MULTIVARIATE POINT PROCESSES

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1. Introduction

We consider in this paper events of two or more types occurring in a one dimensional continuum, usually time. The classification of the events may be by a qualitative variable attached to each event, or by their arising in a common time scale but in different physical locations. Such multivariate point processes, or multitype series of events, are to be distinguished from events of one type occurring in an n dimensional continuum and considered, for example, by Bartlett [2]. It is of course possible to have multivariate point processes in, say, two dimensions, for example, the locations of accidents labelled by day of occurrence, but we do not consider this extension here.

Multivariate series of events arise in many contexts; the following are a few examples.

EXAMPLE 1.1. Queues are a well-known situation in which bivariate point processes arise as the input and output, although interest in the joint properties of the input and output processes is fairly recent (for example, Daley [16] and Brown [7]). The two processes occur simultaneously in time. Many of the variants on simple queueing situations which have been considered give rise to more than two point processes.

EXAMPLE 1.2. An important and rich source of multivariate point processes is neurophysiology (Perkel, Gerstein, and Moore [41]). Information is carried along nerve bundles by spikes which occur randomly in time. (The spikes are extremely narrow and, at least in many situations, their shape and height do not appear to vary or carry information.) The neuronal spike trains of different types may be observations at different locations with no definite knowledge of physical connection, or may be the inputs and outputs to nerve connections (neurons).

EXAMPLE 1.3. When the events are crossings of a given level by a real valued stochastic process in continuous time, the up crossings and down crossings of the

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level constitute a very special bivariate point process in which the two types of events alternate (Leadbetter [27]). However, up crossings of two different levels produce a general type of bivariate point process which is of interest, for example, in reliability investigations.

EXAMPLE 1.4. In reliability studies over time, of continuously operating machines such as computers, the failures are more often than not labelled according to the part of the system in which they occurred, or according to some other qualitative characterization of the failure, for example, mechanical or electrical. One might also be interested in studying interactions between preventive maintenance points and failures occurring during normal operation. Again a comparison between failure patterns in separately located computers (Lewis [29]) might be of interest in determining whether some unknown common variable, such as temperature and/or humidity, influences reliability.

EXAMPLE 1.5. Cox [11] has considered the problem of analyzing events of two types in textile production. The two types of event may be breakdowns in the loom and faults in the cloth, or different types of breakdown of the loom. The continuum is length of thread rather than time.

EXAMPLE 1.6. In the analysis of electrocardiograms the trace is continuous, but both regular heart beats and various types of ectopic heart beats occur. It is therefore of interest to analyze electrocardiograms as bivariate event processes, even though defining the precise time of occurrence of the event (heartbeat) may present some problems.

EXAMPLE 1.7. Traffic studies are a rich source of multivariate point processes. Just two possibilities are that the events may be the passage of cars by a point on a road when the type of event is differentiated by direction of travel, or we may consider passage of cars past two different positions.

EXAMPLE 1.8. Finally, physical phenomena such as volcanoes or earthquakes (Vere-Jones [45], [46], [47]) may have distinguishing features of many kinds—generally highly compacted attributes of the process, for example, the general location of the origin of the earthquake.

Multivariate point processes can be regarded as very special cases of univariate point processes in which a real valued quantity is associated with each point event, that is, special cases of what Bartlett [3] has called, rather generally, line processes. In particular, if the real valued quantity takes only two possible values, we have in effect a bivariate process of events of two types.

Three broad types of problems arise for multivariate point processes. The first are general theoretical and structural problems of which the most outstanding is the problem of characterizing the dependence and interaction between a number of processes. This is the only general theoretical question we will consider in any detail; it is intimately connected with the statistical analysis of bivariate point processes.

The second type of problem is the calculation of the properties of special stochastic models suggested, for example, by physical considerations. This in general is a formidable task even for quite simple models.

Thirdly, there are problems of statistical analysis. These include:

(a) comparing rates in independent processes (Cox and Lewis [14], Chapter 9) from finite samples;

(b) assessing possible dependence between two processes from finite samples;

(c) determining, again from finite samples, the probabilistic structure of a mechanism which transforms one process into a second quite clearly dependent process.

The range of the problems will become clear in the main body of the paper. The topics considered are briefly as follows.

In Section 2, we give some notation and define various types of interevent sequences and counting processes which occur in bivariate point processes. Concepts such as independence of the marginal processes and stationarity and regularity of the complete, bivariate process are defined. The ideas of this section are illustrated by considering two independent renewal processes and also the semi-Markov process (Markov renewal process).

In Section 3, we study dependence and correlation in bivariate point processes, defining complete intensity functions and second order cross intensity functions and cross spectra, giving their relationship to covariance time surfaces. Doubly stochastic bivariate point processes are defined and their cross intensity function is given. Other simple models of bivariate point processes are defined through the complete intensity and cross intensity functions. In this way, various degrees of interaction between events in the bivariate process can be specified. A class of bivariate Markov interval processes is defined.

In Section 4, a simple delay model with marginal Poisson processes is considered in some detail. Other special physical models are considered briefly in Section 5.

General comments on bivariate Poisson processes are given at the end of Section 5; a bivariate Poisson process is defined simply as a bivariate point process whose marginal processes are Poisson processes.

Statistical procedures are considered in Section 6, including the estimation of second order cross intensity functions and cross spectra, as well as covariance time surfaces. Tests for dependence in general and particular situations are considered, and statistical procedures for some special processes are given.

Throughout, emphasis is placed on concepts rather than on mathematical details and a number of open questions are indicated. For the most part we deal with bivariate processes, that is, with events of two types; the generalization of more than two types of events is on the whole straightforward.

2. General definitions and ideas

2.1. *Regularity.* Throughout Section 2, we deal with bivariate processes, that is, processes of events of two types called type a and type b. The process of, say, type a events alone is called a *marginal process*.

In a univariate point process such as a marginal process in a bivariate point process, regularity is defined by requiring that in any interval of length Δt

$$Pr\{(\text{number events in } \Delta t) > 1\} = o(\Delta t). \tag{2.1}$$

Regularity is intuitively the nonoccurrence of multiple events.

For bivariate processes, we say the process is *marginally regular* if its marginal processes, considered as univariate point processes, are both regular. The bivariate process is said to be *regular* if the process of superposed marginal events is regular, that is, if the process of events regardless of type is regular. This type of regularity, of course, implies *marginal regularity*.

A simple, rather degenerate, bivariate process is obtained by taking three Poisson processes, say I, II, and III, and superposing processes I and II to obtain the events of type a and superposing II and III to obtain the events of type b (Marshall and Olkin [33]). Clearly, the bivariate process is marginally regular but not regular. However, if the events of type b are made up of process III events superposed with process II events delayed by a fixed amount, the resulting bivariate process is regular. A commonly used alternative to the word *regular* is *orderly*.

2.2. *Independence and stationarity.* Independence of the marginal processes in a bivariate process is intuitively defined as independence of the number of events (counts) in any two sets of intervals in the marginal processes. The more difficult problem of specifying dependence (and correlation) in the bivariate process is central to this paper and will be taken up in the next section.

In the sequel, we will be primarily concerned with *transient* or *stationary* bivariate point processes, as opposed to *nonhomogeneous* processes. The latter type of process is defined roughly as one with either an evolutionary or cyclic trend, whereas a *transient* process is roughly one whose probabilistic structure eventually becomes stationary (time invariant). There are a number of types of stationarity which need to be defined more carefully.

DEFINITION 2.1 (Simple stationarity). *Let $N^{(a)}(t_1^{(1)}, t_1^{(1)} + \tau_1^{(1)})$ be the number of events of type a in the interval $(t_1^{(1)}, t_1^{(1)} + \tau_1^{(1)}]$ and $N^{(b)}(t_2^{(1)}, t_2^{(1)} + \tau_2^{(1)})$ be the number of events of type b in the interval $(t_2^{(1)}, t_2^{(1)} + \tau_2^{(1)}]$. The bivariate point process is said to have* simple stationarity *if*

$$\begin{aligned} Pr\{N^{(a)}(t_1^{(1)}, t_1^{(1)} + \tau_1^{(1)}) = n^{(a)}; N^{(b)}(t_2^{(1)}, t_2^{(1)} + \tau_2^{(1)}) = n^{(b)}\} \\ = Pr\{N^{(a)}(t_1^{(1)} + y, t_1^{(1)} + \tau_1^{(1)} + y) = n^{(a)}; \\ N^{(b)}(t_2^{(1)} + y, t_2^{(1)} + \tau_2^{(1)} + y) = n^{(b)}\}, \end{aligned} \tag{2.2}$$

for all $t_1, t_2, \tau_1^{(1)}, \tau_2^{(1)}, y > 0$.

In other words, the joint distribution of the number of type a events in a fixed interval and the number of type b events in another fixed interval is invariant under translation.

Simple stationarity of the bivariate process implies an analogous property for the individual marginal processes and for the superposed process.

In the sequel, we assume that for the marginal processes considered individually the probabilities of more than one type a event in τ_1 and more than one type b event in τ_2 are, respectively, $\rho_a\tau_1 + o(\tau_1)$ as $\tau_1 \to 0$ and $\rho_b\tau_2 + o(\tau_2)$ as $\tau_2 \to 0$, where ρ_a and ρ_b are finite.

Simple stationarity and these finiteness conditions imply that the univariate forward recurrence time relationships in the marginal processes and the pooled processes hold (Lawrance [25]).

If in addition the process is regular, Korolyuk's theorem implies that ρ_a, ρ_b, and $\rho_a + \rho_b$ are, respectively, the rates of events of types a, events of types b, and events regardless of type.

DEFINITION 2.2 (Second order stationarity). *By extension, we say that the bivariate point process has* second order stationarity (*weak stationarity*) *if the joint distribution of the number of type a events in two fixed intervals and the number of type b events in another two fixed intervals is invariant under translation.*

This type of stationarity is necessary in the sequel for the definition of a time invariant cross intensity function. Clearly, it implies second order stationarity for the marginal processes considered individually and for the superposed marginal processes.

DEFINITION 2.3 (Complete stationarity). *By extension,* complete stationarity *for a bivariate point process is invariant under translation for the joint distribution of counts in arbitrary numbers of intervals in each process.*

2.3. *Asynchronous counts and intervals.* In specifying stationarity, we did not mention the time origin or the method of starting the process. There are three main possibilities.

(i) The process is started at time $t = 0$ wih initial conditions which produce stationarity, referred to as *stationary initial conditions.*

(ii) The process is transient and is considered beyond $t = 0$ as its start moves off to the left. The process then becomes stationary as the start moves to minus infinity. There is generally a specification of the state of the process at $t = 0$ known as the *stationary equilibrium conditions.*

Note that in both (i) and (ii) stationarity is defined by invariance under shifts to the right.

(iii) In a stationary point process, a time is specified without knowledge of the events and is taken to be the origin, $t = 0$. The time $t = 0$ is said to be an *arbitrary time* in the (stationary) process, selected by an *asynchronous sampling* of the process.

Now there is associated with the stationary bivariate point process a counting process $\mathbf{N}(t_1, t_2) = \{N^{(a)}(t_1), N^{(b)}(t_2)\}$, where

$$N^{(a)}(t_1) \text{ is the number of type } a \text{ events in } (0, t_1], \qquad (2.3)$$
$$N^{(b)}(t_2) \text{ is the number of type } b \text{ events in } (0, t_2],$$

and a bivariate sequence of intervals $\{X^{(a)}(i), X^{(b)}(j)\}$, where, assuming regularity of the process, $X^{(a)}(1)$ is the forward recurrence time in the process of type a events (that is, the time from $t = 0$ to the first type a event), $X^{(a)}(2)$ is the time between the first and second type a events, and so forth; and the $\{X^{(b)}(j)\}$ sequence is defined similarly.

Note that for asynchronous sampling of a stationary process the indices i and j can take negative values; in particular, $\{X^{(a)}(-1), X^{(b)}(-1)\}$ are the bivariate backward recurrence times.

There is a fundamental relationship connecting the bivariate counting processes with the bivariate interval processes; this is a direct generalization of the relationship for the univariate case:

$$N^{(a)}(t_1) < n^{(a)}, \qquad N^{(b)}(t_2) < n^{(b)}, \tag{2.4}$$

if and only if

$$\begin{aligned} S^{(a)}(n^{(a)}) &= X^{(a)}(1) + \cdots + X^{(a)}(n^{(a)}) > t_1, \\ S^{(b)}(n^{(b)}) &= X^{(b)}(1) + \cdots + X^{(b)}(n^{(b)}) > t_2. \end{aligned} \tag{2.5}$$

Probability relationships are written down directly from these identities connecting the bivariate distribution of counts with the bivariate distribution of the sums of intervals $S^{(a)}(n^{(a)})$ and $S^{(b)}(n^{(b)})$.

Equations (2.4) and (2.5) can be used, for example, to prove the asymptotic bivariate normality of $\{N^{(a)}(t_1), N^{(b)}(t_2)\}$ for a broad class of bivariate point processes.

2.4. *Semisynchronous sampling.* In a univariate point process, *synchronous* sampling of the stationary process refers to the placement of the time origin at an arbitrary event and the examination of the counts and intervals following this arbitrary event (Cox and Lewis [14], Chapter 4, and McFadden [35]). In more precise terms (Leadbetter [27], [28], and Lawrance [24]), the notion of an arbitrary event in a stationary point process is the event $\{N(0, \tau) \geqq 1\}$ as τ tends to zero, and the distribution function $F(t)$ of the interval between the arbitrary event and the following event is defined to be

$$1 - F(t) = \lim_{\tau \to 0+} Pr\{N(\tau, \tau + t) = 0 | N(0, \tau) \geqq 1\}. \tag{2.6}$$

In bivariate point processes, the situation is more complex. Synchronous sampling of the marginal process of type a events produces *semisynchronous sampling* of the process of b events from an arbitrary a event, and *vice versa*.

The bivariate counting processes and intervals following these two types of sampling are denoted as follows:

(a) for semisynchronous sampling of b by a,

$N_a^{(a)}(t_1)$ is the number of type a events following an origin at a type a event;

$N_a^{(b)}(t_1)$ is the number of type b events following an origin at a type a event;

$\{X_a^{(a)}(i)\}$ is, for $i = 1$, the time from the origin at a type a event to the next event of type a, and for $i = 2, 3, \cdots$, the intervals between subsequent type a events;

$\{X_a^{(b)}(j)\}$ is, for $j = 1$, the time from the origin at a type a event to the first subsequent type b event, and for $j = 2, 3, \cdots$, the intervals between subsequent type b events;

(b) for semisynchronous sampling of a by b, the subscript becomes b instead of a, in the above expressions, indicating the nature of the origin.

Note that in general (Slivnyak [44]) the sequences $\{X_a^{(a)}(i)\}$ and $\{X_b^{(b)}(j)\}$, being the synchronous interval processes in the marginal point processes, are stationary, whereas $\{X_a^{(b)}(i)\}$ and $\{X_b^{(a)}(j)\}$, the semisynchronous intervals, are in general not stationary. Also for independent processes, the semisynchronous sequences are identical with the asynchronous sequences $\{X^{(a)}(i)\}$ and $\{X^{(b)}(j)\}$.

2.5. *Pooling and superposition of processes.* In discussing regularity, we referred to the superposition of the two marginal processes in the bivariate process. This is the univariate process of events of both types considered without specification of the event type and is referred to simply as the *superposed process.* Study of the superposed process of rate $\rho_a + \rho_b$ is an intimate part of the analysis of the bivariate process. Asynchronous sampling of the superimposed process gives counts and intervals denoted by $N^{(\cdot)}(t_1)$ and $\{X^{(\cdot)}(i)\}$, whereas synchronous sampling, that is, the process considered conditionally (in the Khinchin sense) on the existence of an event of an unspecified type at the origin, gives $N_{\cdot}^{(\cdot)}(t_1)$ and $\{X_{\cdot}^{(\cdot)}(i)\}$.

Semisynchronous sampling of the superposed process by events of type a or type b is also possible and the notation should be clear.

We call the superposed process with specification of the event type the *pooled* process. The original bivariate process can then be respecified in terms of the process

$$\{X_{\cdot}^{(\cdot)}(i), \qquad T_{\cdot}(i)\}, \tag{2.7}$$

where $T_{\cdot}(i)$ is a binary valued process indicating the type of the ith event after the origin in the superposed process with synchronous sampling. Clearly, the marginal processes of event types, that is, $\{T(i)\}$, $\{T_{\cdot}(i)\}$, $\{T_a(i)\}$ and $\{T_b(i)\}$ are themselves of interest. Note that they are in general not stationary processes for all types of sampling and are related to the processes defined in Sections 2.3 and 2.4. Thus, for example,

$$\{X_a^{(\cdot)}(1) \leqq x; T_a(1) = a\} \Leftrightarrow \{X_a^{(a)} < X_a^{(b)}; X_a^{(a)} \leqq x\}, \tag{2.8}$$

with much more complicated statements relating events of higher index i. The binary sequence of event types has no counterpart in univariate point process.

Thus, there are many possible representations of a bivariate point process. Which is the most fruitful is likely to depend on the particular application.

As a very simple practical example of these representations, consider a generalization of the alternating renewal process. We have a sequence of positive random variables $W(1)$, $Z(1)$, $W(2)$, $Z(2)$, $\cdots$, representing operating and repair intervals in a machine. It is natural to assume that $W(i)$ and $Z(i)$ are mutually correlated but independent of other pairs of operating and repair times. Type a events, occurring at the end of the $W(i)$ variables, are machine failures. Type b events occur at the end of $Z(i)$ variables and represent times at which the machine goes back into service.

Specification of the process is straightforward and simple in terms of the pooled process variables $\{X_{\cdot}^{(\cdot)}(i), T_{\cdot}(i)\}$, $\{X_a^{(\cdot)}(i), T_a(i)\}$, and so forth. However, marginally the type b events are a renewal process, whereas the type a events are a nonrenewal, non-Markovian point process and the dependency structure expressed through the intervals in the marginals is complex.

2.6. *Successive semisynchronous sampling.* Finally, we mention the possibility of successive semisynchronous samples of the marginal process of type b events by a events. The origin is at an a event, as in ordinary semisynchronous sampling and connected with this a event is the time forward (or backward) to the next b event. Subsequent a events are associated with the times forward (or backward) to the next b event. It is not clear how generally useful this procedure is in studying bivariate point processes. It has been used, however, by Brown [7] in studying identifiability problems in $M/G/\infty$ queues; see also Section 6.4.

2.7. *Palm-Khinchin formulae.* In the theory of univariate point processes, there are relations connecting the distributions of sums of synchronous intervals and sums of asynchronous intervals. Similar relationships connect the synchronous and asynchronous counting processes. These relationships are sometimes called the Palm-Khinchin formulae and are given, for example, by Cox and Lewis ([14], Chapter 4).

The best known of these relations connects the distributions of the synchronous and asynchronous forward recurrence times in a stationary point process with finite rate ρ (Lawrance [25]). In the context of the marginal process of type a events,

$$\rho_a\{1 - F_{X_a^{(a)}}(t)\} = D_t^+ F_{X^{(a)}}(t), \tag{2.9}$$

where D_t^+ denotes a right derivative. For moments when the relevant moments exist we have

$$E\{(X^{(a)})^r\} = \frac{\rho_a E\{(X_a^{(a)})^{r+1}\}}{r + 1}. \tag{2.10}$$

Palm-Khinchin type formulae for bivariate point processes have been developed by Wisniewski [50], [51]. They are far more complex than those for univariate processes, both in terms of the number of relationships involved and in the analytical problems encountered. Thus, on the first point there are not only interval relationships, but also relationships between the probabilistic structures of the binary sequences $\{T(i)\}$ and $\{T_{\cdot}(i)\}$ and between the probabilistic structures of the binary sequence $\{T(i)\}$ and the binary sequences $\{T_a(i)\}$ and $\{T_b(i)\}$.

On the second point, the analytical problems are illustrated by the following argument. It is easily shown that an arbitrarily selected point in the (univariate) superposed process is of type a with probability $\rho_a/(\rho_a + \rho_b)$. Thus, any probabilistic statement about the variables $\{X_{\cdot}^{(\cdot)}(i), T_{\cdot}(i)\}$, say $g(X_{\cdot}^{(\cdot)}(1), T_{\cdot}(1), \cdots)$, is expressible in terms of the same probabilistic statement for $\{X_a^{(\cdot)}(i), T_a(i)\}$ and $\{X_b^{(\cdot)}(i), T_b(i)\}$.

$$(2.11)\qquad g(X_{\cdot}^{(\cdot)}(1), T_{\cdot}(1), \cdots) = \frac{\rho_a}{\rho_a + \rho_b} g(X_a^{(\cdot)}(1), \cdots) + \frac{\rho_b}{\rho_a + \rho_b} g(X_b^{(\cdot)}(1), \cdots).$$

Now if a relationship between $g(X^{(\cdot)}(1), T(1), \cdots)$, and $g(X_{\cdot}^{(\cdot)}(1), T_{\cdot}(1), \cdots)$ exists, we can relate the asynchronous sequence to the two semisynchronous sequences through (2.11). But the usual univariate Palm-Khinchin formulae relate univariate distributions of *sums* of asynchronous intervals to univariate distributions of sums of synchronous intervals. Clearly, formulae relating *joint* properties of asynchronous intervals and types to joint properties of synchronous intervals $X_{\cdot}(i)$ and types $T_{\cdot}(i)$ are needed if one is, for example, to relate, through (2.11) and generalizations of (2.8), bivariate distributions of asynchronous forward recurrence times $\{X^{(a)}(1), X^{(b)}(1)\}$ to the bivariate distributions of the semisynchronous forward recurrence times $\{X_a^{(a)}(1), X_a^{(b)}(1)\}$ and $\{X_b^{(a)}(1), X_b^{(b)}(1)\}$.

Lawrance [26] has noted this need for extended Palm-Khinchin formulae and conjectured results in the univariate case.

Of Wisniewski's results [50], [51], we cite here only two moment formulae. These relate the moments of the joint asynchronous forward recurrence times $\{X^{(a)}(1), X^{(b)}(1)\}$ with the moments of both of the semisynchronous forward recurrence times, $\{X_a^{(a)}(1), X_a^{(b)}(1)\}$ and $\{X_b^{(a)}(1), X_b^{(b)}(1)\}$. The feature that probabilistic properties of both semisynchronous sequences are needed to determine probabilistic properties of the asynchronous sequence is characteristic of all these relationships, and follows from (2.11).

We have for the bivariate analogues to (2.10), for $r = 1$,

$$(2.12)\qquad \begin{aligned} &\tfrac{1}{2}\rho_a E[\{X_a^{(a)}(1)\}^2] + \tfrac{1}{2}\rho_b E[\{X_b^{(b)}(1)\}^2] \\ &= E\{X^{(a)}\} + E\{X^{(b)}\} \\ &= \rho_a E\{X_a^{(a)}(1)\, X_a^{(b)}(1)\} + \rho_b E\{X_b^{(a)}(1)\, X_b^{(b)}(1)\} \end{aligned}$$

and

$$(2.13)\qquad \begin{aligned} &12E\{X^{(a)}(1)\, X^{(b)}(1)\} \\ &= \rho_a E[3X_a^{(a)}(1)\, \{X_a^{(b)}(1)\}^2 + 3\{X_a^{(a)}(1)\}^2\, X_a^{(b)}(1) - \{X_a^{(a)}(1)\}^3] \\ &\quad + \rho_b E[3X_b^{(b)}(1)\, \{X_b^{(a)}(1)\}^2 + 3\{X_b^{(b)}(1)\}^2\, X_b^{(a)}(1) - \{X_b^{(b)}(1)\}^3]. \end{aligned}$$

The interesting feature of (2.12) is that the correlation between semisynchronous forward recurrence times is a function only of the properties of the marginal processes and not of the dependency structure of the bivariate point process. Moreover, (2.13) shows that if we use correlation between the asynchronous forward recurrence times as a measure of dependence in the bivariate point process, this dependence only affects the third order joint moments of the semisynchronous forward recurrence times.

2.8. *Examples.* To illustrate the definitions and concepts introduced above, we consider two very simple bivariate point processes. The analytical details developed here will be used in Section 6 in considering the statistical analysis of bivariate point processes.

EXAMPLE 2.1. *Independent renewal processes.* Consider two *delayed* renewal processes $\{X^{(a)}(1); X_a^{(a)}(i), i = 2, 3, \cdots\}$ and $\{X^{(b)}(1); X_b^{(b)}(j), j = 2, 3, \cdots\}$, where using a shortened notation,

$$G^{(a)}(x) = Pr\{X^{(a)}(1) \leqq x\} = \frac{\int_0^x \{1 - F^{(a)}(u)\}\, du}{E_a(X)}, \tag{2.14}$$

$$Pr\{X_a^{(a)}(i) \leqq x\} = F^{(a)}(x), \quad E_a(X) = \int_0^\infty x\, dF^{(a)}(x), \tag{2.15}$$

with similar definitions for the process of type b events. The distribution of the variable $X^{(a)}(1)$ in (2.14) and the analogous distribution for $X^{(b)}(1)$ are the stationary initial conditions for the marginal renewal processes, and clearly the independence of the processes implies that these distributions (jointly) give stationarity to the bivariate process. Because of independence there is no difference between semisynchronous and asynchronous sampling; the process is defined completely in terms of the properties of the asynchronous and synchronous intervals.

Properties of intervals in the superposed process, and properties of successive intervals and event types in the pooled process, that is, $\{X^{(\cdot)}(i), T(i)\}$ are very difficult to obtain explicitly. The sequences $X^{(\cdot)}(i)$ and $T(i)$ are neither stationary nor independent, but contain transient effects. We have for example

$$\begin{aligned} Pr\{X^{(\cdot)}(1) > x; T(1) = a\} &= Pr\{X^{(b)}(1) > X^{(a)}(1) > x\} \\ &= \int_x^\infty \{1 - G^{(b)}(y)\}\, dG^{(a)}(y) \end{aligned} \tag{2.16}$$

and, marginally,

$$Pr\{T(1) = a\} = \int_0^\infty \{1 - G^{(b)}(y)\}\, dG^{(a)}(y). \tag{2.17}$$

The only simple case is where the two renewal processes are Poisson processes with parameters ρ_a and ρ_b. Then, of course, $\{X^{(\cdot)}(i)\}$ is a Poisson process of rate $\rho_a + \rho_b$ and $T(i)$ is an independent binomial sequence

$$Pr\{T(1) = a\} = Pr\{T(1) = a \,|\, X^{(\cdot)} > x\} = \frac{\rho_a}{\rho_a + \rho_b}. \tag{2.18}$$

EXAMPLE 2.2. *Semi-Markov processes (Markov renewal processes).* The two state semi-Markov process is the simplest bivariate process with dependent structure and plays, in bivariate process theory, a role similar to that played in univariate process theory by the renewal process. It is, in a sense, the closest one gets in bivariate processes to a regenerative process. The process is defined in terms of the sequences $\{X_a^{(\cdot)}(i), T_a(i)\}$, and $\{X_b^{(\cdot)}(i), T_b(i)\}$, the type processes $\{T_a(i)\}$ and $\{T_b(i)\}$ being Markov chains with transition matrix

$$\mathbf{P} = \begin{pmatrix} p_{aa} & p_{ab} \\ p_{ba} & p_{bb} \end{pmatrix} = \begin{pmatrix} \alpha_1 & 1 - \alpha_1 \\ 1 - \alpha_2 & \alpha_2 \end{pmatrix}, \tag{2.19}$$

while the distributions of the random variables $X_a^{(\cdot)}(i)$, $X^{(\cdot)}(i)$, and $X_b^{(\cdot)}(i)$ depend only on the type of events at i and $(i - 1)$. Thus, illustrating the regenerative nature of the process, we define $F_{aa}(x)$ to be, for $i \geqq 2$,

$$\begin{aligned} F_{aa}(x) &= Pr\{X_a^{(\cdot)}(i) \leqq x \,|\, T_a(i) = a, T_a(i - 1) = a\} \\ &= Pr\{X_\cdot^{(\cdot)}(i) \leqq x \,|\, T_\cdot(i) = a, T_\cdot(i - 1) = a\} \\ &= Pr\{X^{(\cdot)}(i) \leqq x \,|\, T(i) = a, T(i - 1) = a\} \end{aligned} \tag{2.20}$$

with equivalent definitions for $F_{ab}(x)$, $F_{ba}(x)$, $F_{bb}(x)$. Thus, the effect of the initial sampling disappears when the type of the first subsequent events is known.

The joint distributions of the time from the origin to the first event and the type of the first event $(i = 1)$ are either quite arbitrary initial conditions, or initial conditions established by the kind of sampling involved at the origin and denoted by the subscript on the interval random variable. Thus for asynchronous sampling, we get stationary initial conditions which are specified by the joint distribution of $X^{(\cdot)}(i)$ and $T(1)$.

These stationary equilibrium conditions (Pyke and Schaufele [43]) are that $T(1) = a$ and $T(1) = b$ have probabilities p_a and p_b, where

$$\{p_a, p_b\} = \{p_a, p_b\}\,\mathbf{P} = \left\{\frac{1 - \alpha_2}{2 - \alpha_1 - \alpha_2}, \frac{1 - \alpha_1}{2 - \alpha_1 - \alpha_2}\right\}, \tag{2.21}$$

the equilibrium probabilities of the Markov chain, and the time from the origin to an event of type a has distribution function

$$\frac{p_{ba} \int_0^x R_{ba}(u)\,du}{E(X_b^{(b)}(1))} + \frac{p_{aa} \int_0^x R_{aa}(u)\,du}{E(X_a^{(a)}(1))}, \tag{2.22}$$

with a similar definition for the time to an event of type b.

Cinlar [9] has reviewed the properties of semi-Markov processes. Our view of these processes, being related to statistical problems arising in the analysis of bivariate point processes, will be somewhat different from the usual one. Thus, note that in the marginal processes the regenerative property of the semi-Markov process implies that the times between events of type a, $X_a^{(a)}(i)$, $i = 1, 2, \cdots$, are independent and identically distributed, as are the $X_b^{(b)}(j)$, $j = 1, 2, \cdots$. Therefore, the marginal processes are renewal processes and we say that the semi-Markov process is a *bivariate renewal process*. Since the types of successive renewals (events) form a Markov chain, the process is also called a Markov renewal process ([9], p. 130). However, the two marginal renewal processes together with the Markov chain of event types do not determine the process.

The dependency structure of this bivariate renewal process can also be examined through joint properties of forward recurrence times in the process. The joint forward recurrence times $\{X_a^{(a)}(1), X_a^{(b)}(1)\}$ for semisynchronous sampling of b by a are, in the terminology of semi-Markov process theory, the first passage times from state a to state a and from state a to state b, with similar definitions for $\{X_b^{(a)}(1), X_b^{(b)}(1)\}$. Denoting the marginal distributions of these random variables by $F_a^{(a)}(x)$ and so forth, we have the equations

$$F_a^{(a)}(x) = p_{aa}F_{aa}(x) + p_{ab}F_{ab}(x) * F_b^{(a)}(x), \tag{2.23}$$

$$F_a^{(b)}(x) = p_{ab}F_{ab}(x) + p_{aa}F_{aa}(x) * F_a^{(b)}(x), \tag{2.24}$$

$$F_b^{(a)}(x) = p_{ba}F_{ba}(x) + p_{bb}F_{bb}(x) * F_b^{(a)}(x), \tag{2.25}$$

$$F_b^{(b)}(x) = p_{bb}F_{bb}(x) + p_{ba}F_{ba}(x) * F_a^{(b)}(x), \tag{2.26}$$

where $*$ denotes Stieltjes convolution and $F_{aa}(x)$ and so forth, are defined in (2.20).

These equations can be solved using Laplace–Stieltjes transforms. Thus, if $\mathscr{F}_a^{(a)}(s)$ is the Laplace–Stieltjes transform of $F_a^{(a)}(x)$, and so forth, we get

$$\mathscr{F}_a^{(a)}(s) = \alpha_1 \mathscr{F}_{aa}(s) + \frac{(1 - \alpha_1)(1 - \alpha_2)\mathscr{F}_{ab}(s)\mathscr{F}_{ba}(s)}{1 - \alpha_2 \mathscr{F}_{bb}(s)}, \tag{2.27}$$

$$\mathscr{F}_b^{(b)}(s) = \alpha_1 \mathscr{F}_{bb}(s) + \frac{(1 - \alpha_1)(1 - \alpha_2)\mathscr{F}_{ab}(s)\mathscr{F}_{ba}(s)}{1 - \alpha_1 \mathscr{F}_{aa}(s)}. \tag{2.28}$$

From these results, we can write down joint forward recurrence time distributions using the regenerative properties of the process. For example,

$$\begin{aligned} &R_{X_a^{(a)}(1), X_a^{(b)}}(x_1, x_2) \\ &= Pr\{X_a^{(a)}(1) > x_1, X_a^{(b)} > x_2\} \\ &= \begin{cases} \alpha_1 R_{aa}(x_1) + (1 - \alpha_1)\left\{R_{ab}(x_1) + \displaystyle\int_{x_2}^{x_1} [1 - F_b^{(a)}(x_1 - u)]\, dF_{ab}(u)\right\} & \textit{if } x_1 \geqq x_2, \\ (1 - \alpha_1)R_{ab}(x_2) + \alpha_1\left\{R_{aa}(x_2) + \displaystyle\int_{x_1}^{x_2} [1 - F_a^{(b)}(x_2 - u)]\, dF_{aa}(u)\right\} & \text{if } x_2 \geqq x_1. \end{cases} \end{aligned} \tag{2.29}$$

It is actually much simpler, because of the regenerative nature of the process, to express results in terms of the order statistics and order types associated with $R(x_1, x_2)$. These aspects of the process are worked out in greater detail by Wisniewski [50].

Note that the process derived previously as two independent Poisson processes is a very particular form of semi-Markov process. The question then arises whether there are any other semi-Markov processes with Poisson marginals and the answer is clearly yes. For example, when $\alpha_1 = \alpha_2 = 0$ we have the special case of an alternating renewal process and choosing $F_{ab}(x)$ and $F_{ba}(x)$ to be distributions of random variables proportional to chi square variables with one degree of freedom gives Poisson marginals. The example shows in fact that one can produce any desired marginal renewal processes in a semi-Markov process, as is also clear from (2.27) and (2.28).

From equations such as (2.14) and (2.15), it can be shown that no bivariate process of independent renewal marginals is a semi-Markov process unless the marginals are also Poisson processes. The dependency structure in a semi-Markov process is actually better characterized by the second order cross intensity function, which we introduce in the next section, rather than by joint moments of forward recurrence times. This cross intensity function together with the two distributions of intervals in the marginal renewal processes (or equivalently the intensity functions of the marginal renewal processes) completely specifies the semi-Markov process.

3. Dependence and correlation in bivariate point processes

3.1. *Specification.* We now consider in more detail the specification of the structure of bivariate point processes. It is common in the study of particular stochastic processes to find that physically the same process can be specified in several equivalent but superficially different ways. A simple and familiar example is the stationary univariate Poisson process which can be specified as:

(a) a process in which the numbers of events in disjoint sets have independent Poisson distributions with means proportional to the measures of the sets;

(b) a renewal process with exponentially distributed intervals;

(c) a process in which the probability of an event in $(t, t + \Delta t]$ has an especially simple form, as $\Delta t \to 0$.

We call these three specifications, respectively, the counting, the interval, and the intensity specifications. Univariate point processes can in general be specified in these three ways, if the initial conditions are properly chosen.

While the counting specification (a) for bivariate point processes is in principle fundamental, it is often too complicated to be very fruitful. If the joint characteristic functional of the process, defined by an obvious generalization of the univariate case, can be obtained in a useful form, this does give a concise representation of all the joint distributions of counts; even then, such a characteristic functional would usually give little insight into the physical mechanism generating the process.

Often, special processes are most conveniently handled through some kind of interval specification, especially when this corresponds rather closely to the physical origin of the process. In particular, the two state semi-Markov process is most simply specified in this way, as shown in Section 2.8. The two main types of interval specifications discussed in Section 2 were the specifications in terms of the intervals in the marginal processes, or the intervals and event types in the pooled process. The latter is the basic specification for the semi-Markov process. Other processes, such as various kinds of inhibited processes and the bivariate Poisson process of Section 4.1 are specified rather less directly in terms of relations between intervals and event types.

However, in some ways the most convenient general specification is through the intensity. Denote by $\mathscr{H}_t$ the history of the process at time t, that is, a complete specification of the occurrences in $(-\infty, t]$ measured backwards from an origin at t, then two time points t', t'' have the same history if and only if the observed sequences $\{x^{(a)}(-1), x^{(a)}(-2), \cdots\}$, $\{x^{(b)}(-1), x^{(b)}(-2), \cdots\}$ are identical if measured from origins at t' and at t''.

Then a marginally regular process is specified by

$$\lambda^{(a)}(t; \mathscr{H}_t) = \lim_{\Delta t \to 0+} \frac{Pr\{N^{(a)}(t, t + \Delta t) \geqq 1 | \mathscr{H}_t\}}{\Delta t}, \tag{3.1}$$

$$\lambda^{(b)}(t; \mathscr{H}_t) = \lim_{\Delta t \to 0+} \frac{Pr\{N^{(b)}(t, t + \Delta t) \geqq 1 | \mathscr{H}_t\}}{\Delta t}, \tag{3.2}$$

$$\lambda^{(ab)}(t; \mathscr{H}_t) = \lim_{\Delta t \to 0+} \frac{Pr\{N^{(a)}(t, t + \Delta t)N^{(b)}(t, t + \Delta t) \geqq 1 | \mathscr{H}_t\}}{\Delta t}. \tag{3.3}$$

We call these functions the *complete intensity functions* of the process. If the process is regular $\lambda^{(ab)}(t; \mathscr{H}_t) = 0$. Given the functions (3.1) through (3.3) and some initial conditions, we can construct a discretized realization of the process, although this is, of course, a clumsy method of simulation if the interval specification is at all simple.

One advantage of the complete intensity specification is that one can generate families of models of increasing complexity by allowing more and more complex dependency on $\mathscr{H}_t$. This may be useful, for instance, in testing consistency of data with a given type of model, for example, a semi-Markov process. Further, if the main features of $\mathscr{H}_t$ that determine the intensity functions can be found, an appropriate type of model may be indicated.

As an example of a complete intensity specification, consider the two state semi-Markov process. Here the only aspects of $\mathscr{H}_t$ that are relevant, if at least one event has occurred before t, are the backward recurrence time to the previous event and the type of that event $\{x^{(\cdot)}(-1), t(-1)\}$. Any initial conditions disappear once one event has occurred. For convenience, we write the partial history as (u, a), if the preceding event is of type a and (u, b) if it is of type b. Then assuming that the process is regular, that is, that none of the interval distributions has an atom at zero, we have

$$\lambda^{(a)}\{t; (u, a)\} \equiv \lambda_a^{(a)}(u) = \frac{p_{aa} f_{aa}(u)}{p_{aa} R_{aa}(u) + p_{ab} R_{ab}(u)}, \tag{3.4}$$

$$\lambda^{(b)}\{t; (u, a)\} \equiv \lambda_a^{(b)}(u) = \frac{p_{ab} f_{ab}(u)}{p_{aa} R_{aa}(u) + p_{ab} R_{ab}(u)}, \tag{3.5}$$

with similar expressions defining $\lambda_b^{(a)}(u)$ and $\lambda_b^{(b)}(u)$ when the partial history is (u, b). These complete intensities are analogues of the hazard function, or age specific failure rate, which can be used to specify a univariate renewal process.

The semi-Markov process is characterized by the dependence on $\mathscr{H}_t$ being only on u and the type of the preceding event. We can generalize the semi-Markov process in many ways, for instance by allowing a dependence on both of the backward recurrence times $x^{(a)}(-1)$ and $x^{(b)}(-1)$; see Section 3.8.

3.2. *Properties of complete intensity functions.* We now consider briefly some properties of complete intensity functions. It is supposed for simplicity that the process is regular and that it is observed for a time long enough to allow initial conditions to be disregarded. For the semi-Markov process "long enough" is the occurrence of at least one event.

(i) If the process is completely stationary, as defined in Section 2.3, the intensity functions depend only on $\mathscr{H}_t$ and not on t.

(ii) Nonstationary generalizations of a given stationary process can be produced by inserting into the intensity a function either of t, for example, $e^{\gamma t}$, $\exp\{\gamma \cos(\omega_0 t + \phi)\}$, or of the numbers of events that have occurred since the start of the process.

(iii) The intensity specification of a stationary process is unique in the sense that if we have two different intensity specifications and can find a set of histories of nonzero probability such that, say, the first intensity specification gives greater intensity of events of type a than the second, then the two processes are distinguishable from suitable data. Note that this is not the same question as whether two different specifications containing unknown parameters are distinguishable.

(iv) The events of different types are independent if and only if $\lambda^{(a)}$ and $\lambda^{(b)}$ involve $\mathscr{H}_t$ only through the histories of the separate processes of events of type a and type b, denoted, respectively, by $\mathscr{H}_t^{(a)}$ and $\mathscr{H}_t^{(b)}$.

(v) We can call the process purely a dependent if both $\lambda^{(a)}$ and $\lambda^{(b)}$ depend on $\mathscr{H}_t$ only through $\mathscr{H}_t^{(a)}$. In many ways the simplest example of such a process is obtained when both intensities depend only on the backward recurrence time in the process of events of type a, that is, on the time $u^{(a)}$ measured back to the previous event of type a. Denote the intensities by $\lambda^{(a)}(u^{(a)})$ and $\lambda^{(b)}(u^{(a)})$. Then the events of type a form a renewal process; if in particular $\lambda^{(a)}(\cdot)$ is constant, the events of type a form a Poisson process. If simple functional forms are assumed for the intensities, the likelihood of data can be obtained in a fairly simple form and hence an efficient statistical analysis derived.

(vi) A different kind of purely a dependent process is derived from a shot noise process based on the a events, that is, by considering a stochastic process $Z^{(a)}(t)$ defined by

$$Z^{(a)}(t) = \int_0^\infty g(u)\, dN^{(a)}(t - u). \tag{3.6}$$

We then take $\lambda^{(b)}(t)$ and possibly also $\lambda^{(a)}(t)$ to depend only on $Z^{(a)}(t)$. In particular if $g(u) = 1(u < \Delta)$ and $g(u) = 0(u \geqq \Delta)$, the intensities depend only on the number of events of type a in $(t - \Delta, t)$. Hawkes [22] has considered some processes of this type.

(vii) The intensity functions look in one direction in time. This approach is therefore rather less suitable for processes in a spatial continuum, where there may be no reason for picking out one spatial direction rather than another.

(viii) Some simple processes, for example, the bivariate Poisson process of Section 4.1, have intensity specifications that appear quite difficult to obtain.

3.3. *Second order cross intensity functions.* In Section 3.2, we considered the complete intensity functions which specify probabilities of occurrence given the entire history $\mathscr{H}_t$. For some purposes, it is useful with stationary (second order) processes to be less ambitious and to consider probabilities of occurrence conditionally on much less information than the entire history $\mathscr{H}_t$. We then call the functions corresponding to (3.1) through (3.3) incomplete intensity functions. For example, using the notation of (3.4) and (3.5), both

$$\lim_{\Delta t \to 0+} Pr \frac{\{N^{(a)}(t, t + \Delta t) \geqq 1 \mid (u, a)\}}{\Delta t} \tag{3.7}$$

and the corresponding function when the last event is of type b are defined for any regular stationary process, even though they specify the process completely only for semi-Markov processes.

A particularly important incomplete intensity function is obtained when one conditions on the information that an event of specified type occurs at the time origin. Again for simplicity, we consider stationary regular processes. Write

$$h_a^{(a)}(t) = \lim_{\Delta t \to 0+} Pr \frac{\{N^{(a)}(t, t + \Delta t) \geqq 1 \mid \text{type } a \text{ event at } 0\}}{\Delta t}, \tag{3.8}$$

$$h_a^{(b)}(t) = \lim_{\Delta t \to 0+} Pr \frac{\{N^{(b)}(t, t + \Delta t) \geqq 1 \mid \text{type } a \text{ event at } 0\}}{\Delta t}, \tag{3.9}$$

with similar definitions for $h_b^{(a)}(t)$, $h_b^{(b)}(t)$ if an event of type b occurs at 0. We call the function (3.9) a second order cross intensity function. For nonregular processes, it may be helpful to introduce intensities conditionally on events of both types occurring at 0.

Note that the cross intensity functions $h_a^{(b)}(t)$ and $h_b^{(a)}(t)$ will contain Dirac delta functions as components if, for example, there is a nonzero probability that an event of type b will occur exactly τ away from a type a event. For the process to be regular, rather than merely marginally regular, the cross intensity functions must not contain delta functions at the origin.

Note too that $h_a^{(b)}(t)$ is well defined near $t = 0$ and will typically be continuous there.

If following an event of type a at the origin, subsequent events of type a are at times $S_a^{(a)}(1), S_a^{(a)}(2), \cdots$ and those of type b at $S_a^{(b)}(1), S_a^{(b)}(2), \cdots$, we can write

$$h_a^{(a)}(t) = \sum_{r=1}^{\infty} f_{S_a^{(a)}(r)}(t), \tag{3.10}$$

$$h_a^{(b)}(t) = \sum_{r=1}^{\infty} f_{S_a^{(b)}(r)}(t), \tag{3.11}$$

where $f_U(\cdot)$ is the probability density function of the random variable U.

If the process of type a events is a renewal process, (3.10) is a function familiar as the renewal density. For small t the contribution from $r = 1$ is likely to be dominant in all these functions.

The intensities are defined for all t. However, $h_a^{(a)}(t)$ and $h_b^{(b)}(t)$ are even functions of t. Further, it follows from the definition of conditional probability that

$$h_b^{(a)}(t)\rho_b = h_a^{(b)}(-t)\rho_a, \tag{3.12}$$

where ρ_a and ρ_b are the rates of the two processes. For processes without long term effects, we have that as $t \to \infty$ or $t \to -\infty$

$$h_a^{(a)}(t) \to \rho_a, \qquad h_b^{(a)}(t) \to \rho_a, \qquad h_b^{(b)}(t) \to \rho_b. \tag{3.13}$$

Sometimes it may be required to calculate the intensity function of the superposed process, that is, the process in which the type of event is disregarded. Given an event at the time origin, it has probability $\rho_a/(\rho_a + \rho_b)$ of being a type a, and hence the intensity of the superposed process is

$$h_\cdot^{(\cdot)}(t) = \frac{\rho_a}{\rho_a + \rho_b}\{h_a^{(a)}(t) + h_a^{(b)}(t)\} + \frac{\rho_b}{\rho_a + \rho_b}\{h_b^{(a)}(t) + h_b^{(b)}(t)\}. \tag{3.14}$$

This is a general formula for the intensity function of the superposition of two, possibly dependent, processes.

3.4. *Covariance densities.* For some purposes, it is slightly more convenient to work with covariance densities rather than with the second order intensity functions; see, for example, Bartlett [4]. To define the *cross covariance density*, we consider the random variables $N^{(a)}(0, \Delta' t)$ and $N^{(b)}(t, t + \Delta'' t)$ and define

$$\begin{aligned} \gamma_a^{(b)}(t) &= \lim_{\Delta' t, \Delta'' t \to 0+} \frac{\operatorname{Cov}\{N^{(a)}(0, \Delta' t), N^{(b)}(t, t + \Delta'' t)\}}{\Delta' t\, \Delta'' t} \\ &= \rho_a h_a^{(b)}(t) - \rho_a \rho_b. \end{aligned} \tag{3.15}$$

It follows directly from (3.14) or from (3.12) and (3.15), that $\gamma_a^{(b)}(t) = \lambda_b^{(a)}(-t)$. Note that an *autocovariance density* such as $\gamma_a^{(a)}(t)$ can be written

$$\gamma_a^{(a)}(t) = \rho_a\delta(t) + \gamma_{a,\mathrm{cont}}^{(a)}(t) = \rho_a\delta(t) + \rho_a\{h_b^{(a)}(t) - \rho_a\}, \tag{3.16}$$

where the second terms are continuous at $t = 0$ and $\delta(t)$ denotes the Dirac delta function.

We denote by $V^{(ab)}(t_1, t_2)$ the covariance between $N^{(a)}(t_1)$ and $N^{(b)}(t_2)$ in the stationary bivariate process. This is called the *covariance time surface*. Then

$$V^{(ab)}(t_1, t_2) = \mathrm{Cov}\,\{N^{(a)}(t_1), N^{(b)}(t_2)\} = \mathrm{Cov}\left\{\int_0^{t_1} dN^{(a)}(u), \int_0^{t_2} dN^{(b)}(v)\right\} = \int_0^{t_1}\int_0^{t_2} \gamma_b^{(a)}(u - v)\, du\, dv. \tag{3.17}$$

In the special case $t_1 = t_2 = t$, we write $V^{(ab)}(t, t) = V^{(ab)}(t)$. It follows from (3.17) that

$$V^{(ab)}(t) = \int_0^t (t - v)\{\gamma_b^{(a)}(v) + \gamma_a^{(b)}(v)\}\, dv. \tag{3.18}$$

Note that in (3.18) a delta function component at the origin can enter one but not both of the cross covariance densities. If, in (3.18), we take the special highly degenerate case when the type b and type a processes coincide point for point, we obtain the well-known variance time formula

$$V^{(aa)}(t) = \mathrm{Var}\,\{N^{(a)}(t)\} = \rho_a t + 2\int_{0+}^t (t - v)\gamma_a^{(a)}(v)\, dv. \tag{3.19}$$

An interesting question concerns the conditions under which a set of functions $\{\gamma_a^{(a)}(t), \gamma_b^{(a)}(t), \gamma_a^{(b)}(t), \gamma_b^{(b)}(t)\}$ can be the covariance densities of a bivariate point process. Now for all α and β

$$\mathrm{Var}\,\{\alpha N^{(a)}(t_1) + \beta N^{(b)}(t_2)\} = \alpha^2 V^{(aa)}(t_1) + 2\alpha\beta V^{(ab)}(t_1, t_2) + \beta^2 V^{(bb)}(t_2) \geqq 0. \tag{3.20}$$

Thus for all t_1 and t_2,

$$V^{(aa)}(t_1) \geqq 0, \qquad \{V^{(ab)}(t_1, t_2)\}^2 \leqq V^{(aa)}(t_1)V^{(bb)}(t_2), \qquad V^{(bb)}(t_2) \geqq 0. \tag{3.21}$$

The conditions (3.21) can be used to show that certain proposed functions cannot be covariance densities. It would be interesting to know whether corresponding to any functions satisfying (3.21) there always exists a corresponding stationary bivariate point process.

Nothing special is learned by letting t_1 and $t_2 \to 0$ in (3.21). If, however, we let $t_1 = t_2 \to \infty$, we have under weak conditions that

$$V^{(aa)}(t) \sim t\left\{\rho_a + 2\int_{0+}^\infty \gamma_a^{(a)}(v)\, dv\right\} = \rho_a t I^{(aa)}, \tag{3.22}$$

$$V^{(ab)}(t) \sim t\int_0^\infty \{\gamma_b^{(a)}(v) + \gamma_a^{(b)}(v)\}\, dv = (\rho_a\rho_b)^{1/2} t I^{(ab)}, \tag{3.23}$$

$$V^{(bb)}(t) \sim t\left\{\rho_b + 2\int_0^\infty \gamma_b^{(b)}(v)\,dv\right\} = \rho_b t I^{(bb)}, \tag{3.24}$$

where the right sides of these equations define three asymptotic measures of dispersion I. The conditions

$$I^{(aa)} \geqq 0, \qquad \{I^{(ab)}\}^2 \leqq I^{(aa)}I^{(bb)}, \qquad I^{(bb)} \geqq 0 \tag{3.25}$$

must, of course, be satisfied, in virtue of (3.21).

3.5. *Some special processes.* The second order intensity functions, or equivalently the covariance densities, are not the most natural means of representing the dependencies in a point process, if these dependencies take special account of the nearest events of either or both types. Thus for the semi-Markov process, the second order intensity functions satisfy integral equations; see, for example, Cox and Miller [15], pp. 352–356. The relation with the defining functions of the process is therefore indirect. Thus, while in principle the distributions defining the process could be estimated from data via the second order intensity functions, this would be a roundabout approach, and probably very inefficient.

We now discuss briefly two processes for which the second order intensity functions are more directly related to the underlying mechanism of the process.

Consider an arbitrary regular stationary process of events of type a. Let each event of type a be displaced by a random amount to form a corresponding event of type b; the displacements of different points are independent and identically distributed random variables with probability density function $p(\cdot)$. Denote the probability density function of the difference between two such random variables by $q(\cdot)$. Then a direct probability calculation for the limiting, stationary, process shows that (Cox [12])

$$h_a^{(b)}(t) = p(t) + \int_{-\infty}^{\infty} h_a^{(b)}(v)p(t - v)\,dv, \tag{3.26}$$

$$h_b^{(b)}(t) = \int_{-\infty}^{\infty} h_a^{(a)}(v)q(t - v)\,dv. \tag{3.27}$$

In particular, if the type a events form a Poisson process, $h_a^{(a)}(v) = \rho_a$, so that

$$h_a^{(b)}(t) = p(t) + \rho_a, \qquad h_b^{(b)}(t) = \rho_a. \tag{3.28}$$

The constancy of $h_b^{(b)}(t)$ is an immediate consequence of the easily proved fact that the type b events on their own form a Poisson process. The results (3.28) lead to quite direct methods of estimating $p(\cdot)$ from data and to tests of the adequacy of the model. For positive displacements the type a events could be the inputs to an $M/G/\infty$ queue, the type b events being the outputs. Generalizations of this delay process are considered in Sections 4 and 5.

As a second example, consider a *bivariate doubly stochastic Poisson process.* That is, we have an unobservable real valued (nonnegative) stationary bivariate process $\{\mathbf{\Lambda}(t)\} = \{\Lambda_a(t), \Lambda_b(t)\}$. Conditionally on the realized value of this process, we observe two independent nonstationary Poisson processes with rates, respectively, $\Lambda_a(t)$ for the type a events and $\Lambda_b(t)$ for the type b events. Then, by first arguing conditionally on the realized value of $\{\mathbf{\Lambda}(t)\}$, we have a stationary bivariate point process with

$$\begin{aligned}\gamma_a^{(a)}(t) &= \rho_a\delta(t) + c_\Lambda^{(aa)}(t), \qquad \gamma_b^{(a)}(t) = c_\Lambda^{(ab)}(t),\\ \gamma_b^{(b)}(t) &= \rho_b\delta(t) + c_\Lambda^{(bb)}(t),\end{aligned} \tag{3.29}$$

where $E\{\mathbf{\Lambda}(t)\} = (\rho_a, \rho_b)$ and the c_Λ are the auto and cross covariance functions of $\{\mathbf{\Lambda}(t)\}$.

Again there is a quite direct connection between the covariance densities and the underlying mechanism of the process. Two special cases are of interest. One is when $\Lambda_a(t) = \Lambda_b(t)\,\rho_a/\rho_b$, leading to some simplification of (3.29). Another special case is

$$\begin{aligned}\Lambda_a(t) &= \rho_a + R_a\cos(\omega_0 t + \theta + \Phi),\\ \Lambda_b(t) &= \rho_b + R_b\cos(\omega_0 t + \Phi).\end{aligned} \tag{3.30}$$

In this

$$\begin{aligned}&E(R_a) = E(R_b) = 0,\\ E(R_a^2) = \sigma_{aa}, \qquad &E(R_{ab}) = \sigma_{ab}, \qquad E(R_{bb}) = \sigma_{bb},\end{aligned} \tag{3.31}$$

and the random variable Φ is uniformly distributed over $(0, 2\pi)$ independently of R_a and R_b. Further, ρ_a, ρ_b, ω_0, and θ are constants and, to keep the Λ nonnegative, $|R_a| \leqq \rho_a$, $|R_b| \leqq \rho_b$. This defines a stationary although nonergodic process $\{\mathbf{\Lambda}(t)\}$.

Specifications (3.30) and (3.31) yield

$$\begin{aligned}\gamma_a^{(a)}(t) &= \rho_a\delta(t) + \sigma_{aa}\cos(\omega_0 t), \qquad \gamma_b^{(a)}(t) = \sigma_{ab}\cos\{\omega_0(t + \theta)\},\\ \gamma_b^{(b)}(t) &= \rho_b\delta(t) + \sigma_{bb}\cos(\omega_0 t).\end{aligned} \tag{3.32}$$

Of course this process is extremely special. Note, however, that fairly general processes with a sinusoidal component in the intensity can be produced by starting from the complete intensity functions of a stationary process and either adding a sinusoidal component or multiplying by the exponential of a sinusoidal component; the latter has the advantage of ensuring automatically a nonnegative complete intensity function.

3.6. *Spectral analysis of the counting process.* For Gaussian stationary stochastic processes, study of spectral properties is useful for three rather different reasons:

(a) the spectral representation of the process itself may be helpful;

(b) the spectral representation of the covariance matrix may be helpful;

(c) the effect on the process of a stationary linear operator is neatly expressed.

For point processes a general representation analogous to (a) has been discussed by Brillinger [6]. Bartlett [1] has given some interesting second order theory and applications in the univariate case. For doubly stochastic Poisson processes, which are of course very special, we can often use a full spectral representation for the defining $\{\mathbf{\Lambda}(t)\}$ process; indeed the $\{\mathbf{\Lambda}(t)\}$ process may be nearly Gaussian.

If we are content with a spectral analysis of the covariance density, we can write, in particular for the complex valued *cross spectral density function*,

$$g_b^{(a)}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\gamma_b^{(a)}(t)\,dt = g_a^{(b)}(-\omega). \tag{3.33}$$

Because of the mathematical equivalence between the covariance density and the spectral density, the previous general and particular results for covariance densities can all be expressed in terms of the spectral properties. While these will not be given in full here, note first that the measures of dispersion in (3.22) through (3.24) are given by

$$\begin{gathered} \rho_a I^{(aa)} = 2\pi g_a^{(a)}(0), \qquad (\rho_a\rho_b)^{1/2} I^{(ab)} = 2\pi g_b^{(a)}(0), \\ \rho_b I^{(bb)} = 2\pi g_b^{(b)}(0). \end{gathered} \tag{3.34}$$

All the results (3.26) through (3.32) can be expressed simply in terms of the spectral properties. Thus, from (3.29), the spectral analysis of the bivariate doubly stochastic Poisson process leads directly to the spectral properties of the process $\{\mathbf{\Lambda}(t)\}$ on subtracting the "white" Poisson spectra. Thus, spectral analysis of a doubly stochastic Poisson process is likely to be useful whenever the process $\{\mathbf{\Lambda}(t)\}$ has an enlightening spectral form. In the special case (3.32) where $\{\mathbf{\Lambda}(t)\}$ is sinusoidal,

$$g_a^{(a)}(\omega) = \frac{\rho_a}{2\pi} + \frac{\sigma_{aa}}{2\pi}\,\delta(\omega - \omega_0), \tag{3.35}$$

$$g_b^{(a)}(\omega) = \frac{\sigma_{ab}}{2\pi}\,e^{i\omega\theta}\delta(\omega - \omega_0). \tag{3.36}$$

The complex valued cross spectral density can be split in the usual way into real and imaginary components, which indicate the relative phases of the fluctuations in the processes of events of type a and type b. We can also define the coherency as $|g_b^{(a)}(\omega)|^2/\{g_a^{(a)}(\omega)g_b^{(b)}(\omega)\}$; for the doubly stochastic process driven by proportional intensities, $\Lambda_b(t) \propto \Lambda_a(t)$, and the coherency is one for all ω, provided that the "white" Poisson component is removed from the denominator.

The natural analogue for point processes of the stationary linear operators on a real valued process is random translation, summarized in (3.26) and (3.27). It follows directly from these equations and from the relation between covariance densities and intensity functions that

$$g_a^{(b)}(\omega) = \rho_a p^{\dagger}(\omega) + g_a^{(a)}(\omega)p^{\dagger}(\omega), \tag{3.37}$$

$$\left\{g_b^{(b)}(\omega) - \frac{\rho_b}{2\pi}\right\} = q^\dagger(\omega)\left\{g_a^{(a)}(\omega) - \frac{\rho_a}{2\pi}\right\}, \tag{3.38}$$

where $p^\dagger(\omega)$, $q^\dagger(\omega)$ are the Fourier transforms of $p(t)$ and $q(t)$. A more general type of random translation for bivariate processes is discussed in Section 5.

3.7. *Variance and covariance time functions.* For univariate point processes the covariance density or spectral functions are mathematically equivalent to the variance time function $V^{(aa)}(t)$ of (3.19), which gives as a function of t the variance of the number of events in an interval of length t. This function is useful for some kinds of statistical analysis; examination of its behavior for large t is equivalent analytically to looking at the low frequency part of the spectrum.

For bivariate point processes, it might be thought that the variance time function $V^{(aa)}(t)$, $V^{(bb)}(t)$, and the covariance time function $V^{(ab)}(t)$ of (3.18) are equivalent to the other second order specifications. This is not the case, however, because it is clear from (3.18) that only the combinations $\gamma_b^{(a)}\omega) + \gamma_a^{(b)}(\omega)$ can be found from $V^{(ab)}(t)$ and this is not enough to fix the cross covariance function of the process.

The cross covariance density can, however, be found from the covariance time surface, $V^{(ab)}(t_1, t_2)$ of (3.17).

The covariance time function and surface are useful for some rather special statistical purposes.

The variance time function of the superposed process is

$$V^{(\cdot\cdot)}(t) = V^{(aa)}(t) + 2V^{(ab)}(t) + V^{(bb)}(t); \tag{3.39}$$

this is equivalent to the relation (3.14) for intensity functions.

3.8. *Bivariate interval specifications; bivariate Markov interval processes.* As has been mentioned several times in this section, the second order intensity functions and their equivalents are most likely to be useful when the dependencies in the underlying mechanism do not specifically involve nearest neighbors, or other features of the process that are most naturally expressed serially, that is, through event number either in the pooled process or in the marginal processes rather than through real time.

For processes in which an interval specification is more appropriate, there are many ways of introducing functions wholly or partially specifying the dependency structure of the process. For a stationary univariate process we can consider the sequence of intervals between successive events as a stochastic process, indexed by serial number, that is, as a real valued process in discrete time. The second order properties are described by an autocovariance sequence which, say for events of type a, is

$$\gamma_x^{(aa)}(j) = \text{Cov}\,\{X_a^{(a)}(k), X_a^{(a)}(k+j)\}, \qquad j = 0, \pm 1, \cdots. \tag{3.40}$$

McFadden [35] has shown that the autocovariance sequence is related to the distribution of counts by the simple, although indirect, formula

$$\gamma_x^{(aa)}(j) = \frac{1}{\rho_a}\left[\int_0^\infty Pr\{N^{(a)}(t) = j\}\,dt - \frac{1}{\rho_a}\right]. \tag{3.41}$$

Stationarity of the $X_a^{(a)}(i)$ sequence is discussed in Slivnyak [44].

If the distribution of $N^{(a)}(t)$ is given by the probability generating function

$$\phi^{(a)}(z, t) = \sum_{j=0}^{\infty} z^j Pr\{N^{(a)}(t) = j\} \tag{3.42}$$

with Laplace transform $\phi^{(a)*}(z, s)$, it will be convenient to substitute in (3.41) the result

$$\int_0^\infty Pr\{N^{(a)}(t) = j\}\,dt = \left[\frac{1}{j!}\,\frac{\partial^j \phi^{(a)*}(z, s)}{\partial z^j}\right]_{z=0,s=0+}. \tag{3.43}$$

The sequence (3.40) and the analogous one for events of type b summarize the second order marginal properties. To study the joint properties of various kinds of intervals between events, the following are some of the possibilities.

(i) The two sets of intervals $\{X_a^{(a)}(r), X_b^{(b)}(r)\}$ may be considered as a bivariate process in discrete time, that is, we may use serial number in each process as a common discrete time scale. Cross covariances and cross spectra can then be defined in the usual way. While this may occasionally be fruitful, it is not a useful general approach, because for almost all physical models events in the two processes with a common serial number will be far apart in real time. Another problem is that if the process is sampled semisynchronously, say on a type a event, the sequence $X_a^{(b)}(r)$ is not a stationary sequence, although it will generally "converge" to the sequence $X_b^{(b)}(r)$. Again, sufficiently far out from the sampling point, events in the two processes with common serial number will be far apart in real time.

(ii) We may consider the intervals between successive events in the process taken regardless of type, that is, the superposed process. This gives a third covariance sequence, namely, $\gamma_x^{(\cdot\cdot)}(j)$. For particular processes this can be calculated from (3.41) applied to the pooled process, particularly if the joint distribution of the count $N^{(a)}(t)$ and $N^{(b)}(t)$ are available.

In fact, if the joint distribution of $\{N^{(a)}(t), N^{(b)}(t)\}$ is specified by the joint probability generating function $\phi^{(ab)}(z_a, z_b, t)$ with Laplace transform $\phi^{(ab)*}(z_a, z_b, s)$, we have from (3.41) and (3.43) that

$$\gamma_x^{(\cdot\cdot)}(j) = \frac{1}{\rho_a + \rho_b}\left\{\frac{1}{j!}\,\frac{\partial^j \phi^{(ab)*}(z, z, s)}{\partial z^j} - \frac{1}{\rho_a + \rho_b}\right\}, \tag{3.44}$$

the derivative being evaluated at $z = 0$, $s = 0+$.

A limitation of this approach is, however, that independence of the type a and type b events is not reflected in any simple general relation between the three covariance sequences; this is clear from (3.44). Consider also the process of two independent renewal processes of Section 2.3; the covariance sequence for the superposed sequence is complex and not directly informative.

(iii) The discussion of (i) and (ii) suggests that we consider some properties of the intervals in the pooled sequence, that is, the superposed process with the type of each event being distinguished. Possibilities and questions that arise include the following.

(a) The sequence of event types can be considered as a binary time series. In particular, it might be useful to construct a simple test of dependence of the two series based on the nonrandomness of the sequence of event types. Such a test would, however, at least in its simplest form, require the assumption that the marginal processes are Poisson processes.

(b) We can examine the distributions and in particular the means of the backward recurrence times from events of one type to those of the opposite type, that is, $X_b^{(a)}(-1)$ and $X_a^{(b)}(-1)$. If the two types of events are independently distributed, the two "mixed" recurrence times should have marginal distributions corresponding to the equilibrium recurrence time distributions in the marginal process of events of types a and b.

(c) A more symmetrical possibility similar in spirit to (b) is to examine the joint distribution of the two backward recurrence times measured from an arbitrary time origin, that is, of $X^{(a)}(-1)$ and $X^{(b)}(-1)$; the marginal distributions are, of course, the usual ones from univariate theory. If the events of the two types are independent, the two recurrence times are independently distributed with the distribution of the equilibrium recurrence times. Note, however, the discussion following (2.13). It would be possible to adapt (b) and (c) to take account of forward as well as of backward recurrence times.

(d) Probably the most useful general procedure for examining dependence in a bivariate process through intervals is to consider intensities conditional on the two separate asynchronous backward recurrence times. This is not quite analogous to the use of second order intensities of Section 3.3. Denote the realized backward recurrence times from an arbitrary time origin in the stationary process by u_a, u_b, that is, $u_a = x^{(a)}(-1)$, $u_b = x^{(b)}(-1)$. We then define the *serial intensity functions* for a stationary regular process by

$$\lambda^{(a)}(u_a, u_b) = \lim_{\Delta t \to 0+} \frac{Pr\{N^{(a)}(\Delta t) \geqq 1 \mid U_a = u_a, U_b = u_b\}}{\Delta t}, \tag{3.45}$$

with an analogous definition for $\lambda^{(b)}(u_a, u_b)$. These are, in a sense, third order rather than second order functions, since they involve occurrences at three points.

Now these two serial intensities are defined for all regular stationary processes, but they are complete intensity functions in the sense of Section 3.1 only for a very special class of process that we shall call *bivariate Markov interval processes*. These processes include semi-Markov processes and independent renewal processes; note, however, that in general the marginal processes associated with a bivariate Markov interval process are not univariate renewal processes (as with semi-Markov processes) and this is why we have not called the processes bivariate renewal processes.

As an example of this type of process, consider the alternating process of Section 2.5 with disjoint pairwise dependence. Denote the marginal distribution of the $W(i)$ by $G(x)$, and the conditional distribution of the $Z(i)$, given $W(i) = w$, by $F(z|w)$. If $\bar{G}(x) = 1 - G(x)$, $\bar{F}(z|w) = 1 - F(z|w)$, and the probability densities $g(x)$ and $f(z|w)$ exist, then

$$(3.46) \qquad \gamma^{(a)}(u_a, u_b) = \frac{g(u_a)}{\bar{G}(u_a)}, \qquad \gamma^{(b)}(u_a, u_b) = \frac{f(u_a|u_b - u_a)}{\bar{F}(u_a|u_b - u_a)}.$$

These are essentially hazard (failure rate) functions.

Thorough study of bivariate Markov interval processes would be of interest. The main properties can be obtained in principle because of the fairly simple Markov structure of the process. In particular if $p(u, v)$ denotes the bivariate probability density function of the backward recurrence times from an arbitrary time, that is, of (U_a, U_b) or $(X^{(a)}(-1), X^{(b)}(-1))$, then

$$(3.47) \qquad \frac{\partial p(u, v)}{\partial u} + \frac{\partial p(u, v)}{\partial v} = -\{\lambda^{(a)}(u, v) + \lambda^{(b)}(u, v)\}p(u, v),$$

and

$$(3.48) \qquad p(0, v) = \int_0^\infty p(u, v)\lambda^{(a)}(u, v)\, du, \qquad p(u, 0) = \int_0^\infty p(u, v)\lambda^{(b)}(u, v)\, dv.$$

From the normalized solution of these equations, some of the simpler properties of the process can be deduced.

More generally, (3.45) may be a useful semiqualitative summary of the local serial properties of a bivariate point process. It does not seem possible to deduce the properties of the marginal processes given just $\lambda^{(a)}(u_a, u_b)$ and $\lambda^{(b)}(u_a, u_b)$, except for very particular processes such as the Markov interval process. For the alternating process with pairwise disjoint dependence, we indicated this difficulty in Section 2.5.

4. A bivariate delayed Poisson process model with Poisson noise

In previous sections, we defined bivariate Poisson processes to be those processes whose marginal processes (processes of type a events and type b events) are Poisson processes. Bivariate Poisson processes with a dependency structure which is completely specified by the second order intensity function arise from semi-Markov (Markov renewal) processes. The complete intensity function is also particularly simple.

Other bivariate Poisson processes can be constructed and in the present section we examine in some detail one such process. Its physical specification is very simple, although the specification of its dependency structure via a complete intensity function is difficult. The details of the model also illustrate the definitions introduced in Section 2.

General considerations on bivariate Poisson processes will be given in the next section.

4.1. *Construction of the model.* Suppose we have an unobservable main or generating Poisson process of rate μ. Events from the main process are delayed (independently) by random amounts Y_a with common distribution $F_a(t)$ and superposed on a "noise" process which is Poisson with rate λ_a. The resulting process is the observed marginal process of type a events. Similarly, the events in the main process are delayed (independently) by random amounts with common distribution $F_b(t)$ and superposed with another independent noise process which is Poisson with rate λ_b. The resulting process is then the marginal process of type b events. It is not observed which type a and which type b events originate from common main events.

In what follows, we assume for simplicity that the two delays associated with each main point are independent and positive random variables. The process has a number of possible interpretations. One is as an immigration death process with immigration consisting of couples "arriving" and type a events being deaths of men and type b events being deaths of women. Other queueing or service situations should be evident. The Poisson noise processes are added for generality and because they lead to interesting complications in inference procedures. In particular applications, it might be known that one or both noise processes are absent.

Various special cases are of interest. Thus, if delays of both types are equal with probability one, we have the Marshall–Olkin process [34] mentioned in Section 2. Without the added noise and if delays on one side (say, the a event side) are zero with probability one, we have the delay process of Section 3.5 or, equivalently, an $M/G/\infty$ queue, where type a events are arrivals and type b events are departures. The noise process on the a event side would correspond to independent balking in the arrival process.

4.2. *Some simple properties of the model.* If we consider the transient process from its initiation, it is well known (for example, Cox and Lewis [14], p. 209) that the processes are nonhomogeneous Poisson processes with rates that are, respectively,

$$\rho_a(t_1) = \lambda_a + \mu F_a(t_1), \tag{4.1}$$

$$\rho_b(t_2) = \lambda_b + \mu F_b(t_2). \tag{4.2}$$

Furthermore, the superposed process is a generalized branching Poisson process whose properties are given by Lewis [30] and Vere-Jones [47]. Thus, at each point in the main or generating process there are, with probability $(\lambda_a + \lambda_b)/$

$(\lambda_a + \lambda_b + \mu)$, no subsidiary events, and, with probability $\mu/(\lambda_a + \lambda_b + \mu)$, two subsidiary events. In the second case, the two subsidiary events are independently displaced from the main or parent event by amounts having distributions $F_a(\cdot)$ and $F_b(\cdot)$.

It is also known (Doob [17]) that as $t \to \infty$, or the origin moves off to the right, the marginal processes become simple stationary Poisson processes of rates

$$\rho_a = \lambda_a + \mu, \qquad \rho_b = \lambda_b + \mu, \tag{4.3}$$

respectively, for any distributions $F_a(u)$ and $F_b(u)$. The superposed process is then a stationary generalized branching Poisson process of rate $\lambda_a + \lambda_b + 2\mu$. The bivariate process is unusual in this respect, since there are very few dependent point processes whose superposition has a simple structure. The properties of the process of event types $\{T(i)\}$ or $\{T.(i)\}$ are, however, by no means simple to obtain, as will be evident when we consider bivariate properties below. Note too that stationarity of the marginal and superposed process does not imply stationarity of the bivariate process. A counterexample will be given later when initial conditions are discussed.

Asymptotic results for the bivariate counting process $\{N^{(a)}(t_1), N^{(b)}(t_2)\}$ can be obtained by a simple generalization of the methods of Lewis [30]. If, for simplicity, $t_1 = t_2 = t$, the intuitive basis of the method is that when t is very large, the proportion of events that are delayed from the generating process until after t goes (in some sense) to zero and the process behaves as though all events are concentrated at their generating event, that is, like the Marshall–Olkin process. Thus,

$$E\{N^{(a)}(t)\} = \text{Var}\,\{N^{(a)}(t)\} = V^{(aa)}(t) \sim \rho_a t, \tag{4.4}$$

$$E\{N^{(b)}(t)\} = \text{Var}\,\{N^{(b)}(t)\} = V^{(bb)}(t) \sim \rho_b t, \tag{4.5}$$

$$\text{Var}\,\{N^{(\cdot)}(t)\} \sim (\rho_a + \rho_b + 2\mu)t, \tag{4.6}$$

and therefore

$$\text{Cov}\,\{N^{(a)}(t), N^{(b)}(t)\} = V^{(ab)}(t) \sim \mu t. \tag{4.7}$$

The asymptotic measures of dispersion $I^{(aa)}$, $I^{(ab)}$, and $I^{(bb)}$ defined in equations (3.22) to (3.24) are therefore 1, $\mu/(\rho_a\rho_b)^{1/2}$, and 1. Result (4.7) will be useful in a statistical analysis of the process. By similar methods (Lewis, [30]), one can establish the joint asymptotic normality of the bivariate counting process.

Another property of the process which is simple to derive is the second order cross intensity function (3.8) or the covariance density function (3.14). In fact because of the Poisson nature of the main process and the independence of the noise processes from the main process, there is a contribution to the covariance density only if the type b event is a delayed event and the event at τ is the same event appearing in the type a event process with its delay of Y_a. Thus using (3.10), we get the cross covariance function

$$\gamma_b^{(a)}(\tau) = \mu f_{Y_a - Y_b}(\tau) = \gamma_a^{(b)}(-\tau) \tag{4.8}$$

if $F_{Y_a}(\cdot)$ and $F_{Y_b}(\cdot)$ are absolutely continuous.

If $F_{Y_a}(\cdot)$ and $F_{Y_b}(\cdot)$ have jumps, there will be delta function components in the cross intensity. In particular, when Y_a and Y_b are zero with probability one, there is a delta function component at zero and the process is marginally regular but not regular.

Result (4.8) will be verified from the more detailed results we derive next for the asynchronously sampled, stationary bivariate process. For this we must first consider detailed results for the transient process.

4.3. *The transient counting process.* The number of events of type a in an interval $(0, t_1]$ following the start of the process is denoted by $N_0^{(a)}(t_1)$ and the number of events of type b in $(0, t_2]$ by $N_0^{(b)}(t_2)$.

Assume first that $t_2 \geqq t_1 > 0$.

Now if a main event occurs at time v in the interval $(0, t_1]$, then it contributes either one or no events to the type a event process in $(0, t_1]$ and one or no events to the type b event process in $(0, t_2]$. This bivariate binomial random variable has generating function

$$\begin{aligned} 1 &+ (1 - z_1)(1 - z_2)F_a(t_1 - v)F_b(t_2 - v) \\ &+ (z_2 - 1)F_b(t_2 - v) + (z_1 - 1)F_a(t_1 - v). \end{aligned} \tag{4.9}$$

Since we will be using the conditional properties of Poisson processes in our derivation, we require the time v to be uniformly distributed over $(0, t_1]$ and the resulting generating function for the contribution of each main point is obtained by integrating (4.9) with respect to v from 0 to t_1 and dividing by t_1. After some manipulation, this gives

$$\begin{aligned} 1 &+ \frac{(1 - z_1 - z_2 + z_1 z_2)}{t_1} \int_0^{t_1} F_a(v)F_b(t_2 - t_1 + v)\,dv \\ &+ \frac{(z_2 - 1)}{t_1} \int_0^{t_1} F_b(t_2 - t_1 + v)\,dv + \frac{(z_1 - 1)}{t_1} \int_0^{t_1} F_a(v)\,dv \\ &= Q(z_1, z_2, t_2, t_1). \end{aligned} \tag{4.10}$$

Now assume that there are k_1 events from the main Poisson process of rate μ in $(0, t_1]$, and k_2 main events in $(t_1, t_2]$. Then using the conditional properties of the Poisson process and the independence of the number of main events in $(0, t_1]$ and $(t_1, t_2]$, we get for the conditional generating function of $N_0^{(a)}(t_1)$ and $N_0^{(b)}(t_2)$

$$\begin{aligned} \exp\{\lambda_a t_1(z_1 - 1) &+ \lambda_b t_2(z_2 - 1)\}\{Q(z_1, z_2, t_2, t_1)\}^{k_1} \\ &\cdot \left\{1 + \frac{(z_2 - 1)}{(t_2 - t_1)} \int_0^{t_2 - t_1} F_b(u)\,du\right\}^{k_2}. \end{aligned} \tag{4.11}$$

Removing the conditioning on the independently Poisson distributed number of events k_1 and k_2, we have for the logarithm of the joint generating function of $N_0^{(a)}(t_1)$ and $N_0^{(b)}(t_2)$

$$(4.12)\quad \begin{aligned}\psi_0(z_1, z_2; t_1, t_2) &= \log \phi(z_1, z_2; t_1, t_2)\\ &= \rho_a t_1(z_1 - 1) + \rho_b t_2(z_2 - 1)\\ &\quad - \mu(z_2 - 1)\int_0^{t_2} R_b(u)\,du - \mu(z_1 - 1)\int_0^{t_1} R_a(u)\,du\\ &\quad + \mu(1 - z_1 - z_2 + z_1 z_2)\int_0^{t_1} F_a(u)F_b(t_2 - t_1 + u)\,du.\end{aligned}$$

where $\rho_b = \lambda_a + \mu$, $\rho_b = \lambda_b + \mu$, $R_b(u) = 1 - F_b(u)$, and we still have $t_2 \geqq t_1$.

A similar derivation gives the result for $t_1 \geqq t_2$ and we can write for the general case

$$(4.13)\quad \begin{aligned}\psi_0(z_1, z_2; t_1, t_2) &= \rho_a t_1(z_1 - 1) + \rho_b t_2(z_2 - 1)\\ &\quad - \mu(z_2 - 1)\int_0^{t_2} R_b(v)\,dv - \mu(z_1 - 1)\int_0^{t_1} R_a(v)\,dv\\ &\quad + \mu(1 - z_1)(1 - z_2)\int_0^{\min(t_1, t_2)} F_a(t_1 - v)F_b(t_2 - v)\,dv.\end{aligned}$$

The expected numbers of events (4.1) and (4.2) in the marginal processes also come out of (4.13), as do the properties of the transient, generalized branching Poisson process obtained by superposing events of type a and type b. Moreover, when the random variables Y_a and Y_b have fixed values, $\psi_0(0, 0; t_1, t_2)$ gives the logarithm of the survivor function of the bivariate exponential distribution of Marshall and Olkin [33], [34].

Note that $\psi_0(z_1, z_2; t_1, t_2)$ is the generating function of a bivariate Poisson variate, that is, a bivariate distribution with Poisson marginals. It is, in fact, the bivariate form of the multivariate distribution which Dwass and Teicher [18] showed to be the only infinitely divisible Poisson distribution:

$$(4.14)\quad \phi(\mathbf{z}) = \exp\left\{\sum_{i=1}^{n} a_i(z_i - 1) + \sum_{i<j} a_{ij}(z_i - 1)(z_j - 1) + \cdots + a_{1,2,\cdots,n}\prod_{i=1}^{n}(z_i - 1)\right\}.$$

However, since the coefficients in (4.13) depend on t_1 and t_2, the joint distribution of events of type a in two disjoint intervals and events of type b in another two disjoint intervals will not have the form (4.14). This is clearly only true for the highly degenerate Marshall–Olkin process of Section 2.

4.4. *The stationary asynchronous counting process.* To derive the properties of the generating function of counts in the stationary limiting process, or equivalently the asynchronously sampled stationary process, we consider first the number of events of type a in $(t, t_1]$ and of type b in $(t, t_2]$. Because of the independent interval properties of the main and noise Poisson processes of rates λ_a, λ_b, and μ, respectively, this is made up independently from noise and main events occurring in $(t, \max(t_1, t_2)]$, whose generating function is given by (4.13), and by main events occurring in $(0, t]$ and delayed into $(t, t_1]$ or $(t, t_2]$.

Consider, therefore, the generating function of the latter type of events. A main event at v in $(0, t]$ generates either one or no type a events in $(t, t_1]$ and either one or no events of type b in $(t, t_2]$. The generating function of this bivariate binomial random variable is

$$1 + (z_1 - 1)p_a + (z_2 - 1)p_b + (z_1 - 1)(z_2 - 1)p_a p_b, \tag{4.15}$$

where

$$p_a = p_a(1; t_1; t; v) = R_a(t - v) - R_a(t + t_1 - v) \tag{4.16}$$

and

$$p_b = p_b(1; t_2; t; v) = R_b(t - v) - R_b(t + t_2 - v). \tag{4.17}$$

If the start time v is assumed to be uniformly distributed over $(0, t]$, then the generating function becomes

$$1 + (z_1 - 1)\frac{\bar{p}_a}{t} + (z_2 - 1)\frac{\bar{p}_b}{t} + (z_1 - 1)(z_2 - 1)\frac{\overline{p_a p_b}}{t}, \tag{4.18}$$

where

$$\bar{p}_a = \int_0^t \{R_a(v) - R_a(v + t_1)\}\, dv, \tag{4.19}$$

$$\bar{p}_b = \int_0^t \{R_b(v) - R_b(v + t_2)\}\, dv, \tag{4.20}$$

$$\overline{p_a p_b} = \int_0^t \{R_a(v) - R_a(v + t_1)\}\{R_b(v) - R_b(v + t_2)\}\, dv. \tag{4.21}$$

It follows from (4.19) and (4.20) that if t_1 and t_2 are finite, we have, even if $E(Y_a)$ and $E(Y_b)$ are infinite,

$$\lim_{t\to\infty} \bar{p}_a = \int_0^{t_1} R_a(v)\, dv, \qquad \lim_{t\to\infty} \bar{p}_b = \int_0^{t_2} R_b(v)\, dv; \tag{4.22}$$

and since $\overline{p_a p_b} \leqq \bar{p}_b$ for all t, t_1, t_2, we have that

$$\lim_{t\to\infty} \overline{p_a p_b} = \int_0^{\infty} \{R_a(v) - R_a(v + t_1)\}\{R_b(v) - R_b(v + t_2)\}\, dv \tag{4.23}$$

exists for finite t_1 and t_2.

The results (4.19) through (4.23) are used, as in the derivation of (4.12), to obtain the cumulant generating function of the contribution of delayed events of type a and type b to $(t, t_1]$ and $(t, t_2]$ when $t \to \infty$. This is

$$(4.24)\quad \psi^+(z_1, z_2; t_1, t_2) = \mu(z_1 - 1)\int_0^{t_1} R_a(v)\,dv + \mu(z_2 - 1)\int_0^{t_2} R_b(v)\,dv + \mu(z_1 - 1)(z_2 - 1)\int_0^{\infty} \{R_a(v) - R_a(v + t_1)\}\{R_b(v) - R_b(v + t_2)\}\,dv.$$

Combined with (4.13), we have for the stationary bivariate process the result

$$(4.25)\quad \psi(z_1, z_2; t_1, t_2) = \rho_a t_1(z_1 - 1) + \rho_b t_2(z_2 - 1) + \mu(z_1 - 1)(z_2 - 1) \cdot\left[\int_0^{\min(t_1,t_2)} F_a(t_1 - v)F_b(t_2 - v)\,dv + \int_0^{\infty} \{R_a(v) - R_a(v + t_1)\}\{R_b(v) - R_b(v + t_2)\}\,dv\right].$$

Note that this is the cumulant generating function of a bivariate Poisson distribution and that the covariance time function (3.17) is the term in (4.25) multiplying $(z_1 - 1)(z_2 - 1)$;

$$(4.26)\quad V^{(ab)}(t_1, t_2) = \mu\int_0^{\min(t_1,t_2)} R_a(t - v_1)R_b(t_2 - v)\,dv + \mu\int_0^{\infty} \{R_a(v) - R_a(v + t_1)\}\{R_b(v) - R_b(v + t_2)\}\,dv.$$

Differentiation of this expression with respect to t_1 and t_2 gives, after some manipulation, the covariance density (4.8), as predicted by the general formula (3.17).

Thus, if the densities associated with $R_a(\cdot)$ and $R_b(\cdot)$ exist, we can express (4.25) as

$$(4.27)\quad \psi(z_1, z_2; t_1, t_2) = \rho_a t_1(z_1 - 1) + \rho_b t_2(z_2 - 1) + (z_1 - 1)(z_2 - 1)\mu\int_0^{t_1}\int_0^{t_2} f_{Y_a - Y_b}(u - v)\,du\,dv.$$

There are a number of alternative forms for and derivations of this distribution.

The behavior of $V^{(ab)}(t_1, t_2)$, although it is clearly a monotone nondecreasing function of both t_1 and t_2, is complex and will not be studied further here. In (4.7), we saw that along the line $t_1 = t_2 = t$ it is asymptotically μt.

We have also not established the complete stationarity of the limiting bivariate process; this follows from the fact that the delay depends only on the distance from the Poisson generating event, and can be established rigorously using bivariate characteristic functionals.

The complete intensity functions for this process (3.1) cannot be written down and although the second order intensity function is simple it does not specify the dependency structure of the process completely, as it does for the bivariate semi-Markov process. Note too that the cross covariance function (4.8) is always positive, so that there is in effect no inhibition of type a events by b events. In fact from the construction of the process, it is clear that just the opposite effect takes place. We examine the dependency structure of the delay process in more detail here by looking at the joint asynchronous forward recurrence time distribution. This distribution is of some interest in itself.

4.5. *The joint asynchronous forward recurrence times.* In the asynchronous process of the previous section, the time to the kth event of type a, $S^{(a)}(k)$, has a gamma distribution with parameter k and $S^{(b)}(h)$ has a gamma distribution with parameter h. Thus, the joint distribution of these random variables is a bivariate gamma distribution of mixed marginal parameters k and h which is obtained from the generating function (4.25) via the fundamental relationship (2.5). We consider only the joint forward recurrence times $S^{(a)}(1) = X^{(a)}(1)$ and $S^{(b)}(1) = X^{(b)}(1)$ which have a bivariate exponential distribution:

$$
\begin{aligned}
(4.28) \qquad R_{ab}(t_1, t_2) &= Pr\{X^{(a)}(1) > t_1, X^{(b)}(1) > t_2\} \\
&= \exp \{\psi(0, 0: t_1, t_2)\} \\
&= \exp \{-\rho_a t_1 - \rho_b t_2 + V^{(ab)}(t_1, t_2)\}.
\end{aligned}
$$

Clearly, this bivariate exponential distribution reduces to the distribution discussed by Marshall and Olkin [33] in the degenerate case when there are no delays (or fixed delays [34]). For no delays

$$
(4.29) \qquad R_{ab}(t_1, t_2) = -\rho_a t_1 - \rho_b t_2 + \mu \min (t_1, t_2).
$$

The bivariate exponential distribution (4.27) is not the same as the infinitely divisible exponential distribution discussed by Gaver [20], Moran and Vere-Jones [38], and others. Whenever the delay distributions $R_a(\cdot)$ and $R_b(\cdot)$ have jumps, $R_{ab}(t_1, t_2)$ will have singularities.

For the correlation coefficient, we have

$$
(4.30) \qquad \rho_a \rho_b \operatorname{Corr} \{X^{(a)}(1), X^{(b)}(1)\} = \int_0^\infty \int_0^\infty R_{ab}(t_1, t_2)\, dt_1\, dt_2 - \frac{1}{\rho_a \rho_b}.
$$

It is not possible to integrate this expression explicitly except in special cases. However, since $V^{(ab)}(t_1, t_2) \geqq 0$, we clearly have that the correlation coefficient is greater than zero.

For the special case (4.29) the correlation (4.30) is $1/\{(1 + (\lambda_a + \lambda_b)/\mu)\}$.

We do not pursue further here the properties of the process obtainable from the joint distribution of counts (4.25) of the synchronous counting process $\{N^{(a)}(t_1), N^{(b)}(t_2)\}$. However, it is useful to summarize what useful properties can be derived for this or any other bivariate process such as those given in the next section, from this bivariate distribution.

(i) The marginal generating functions ($z_1 = 0$ or $z_2 = 0$) give the correlation structure of the marginal interval process through equation (3.43). This is trivial for the delay process.

(ii) The generating function with $z_1 = z_2$ gives the correlation structure of the intervals in the superposed process through (3.44). For the delay process this is the interval correlation structure of a clustering (branching) Poisson process.

(iii) The covariance time surface and cross intensity and marginal intensity functions can be obtained. Again for the delay process this is trivial.

(iv) The joint distribution of the asynchronous forward recurrence times $\{X^{(a)}(1), X^{(b)}(1)\}$ can be calculated. Other functions of interest are the smaller and larger of $X^{(a)}(1)$ and $X^{(b)}(1)$, and the conditional distributions and expectations, for example, $E\{X^{(a)}(1)|X^{(b)}(1) = x\}$. The latter is difficult to obtain for the delay process, the regression being highly nonlinear.

(v) In principle, one can obtain not only the distributions of the smaller and larger of $X^{(a)}(1)$ and $X^{(b)}(1)$, but also the order type (jointly or marginally) since

$$Pr\{T(1) = a\} = Pr\{X^{(a)}(1) < X^{(b)}(1)\}. \tag{4.31}$$

(vi) It is not possible to obtain the complete distributions of types, for example, $Pr\{T(1) = a; T(2) = a\}$ from the bivariate distribution of asynchronous counts, since these counts are related to the sums of intervals by (2.5). For this information, we need more complete probability relationships, that is, for the pooled process $\{X^{(\cdot)}(1), T(1); X^{(\cdot)}(2), \cdots\}$. Note too that $\{T(i)\}$ is not a stationary binary sequence.

It is possible to obtain distributions of semisynchronous counting processes for the delay processes although we do not do this here. One reason for doing this is to obtain information on the distribution of the stationary sequences $T_{.}(i)$. Thus,

$$Pr\{T_{.}(0) = a, T_{.}(1) = b\} = \frac{\rho_a}{\rho_a + \rho_b} Pr\{X_a^{(b)}(1) < X_a^{(a)}(1)\}, \tag{4.32}$$

and so forth, from which the correlation coefficient of lag one is obtained. It is not possible to carry the argument to lags of greater than one solely with joint distributions of sums of semisynchronous intervals.

4.6. *Stationary initial conditions.* We discuss here briefly the problem of obtaining stationary initial conditions for the delay process, since this has some bearing on the problems considered in this paper.

Note that for the marginal processes in the delayed Poisson process the numbers of events generated before t which are delayed beyond t have, if $E(Y_a) < \infty$ and $E(Y_b) < \infty$, Poisson distributions with parameters $\mu E(Y_a)$ and $\mu E(Y_b)$, respectively, when $t \to \infty$. Denote these random variables by $Z^{(a)}$ and $Z^{(b)}$. If the transient process of Section 4.3 is started with an additional number

Z_a of type a events which occur independently at distances $\bar{Y}_a(1), \cdots, \bar{Y}_a(Z_a)$ from the origin, where

$$Pr\{\bar{Y}_a(i) \leqq t\} = \int_0^t \frac{R_a(u)\, du}{\{E(Y_a)\}}, \tag{4.33}$$

and with an additional number Z_b of type b events which occur independently at distances $\bar{Y}_b(1), \cdots, \bar{Y}_b(z_b)$ from the origin, where the common distribution of the $\bar{Y}_b(j)$ is directly analogous to that of the $\bar{Y}_a(i)$, then the marginal processes are stationary Poisson processes (Lewis [30]). However, the bivariate process is not stationary. This can be verified, for instance, by obtaining the covariance density from the resulting generating function and noting that it depends on t_1 and t_2 separately, and not just on their difference.

In obtaining stationary initial conditions, the joint distribution of Z_a and Z_b is needed. Without going into the details of the limiting process, the generating function for these random variables is clearly (4.24) when $t_1 \to \infty$ and $t_2 \to \infty$. Thus,

$$\begin{aligned} \psi_{Z_a,Z_b}(z_1, z_2) &= (z_1 - 1)\mu E(Y_a) + (z_2 - 1)\mu E(Y_b) + (z_1 - 1)(z_2 - 1)\mu \\ &\quad \cdot E\{\min(Y_a, Y_b)\}, \end{aligned} \tag{4.34}$$

where $E\{\min(Y_a, Y_b)\} = \int_0^\infty R_a(v)R_b(v)\, dv$. This is the generating function (4.14) of a bivariate Poisson distribution.

Further details of this model, including the complete stationary initial conditions, will be given in another paper.

5. Some other special processes

We discuss here briefly several important models for bivariate point processes. The specification of the models is through the structure of intervals and is based on direct physical considerations, unlike, say, the bivariate Markov process with its specification of degree of dependence through the complete intensity functions. At the end of the section we consider the general problem of specifying the form of bivariate Poisson processes.

5.1. *Single process subject to bivariate delays.* The bivariate delayed Poisson process of the previous section can be generalized in several ways. First, the delays Y_a and Y_b might be correlated since, for instance, in the example of a man and wife in a bivariate immigration death process, their residual lifetimes would be correlated. Again Y_a and Y_b may take both positive and negative values. The stationary analysis of the previous section goes through essentially unchanged although specifying initial conditions is difficult. The covariance function (4.8) is the same except that $f_{Y_a - Y_b}(t)$ is, of course, no longer a simple convolution.

Another extension is to consider main processes which are, say, regular stationary point processes with rate μ and intensity function $h_\mu(t)$. Then the cross intensity function for the bivariate process, $h_a^{(b)}(t)$, becomes

$$(5.1)\qquad \frac{\lambda_a}{\lambda_a + \mu}\rho_b + \frac{\mu}{\lambda_a + \mu}\left\{\lambda_b + f_{Y_a - Y_b}(t) + \int_{-\infty}^{\infty} h_\mu(u) f_{Y_a - Y_b}(t - u)\, du\right\},$$

with a similar expression for $h_b^{(a)}(t)$. These should be compared with (3.26). Except when the main process is a renewal process, explicit results beyond the intensity function are difficult to obtain. For the renewal case an integral equation can be written down, as also for branching renewal processes (Lewis, [32]); from the integral equation higher moments of the bivariate counting process can be derived.

5.2. *Bivariate point process subject to delays.* Instead of having a univariate point process in which each point (say the ith) is delayed by two different amounts $Y_a(i)$ and $Y_b(i)$ to form the bivariate process, one can have a main bivariate point process in which the ith type a event is delayed by $Y_a(i)$ and the jth type b event is delayed by $Y_b(j)$, thus forming a new bivariate point process. This does not reduce to the bivariate delay process of Section 5.1 although it is conceptually similar.

The simplest illustration is where there is error (jitter) in recording the positions of the points. Usually the errors are taken to be independently distributed, although Y_a and Y_b may have different distributions. Another situation is an immigration death process with two different types of immigrants.

If the main process has cross intensities $\bar{h}_a^{(b)}(t)$ and $\bar{h}_b^{(a)}(t)$, then the delayed bivariate process (with no added Poisson noise) has cross intensity

$$(5.2)\qquad h_a^{(b)}(t) = \int_{-\infty}^{\infty} \bar{h}_a^{(b)}(t - v) f_{Y_a - Y_b}(v)\, dv.$$

It will not be possible from data to separate properties of the jitter process from those of the underlying main process, unless strong special assumptions are made.

An interesting situation occurs when the main process is a semi-Markov process with marginal processes which are Poisson processes as, for example, in Section 2. Then the delayed bivariate point process is, in equilibrium, a bivariate Poisson process.

5.3. *Clustering processes.* Univariate clustering processes (Neyman and Scott [39], Vere-Jones [47], and Lewis [30]) are important. Each main event generates one subsidiary sequence of events and the subsidiary sequences have a finite number of points with probability one. The subsidiary processes are independent of one another but can be of quite general structure. When the subsidiary processes are finite renewal processes, the clustering process is known as a Bartlett–Lewis process; when the events are generated by independent delays from the initiating main event, the process is known as a Neyman–Scott cluster process.

The bivariate delay process and delayed bivariate process described in the previous two subsections are special cases of bivariate cluster processes and clearly both types of main process are possible for these cluster processes. As an example of a bivariate main process generating two different types of subsidiary process, Lewis [29] considered computer failure patterns and discussed the possibility of two types of subsidiary sequences, one generated by permanent component failures and the other by intermittent component failures.

There are many possibilities that will not be discussed here. Some general points of interest are, however, the following.

(i) When the main process is a univariate Poisson process, producing a bivariate clustering Poisson process, bivariate superposition of such processes again produces a bivariate clustering Poisson process. The process is thus infinitely divisible.

(ii) Both the marginal processes and the superposed processes are (generalized) cluster processes. Thus, we can use known results for these processes and expressions such as (3.39) and (3.14) to find variance time curves and cross intensities for the bivariate process. When the main process is a semi-Markov process, the marginal processes are clustering (or branching) renewal processes (Lewis, [32]).

(iii) The analysis in Section 4 can be used for these processes when the main process is a univariate Poisson process. Bivariate characteristic functionals are probably also useful.

5.4. *Selective inhibition.* A simple, realistic and analytically interesting model arises in neurophysiological contexts. We have two series of events, the first called the inhibitory series of events and the second the excitatory series of events, occurring on a common time scale. Each event in the inhibitory series blocks only the next excitatory event (and blocks it only if no following inhibitory event occurs before the excitatory event). This is the simplest of many possibilities.

Although only the sequence of noninhibited excitatory events (the responses) is usually studied, Lawrance has pointed out that there are a number of bivariate processes generated by this mechanism, in particular the inhibitory events and the responses [24], [25]. These may constitute the input and output to a neuron, and are the only pair we consider here. In particular, we take the excitatory process to be a Poisson process with rate ρ_a and the inhibiting process to be a renewal process with interevent probability distribution function $F_b(x)$. The response process has dependent intervals unless the inhibitory process also is a Poisson process.

When the excitatory process is a renewal process with interevent probability distribution function $F_a(x)$ and the inhibitory process is Poisson with rate ρ_b, the process of responses is a renewal process. This follows because the original renewal process is in effect being thinned at a rate depending only on the time since the last recorded response and such an operation preserves the renewal property. This bivariate renewal process is not a semi-Markov process, as can be seen by attempting to write down the complete intensity functions (3.4) and (3.5). The complete intensity functions become simple only for the trivariate process of inhibitory events, responses, and nonresponses.

Coleman and Gastwirth [10] have shown that it is possible to pick $F_a(x)$ so that the responses also form a Poisson process. The covariance density of this bivariate Poisson process can be obtained: it is always negative (personal communication, T. K. M. Wisniewski).

Other forms of selective inhibition can be postulated: some have been discussed by Coleman and Gastwirth [10]. Another possibility is the simultaneous inhibition, as above, of two excitatory processes by a single, unobservable inhibitory process. When the inhibitory process is Poisson and the excitatory processes are renewal processes, the two response processes are a bivariate renewal process.

There are, of course, many other neurophysiological models, generally more complicated than the selective inhibition models and many times involving the doubly stochastic mechanism discussed in Section 3.5. An interesting example is given by Walloe, Jansen, and Nygaard [48].

5.5. *General remarks on bivariate Poisson processes.* In this and previous sections, we have encountered several examples of bivariate Poisson processes, defined as bivariate point processes in which the marginal processes are Poisson processes.

(i) The degenerate Poisson process of Marshall and Olkin was discussed in Section 2.

(ii) The process in (i) is a special case of a broad family of bivariate Poisson processes generated by bivariate delays on univariate Poisson processes. Several other examples arise in considering delays on bivariate Poisson processes.

(iii) Semi-Markov processes have renewal marginals and a broad class of bivariate Poisson processes is obtained by choosing the marginal processes to be Poisson processes. Delays added to these particular semi-Markov processes again produce bivariate Poisson processes.

(iv) A rather special case arises when a Poisson process inhibits a renewal process.

Another example is mentioned because it illustrates the problem considered in Section 3.8 of specifying dependency structure in terms of the bivariate, discrete time sequence of marginal intervals. Thus, we can start the process and require that the intervals in the marginals with the same serial index be bivariate exponentials. Any bivariate exponential distribution may be used, such as (4.28) or those of Gaver [20], Plackett [42], Freund [19], and Griffiths [21]. The interval structure is stationary, as is the counting process of the marginals, which are Poisson processes. The bivariate counting process is, however, not stationary. It is not clear whether one gets the counting process to be stationary, as defined in Section 2, when moving away from the origin, but since the time lag between the dependent intervals increases indefinitely as $n \to \infty$ the process is degenerate and tends to almost independent Poisson processes a long time from the origin.

No general structure is known for bivariate Poisson processes. There follow some general comments and some open questions.

(i) The bivariate Poisson process as defined is infinitely divisible in that bivariate superposition produces bivariate Poisson processes. However, of the above models of bivariate Poisson processes, only the bivariate delayed Poisson process keeps the same dependency structure under bivariate superposition.

(ii) Does unlimited bivariate superposition produce two independent Poisson processes? The answer is, generally, yes (Cinlar [8]).

(iii) It can be shown that successive independent delays on bivariate Poisson processes (and most bivariate processes) produces in the limit a process of independent Poisson processes. This can be seen from (5.2) and (5.3), but needs bivariate characteristic functionals for a complete proof.

(iv) The numbers of events of the two types in an interval $(0, t]$ in a bivariate Poisson process have a bivariate Poisson distribution. Some general properties of such distributions are known (Dwass and Teicher [18]); the bivariate Poisson distribution (4.33) is the only infinitely divisible bivariate Poisson distribution. An open question of interest in investigating bivariate Poisson processes is whether, when Z_1, Z_2, and $Z_1 + Z_2$ have marginally Poisson distributions of means μ_1, μ_2, and $\mu_1 + \mu_2$, Z_1 and Z_2 are independent. If this is so, a bivariate Poisson process in which the superposed marginal process is a Poisson process must have the events of two types independent.

(v) The broad class of stationary bivariate Poisson processes arising from delay mechanisms have positive cross covariance densities, that is, no "inhibitory effect." For the semi-Markov process with Poisson marginals, it is an open question as to whether cross covariance densities which take on negative values exist. In particular, for the alternating renewal process with identical gamma distributions of index one for up and down times, the cross covariance is strictly positive. The only model which is known to produce a bivariate Poisson process with strictly negative covariance density is the Poisson inhibited renewal process described earlier in this section.

6. Statistical analysis

6.1. *General discussion.* We now consider in outline some of the statistical problems that arise in analyzing data from a bivariate point process. If a particular type of model is suggested by physical considerations, it will be required to estimate the parameters and test goodness of fit. In some applications, a fairly simple test of dependence between events of different types will be the primary requirement. In yet other cases, the estimation of such functions as the covariance densities will be required to give a general indication of the nature of the process, possibly leading to the suggestion of a more specific model. In all cases, the detection and elimination of nonstationarity may be required.

There is one important general distinction to be drawn, parallel to that between correlation and regression in the analysis of quantitative data. It may be that both types of event are to be treated symmetrically, and that in particular the stochastic character of both types needs analysis. This is broadly the attitude implicit in the previous sections. Alternatively one type of event, say b, may be causally dependent on previous a events, or it may be required to predict events of type b given information about the previous occurrences of events of both types. Then it will be sensible to examine the occurrence of the b's conditionally on the observed sequence of a's and not to consider the stochastic mechanism generating the a's; this is analogous to the treatment of the independent variable in regression analysis as "fixed." Note in particular that the pattern of the a's might be very nonstationary and yet if the mechanism generating the b's is stable, simple "stationary" analyses may be available.

In the rest of this section, we sketch a few of the statistical ideas required in analyzing this sort of data.

6.2. *Likelihood analyses.* If a particular probability model is indicated as the basis of the analysis when the model is specified except for unknown parameters, in principle it will be a good thing to obtain the likelihood of the data from which exactly or asymptotically optimum procedures of analysis can be derived, for example, by the method of maximum likelihood; of course, this presupposes that the usual theorems of maximum likelihood theory can be extended to cover such applications. Unfortunately, even for univariate point processes, there are relatively few models for which the likelihood can be obtained in a useful form. Thus, one is often driven to rather *ad hoc* procedures.

Here we note a few very particular processes for which the likelihood can be calculated.

In a semi-Markov process, the likelihood can be obtained as a product of a factor associated with the two state Markov chain and factors associated with the four distributions of duration; if sampling is for a fixed time there will be one "censored" duration. Moore and Pyke [36] have examined this in detail with particular reference to the asymptotic distributions obtained when sampling is for a fixed time, so that the numbers of intervals of various types are random variables.

A rather similar analysis can be applied to the bivariate Markov process of intervals of Section 3.8, although a more complex notation is necessary. Let

$$L^{(a)}(x; v, w) = \exp\left\{-\int_0^x \lambda^{(a)}(z + v, z + w)\, dz\right\}, \tag{6.1}$$

$$L^{(b)}(x; v, w) = \exp\left\{-\int_0^x \lambda^{(b)}(z + v, z + w)\, dz\right\}. \tag{6.2}$$

We can summarize the observations as a sequence of intervals between successive events in the pooled process, where the intervals are of type aa, ab, ba, or bb. We characterize each interval by its length x and by the backward recurrence time at the start of the interval measured to the event of opposite type. Denote this by v if measured to a type a event and by w if measured to a type b event.

Then the contribution to the likelihood of the length of the interval and the type of the event at the end of the interval is

(6.3) $\lambda^{(a)}(x, w + x)\, L^{(a)}(x; 0, w)\, L^{(b)}(x; 0, w)$ for an aa interval,

(6.4) $\lambda^{(b)}(x, w + x)\, L^{(a)}(x; 0, w)\, L^{(b)}(x; 0, w)$ for an ab interval,

(6.5) $\lambda^{(a)}(v + x, x)\, L^{(a)}(x; v, 0)\, L^{(b)}(x; v, 0)$ for a ba interval,

(6.6) $\lambda^{(b)}(v + x, x)\, L^{(a)}(x; v, 0)\, L^{(b)}(x; v, 0)$ for a bb interval.

Thus, once the intensities are specified parametrically the likelihood can be written down and, for example, maximized numerically.

Now the above discussion is for the "correlational" approach in which the two types of event are treated symmetrically. If, however, we treat the events of type b as the dependent process and argue conditionally on the observed sequence of events of type a, the analysis is simplified, in effect by replacing $\lambda^{(a)}(\cdot, \cdot)$ and $L^{(a)}(\cdot, \cdot, \cdot)$ by unity.

A particular case of interest is when the intensities are linear functions of their arguments. This, of course, precludes having the semi-Markov process as a special case.

A further example of when a likelihood analysis is feasible is provided by the bivariate sinusoidal Poisson process of (3.30) with R_a, R_b, and Φ regarded as unknown parameters. An analysis in terms of exponential family likelihoods is obtained by taking (3.20) to refer to the log intensity; for the univariate analysis see Lewis [31].

In the bivariate case, we reparametrize and have

$$\lambda_a(t) = \frac{\rho_a \exp\{R_a \cos(\omega_0 t + \theta + \Phi)\}}{I_0(R_a)} \tag{6.7}$$

and

$$\lambda_b(t) = \frac{\rho_b \exp\{R_b \cos(\omega_0 t + \Phi)\}}{I_0(R_b)}, \tag{6.8}$$

where $I_0(R_b)$ is a zero order modified Bessel function of the first kind. It is convenient to assume that observation on both processes is for a common fixed period t_0, where $\omega_0 t_0$ is an integral multiple of 2π, say $2\pi p$. Then $\int_0^{t_0} \lambda_a(u)\, du = \rho_a t_0$ and $\int_0^{t_0} \lambda_b(u)\, du = \rho_b t_0$.

If $n^{(a)}$ type a events are observed in $(0, t_0]$ at times $t_1^{(a)}, \cdots, t_{n^{(a)}}^{(a)}$ and $n^{(b)}$ type b events at times $t_1^{(b)}, \cdots, t_{n^{(b)}}^{(b)}$, then, using the likelihood for the nonhomogeneous bivariate Poisson process

$$\prod_{i=1}^{n^{(a)}} \lambda_a(t_i^{(a)}) \prod_{j=1}^{n^{(b)}} \lambda_b(t_j^{(b)}) \exp\{-\rho_a t_0 - \rho_b t_0\}, \tag{6.9}$$

we find that the set of sufficient statistics for $\{\rho_a, \rho_b, R_a \cos(\theta + \Phi), R_a \sin(\theta + \Phi), R_b \cos \Phi, R_b \sin \Phi\}$ are $\{n^{(a)}, n^{(b)}, \mathscr{A}_a(\omega_0), \mathscr{B}_a(\omega_0), \mathscr{A}_b(\omega_0), \mathscr{B}_b(\omega_0)\}$, where

$$(6.10) \qquad \mathscr{A}_a(\omega_0) = \sum_{i=1}^{n^{(a)}} \cos(\omega_0 t_i^{(a)}), \qquad \mathscr{B}_a(\omega_0) = \sum_{j=1}^{n^{(b)}} \sin(\omega_0 t_j^{(a)}),$$

with similar definitions for $\mathscr{A}_b(\omega_0)$ and $\mathscr{B}_b(\omega_0)$.

Typically, if $R_a = R_b = R$, maximum likelihood estimates of R and tests of $R = 0$ are based on monotone functions of $\mathscr{A}_a(\omega_0)$, $\mathscr{B}_a(\omega_0)$, $\mathscr{A}_b(\omega_0)$, and $\mathscr{B}_b(\omega_0)$. The estimation and testing procedures are formally equivalent to tests for directionality on a circle from two independent samples when the direction vector has a von Mises distribution (Watson and Williams, [49]).

Other trend analyses can be carried out with a similar type of likelihood analysis if the model is a nonhomogeneous bivariate Poisson process.

For most other special models, including quite simple ones such as the delayed Poisson process of Section 4, it does not seem possible to obtain the likelihood in usable form; it would be helpful to have ways of obtaining useful pseudo-likelihoods for such processes.

For testing goodness of fit, it may sometimes be possible to imbed the model under test in some richer family; for instance, agreement with a parametric semi-Markov model could be tested by fitting some more general bivariate Markov interval process and comparing the maximum likelihoods achieved. More usually, however, it will be a case of finding relatively *ad hoc* test statistics to examine various aspects of the model.

In situations in which the model of independent renewal processes or the semi-Markov model may be relevant, the following procedures are likely to be useful. To test consistency with an independent renewal process model, we may:

(a) examine for possible nonstationarity,

(b) test the marginal processes for consistency with a univariate renewal model (Cox and Lewis [14], Chapter 6),

(c) test for dependence using the estimates of the cross intensity given in the next section, or test that the event types do not have the first order Markov property.

If dependence is present, it may be natural to see whether the data are consistent with a semi-Markov process. (Note, however, that the family of independent renewal models is not contained in the family of semi-Markov models.) To test for the adequacy of an assumed parametric semi-Markov model, we may, for example, proceed as follows:

(a) examine for possible nonstationarity,

(b) test the sequence of event types for the first order Markov property (Billingsley [5]),

(c) examine the distributional form of the four separate types of interval,

(d) examine the dependence of intervals on the preceding interval and the preceding event type.

6.3. *Estimation of intensities and associated functions.* If a likelihood based analysis is not feasible, we must use the more empirical approach of choosing aspects of the process thought to be particularly indicative of its structure and estimating these aspects from the data. In this way we may be able to obtain estimates of unknown parameters and tests of the adequacy of a proposed model.

In the following discussion, we assume that the process is stationary. With extensive data, it will be wise first to analyze the data in separate sections, pooling the results only if the sections are reasonably consistent.

The main aspects of the process likely to be useful as a basis for such procedures are the frequency distributions of intervals of various kinds, the second order functions of Section 3.3 through 3.7 and, the bivariate interval properties, in particular the serial intensity functions (3.42). As stressed in Section 3, it will often happen that one or other of the above aspects is directly related to the underlying mechanism of the process and hence is suitable for statistical analysis.

Estimation of the univariate second order functions does not need special discussion here. We therefore merely comment briefly on the estimation of the serial intensity functions and the cross properties; for the latter the procedures closely parallel the corresponding univariate estimation procedures.

6.3.1. *Cross intensity function.* To obtain a smoothed estimate of the cross intensity function $h_a^{(b)}(t)$, choose a grouping interval Δ and count the total number of times a type b event occurs a distance between t and $t + \Delta$ to the right of a type a event; let the random variable corresponding to this number be $R_a^{(b)}(t, t + \Delta)$. In practice, we form a histogram from all possible intervals between events of type a and events of type b. We now follow closely the argument of Cox and Lewis ([14], p. 122) writing, for observations over $(0, t_0)$,

$$(6.11)\quad R_a^{(b)}(t, t + \Delta) = \left\{\int_{u=0}^{t_0-t-\Delta} \int_{x=t}^{t+\Delta} + \int_{u=t_0-t-\Delta}^{t_0-t} \int_{x=t}^{t_0-u}\right\} dN^{(a)}(u)\, dN^{(b)}(u + x).$$

Now for a stationary process

$$(6.12)\quad E\{dN^{(a)}(u)\, dN^{(b)}(u + x)\} = \rho_a h_a^{(b)}(x)\, du\, dx,$$

and a direct calculation, plus the assumption that $h_a^{(b)}(x)$ varies little over $(t, t + \Delta)$, gives

$$(6.13)\quad E\{R_a^{(b)}(t, t + \Delta)\} = (t_0 - t - \tfrac{1}{2}\Delta)\rho_a \int_t^{t+\Delta} h_a^{(b)}(x)\, dx,$$

thus leading to a nearly unbiased estimate of the integral of the cross intensity over $(t, t + \Delta)$.

If the type b events are distributed in a Poisson process independently of the type a events, we can find the exact moments of $R_a^{(b)}(t, t + \Delta)$, by arguing conditionally both on the number of type b events and on the whole observed process of type a events (see Section 6.4). To a first approximation, $R_a^{(b)}(t, t + \Delta)$

has (conditionally) a Poisson distribution of mean $n^{(a)}n^{(b)}\Delta/t_0$ provided that Δ is small and, in particular, that few type a events occur within Δ of one another. This provides the basis for a test of the strong null hypothesis that the type b events follow an independent Poisson process; it would be interesting to study the extent to which the test is distorted if the type b events are distributed independently of the type a events, although not in a Poisson process.

6.3.2. *Cross spectrum.* Estimation of the cross spectrum is based on the cross periodogram, defined as follows. For each marginal process, we define the finite Fourier-Stieltjes transforms of $N^{(a)}(t)$ and $N^{(b)}(t)$ (Cox and Lewis [14], p. 124) to be

$$H_{t_0}^{(a)}(\omega) = (2\pi t_0)^{-1/2} \sum_{\ell=1}^{n^{(a)}} \exp\{i\omega t_\ell^{(a)}\} = (2\pi t_0)^{-1/2}\{\mathscr{A}_{t_0}^{(a)}(\omega) + i\mathscr{B}_{t_0}^{(a)}(\omega)\}, \tag{6.14}$$

$$H_{t_0}^{(b)}(\omega) = (2\pi t_0)^{-1/2} \sum_{j=1}^{n^{(b)}} \exp\{i\omega t_j^{(b)}\} = (2\pi t_0)^{-1/2}\{\mathscr{A}_{t_0}^{(b)}(\omega) + i\mathscr{B}_{t_0}^{(b)}(\omega)\}. \tag{6.15}$$

The cross periodogram is then

$$\mathscr{I}_{t_0}^{(ab)}(\omega) = H_{t_0}^{(a)}(\omega)\, \bar{H}_{t_0}^{(b)}(\omega) \tag{6.16}$$

(Jenkins [23]). Thus, the estimates of the amplitude and phase of harmonic components of fixed frequencies in a nonhomogeneous bivariate Poisson model considered in the previous section are functions of the empirical spectral components. It can also be shown, as for the univariate case (Lewis [31]), that $\mathscr{I}_{t_0}^{(ab)}(\omega)$ is the Fourier transform of the unsmoothed estimator of the cross intensity function obtained from all possible intervals between events of type a and events of type b.

The distribution theory of $\mathscr{I}_{t_0}^{(ab)}(\omega)$ for independent Poisson processes follows simply from the conditional properties of the Poisson processes. Thus, we find that $A_{t_0}^{(a)}(\omega)$ and $B_{t_0}^{(a)}(\omega)$ have the (conditional) joint generating function

$$[I_0\{(\xi_a^2 + \xi_b^2)^{1/2}\}]^{n^{(a)}} \tag{6.17}$$

if $\omega t_0 = 2\pi p$, from which it can be shown, for example, that $\mathscr{A}_{t_0}^{(a)}(\omega)$ and $\mathscr{B}_{t_0}^{(b)}(\omega)$ go rapidly to independent normal random variables with means 0 and standard deviations $\frac{1}{2}t_0\rho_a$ as $n^{(a)}$ becomes large. Consequently, the real and imaginary components of the cross periodogram have double exponential distributions centered at zero with a variance which does not decrease as t_0 increases.

At two frequencies ω_1 and ω_2 such that $\omega_1 t_0 = 2\pi p_1$ and $\omega_2 t_0 = 2\pi p_2$, the real components of $\mathscr{I}_{t_0}^{(ab)}(\omega_1)$ and $\mathscr{I}_{t_0}^{(ab)}(\omega_2)$ are asymptotically uncorrelated, as are the imaginary components. Consequently, smoothing of the periodogram is required to get consistent estimates of the in phase and out of phase components of the cross spectrum. The problems of bias, smoothing, and computation of the spectral estimates are similar to those for the univariate case discussed in detail by Lewis [31].

Note that the smoothed intensity function or the smoothed spectral estimates can be used to estimate the delay probability density function in the one sided Poisson delay model (see equations (3.26) and (3.27)) and the difference of the delays in the two sided (bivariate) Poisson delay model. In the first case, the estimation procedure is probably much more efficient, in some sense, than the procedure discussed by Brown [7] unless the mean delay is much shorter than the mean time between events in the main Poisson process.

6.3.3. *Covariance time function.* Another problem that arises with the bivariate Poisson delay process is to test for the presence of the Poisson noise and to estimate the rate μ of the unobservable main process. Since the covariance time curve $V^{(ab)}(t) \sim \mu t$, we can estimate μ by estimating $V^{(ab)}(t)$ and also test for Poisson noise by comparing the estimated measures of dispersion $I^{(aa)}$, $I^{(bb)}$, and $I^{(ab)}$, defined in (3.22), (3.23), and (3.24). Care will be needed over possible nonstationarity.

The simplest method for estimating $V^{(ab)}(t)$ is to estimate the variance time curves $V^{(aa)}(t)$, $V^{(bb)}(t)$, and $V^{(\cdot\cdot)}(t)$ with the procedures given by Cox and Lewis ([14], Chapter 5) and to use (3.39) to give an estimate of $V^{(ab)}(t)$.

There is no evident reason for estimating the covariance time surface $C(t_1 t_2)$ along any line except $t_1 = t_2$.

6.3.4. *Serial intensity function.* Estimation of the serial intensity functions raises new problems, somewhat analogous to the analysis of life tables. Consider the estimation of $\lambda^{(a)}(u_a, u_b)$ of (3.42). One approach is to pass to discrete time, dividing the time axis into small intervals of length Δ. Each such interval is characterized by the values of (u_a, u_b) measured from the center of the interval if no type a event occurs within the interval, and by the values of (u_a, u_b) at the type a event in question if one such event occurs; we assume for simplicity of exposition that the occurrence of multiple type a events can be ignored. Thus, each time interval contributes a binary response plus the values of two explanatory variables (u_a, u_b); the procedure extends to the case of more than two explanatory variables, and to the situation in which multiple type a events occur within the intervals Δ.

We can now do one or both of the following:

(a) assume a simple functional form for the dependence on (u_a, u_b) of the probability $\lambda^{(a)}(u_a, u_b)\Delta$ of a type a event and fit by weighted least squares or maximum likelihood (Cox [13]);

(b) group into fairly coarse "cells" in the (u_a, u_b) plane and find the proportion of "successes" in each cell.

It is likely that standard methods based on an assumption of independent binomial trials are approximately applicable to such data and, if so, specific assumptions about the form of the serial intensities can be tested. In particular, we can test the hypothesis that the process is, say, purely a dependent, making the further assumption to begin with that the dependence is only on u_a.

By extensions of this method, that is, by bringing in dependencies on more aspects of the history at time t than merely u_a and u_b, it may be possible to build up empirically a fairly simple model for the process.

6.4. *Simple tests for dependence.* As noted previously, it may sometimes be required to construct simple tests of the null hypothesis that the type a and type b events are independent, as defined in Section 2.2. This may be done in various ways. Much the simplest situation arises when we consider the dependence of, say, the type b events on the type a events, argue conditionally on the observed type a process, and consider the strong null hypothesis that the type b events form an independent Poisson process. Then, conditionally on the total number of events of type b, the positions of the type b events, $t_1^{(b)}, \cdots, t_{n(a)}^{(b)}$ are independently and uniformly distributed over the period of observation. Thus in principle, the exact distribution of any test statistic can be obtained free of nuisance parameters.

The two simplest of the many possible test statistics are probably:

(a) particular ordinates of the cross intensity function, usually that near the origin; equivalently we can use the statistic $R_a^{(b)}(0, \Delta)$ of Section 6.3, directly;

(b) the sample mean recurrence time backwards from a type b event to the nearest preceding type a event.

The null distribution of $R_a^{(b)}(0, \Delta)$ can be found as follows. Place an interval of length Δ to the right of each type a event. (It is assumed for convenience that either there is a type a event at the origin, or that the position of the last type a event before the origin is available.) Let $\pi_0, \pi_1, \pi_2, \cdots, \pi_{n(a)}$ be the proportion of the observed interval $(0, t_0)$ covered jointly by $0, 1, 2, \cdots, n^{(a)}$ of these intervals Δ. Then, if there are $n^{(b)}$ events of type b in all, the null distribution of $R_a^{(b)}(0, \Delta)$ is that of the sum of $n^{(b)}$ independent random variables each taking the value i with probability π_i, $i = 1, \cdots, n^{(a)}$.

Similarly, for the second test statistic, we can find the null distribution as follows. Regard the sequence of intervals between successive type a events as a finite population $x = \{x_1, \cdots, x_N\}$, say. This includes the intervals from 0 to the first type a event and from the last type a event to t_0. If t_0 is preassigned, $N = n^{(a)} + 1$. Note that $\Sigma\, x_i = t_0$. Then the null distribution of the test statistic is that of the mean of n_b independent and identically distributed random variables each with probability density function

$$\frac{1}{t_0} \sum_{i=1}^{N} U(x; x_i), \tag{6.18}$$

where

$$U(x; x_i) = \begin{cases} 1, & 0 \leqq x \leqq x_i, \\ 0, & \text{otherwise}. \end{cases} \tag{6.19}$$

Thus, in particular, the null mean and variance of the test statistic are

$$\frac{\sum x_i^2}{2t_0}, \qquad \frac{1}{n^{(b)}} \left\{ \frac{1}{3t_0} \sum x_i^3 - \frac{(\sum x_i^2)^2}{4t_0^2} \right\}. \tag{6.20}$$

A strong central limit effect may be expected.

The tests derived here may be compared with similar ones in which the null distribution is derived by computer simulation, permuting at random the observed sequences of intervals (Perkel [40]; Moore, Perkel, and Segundo [37]; Perkel, Gerstein, and Moore [41]). In both types of procedure, it is not clear how satisfactory the tests are in practice as general tests of independence, when the type b process is not marginally Poisson. Note, however, that in order to obtain a null distribution for (a) and (b) above it is necessary to assume only that one of the marginal processes is a Poisson process.

If it is required to treat the two processes symmetrically, taking the null hypothesis that there are two mutually independent Poisson processes, there are many possibilities, including the use of the estimated cross spectral or cross intensity functions or of a two sample test based on the idea that, conditionally on $n^{(a)}$ and $n^{(b)}$, the times to events in the two processes are the order statistics from two independent populations of uniformly distributed random variables. Again, in the symmetrical case when both marginal processes are clearly not Poisson processes, tests of independence based on the cross spectrum are probably the best broad tests. For this purpose, investigation of the robustness of the distribution theory given in Section 6.3 would be worthwhile.

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We are indebted to Mr. T. K. M. Wisniewski, Dr. A. J. Lawrance, and Professor D. P. Gaver for helpful discussions during the growth of this paper.

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