION 2.5. *Let δ be any number of the form 2^{-k}, $k = 0, 1, \cdots$. Let A_δ (respectively, A^δ) be the set of all functions α_δ(respectively, α^δ) $: Y \twoheadrightarrow X$ such that if y, y' in Y satisfy*

$$y(\tau) = y'(\tau), \qquad 0 \leqq \tau \leqq i\delta, \tag{2.8}$$

then

$$\alpha_\delta(y)(\tau) = \alpha_\delta(y)(\tau), \qquad 0 \leqq \tau \leqq (i + 1)\delta, i = 1, \cdots, \frac{1}{\delta} - 1, \tag{2.9}$$

{respectively, $\alpha^\delta(y)(\tau) = \alpha^\delta(y)(\tau)$, $0 \leqq \tau \leqq i\delta$, $i = 1, \cdots, 1/\delta$}.

Similarly, we define the sets B_δ and B^δ. Note that,

$$\begin{aligned} A_{\delta_2} \subset A_{\delta_1} \subset A \subset A^{\delta_1} \subset A^{\delta_2}, \\ B_{\delta_2} \subset B_{\delta_1} \subset B \subset B^{\delta_1} \subset B^{\delta_2} \end{aligned} \tag{2.10}$$

for $\delta_1 \leqq \delta_2$.

The game G^δ is played as follows. Player I chooses $\alpha^\delta \in A_\delta$ and II chooses $\beta_\delta \in B_\delta$. The outcome is a unique pair (x, y) in $X \times Y$, which we write $o(\alpha^\delta, \beta_\delta)$, such that $\alpha^\delta(y) = x$ and $\beta_\delta(x) = y$. We define

$$V^\delta = \inf_{\beta_\delta \in B_\delta} \sup_{\alpha^\delta \in A^\delta} P(o(\alpha^\delta, \beta_\delta)). \tag{2.11}$$

Dually, we define the game $G_\delta = (A_\delta, B^\delta, P)$ and the number

$$V_\delta = \sup_{\alpha_\delta \in A_\delta} \inf_{\beta^\delta \in B^\delta} P\big(o(\alpha_\delta, B^\delta)\big). \tag{2.12}$$

It is useful and elementary to observe that we also have

$$V^\delta = \sup_{A^\delta} \inf_{B_\delta} P\big(o(\alpha^\delta, \beta_\delta)\big) \tag{2.13}$$

and

$$V_\delta = \inf_{B^\delta} \sup_{A_\delta} P\big(o(\alpha_\delta, \beta^\delta)\big). \tag{2.14}$$

Also from (2.10), we can immediately conclude that $V_{\delta_2} \leqq V_{\delta_1} \leqq V^{\delta_1} \leqq V^{\delta_2}$ when $\delta_1 \leqq \delta_2$, so that we can define

$$\underline{V} = \lim_{\delta \to 0} V_\delta, \qquad \bar{V} = \lim_{\delta \to 0} V^\delta. \tag{2.15}$$

We can now see that the original game $G = (A, B, P)$ has a saddle value if $\underline{V} = \bar{V}$.

The crucial observation at this point is to note that when the dynamics are modeled by (2.1) and (2.2), the advantage due to the information bias in favor of player I in the game G^δ, and to II in G_δ, disappears as δ approaches 0. Specifically, we can prove the following proposition [16].

PROPOSITION 2.2. *For every $\varepsilon > 0$, there is a $\delta > 0$ and a map $\pi_\delta : X \twoheadrightarrow X$ such that*

$$\|\pi_\delta(x) - x\| \leqq \varepsilon \qquad \text{for all } x \in X, \tag{2.16}$$

and if x and x' in X satisfy

$$x(\tau) = x'(\tau) \qquad \text{for } 0 \leqq \tau \leqq t, \tag{2.17}$$

then, $\pi_\delta(x)(\tau) = \pi_\delta(x')(\tau)$ for $0 \leqq \tau \leqq t + \delta$.

As a corollary we get the following crucial result.

COROLLARY 2.1. *If $\alpha^\delta \in A^\delta$, and $\beta^\delta \in B^\delta$, then $(\pi_\delta \circ \alpha^\delta) \in A_\delta$ and $(\beta^\delta \circ \pi_\delta) \in B_\delta$. Furthermore,*

$$\|(\pi_\delta \circ \alpha^\delta)(y) - \alpha^\delta(y)\| \leqq \varepsilon, \qquad y \in Y, \alpha^\delta \in A^\delta \tag{2.18}$$

whenever π_δ satisfies (2.16).

Now let $\eta > 0$ be arbitrary and let $\hat{\varepsilon} > 0$ be such that

$$|P(x, y) - P(x', y')| < \tfrac{1}{2}\eta, \tag{2.19}$$

whenever

$$\begin{aligned} |x - x'| &\leqq \hat{\varepsilon}, \\ |y - y'| &\leqq \hat{\varepsilon}, \end{aligned} \tag{2.20}$$

for $x, x' \in X$, $y, y' \in Y$. Next let δ be small enough so that (2.16) is satisfied with $\varepsilon = \hat{\varepsilon}$. Let $\underline{\alpha}^\delta \in A^\delta$ be such that (see (2.13))

$$P\big(o(\underline{\alpha}^\delta, \beta_\delta)\big) \geqq V^\delta - \tfrac{1}{2}\eta \qquad \text{for all } \beta_\delta \in B^\delta \tag{2.21}$$

and define

$$\underset{\sim}{\alpha}_\delta = \pi_\delta \circ \underset{\sim}{\alpha}^\delta. \tag{2.22}$$

Finally, let β^δ in B^δ be arbitrary and let $(x, y) = o(\underset{\sim}{\alpha}_\delta, B^\delta)$, that is,

$$\begin{aligned} x &= \underset{\sim}{\alpha}_\delta(y), \\ y &= \beta^\delta(x). \end{aligned} \tag{2.23}$$

But, if we let $\underset{\sim}{\beta}_\delta = \beta^\delta \circ \pi_\delta$ and $\underset{\sim}{x} = \underset{\sim}{\alpha}^\delta(y)$, we see from (2.33) that $(\underset{\sim}{x}, y) = o(\underset{\sim}{\alpha}^\delta, \underset{\sim}{\beta}_\delta)$ so that $P(\underset{\sim}{x}, y) \geqq V^\delta - \frac{1}{2}\eta$ by (2.21). However, $|x - x'| \leqq \hat{\varepsilon}$ by (2.18), so that we get

$$P(o(\underset{\sim}{\alpha}_\delta, \beta^\delta)) = P(x, y) \geqq P(\underset{\sim}{x}, y) - \tfrac{1}{2}\eta \geqq V^\delta - \eta \tag{2.24}$$

from (2.19). Since β^δ is arbitrary, we can take the infinum over β^δ on the left and conclude that $V_\delta \geqq V^\delta - \eta$. This proves that the two limits in (2.15) coincide and the game (A, B, P) has a saddle value. It is fairly straightforward to show from this fact that the game also has a saddle point. See [16] for details.

Subsequently, in a series of papers [5], [6], [7], Friedman extended Theorem 2.2 to the case where the dynamics are not separated but are mixed as in

$$\dot{z} = h(t, z(t), u(t), v(t)), \qquad x(0) = z_0. \tag{2.25}$$

The payoff function P is assumed to be of the form

$$P(u, v) = \int_0^1 h_0(t, z(t), u(t), v(t))\, dt. \tag{2.26}$$

If we let U be the set of all admissible controls of I and V be the set of all admissible controls of II, then for this case strategies of I are defined as causal maps $\alpha : V \twoheadrightarrow U$, and strategies for II are causal maps $\beta : U \twoheadrightarrow V$, just as in Definition 2.1. Using approximations, similar to the games G^δ, G_δ, Friedman shows that a saddle value exists *provided* that the functions h, h_0 in (2.25) and (2.26) separate in the form

$$h(t, z, u, v) = h^1(t, z, u) + h^2(t, z, v), \tag{2.27}$$

$$h_0(t, z, u, v) = h_0^1(t, z, u) + h_0^2(t, z, v). \tag{2.28}$$

Such a separation is crucial in showing that the advantage to players in the games G^δ, G_δ vanishes as δ approaches 0. This point is easily demonstrated by considering the following example. Take $\dot{z}(t) = (u(t) - v(t))^2$, $z(0) = 0$, $0 \leqq t \leqq 1$, and $P(u, v) = z(1)$. We require that $u(t) \in [0, 1]$, $v(t) \in [0, 1]$. Now suppose player II chooses a strategy first. Player I then chooses his strategy according to,

$$u(t) = \begin{cases} 1 & \text{if } 0 \leqq v(t) \leqq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < v(t) \leqq 1. \end{cases} \tag{2.29}$$

Clearly then, independent of the choice of II, $P \geqq \frac{1}{4}$. On the other hand, if I chooses a strategy first and II follows according to the formula $v(t) \equiv u(t)$, evidently $P = 0$. Hence a saddle value cannot exist.

3. The synthesis problem

Let us consider games with dynamics of the form (2.25) and payoff in the form (2.26). We assume that (2.27) and (2.28) hold. Furthermore, h and h_0 are required to be continuous, $u(t) \in U$, $v(t) \in V$, where U and V are compact. Instead of starting in a fixed initial condition z_0 at time 0, let us consider games starting in different initial states z at times t in $[0, 1]$, the game being defined over $[t, 1]$. Let $\pi(z, t)$ be the value of the game starting in the initial condition (z, t). It can be shown [5] that π is continuously differentiable and satisfies the partial differential equation

$$(3.1) \qquad 0 = \frac{\partial \pi(z, t)}{\partial t} + \min_{v \in V} \max_{u \in U} \left[\left\langle \frac{\partial v}{\partial z}(z, t), h(t, z, u, v) \right\rangle + h_0(t, z, u, v) \right].$$

Of course, we have the boundary data

$$(3.2) \qquad \pi(z, 1) \equiv 0.$$

Note that because of (2.26) and (2.27) the min and max in (3.1) can be interchanged. Equations (3.1) and (3.2) have been solved formally in numerous cases by Isaacs [10]. Indeed, he was the first to discover this equation and it is called Isaacs' equation. In the case of one player, it reduces to the Hamilton–Jacobi equation of the calculus of variation.

In a significant paper, Berkovitz [1] considers solving (3.1) and obtaining a saddle point as a pair of "feedback" strategies

$$(3.3) \qquad \begin{aligned} u(t) &= u(t, z(t)), \\ v(t) &= v(t, z(t)), \end{aligned}$$

by constructing a field of trajectories.

In a different direction, Fleming [4] studies the parabolic equation obtained from (3.1) by adding the term

$$(3.4) \qquad \varepsilon \sum_i \frac{\partial^2 \pi}{\partial z_i^2}(z, t), \qquad \varepsilon > 0,$$

to the right side. If we denote by π^ε the solution to this equation, then Fleming shows that π^ε converges to the value π as ε approaches 0, uniformly on compact sets. Furthermore, he shows that π^ε can be regarded as the value of the stochastic game with dynamics

$$(3.5) \qquad dz(t) = h(t, z, u, v)\, dt + (2\varepsilon)^{1/2}\, dB(t),$$

and with payoff

$$P(u, v) = E \int_0^1 h_0(t, z, u, v)\, dt. \tag{3.6}$$

In (3.5), $B(t)$ is a standard n dimensional Brownian motion, and in (3.6), E denotes expectation.

This brings us to the final part of this section. Consider the stochastic game defined by the differential equation

$$dx(t) = f\big(t, x(t), u(t), v(t)\big)\, dt + dB(t), \qquad x(0) = x_0,\ 0 \leqq t \leqq 1, \tag{3.7}$$

where f is of the form

$$f(t, x, u, v) = \begin{pmatrix} f_1(t, x, u) \\ f_2(t, x, v) \end{pmatrix}. \tag{3.8}$$

In (3.7), $B(t)$ is an n dimensional Brownian motion process. We shall define the solution of (3.7) in such a way that $x(t)$ has continuous sample paths, so that we will suppose that the sample paths of x belong to C, the Banach space of continuous functions from $[0, 1]$ into R^n. For each $t \in [0, 1]$, let $\mathscr{C}_t$ be the σ-field generated by all subsets of the form

$$\{z \mid z \in C, z(\tau) \in A\}, \tag{3.9}$$

where $0 \leqq \tau \leqq t$, and A is a Borel subset of R^n.

Suppose that u and v are m dimensional and U and V are compact subsets of R^m.

DEFINITION 3.1. *A strategy for player I is any function* $\alpha : [0, 1] \times C \twoheadrightarrow U$ *such that:*

(i) α *is measurable with respect to the product* σ*-algebra* $\mathscr{L} \otimes \mathscr{C}_1$ *on* $[0, 1] \times C$ (*here* $\mathscr{L}$ *is the set of Lebesgue measurable subsets of* $[0, 1]$);

(ii) *for each fixed t in* $[0, 1]$, $\alpha(t, \cdot)$ *is measurable with respect to* $\mathscr{C}_t$.

Let A denote the set of all strategies of player I. Similarly, we define B as the set of all strategies of player II consisting of all jointly measurable, causal maps $\beta : [0, 1] \times C \twoheadrightarrow V$.

We impose the following conditions on f:

(i) $f(t, x, u, v)$ is measurable in (t, x, u, v), continuous in (u, v) for fixed (t, x);

(ii) $f(t, x, U, v)$ and $f(t, x, u, V)$ are convex;

(iii) there is an increasing function $f_0 : R \twoheadrightarrow R$ such that $|f(t, x, u, v)| \leqq f_0(|x|)$ for all $x \in R^n$, $u \in U$, $v \in V$, $t \in [0, 1]$.

The following fundamental existence result is due to Girsanov [8].

THEOREM 3.1. *For each* $\alpha \in A$, $\beta \in B$, *there exists a solution* $x(t)$ *of* (3.7) *with sample paths in C such that the measure* $\mu_{(\alpha, \beta)}$ *induced by x on* $(C, \mathscr{C}_1)$ *is mutually* *absolutely continuous with respect to the Wiener measure* μ *on* $(C, \mathscr{C}_1)$ *and the density* $\eta_{(\alpha,\beta)} = d\mu_{(\alpha,\beta)}/d\mu$ *is given by*

$$\eta_{(\alpha,\beta)} = \exp\left\{\int_0^1 \langle f(t, B(t), \alpha(t, B), \beta(t, B)), dB(t)\rangle - \tfrac{1}{2}\int_0^1 |f(t, B(t), \alpha(t, B), \beta(t, B))|^2\, dt\right\}. \tag{3.10}$$

(In the above formula the first integral should be interpreted as a stochastic integral.)

When the function f has the form (3.8), we see that (3.10) becomes

$$\eta_{(\alpha,\beta)}(x) = \eta_\alpha(x)\eta_\beta(x), \tag{3.11}$$

where

$$\eta_\alpha = \exp\left\{\int_0^1 \langle f_1(t, B(t). \alpha(t, B)), dB_1(t)\rangle - \tfrac{1}{2}\int_0^1 |f_1(t, B(t), \alpha(t, B))|^2\, dt\right\} \tag{3.12}$$

and

$$\eta_\beta = \exp\left\{\int_0^1 \langle f_2(t, B(t), \beta(t, B)), dB_2(t)\rangle - \tfrac{1}{2}\int_0^1 |f_2(t, B(t), \beta(t, B))|^2\, dt\right\}. \tag{3.13}$$

Let $P : C \twoheadrightarrow R$ be any bounded function measurable with respect to $\mathscr{C}_1$. For any $\alpha \in A$, $\beta \in B$, we define the payoff to I as $E_{(\alpha,\beta)}(P)$, where $E_{(\alpha,\beta)}(P)$ is the expectation of P with respect to the measure $\mu_{(\alpha,\beta)}$ induced on C by (α, β) via x. In view of (3.11), we see that

$$E_{(\alpha,\beta)}(P) = \int_C P(x)\eta_\alpha(x)\eta_\beta(x)\, d\mu(x). \tag{3.14}$$

From (3.14), we see that the choice of α or β affects the payoff only through the densities η_α, η_β. Let

$$\begin{aligned} D_I &= \{\eta_\alpha \mid \alpha \in A\}, \\ D_{II} &= \{\eta_\beta \mid \beta \in B\}. \end{aligned} \tag{3.15}$$

The following result is a useful characterization.

THEOREM 3.2 (Duncan and Varaiya [2]). *The sets* D_I *and* D_{II} *are strongly closed, convex subsets of* $L_1(C, \mathscr{C}, \mu)$.

As a corollary of Theorems 1.1 and 3.2, we have the following existence result.

THEOREM 3.3. *There exists a saddle point for the above stochastic game.*

PROOF. The sets D_I and D_{II} are weakly compact. The payoff (3.14) is linear and continuous in η_α for fixed η_β and linear and continuous in η_β for fixed η_α. By Theorem 1.1, there exists a saddle point.

4. Generalizations and comments

In the results presented in the previous sections, the game started at a fixed initial state and ended at a fixed final time. It is important in many cases to have a variable end time. Usually, this situation is formulated by requiring that the termination of the game occur when the state of the players enters a specific target set. A specific game of this kind—the pursuit-evasion game—has been considered in [16]. Generalizations have been studied in [6] and [7].

Essentially, the only class of games for which explicit solutions are available is the case where the dynamics in (2.25) are linear and the integrand of the payoff function (2.26) is quadratic. Numerous special cases have been solved and the literature on differential games is growing rapidly. An exhaustive bibliography classifying the literature, up to October 1969, appears in [11].

As is well known, unlike the situation in zero sum games where saddle point is the natural solution concept, in the case of more than two players or where the payoff is not zero sum, there are many solution concepts. Roughly speaking, these concepts separate into two classes, cooperative and noncooperative, but the distinction becomes less clear in dealing with a dynamic situation. Although most results in general (as opposed to two person, zero sum) differential games are concerned with noncooperative solutions, there are the beginnings of a theory (or theories) in the richer area of cooperative solutions. Reference [15] exhibits the wealth of possible solution concepts in the general case, and [9] is an attempt to place differential games within the context of decision theory.

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