STOCHASTIC INTEGRALS AND PROCESSES WITH STATIONARY INDEPENDENT INCREMENTS

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1. Introduction

Let $X = \{X(t); 0 \leqq t \leqq 1\}$ be a real valued stochastic process with stationary independent increments and right continuous paths $X(0) = 0$. The characteristic function of $X(t)$ then has the form $\exp\{t\psi(u)\}$, where

$$(1.1) \qquad \psi(u) = iug - \tfrac{1}{2}\sigma^2 u^2 + \int \left[e^{iux} - 1 - \frac{iux}{1 + x^2} \right] \nu(dx).$$

The measure ν is called the Lévy measure of X, and ψ is called the exponent. It will be assumed throughout that $\sigma^2 = 0$. The index $\beta(X)$ of the process X is

$$(1.2) \qquad \beta(X) = \inf\left\{p > 0; \int_{|x| < 1} |x|^p \nu(dx) < \infty\right\}.$$

If $\int_{|x|<1} |x| \nu(dx) < \infty$, then by subtracting a linear term from X one may write the exponent ψ as

$$(1.3) \qquad \psi(u) = \int [1 - e^{iux}] \nu(dx);$$

it will be assumed from now on that the exponent is in this form whenever $\int_{|x|<1} |x| \nu(dx) < \infty$.

This paper studies the sample function behavior of processes $Y = \{Y(t); 0 \leqq t \leqq 1\}$, where $Y(t)$ has the form $Y(t) = \int_0^t v(s)\, dX(s)$ and where $v = \{v(s); 0 \leqq s \leqq 1\}$ is a stochastic process of a special type described below. Section 2 contains a development of the theory of such stochastic integrals, together with conventions and notations prerequisite for the rest of the paper. The construction of the stochastic integral is made to depend on an inequality of L. E. Dubins and J. L. Savage, and has applications to more general theories of stochastic integration. In Section 3 a local limit theorem is proved. If $|v(s)| \leqq 1$ and if $p > \beta(X)$, then $|Y(t)| t^{-1/p}$ converges to zero a.s. as $t \downarrow 0$. This generalizes (with different proof) a result known for the case $v(s) \equiv 1$ (see [2]). To state the results of Section 4, let $\pi_n : 0 = t_{n,1} < \cdots < t_{n,k_n} = 1$ be a sequence of partitions of $[0, 1]$ satisfying

$$(1.4) \qquad \lim_{n \to \infty} \max_k [t_{n,k+1} - t_{n,k}] = 0;$$

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and let

$$V(\pi_n, Y, p) = \sum_k |Y(t_{n,k+1}) - Y(t_{n,k})|^p. \tag{1.5}$$

If $|v(s)| \leqq 1$, and if $\int_{|x|<1} |x|^p \nu(dx) < \infty$, then $V(\pi_n, Y, p)$ converges in probability to $\Sigma_{0 \leqq s \leqq 1} |v(s)|^p |j(X, s)|^p$, where $j(X, s) = X(s) - X(s-)$. This generalizes and improves (with different proof) the result obtained in [18] under the additional assumptions $p > \beta(X)$ and $v(s) \equiv 1$. Moreover, if the Lévy measure is concentrated on a finite interval, convergence is L_r for every $r < \infty$. Section 5 is devoted to the concept of zero jumps. A zero jump is experienced by X at time t if either $X(t, \omega) < 0 < X(t-, \omega)$ or $X(t-, \omega) < 0 < X(t, \omega)$. This concept appears to be of some interest in describing the sample function behavior of X. For example, as shown in Section 5, X may jump over zero infinitely often as $t \downarrow 0$, yet without ever hitting zero itself. Section 5 begins by establishing a stochastic integral formula analogous to the famous Itô formula for the case of Brownian motion. This is eventually made to yield an inequality relating the zero jumps up to time t to other more tractable quantities. This development draws on the results established in preceding sections.

2. Stochastic integral

Throughout this paper, the notation $X = \{X(t); 0 \leqq t \leqq 1\}$ will always denote a process with stationary independent increments, as in Section 1. Let $X^a(t)$ be the sum up to time t of all the jumps of X having absolute magnitude greater than a. Define $X_a(t) = X(t) - X^a(t)$. Then X^a and X_a are independent processes with independent increments. The process X_a has moments of all orders, its Lévy measure is concentrated on $(-a, a)$, and $\{X_a(t) - ct; t \geqq 0\}$ is a martingale if $c = EX_a(1)$. The exponent of X^a is

$$\psi^a(u) = \int_{|x|>a} [e^{iux} - 1]\nu(dx), \tag{2.1}$$

and that of X_a is $\psi_a(u) = \psi(u) - \psi^a(u)$. In the interval $[0, 1]$, the path $X(\cdot, \omega)$, for each ω, will experience only a finite number of jumps exceeding a. Hence, if a is large, $X(t) = X_a(t)$, $0 \leqq t \leqq 1$, for all ω in a set Ω_a, where $\Omega_a \uparrow 1$ as $a \to \infty$. For this reason we can (and will) often replace the process X in arguments below with the truncated process X_a. (See also [18], where this point of view is further explained.) Let us proceed now with the development of the stochastic integral.

Let $\{F(t); t \geqq 0\}$ denote the family of sigma fields given by $F(t) = F\{X(s): s \leqq t\}$. All processes are henceforth automatically assumed to be adapted to the family $F(t)$. Let G be the sigma field on $[0, \infty) \times \Omega$ generated by all the processes with left continuous paths. A process $v = \{v(t); t \geqq 0\}$ is "previsible" if it is measurable when regarded as a map of $([0, \infty) \times \Omega, G) \to$

$((-\infty, \infty), B)$. Here B denotes the Borel sigma field. A previsible process v is a step function if

$$(2.2) \qquad v(t) = \sum_{1 \leqq k \leqq n} I_{(t_k, t_{k+1}]}(t) v_k, \qquad 0 \leqq t_1 \leqq \cdots \leqq t_n \leqq 1,$$

and v_k is an $F(t_k)$ measurable random variable. (For any set A, I_A will always denote the indicator function of A.) If v is a step function, then the stochastic integral of v with respect to X is the right continuous process $Y = \{Y(t); 0 \leqq t \leqq 1\}$ defined by

$$(2.3) \quad Y(t) = \int_0^t v(s)\, dX(s) = v_1[X(t_2) - X(t_1)] + \cdots + v_{k(t)}[X(t) - X(t_{k(t)})],$$

where $k(t) = \max\{k : t_k < t\}$.

The problem now is to extend this definition to more general processes v. There are several ways of doing this. The following approach, which is applicable in much wider contexts (see the remark following the proof of Theorem 2.2), extends the version by H. McKean, Jr. [13] of Itô's original approach to Brownian motion through the use of different martingale inequalities. For the present paper, it was desirable to have the integral defined in a canonical manner as an a.s. limit uniformly in t, $0 \leqq t \leqq 1$. For other methods, consult P. A. Meyer [15]. The present method yields in a natural way a larger class of v that may serve as integrands than do the methods of [15].

Let us begin with an inequality of Dubins and Savage [10]. Let $\{F_n; n \geqq 0\}$ be an increasing family of sigma fields; let $\{d_k; k \geqq 1\}$ be a sequence of martingale differences, and let $\{w_n; n \geqq 1\}$ be a sequence of random variables such that w_k is F_{k-1} measurable. Finally, let $h_n = E(d_n^2 | F_{n-1})$. Then, assuming d_k is F_k measurable for $k \geqq 1$,

$$(2.4) \qquad P\left\{\sum_{k=1}^n d_k \geqq a \sum_{k=1}^n h_k + b \text{ for some } n\right\} \leqq (1 + ab)^{-1}.$$

This bound is known to be sharp, and implies that

$$(2.5) \qquad P\left\{\left|\sum_{k=1}^n d_k\right| \geqq a \sum_{k=1}^n h_k + b; \text{ for some } n\right\} \leqq 2(1 + ab)^{-1}.$$

It then follows that

$$(2.6) \quad P\left\{\left|\sum_{k=1}^n w_k\, d_k\right| \geqq a \sum_{k=1}^n w_k^2 h_k + b \ \text{ for some } n\right\} \leqq 2(1 + ab)^{-1}.$$

Hence, if v is any step function, and X is any integrable process with stationary independent increments, then

$$P\left\{\left|\int_0^t v(s)\, d[X(s) - cs]\right| \geqq a \int_0^t v(s)^2 r\, ds + b;\ \text{for some } t, 0 \leqq t \leqq 1\right\} \leqq 2(1 + ab)^{-1}, \tag{2.7}$$

where $c = EX(1)$ and $r = \text{Var}\, X(1)$; or

$$P\left\{\left|\int_0^t v(s)\, dX(s)\right| \geqq ar \int_0^t v(s)^2\, ds + |c| \int_0^t |v(s)|\, ds + b;\ \text{for some } t\right\} \leqq 2(1 + ab)^{-1}. \tag{2.8}$$

The following theorem gives the existence of the stochastic integral.

THEOREM 2.1. *Let v_n be a sequence of step functions, and let v be a measurable process, not necessarily previsible, such that*

$$P\left\{\int_0^1 [v(s) - v_n(s)]^2\, ds \leqq 2^{-n};\ n \uparrow \infty\right\} = 1. \tag{2.9}$$

Then the sequence of step function integrals $\int_0^t v_n(s)\, dX(s)$ converges for every ω, uniformly in t, $0 \leqq t \leqq 1$. Any process v satisfying

$$P\left\{\int_0^1 [v(s)]^2\, ds < \infty\right\} = 1 \tag{2.10}$$

may be represented as a limit of step functions as in (2.9).

Of course, one defines

$$Y(t) = \int_0^t v(s)\, dX(s) = \lim_{n\to\infty} \int_0^t v_n(s)\, dX(s), \tag{2.11}$$

and so the existence of $Y(t)$ is established as an a.s. limit for all $t \in [0, 1]$, simultaneously. By virtue of the uniform convergence, one automatically has the following corollary.

COROLLARY 2.1. *The process $\{Y(t) : t \geqq 0\}$ is right continuous.*

PROOF OF THEOREM 2.1. Because of the facts stated at the beginning of this section, there will be no loss in generality in assuming the Lévy measure of X concentrated on a finite interval $[-a, a]$ (If

$$\Omega_a = \{\omega : X(t) = X_a(t);\ 0 \leqq t \leqq 1\}, \tag{2.12}$$

then $\Omega_a \uparrow \Omega$ as $a \uparrow \infty$; the step function integrals relative to X and to X_a are then exactly the same for each $\omega \in \Omega_a$.) Let $c = EX(1)$ and $r = \text{Var}\, X(1)$. Suppose first that v_n is a sequence of step functions such that

$$P\left\{\int_0^1 [v_n(s)]^2\, ds \leqq 2^{-n};\ n \uparrow \infty\right\} = 1. \tag{2.13}$$

Then from (2.8);

$$\left\{\sup_{0 \leqq t \leqq 1} \left|\int_0^t v_n(s)\, dX(s)\right| \leqq 2^{-n/3};\ n \uparrow \infty\right\} = 1. \tag{2.14}$$

For, let $a_n = [2^n n^4]^{1/2}$ and $b_n = 2^{-n/2}$, so that $(1 + a_n b_n)^{-1}(1 + n^2)^{-1}$ is the general term of a convergent series. Then from the Borel–Cantelli lemma,

$$P\left\{\left|\int_0^t v_n(s)\, dX(s)\right| \geqq a_n r \int_0^t [v_n(s)]^2\, ds + |c| \int_0^t |v_n(s)|\, ds + b_n;\right. \tag{2.15}$$

$$\left.\text{for some } t,\ n \uparrow \infty\right\} = 0.$$

But $\int_0^t v_n^2(s)\, ds \leqq \int_0^1 v_n^2(s)\, ds \leqq 2^{-n}$ for all but a finite number of n, and so also

$$\int_0^t |v_n(s)|\, ds \leqq \left[\int_0^1 v_n^2(s)\, ds\right]^{1/2} \leqq 2^{-n/2} \tag{2.16}$$

for all but a finite number of n, implying that

$$P\left\{\sup_{0 \leqq t \leqq 1} \left|\int_0^t v_n(s)\, dX(s)\right| \geqq 2^{-n/2}(n^2 r + |c| + 1);\ n \uparrow \infty\right\} = 0. \tag{2.17}$$

Therefore, (2.14) holds.

Next, let $\{v_n\}$ be a sequence of step functions satisfying (2.9). By (2.14), $\sup_{0 \leqq t \leqq 1} |\int_0^t (v_{n+1} - v_n)\, dX|$ goes to zero geometrically fast, so that $\int_0^t v_n(s)\, dX(s)$ converges uniformly in t for almost every ω. This completes the proof of the existence of $\int_0^t v(s)\, dX(s)$. The last statement of the theorem is proved as in McKean's book [13].

Remark 2.1. Let $\{X(t),\ t \geqq 0\}$ be an L_2 martingale, and $A(t)$ the natural increasing process associated with it. These concepts are defined in [14]. The analogue of (2.7) then becomes, if v is a step function,

$$P\left\{\left|\int_0^t v(s)\, dX(s)\right| \geqq a \int_0^t v^2(s)\, dA(s) + b;\ \text{for some } t\right\} \leqq 2(1 + ab)^{-1}. \tag{2.18}$$

Using the argument of Theorem 2.1, one can obtain the existence of $\int_0^t v(s)\, dX(s)$ as an a.s. limit, uniformly in t, whenever there exist step functions v_n such that

$$P\left\{\int_0^1 [v(s) - v_n(s)]^2\, dA(s) < 2^{-n};\ n \uparrow \infty\right\} = 1. \tag{2.19}$$

In particular, one can see directly that this is possible whenever

$$P\left\{\int_0^1 v^2(s)\, dA(s) < \infty\right\} = 1 \tag{2.20}$$

and A has continuous paths (that is, X is quasileft continuous), and so establishes the existence of $\int_0^t v(s)\, dX(s)$ for all v satisfying (2.20). This seems to be an improvement on P. Courrège's approach to this problem [7].

The process $Y(t) = \int_0^t v(s)\, dX(s)$ is now defined for any process v satisfying (2.9). One can then enquire of the existence of the stochastic integral $\int_0^t w(s)\, dY(s)$. If w is a step function, then

$$\begin{aligned}\int w(s)\, dY(s) &= \sum w_k I_{(t_k, t_{k+1}]}[Y(_{k+1}) - Y(t_k)] \\ &= \sum w_k I_{(t_k, t_{k+1}]} \int_{t_k}^{t_{k+1}} v(s)\, dX(s) = \int w(s) v(s)\, dX(s).\end{aligned} \tag{2.21}$$

Using the development of Theorem 2.1, one checks that if $P\{\int_0^1 w^2(s)\, ds < \infty\} = 1$, then there are step functions w_n such that

$$P\left\{\int_0^1 [w(s) - w_n(s)]^2\, ds < 2^{-n};\, n \uparrow \infty\right\} = 1, \tag{2.22}$$

and $\int_0^t w_n(s)\, dY(s)$ converges to $\int_0^t w(s)v(s)\, dX(s)$ for each ω, uniformly in $t, 0 \leqq t \leqq 1$.

Several properties of the process $Y(t) = \int_0^t v(s)\, dX(s)$ will now be recorded for future reference.

PROPERTY 2.1. *Suppose $X = X_1 + X_2$, where the X_i are processes with independent increments. Then*

$$\int_0^t v(s)\, dX(s) = \int_0^t v(s)\, dX_1(s) + \int_0^t v(s)\, dX_2(s), \qquad 0 \leqq t \leqq 1. \tag{2.23}$$

PROPERTY 2.2. *Let the Lévy measure of X be concentrated on a finite interval. There is a constant K depending only on $EX(1)$ and* Var $X(1)$ *such that*

$$E\left[\int_0^t v(s)\, dX(s)\right]^2 \leqq KE \int_0^t v^2(s)\, ds, \tag{2.24}$$

$$E\left[\sup_{0 \leqq t \leqq 1} \int_0^t v(s)\, dX(s)\right]^2 \leqq KE \int_0^1 v^2(s)\, ds. \tag{2.25}$$

This follows upon observing that the following hold for step functions v:

$$E\left|\int_0^t v(s)\, d[X(s) - cs]\right|^2 = rE \int_0^t v^2(s)\, ds, \tag{2.26}$$

and

$$E\left|\sup_{0 \leqq t \leqq 1} \int_0^t v(s)\, d[X(s) - cs]\right|^2 \leqq 2rE \int_0^t v^2(s)\, ds. \tag{2.27}$$

These, of course are well-known martingale results. Here, as usual, $c = EX(1)$ and $r = \text{Var}\, X(1)$.

PROPERTY 2.3. *If the Lévy measure of X is concentrated on a finite interval and* (2.9) *holds, then*

$$P\left\{\left|\int_0^t v(s)\,dX(s)\right| \geqq ar\int_0^t v^2(s)\,ds + |c|\int_0^t |v(s)|\,ds + b;\ \text{for some } t\right\} \leqq 2(1+ab)^{-1}. \tag{2.28}$$

PROPERTY 2.4. *If the Lévy measure of X is concentrated on a finite interval, if* $c = EX(1)$, *and if* $|v(t)| \leqq 1$, $0 \leqq t \leqq 1$, *then for* $p \geqq 1$,

$$E\sup_{0\leqq s\leqq t}\left|\int_0^t v(s)\,dX(s)\right|^p \leqq c_pE|X(t)|^p + c_pct^p. \tag{2.29}$$

Here c_p is a constant depending on p only.

If $p > 1$, Theorem 9 of D. Burkholder [5] (see also [16]) implies that

$$E\sup_{0\leqq s\leqq t}\left|\int_0^t v(s)\,d[X(s) - cs]\right|^p \leqq c_pE|X(t) - ct|^p \tag{2.30}$$

whenever v is a step function. That (2.30) holds for $p = 1$ follows from Theorem 1.1 of [17]. Property 2.4 is a consequence of (2.30).

PROPERTY 2.5. *Suppose that the Lévy measure of X is concentrated on* $[-a, a]$ *and that* (2.9) *holds. If*

$$\psi(u) = iug + \int_{|x|<a}\left[e^{iux} - 1 - \frac{iux}{1+x^2}\right]\nu(dx), \tag{2.31}$$

let X^n *be the process with exponent*

$$iug + \int_{s_n<|x|<a}\left[e^{iux} - i - \frac{iux}{1+x^2}\right]\nu(dx). \tag{2.32}$$

Let Y^n *be the process*

$$Y^n(t) = \int_0^t v(s)\,dX^n(s). \tag{2.33}$$

Choose s_n *so that*

$$\int_{|x|\leqq s_n}|x|^2\nu(dx) < 2^{-n}. \tag{2.34}$$

Then for each ω, $Y^n(t) \to Y(t)$, *uniformly for* $0 \leqq t \leqq 1$.

PROOF. The process $X(t) - X^n(t)$ has stationary independent increments. According to Property 2.3, if $c_n = E[X(1) - X^n(1)]$ and $r_n = \text{Var}[X(1) - X^n(1)]$, then for constants a_n, b_n,

$$P\left\{\sup_{0 \leqq t \leqq 1} \left|\int_0^t v(s)\, dX(s) - \int_0^t v(s)\, dX^n(s)\right| > a_n r_n \int_0^t v^2(s)\, ds + |c_n| \int_0^t |v(s)|\, ds + b_n\right\} \leqq 2(1 + a_n b_n)^{-1}. \tag{2.35}$$

Since the exponent of the process $X(t) - X^n(t)$ is

$$\int_{0 < |x| \leqq s_n} \left[e^{iux} - 1 - \frac{iux}{1 + x^2}\right] \nu(dx), \tag{2.36}$$

$r_n \leqq 2^{-n}$ and $c_n \leqq 2^{-n/2}$. Set $a_n = n^3$ and $b_n = n^{-1}$, and conclude Property 2.5 from the Borel–Cantelli theorem.

PROPERTY 2.6. *Suppose almost all paths of X are of bounded variation. (A necessary and sufficient condition for this is $\int_{|x|<1} |x| \nu(dx) < \infty$.) Let v be previsible and satisfy* (2.10). *Let $L - \int_0^t v(s)\, dX(s)$ be the ordinary Lebesgue–Stieltjes integral (calculated for each ω), and $\int_0^t v(s)\, dX(s)$ the stochastic integral. Then*

$$\int_0^t v(s)\, dX(s) = L - \int_0^t v(s)\, dX(s) \qquad \text{a.s., } 0 \leqq t \leqq 1, \tag{2.37}$$

(*the exceptional set does not depend on* t).

PROOF. Suppose without any loss that the Lévy measure of X is on a finite interval. Let C be the class of bounded step functions and H the class of bounded previsible processes for which (2.37) holds. Then C is a vector space closed under $\wedge$, and H is a vector space of real functions on $\Omega \times [0, 1]$ containing C. Let $v_n \in H$, $0 \leqq v_n \leqq M$, and $v_n \uparrow v$. From Property 2.2,

$$\lim_{n \to \infty} \int_0^t v_n(s)\, dX(s) = \int_0^t v(s)\, dX(s) \tag{2.38}$$

in L_2 and from monotone convergence,

$$\lim_{n \to \infty} L - \int_0^t v_n(s)\, dX(s) = L - \int_0^t v(s)\, dX(s) \tag{2.39}$$

a.e. and in L_2. Hence, there is a null set A such that if $\omega \notin A$, then

$$\int_0^t v(s)\, dX(s) = L - \int_0^t v(s)\, dX(s) \tag{2.40}$$

for every rational t. Since both sides are right continuous, equality holds for all t, and $\omega \notin A$; it follows that H is closed under monotone limits. Therefore, H contains all bounded previsible v by T20 of Meyer [14], or rather the remark following it. Next, let v be nonnegative, previsible. There then exist bounded previsible v_n such that $v_n \uparrow v$, and we can choose v_n so that

$$P\left\{\int_0^1 (v - v_n)^2\, ds < 2^{-n},\ n \uparrow \infty\right\} = 1. \tag{2.41}$$

Arguing with Property 2.3 in a manner now familiar, one sees that for almost all ω, $\int_0^t v_n(s)\, dX(s)$ converges to $\int_0^t v(s)\, dX(s)$, uniformly in t. Since

$$\int_0^t v_n(s)\, dX(s) = L - \int_0^t v_n(s)\, dX(s) \tag{2.42}$$

by what has already been proved, it need be proved only that

$$\lim_{n\to\infty} L - \int_0^t v_n(s)\, dX(s) = L - \int_0^t v(s)\, dX(s) \qquad \text{a.e.} \tag{2.43}$$

Write $X = X_1 - X_2$, where X_1, X_2 are independent processes with stationary independent increments, each with increasing paths. Since $0 \leqq v_n \uparrow v$, clearly,

$$\lim_{n\to\infty} L - \int_0^t v_n(s)\, dX_i(s) = L - \int_0^t v(s)\, dX_i(s) \leqq \infty, \qquad i = 1, 2, \tag{2.44}$$

for every ω, so it is necessary only to show that $L - \int_0^t v(s)\, dX_i(s) < \infty$. But by what has already been proved,

$$\infty > \lim_{n\to\infty} \int_0^t v_n(s)\, dX_i(s) = \lim L - \int_0^t v_n(s)\, dX_i(s), \tag{2.45}$$

since X_i is a process with stationary independent increments. For general previsible v satisfying (2.9), write $v = v^+ - v^-$.

REMARK 2.2. A theorem of this nature has already been proved for L_2 martingales by Meyer and C. Doléans-Dade; the first half of the present proof follows theirs [8].

The following property is a convenient technicality.

PROPERTY 2.7. *Let v be a (measurable) process satisfying* (2.10), *and let v_n be a sequence of step functions satisfying* (2.9). *Then there is a previsible process $\hat{v}$ such that*

$$P\left\{\int_0^1 [\hat{v}(s) - v_n(s)]^2\, ds < 2^{-n},\ n \uparrow \infty\right\} = 1. \tag{2.46}$$

In particular, the stochastic integrals $\int_0^t v(s)\, dX(s)$ and $\int_0^t \hat{v}(s)\, dX(s)$ will be the same.

It follows from Property 2.7 that in discussion of stochastic integrals, there is no loss in generality in assuming v previsible. Accordingly, this assumption will be in effect throughout the remainder of the paper.

3. Local limit theorems

The goal of this section is to prove the following theorem.

THEOREM 3.1. *Let $X = \{X(t); 0 \leqq t\}$ be a process with stationary, independent increments and index $\beta(X)$. Let $v = \{v(t); 0 \leqq t\}$ be a stochastic process with $\sup_{0 \leqq t \leqq 1} |v(t)| \leqq 1$. Let $Y = \{Y(t); 0 \leqq t\}$ be the process $Y(t) = \int_0^t v(s)\, dX(s)$. If $p > \beta(X)$, then as $t \downarrow 0$,*

$$\frac{Y(t)}{t^{1/p}} \to 0 \qquad \text{a.s.} \tag{3.1}$$

This theorem is known for the case $v(s) \equiv 1$. The first proof was given by R. M. Blumenthal and R. K. Getoor [2]; in [18], a second proof is obtained as a consequence of a more general theorem. The method of proof below is new, even for the case $v \equiv 1$; since stochastic integration destroys stationarity and independence of increments, the methods used up to now had to be replaced in order to obtain the present theorem. A standard truncation argument shows that the hypothesis $|v(t)| \leqq 1$, all t, can be replaced by the hypothesis that almost all paths of v be bounded on finite intervals.

Before proceeding, it will be convenient to record as lemmas two results proved in [18] (Theorems 3.1 and 2.1, respectively).

LEMMA 3.1. *If $\int_{|x|<1} |x|^p \nu(dx) < \infty$, then*

$$\sum_{s \leqq 1} |j(X, s)|^p < \infty \qquad \text{a.s.} \tag{3.2}$$

Here, $j(X, s) = X(s) - X(s-)$, as usual.

LEMMA 3.2. *Let ν be concentrated on a finite interval and $\int_{|x|<1} |x|^p \nu(dx) < \infty$. Then if $0 < p \leqq 2$,*

$$E|X(t)|^p \leqq Ct, \qquad 0 \leqq t \leqq 1. \tag{3.3}$$

The constant

$$C \leqq 2^p\left[\int |x|^p \nu(dx) + et^{p-1}\right] \tag{3.4}$$

if $p > 1$ and $e = EX(1)$; if $p \leqq 1$, then $C \leqq 2 \int |x|^p \nu(dx)$.

PROOF OF THEOREM 3.1. The proof falls into several parts.

(a) Suppose that $1 < p < 2$. Without loss of generality suppose the Lévy measure is concentrated on $[-a, a]$. Moreover, there is also no loss in assuming that X is a martingale. For, if X is not a martingale and if $c = EX(1)$, then $X(t) - ct$ and $Y(t) - c\int_0^t v(s)\, ds$ are martingales; and, since $p > 1$,

$$\frac{\left|\int_0^t v(s)\, ds\right|}{t^{1/p}} \leqq t^{(p-1)/p} \to 0 \qquad \text{a.s.} \tag{3.5}$$

as $t \to 0$.

Next, pick p' subject to $\beta(X) < p' < p, p' > 1$. Then if $\varepsilon > 0$,

$$P\{|Y(2^{-n})| > \varepsilon 2^{-n/p}\} \leqq E|Y(2^{-n})|^{p'}\varepsilon^{-p'}2^{np'/p} \leqq CE|X(2^{-n})|^{p'}\varepsilon^{-p'}2^{np'/p}, \leqq C\varepsilon^{-p'}2^{-n[1-(p'/p)]}, \tag{3.6}$$

using (2.29) and (3.3). Hence,

$$\sum_n P\{|Y(2^{-n})| > \varepsilon 2^{-n/p}\} < \infty, \tag{3.7}$$

so that

$$\lim_{n\to\infty} |Y(2^{-n})|2^{n/p} = 0 \qquad \text{a.s.} \tag{3.8}$$

by the Borel–Cantelli theorem. Moreover,

$$\sup_{t:2^{-n-1}\leqq t\leqq 2^{-n}} |Y(t) - Y(2^{-n-1})|2^{n/p} \to 0 \qquad \text{a.s.} \tag{3.9}$$

as $n \to \infty$. This is verified as follows. Since $\{Y(t) - Y(2^{-n-1}), t \geqq 2^{-n-1}\}$ is a martingale, Doob's submartingale maximal inequality applied to the submartingale $\{|Y(t) - Y(2^{-n-1})|^{p'}; t \geqq 2^{-n-1}\}$ yields

$$\begin{aligned} P\{\sup_{t:2^{-n-1}\leqq t\leqq 2^{-n}} |Y(t) - Y(2^{-n-1})| > \varepsilon 2^{-n/p}\} &= P\{\sup_{t:2^{-n-1}\leqq t\leqq 2^{-n}} |Y(t) - Y(2^{-n-1})|^{p'} > \varepsilon^{p'}2^{-np'/p}\} \\ &\leqq E|Y(2^{-n}) - Y(2^{-n-1})|^{p'}\varepsilon^{-p'}2^{np'/p}. \end{aligned} \tag{3.10}$$

By inequality (2.29) (Property 2.4) and Lemma 3.2, the last expression does not exceed

$$C[2^{-n} - 2^{-n-1}]2^{np'/p} = C2^{-n[1-(p'/p)]}. \tag{3.11}$$

The Borel–Cantelli theorem then yields (3.9).

Finally, one deduces that $\lim_{t\to 0} |Y(t)|t^{-1/p} = 0$ a.s. for the case $1 < p < 2$ by means of the following calculation. If t is given, then there exists an integer $n(n = n(t))$ such that $2^{-n-1} \leqq t \leqq 2^{-n}$. Then

$$|Y(t)|t^{-1/p} \leqq |Y(2^{-n-1})|2^{(n+1)/p} + \sup_{t:2^{-n-1}\leqq t\leqq 2^{-n}} |Y(t) - Y(2^{-n})|2^{(n+1)/p}. \tag{3.12}$$

By (3.8) and (3.9) the right side goes to zero as $t \to 0$ $(n \to \infty)$.

(b) Next suppose $\beta(X) = 2$. Using the same inequalities of (a), one sees that

$$P\{|Y(2^{-n})| > \varepsilon n 2^{-n/2}\} \leqq Cn^{-2}, \tag{3.13}$$

implying by the Borel–Cantelli lemma that $|Y(2^{-n})|2^{n/2}n^{-1} \to 0$ a.s. Hence, if $p > 2$, $|Y(2^{-n})|2^{n/p} \to 0$ a.s. Similarly, it is easy to see that

$$\sup_{t: 2^{-n-1} \leqq t \leqq 2^{-n}} |Y(t) - Y(2^{-n-1})|2^{n/p} \to 0 \qquad \text{a.s.} \tag{3.14}$$

as $n \to \infty$, and the argument is completed as in (a).

(c) Suppose $p \leqq 1$. According to the conventions of Section 1, the exponent of X is of the form $\int [e^{iux} - 1]\nu(dx)$. This implies that $X(t) = T(t) - S(t)$, where T and S are independent processes, each with increasing paths. It is then enough to prove the result for $X(t) = T(t)$. But in this case, $|\int_0^t v(s)\, dX(s)| \leqq |X(t)|$, so the result follows from the known fact for X.

Remark 3.1. A modification of the argument of Millar [18] shows that if we assume only

$$\int_{|x|<1} |x|^p \nu(dx) < \infty, \qquad |v(t)| \leqq 1 \qquad \text{for all } t, \tag{3.15}$$

then $|\int v(s)\, dX(s)|t^{-1/p} \to 0$ in probability. Condition (3.15) is slightly weaker than $p > \beta(X)$, and it is known that under condition (3.15) one cannot assert a.s. convergence in general.

Remark 3.2. If $|v(t, \omega)| > \delta$ all (t, ω), and if $p < \beta(X)$, then

$$\limsup_{t \to 0} \left| \int_0^t v(s)\, dX(s) \right| t^{-1/p} = +\infty \text{ a.s.} \tag{3.16}$$

The proof is similar to the proof of the case $v \equiv 1$ in [18], and is omitted.

4. Variation of sample functions

Throughout this section, $\{\pi_n; n \geqq 1\}$ will be a sequence of partitions of $[0, 1]$ with $\pi_n: 0 = t_{n,1} < \cdots < t_{n,k_n} = 1$, and will satisfy

$$\lim_{n \to \infty} \max_k [t_{n,k+1} - t_{n,k}] = 0. \tag{4.1}$$

The partition π_{n+1} is not necessarily a refinement of π_n. If $Y = \{Y(t); 0 \leqq t \leqq 1\}$ is a stochastic process, define

$$V(\pi_n, Y, p) = V_n(Y, p) = \sum_k |Y(t_{n,k+1}) - Y(t_{n,k})|^p. \tag{4.2}$$

Theorem 4.1. *Let $v = \{v(t); t \geqq 0\}$ satisfy $\sup_t |v(t)| \leqq 1$. Let $X = \{X(t); t \geqq 0\}$ be a process with stationary independent increments such that $\int_{|x|<1} |x|^p \nu(dx) < \infty$. Let $Y(t) = \int_0^t v(s)\, dX(s)$. Then $V(\pi_n, Y, p)$ converges in probability to $\Sigma_{0 \leqq s \leqq 1} |v(s)|^p |j(X, s)|^p < \infty$, where $j(X, s) = X(s) - X(s-)$.*

This theorem constitutes an improvement of Theorem 3.2 of [18], which asserts the convergence in probability under the hypothesis $v \equiv 1$ and the somewhat stronger assumption that $p > \beta(X)$. The present proof is different and rather simpler than the proof of [18]. It will reveal that in the case $p \leqq 1$, almost everywhere convergence of $V_n(Y, p)$ holds, even if π_{n+1} is not a refinement of π_n; this is a further improvement of the results of [18].

PROOF. (a) The case $p \leqq 1$. Since v is previsible and $p \leqq 1$, $\int_0^t v(s)\,dX(s)$ is an ordinary Lebesgue–Stieltjes integral for each ω (Property 6, Section 2), so that in fact

$$\int_0^t v(s)\,dX(s) = \sum_{0 \leqq s \leqq t} v(s)[X(s) - X(s-)]. \tag{4.3}$$

(According to the conventions of Section 1, X will have no "linear part" if $p \leqq 1$.) It is clear that

$$\liminf_n V_n \geqq \sum_{0 \leqq s \leqq 1} |v(s)|^p |j(X, s)|^p. \tag{4.4}$$

However, since

$$V_n(Y, p) = \sum_k \left| \int_{t_{n,k}}^{t_{n,k+1}} v(s)\,dX(s) \right|^p, \tag{4.5}$$

and since

$$\begin{aligned} \left| \int_{t_{n,k}}^{t_{n,k+1}} v(s)\,dX(s) \right|^p &= \Big| \sum_{t_{n,k} \leqq s \leqq t_{n,k+1}} v(s) j(X, s) \Big|^p \\ &\leqq \Big[\sum_{t_{n,k} \leqq s \leqq t_{n,k+1}} |v(s)|\,|j(X, s)| \Big]^p \\ &\leqq \sum_{t_{n,k} \leqq s \leqq t_{n,k+1}} |v(s)|^p |j(X, s)|^p \end{aligned} \tag{4.6}$$

(because $p \leqq 1$), it follows that

$$V_n(Y, p) \leqq \sum_{0 \leqq s \leqq 1} |v(s)|^p |j(X, s)|^p \tag{4.7}$$

for all n. This proves (a), since $|v(s)| \leqq 1$ and $\Sigma_{0 \leqq s \leqq 1} |j(X, s)|^p$ is finite a.s. by Lemma 3.1.

(b) Consider the case $p > 1$. Suppose without loss of generality that the Lévy measure of X concentrates on $[-1, 1]$. If the exponent of X is of the form

$$\psi(u) = iug + \int_{|x| \leqq 1} [e^{iux} - 1 - iux/(1 + x^2)]\nu(dx), \tag{4.8}$$

then write $X = X_n + X^n$, where $X^n = \{X^n(t); t \geqq 0\}$ is the process with stationary independent increments having exponent

$$iug + \int_{1 \geqq |x| \geqq (1/n)} \left[e^{iux} - 1 - \frac{iux}{1 + x^2} \right] \nu(dx). \tag{4.9}$$

Then

$$Y(t) = \int_0^t v(s)\,dX(s) = \int_0^t v(s)\,dX_n(s) + \int_0^t v(s)\,dX^n(s) = Y_n(t) + Y^n(t), \tag{4.10}$$

and also

$$\sum_{0 \leqq s \leqq 1} |v(s)|^p |j(X, s)|^p = \sum_{0 \leqq s \leqq 1} |v(s)|^p |j(X_n, s)|^p + \sum_{0 \leqq s \leqq 1} |v(s)|^p |j(X^n, s)|^p = J_n + J^n. \tag{4.11}$$

Then

$$\begin{aligned} E|V_j(Y, p) - V_j(Y^n, p)| &= E\Big|\sum_k |Y_n(t_{j,k+1}) - Y_n(t_{j,k}) + \{Y^n(t_{j,k+1}) - Y^n(t_{j,k})\}|^p \\ &\qquad - \sum_k |Y^n(t_{j,k+1}) - Y^n(t_{j,k})|^p\Big| \\ &\leqq pE \sum_k |Y^n(t_{j,k+1}) - Y^n(t_{j,k})|^{p-1} |Y_n(t_{j,k+1}) - Y_n(t_{j,k})| \\ &\qquad + pEV_j(Y_n, p) \\ &\leqq p \sum_k E^{(p-1)/p}|Y^n(t_{j,k+1}) - Y^n(t_{j,k})|^p E^{1/p}|Y_n(t_{j,k+1}) - Y_n(t_{j,k})|^p \\ &\qquad + pEV_j(Y_n, p). \end{aligned} \tag{4.12}$$

In the calculation (4.12), we have used the following elementary inequality where $1 < p \leqq 2$ and $0 < s < 1$,

$$\begin{aligned} \Big||x + y|^p - |x|^p\Big| &= p|x + sy|^{p-1}|y| \\ &\leqq p(|x|^{p-1} + |y|^{p-1})|y| \\ &= p|x|^{p-1}|y| + p|y|^p. \end{aligned} \tag{4.13}$$

Next, observe that if $e_n = EX_n(1)$ then by (2.29)

$$\begin{aligned} E|Y_n(t) - Y_n(s)|^p &\leqq c_p E|X_n(t) - X_n(s)|^p + c_p|e_n|^p|t - s|^p \\ &\leqq K\left[\int_{|x| < (1/n)} |x|^p \nu(dx) + |e_n|^p|t - s|^{p-1}\right](t - s), \end{aligned} \tag{4.14}$$

using Lemma 3.2, and where K is an absolute constant. Similarly, if $e^n = EX^n(1)$, then

$$\begin{aligned} E|Y^n(t) - Y^n(s)|^p &\leqq K\left[\int_{(1/n) \leqq |x| < 1} |x|^p \nu(dx) + |e^n|^p|t - s|^{p-1}\right](t - s). \\ &\leqq M|t - s|, \end{aligned} \tag{4.15}$$

where M is a finite constant independent of n. Let

$$\|\pi_j\| = \max_k |t_{j,k+1} - t_{j,k}|, \qquad b = \sup_n \{|e_n|^p\} < \infty; \tag{4.16}$$

and let

$$(4.17)\qquad c_n(j) = K\left[\int_{|x|<(1/n)} |x|^p \nu(dx) + b\|\pi_j\|^{p-1}\right].$$

Then, using (4.14) and (4.15) in (4.12), we have

$$(4.18)\qquad \begin{aligned} &E|V_j(Y, p) - V_j(Y^n, p)| \\ &\leqq p \sum_k M^{(p-1)/p}(t_{j,k+1} - t_{j,k})^{(p-1)/p}[c_n(j)]^{1/p}(t_{j,k+1} - t_{j,k})^{1/p} \\ &\leqq \text{const } c_n(j)^{1/p} \equiv Cc_n(j)^{1/p}. \end{aligned}$$

Since p > 1, it is obvious that

$$(4.19)\qquad V_j(Y^n, p) \to J^n \qquad \text{a.s.}$$

as $j \to \infty$, because X^n is piecewise linear with a finite number of jumps in any interval. Let $\varepsilon > 0$. Then

$$(4.20)\qquad \begin{aligned} &P\{|V_j(Y) - J| > \varepsilon\} \\ &\leqq P\{|V_j(Y^n) - J^n| > \tfrac{1}{3}\varepsilon\} + P\{|V_j(Y) - V_j(Y^n)| > \tfrac{1}{3}\varepsilon\} + P\{J_n > \tfrac{1}{3}\varepsilon\}. \end{aligned}$$

As $n \to \infty$, the third term above goes to zero. The middle term is dominated by $C\int_{|x|<1/n} |x|^p \nu(dx) + C\|\pi_j\|^{p-1}$, because of (4.18) and Chebyshev's inequality. Therefore, if δ is given, one may choose n so large that

$$(4.21)\qquad P\{|V_j(Y) - J| > \varepsilon\} \leqq P\{|V_j(Y^n) - J^n| > \tfrac{1}{3}\varepsilon\} + C\|\pi_j\|\varepsilon^{-1} + \delta.$$

Letting $j \to \infty$ and applying (4.19), now completes the proof that $V_j(Y)$ converges to J in probability.

THEOREM 4.2. *If the Lévy measure of X concentrates on a finite interval, and if* $\int_{|x|<1} |x|^p \nu(dx) < \infty$, *then* $V_n(Y, p)$ *converges in* L_r *norm for every* r, $1 \leqq r < \infty$. *(The process Y satisfies the hypotheses of Theorem 4.1.)*

In order to prove this, it will be convenient to establish the following lemma. Let $\{d_k; k \geqq 1\}$ be a sequence of martingale differences, with $E|d_k|^p < \infty$, and let $f_n = \sum_{k=1}^n d_k$.

LEMMA 4.1. *There are positive constants C, c depending on p only such that for every* $n \geqq 1$:

(i) *if* $1 \leqq p \leqq 2$, *then* $E|f_n|^p \leqq C \sum_{k=1}^n E|d_k|^p$;

(ii) *if* $p \geqq 2$, *then* $E|f_n|^p \geqq c \sum_{k=1}^n E|d_k|^p$.

PROOF. Part (i) is known (see C. G. Esseen and B. von Bahr [11]). Part (ii) is established by a simple modification of S. D. Chatterji's proof [6] of the Esseen–von Bahr result. Since

$$(4.22)\qquad \inf_x \left\{\frac{(|1 + x|^p - 1 - px)}{|x|^p}\right\} = c_p > 0.$$

it follows that

$$|A + B|^p \geqq |A|^p + c_p|B|^p - \operatorname{sgn}[A]|A|^{p-1}B, \tag{4.23}$$

so that

$$E|f_{n+1}|^p = E|f_n + d_{n+1}|^p \geqq E|f_n|^p + c_pE|d_{n+1}|^p. \tag{4.24}$$

The result now follows by induction.

LEMMA 4.2. *If the Lévy measure of X is concentrated on a finite interval, if* $\int_{|x|<1} |x|^p \nu(dx) < \infty$ *for some* p, $0 < p \leqq 2$, *and if* $\sup_t |v(t)| \leqq 1$, *then*

$$E\left\{\left|\int_s^t v(u)\, dX(u)\right|^p \middle| F(s)\right\} \leqq c_p(t - s), \qquad 0 \leqq t - s \leqq 1. \tag{4.25}$$

PROOF. Let $\{d_k; k \geqq 1\}$ be as in Lemma 4.1; let $\{v_k; k \geqq 1\}$ be a sequence of bounded random variables, $|v_k| \leqq 1$ and v_k measurable with respect to $F_{k-1} = F(d_1, \cdots, d_{k-1})$. Results of Burkholder [5] then imply if $p > 1$ that $E|\Sigma_{k=1}^n v_kd_k|^p \leqq c_pE|\Sigma_{k=1}^n d_k|^p$. Let m be given and $A \in F_m$. Then

$$\int_A \left|\sum_{m+1}^n v_kd_k\right|^p = \int \left|\sum_{m+1}^n (v_kI_A)d_k\right|^p \leqq c_p \int_A \left|\sum_{m+1}^n d_k\right|^p, \tag{4.26}$$

implying that

$$E\left\{\left|\sum_{m+1}^n v_kd_k\right|^p \middle| F_m\right\} \leqq c_pE\left\{\left|\sum_{m+1}^n d_k\right|^p \middle| F_m\right\}. \tag{4.27}$$

If X is a martingale, and if v is a step function, it now follows that

$$E\left\{\left|\int_s^t v(u)\, dX(u)\right|^p \middle| F(s)\right\} \leqq c_pE\{|X(t) - X(s)|^p|F(s)\} \leqq c_p(t - s), \tag{4.28}$$

using Lemma 3.2. Conclusion (4.25) now follows for all bounded v by passage to the limit. To handle the case when X is not a martingale, consider $X(t) - tEX(1)$. The case $p \leqq 1$ is easy.

PROOF OF THEOREM 4.2. To show convergence in L_r, $1 \leqq r < 2$, it suffices to show $\sup_j E[V_j(Y, p)]^2 < \infty$. The case $1 < p \leqq 2$ will be discussed first. Here there is no loss in generality in assuming that X is a martingale (subtract off a multiple of t if necessary). Then, in obvious notation,

$$\begin{aligned} E[V_j(Y, p)]^2 &= E\left[\sum_k |Y(t_{j,k+1}) - Y(t_{j,k})|^p\right]^2 \\ &= E\left[\sum_k |\Delta_k|^p\right]^2 \\ &= E\sum_k |\Delta_k|^{2p} + E\sum_{i \neq k} |\Delta_k|^p|\Delta_i|^p. \end{aligned} \tag{4.29}$$

But $E\,\Sigma\,|\Delta_k|^{2p} \leqq cE|Y(1)|^{2p} < \infty$, from Lemma 4.1(ii). Also, suppressing the

j in $t_{j,k}$, if $i < k$,

$$E|\Delta_k|^p|\Delta_i|^p = E|Y(t_{i+1}) - Y(t_i)|^p E\{|Y(t_{k+1}) - Y(t_k)|^p \,|F(t_{i+1})\} \tag{4.30}$$

and

$$E\{|Y(t_{k+1}) - Y(t_k)|^p|F(t_{i+1})\} \leqq c(t_{k+1} - t_k), \tag{4.31}$$

using Lemma 4.2. Therefore,

$$E \sum_{i \neq k} |\Delta_k|^p|\Delta_i|^p \leqq c \sum_{i \neq k} (t_{i+1} - t_i)(t_{k+1} - t_k) \leqq c. \tag{4.32}$$

It follows that $\sup_j E[V_j(Y, p)]^2 < \infty$, proving convergence in L_r, $1 \leqq r < 2$. One proves convergence in L_r, $1 \leqq p < 2n$ by showing in a similar manner that $\sup_j E[V_j(Y, p)]^{2n} < \infty$.

If p $\leqq$ 1, then from the proof of part (a) of Theorem 4.1, $V_j(Y, p) \leqq \Sigma_{0 \leqq s \leqq 1} |j(X, s)|^p$. The Lévy measure of the latter random variable is concentrated on a finite interval (see [18]), so it has moments of all orders.

Let us conclude this section with a few remarks about the a.s. convergence of $V_j(Y, p)$. The case where $v \equiv 1$ was discussed in [18], where some open problems were listed. For general v, we give only the following rather limited results (compare with [4] and [21]).

THEOREM 4.3. *Assume* $\sup_t |v(t)| \leqq 1$ *and* $Y(t) = \int_0^t v(s)\, dX(s)$. *If* $\Sigma_n [\Sigma_k (t_{n,k+1} - t_{n,k})^2]^{1/2} < \infty$, *then* $V_j(Y, 2)$ *converges a.s.*

PROOF. Assume without loss of generality that the Lévy measure concentrates on a finite interval. Then if $c = EX(1)$, $X(t) - ct$ and $Y'(t) = Y(t) - c\int_0^t v(s)\, ds$ are martingales. Moreover,

$$\begin{aligned}\sum_k [Y(t_{n,k+1}) - Y(t_{n,k})]^2 &= \sum_k [Y'(t_{n,k+1}) - Y'(t_{n,k})]^2 \\ &\quad + \sum_k c^2 \left[\int_{t_{n,k}}^{t_{n,k+1}} v(s)\, ds\right]^2 \\ &\quad + 2c \sum_k [Y'(t_{n,k+1}) - Y'(t_{n,k})] \int_{t_{n,k}}^{t_{n,k+1}} v(s)\, ds.\end{aligned} \tag{4.33}$$

Since the second term on the right obviously converges to zero a.s. as $n \to \infty$, and since the expectation of the third term is less than

$$\begin{aligned}E \sum_k{}^{1/2} |Y'(t_{n,k+1}) - Y'(t_{n,k})|^2 \sum_k{}^{1/2} (t_{n,k+1} - t_{n,k})^2 \\ \leqq \text{const} \sum_k{}^{1/2} (t_{n,k+1} - t_{n,k})^2\end{aligned} \tag{4.34}$$

(by Lemma 3.2), it follows from the Borel–Cantelli theorem that there is no loss in generality in assuming that X and Y are martingales.

In Section 5, the following formula will be established:

$$\sum_{0 \leqq s \leqq t} [v(s)]^2 [j(X, s)]^2 = \left[\int_0^t v(s)\, dX(s)\right]^2 - 2\int_0^t v(s) Y(s)\, dX(s). \tag{4.35}$$

Assuming (4.35) and using in the third equality that X is a martingale,

$$\begin{aligned} (4.36) \qquad & E\left|V_n(Y, 2) - \sum_{0 \leqq s \leqq 1} [v(s)]^2 [j(X, s)]^2\right|^2 \\ & = E\left|\sum_k \left\{\left[\int_{t_{n,k}}^{t_{n,k+1}} v(s)\, dX(s)\right]^2 - \sum_{t_{n,k} \leqq s \leqq t_{n,k+1}} [v(s)]^2 [j(X, s)]^2\right\}\right|^2 \\ & = 4E\left|\sum_k \int_{t_{n,k}}^{t_{n,k+1}} v(s)[Y(s) - Y(t_{n,k})]\, dX(s)\right|^2 \\ & = 4\sum_k E\left|\int_{t_{n,k}}^{t_{n,k+1}} v(s)[Y(s) - Y(t_{n,k})]\, dX(s)\right|^2 \\ & = 4c\sum_k \int_{t_{n,k}}^{t_{n,k+1}} Ev^2(s)[Y(s) - Y(t_{n,k})]^2\, ds \\ & \leqq \text{const} \sum_k (t_{n,k+1} - t_{n,k})^2. \end{aligned}$$

The result now follows from the Borel–Cantelli theorem. It is clear from the proof that if X is known to be a martingale, or if every truncation of X is a martingale (which happens if X is symmetric), then one needs only $\Sigma_n \Sigma_k (t_{n,k+1} - t_{n,k})^2 < \infty$.

5. Zero jumps

In this section, some of the preceding ideas will be used to study certain sample function properties of X itself, in particular the zero jumps (defined below) of the process X. Before doing this, however, it is necessary to have an analogue of Itô's formula for processes $Y(t) = \int_0^t v(s)\, dX(s)$.

THEOREM 5.1. *Suppose* $\int_{|x|<1} |x|^p \nu(dx) < \infty$, *and* $\sup_t |v(s)| \leqq 1$. *Let F be a real function such that* (*i*) *F' exists and is continuous;* (*ii*) *for every neighborhood N of 0 there is a constant M* (*depending on N perhaps*) *such that*

$$|F(x + h) - F(x) - F'(x)h| \leqq Mh^p, \qquad x \in N, x + h \in N. \tag{5.1}$$

Then

$$\begin{aligned} (5.2) \qquad F(Y(t)) = & \int_0^t F'[Y(s-)]\, dY(s) \\ & + \sum_{0 \leqq s \leqq t} \{F[Y(s)] - F[Y(s-)] - F'[Y(s-)]j(Y, s)\}. \end{aligned}$$

Theorems of this type have been established in great generality for quasi-martingales (of which Y is an example) by Meyer [15], but under more restrictive hypotheses on F (F is assumed twice continuously differentiable). The special nature of the process Y and the theory of sample variation in Section 4 permit the stronger conclusion here. Theorem 5.1 yields, for example, the following conclusion.

COROLLARY 5.1. *Let Y satisfy the hypotheses of Theorem 5.1, and assume in addition that $p > 1$. Then $|Y(t)|^p$ is a quasimartingale.*

Also, choosing $F(x) = x^2$ yields formula (4.35), required in the preceding section.

PROOF. As usual, one may suppose that the Lévy measure of X is on $[-1, 1]$. Let X^n, Y^n be defined as in the proof of Theorem 4.1, part (b). Then $Y^n(t) = \int_0^t v(s)\, dX^n(s)$ is an ordinary Lebesgue–Stieltjes integral for each ω. Let $S_0 \equiv 0$, and $S_1 < S_2 < \cdots$ be an enumeration of the jumps of Y^n (since X^n has only a finite number of jumps in any finite interval, the same is true of Y^n). If t is fixed, let $T_k = \min\{S_k, t\}$. Then

$$
\begin{aligned}
(5.3)\quad & F[Y^n(t)] \\
&= \sum_k \{F[Y^n(T_{k+1}-)] - F[Y^n(T_k)]\} + \sum_k F[Y^n(T_k)] - F[Y^n(T_k-)] \\
&= \sum_k \{F[Y^n(T_{k+1}-)] - F[Y^n(T_k)]\} + \sum_k F'[Y^n(T_k-)]j(Y^n, T_k) \\
&\qquad + \sum_k \{F[Y^n(T_k)] - F[Y^n(T_k-)] - F'[Y^n(T_k-)]j(Y^n, T_k)\} \\
&= \sum_k \int_{(T_k, T_{k+1})} F'[Y^n(s)]\, dY^n(s) + \sum_k F'[Y^n(T_k-)]j(Y^n, T_k) \\
&\qquad + \sum_{0 \leqq s \leqq t} \{F[Y^n(s)] - F[Y^n(s-)] - F'[Y^n(s-)]j(Y^n, s)\} \\
&= \int_0^t F'[Y^n(s-)]\, dY^n(s) \\
&\qquad + \sum_{0 \leqq s \leqq t} \{F[Y^n(s)] - F[Y^n(s-)] - F'[Y^n(s-)]j(Y^n, s)\},
\end{aligned}
$$

using the fact that Y^n is continuous on (T_k, T_{k+1}). By choosing a subsequence if necessary, we may suppose that for each ω the paths of Y^n converge uniformly for $0 \leqq t \leqq 1$ to the paths of Y (see Property 2.5). For each ω, the paths of Y are bounded on $[0, 1]$. Therefore, there exists $M(\omega)$ such that

$$
\begin{aligned}
(5.4)\quad & |F[Y^n(s)] - F[Y^n(s-)] - F'[Y^n(s-)]j(Y^n, s)| \\
&\qquad \leqq M(\omega)|j(Y^n, s)|^p \leqq M(\omega)|j(X, s)|^p.
\end{aligned}
$$

Since $\Sigma_{0 \leqq s \leqq 1} |j(X, s)|^p < \infty$ (Lemma 3.1), it follows that

$$
(5.5)\qquad \sum_{0 \leqq s \leqq t} F[Y^n(s)] - F[Y^n(s-)] - F'[Y^n(s-)]j(Y^n, s)
$$

converges for the given ω to

$$
(5.6)\qquad \sum_{0 \leqq s \leqq t} F[Y(s)] - F[Y(s-)] - F'[Y(s-)]j(Y, s),
$$

a process having paths of bounded variation. Finally, one checks that for each ω, $\int_0^t F'[Y^n(s-)]\,dY^n(s)$ converges to $\int_0^t F'[Y(s-)]\,dY(s)$; by taking a further subsequence if necessary, one can ensure that for each ω the convergence is uniform in t, $0 \leqq t \leqq 1$.

REMARK 5.1. The proof shows that if the Lévy measure of X is concentrated on a set bounded away from zero, then the conclusion of Theorem 5.1 holds with no further hypotheses other than the existence and continuity of F'.

DEFINITION 5.1. *A process X is said to have a zero jump at s if either $X(s) \leqq 0 < X(s-)$ or $X(s) > 0 \geqq X(s-)$. The notation $Z(t) = \Sigma'_{0 \leqq s \leqq t} |X(s)|$ will denote the sum of all $|X(s)|$, $s \leqq t$, over only those s at which a zero jump occurs.*

The rest of this section will be devoted to a study of the process $Z(t)$. The asymmetry in the definition of zero jump was introduced in order to keep the statement of Theorem 5.2 simple. In all cases of real interest (specifically when $\nu(R) = \infty$), the definition can be replaced by the condition that either $X(s) < 0 < X(s-)$ or $X(s-) < 0 < X(s)$, as shown in the proof of Theorem 5.2.

Of course if may happen that the process Z is rather trivial, for example when X is a subordinator. On the other hand, many examples exist of processes X which experience infinitely many zero jumps as t varies in any interval of the form $[0, \varepsilon]$, $\varepsilon > 0$. For example, if the Lévy measure of X is concentrated on $(0, \infty)$, then X has only upward jumps; however, if $\int_{-\infty}^{\infty} [\lambda + Re\,\psi(x)]^{-1}\,dx < \infty$ for all positive λ, then it is known that $\inf \{t > 0; X(t) = 0\} = 0$ a.e. (Blumenthal and Getoor, [3], p. 64). It is easy to see that this implies that X has infinitely many zero jumps. In such cases, it is not even clear *a priori* that the process $Z(t)$ is finite—this will be the conclusion of Theorem 5.2, where a stochastic upper bound for Z is derived.

It is interesting to notice also that as $t \downarrow 0$, a process may jump over zero infinitely often, but without hitting 0 itself infinitely often. Here is an example. Let X be the symmetric Cauchy process. It is then known that 0 is not regular for $\{0\}$; that is, the process does not pass through 0 infinitely often as $t \downarrow 0$ (see Port [19]). However, it is also known (see Blumenthal and Getoor [2], or Millar [18]) that

$$\limsup_{t \to 0} |X(t)| t^{-1/2} = +\infty, \tag{5.7}$$

implying by symmetry and the zero-one law that

$$\limsup_{t \to 0} X(t) t^{-1/2} = +\infty \tag{5.8}$$

and

$$\liminf_{t \to 0} X(t) t^{-1/2} = -\infty. \tag{5.9}$$

In particular. X must pass from above to below zero infinitely often as $t \downarrow 0$. Since the process does not hit zero while doing this, it therefore must jump over 0 infinitely often.

THEOREM 5.2. *The process sum of jumps $Z(t)$ satisfies the relation*

$$Z(t) \leqq X(t)^+ - \int_0^t I_{(0,\infty)}[X(s-)]\, dX(s). \tag{5.10}$$

PROOF. Suppose without loss of generality that the Lévy measure concentrates on $[-1, 1]$.

(a) Suppose that the exponent of X is of the form

$$\psi(u) = iug + \int [e^{iux} - 1]\nu(dx), \tag{5.11}$$

where $\nu(R) < \infty$. Then the paths of X are piecewise linear (see, for example, [12], p. 274). Let $\varepsilon > 0$ and $F(x) = \int_0^x (\varepsilon \wedge y)^+ \, dy$. By Theorem 5.1 (or rather, Remark 5.1),

$$\begin{aligned} F[X(t)] = \int_0^t & F'[X(s-)]\, dX(s) \\ &+ \sum_{0 \leqq s \leqq t} F[X(s)] - F[X(s-)] - F'[X(s-)]j(X, s), \end{aligned} \tag{5.12}$$

where the sum on the right is only a finite sum for each ω, and the integral on the right is an ordinary Lebesgue–Stieltjes integral (Property 2.6). Divide (5.2) by ε, and let $\varepsilon \downarrow 0$. Then

$$\varepsilon^{-1}F[X(t)] = \varepsilon^{-1} \int_0^{X(t)} (\varepsilon \wedge y)^+ \, dy, \tag{5.13}$$

so that

$$\begin{aligned} \lim_{\varepsilon \to 0} \varepsilon^{-1}F[X(t)] &= 0, \text{ if } X(t) \leqq 0 \\ &= \lim_{\varepsilon \to 0} \left[\varepsilon^{-1} \int_0^{\varepsilon} y \, dy + \varepsilon^{-1} \int_{\varepsilon}^{X(t)} \varepsilon \, dy \right] = X(t), \end{aligned} \tag{5.14}$$

if $X(t) > 0$. That is, $\lim_{\varepsilon \to 0} \varepsilon^{-1}F[X(t)] = X(t)^+$. Since $\int_0^t F'[X(s-)]\, dX(s) = \int_0^t [\varepsilon \wedge X(s-)]^+ \, dX(s)$ is a Lebesgue–Stieltjes integral, and since $0 \leqq (\varepsilon \wedge X(s-))^+\varepsilon^{-1} \leqq 1$ with

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1}[\varepsilon \wedge X(s-)]^+ = I_{(0,\infty)}[X(s-)], \tag{5.15}$$

we may apply the dominated convergence theorem to obtain

$$\lim_{\varepsilon \downarrow 0} \int_0^t F'[X(s-)]\, dX(s) = \int_0^t I_{(0,\infty)}[X(s-)]\, dX(s). \tag{5.16}$$

Finally, to evaluate the remaining term, it suffices to evaluate only

$$\lim_{\varepsilon \downarrow 0} \left[\int_{X(s-)}^{X(s)} (\varepsilon \wedge y)^+ \, dy - [\varepsilon \wedge X(s-)]^+ j(X, s) \right] \tag{5.17}$$

for each s, since there is involved only a finite sum. There are several cases. If

$X(s) > 0 \geqq X(s-)$, and if $\varepsilon < X(s)$, then

$$(5.18)\qquad \varepsilon^{-1}\left[\int_{X(s-)}^{X(s)} (\varepsilon \wedge y)^+ \, dy - [\varepsilon \wedge X(s-)]^+ j(X, s)\right] = \varepsilon^{-1}\left[\int_0^{\varepsilon} y \, dy + \int_{\varepsilon}^{X(s)} \varepsilon \, dy\right] \to X(s).$$

Similarly, if $X(s) \leqq 0 < X(s-)$, then the resulting limit is $-X(s)$. If either $X(s), X(s-) > 0$, or $X(s), X(s-) < 0$, then the resulting limit is 0. Hence, considering all cases, one finds that

$$(5.19)\qquad \lim_{\varepsilon \downarrow 0}\left[\int_{X(s-)}^{X(s)} (\varepsilon \wedge y)^+ \, dy - [\varepsilon \wedge X(s-)]^+ j(X, s)\right] = |X(s)|$$

if there is a zero jump at s, and the limit is zero if not. This establishes the theorem for the special case. Note that in this case, the theorem is true with equality.

(b) The process X is said to have a strict zero jump at s if either $X(s-) < 0 < X(s)$ or $X(s) < 0 < X(s-)$. Suppose that $\nu(R) = \infty$. Then for almost every ω, all the zero jumps of the path $X(t)$ are strict. Here is the proof of this fact. Let $\{T_k, k \geqq 1\}$ be a sequence of stopping times that enumerates all the jumps of X. For example, let

$$(5.20)\qquad T_{n,1} = \inf\left\{t > 0; |X(t) - X(t-)| \in \left[\frac{1}{n}, \frac{1}{n-1}\right)\right\},$$

and

$$(5.21)\qquad T_{n,k+1} = \inf\left\{t > T_{n,k}; |X(t) - X(t-)| \in \left[\frac{1}{n}, \frac{1}{n-1}\right)\right\}.$$

Then $\{T_{n,k}; n \geqq 1, k \geqq 1\}$ enumerates the jumps.

The desired assertion will follow if we show that $X(T_k)$ and $X(T_k-)$ are nonatomic. Let $\varepsilon > 0$. The process $Y(t) = X(t + \varepsilon) - X(\varepsilon)$ is independent of $X(\varepsilon)$, and $Y \sim X$. Let $\{S_n; n \geqq 1\}$ be an enumeration of the jumps of $\{Y(t)\}$. Then

$$\begin{aligned}(5.22)\qquad & P\{\varepsilon < T_k, X(T_k) = 0\} \\ &= \sum_j P\{\varepsilon < T_k, T_k = S_j + \varepsilon, X(\varepsilon) + [X(S_j + \varepsilon) - X(\varepsilon)] = 0\} \\ &= \sum_j P\{\varepsilon < T_k, T_k = S_j + \varepsilon, X(\varepsilon) + Y(S_j) = 0\}.\end{aligned}$$

But $P\{\varepsilon < T_k, T_k = S_j + \varepsilon, X(\varepsilon) + Y(S_j) = 0\} \leqq P\{X(\varepsilon) + Y(S_j) = 0\}$, and $X(\varepsilon)$ is independent of $Y(S_j)$. Also, since $\nu(R) = \infty$, $X(\varepsilon)$ is nonatomic (A. Wintner and P. Hartman [20]; see also J. R. Blum and M. Rosenblatt [1]).

Since it is well known that the convolution of two measures is nonatomic if at least one of them is, $P\{X(\varepsilon) + Y(S_j) = 0\} = 0$, and so $P\{\varepsilon < T_k, X(T_k) = 0\} = 0$. Let $\varepsilon \downarrow 0$ to get the result, since $P\{T_k > 0\} = 1$ (see the construction of T_k above). The case of $X(T_k -)$ is treated similarly.

(c) Theorem 5.2 will now be verified under the assumption $\nu(R) = \infty$, and this will complete the proof. By part (b), one need consider only strict zero jumps. Let X^n be the process defined in the proof of Theorem 4.1, part (b). By choosing a subsequence if necessary, one may assume that for each ω the paths of X^n on $[0, 1]$ converge uniformly to those of X (see Property 2.5). By part (a) of the present proof,

$$\sum_{0 \leqq s \leqq t}{}' |X^n(s)| = X^n(t)^+ - \int_0^t I_{(0,\infty)}[X^n(s-)]\, dX^n(s). \tag{5.23}$$

Now let $n \to \infty$ (through the subsequence, if necessary).

Then for every ω, $X^n(t)^+ \to X(t)^+$, $0 \leqq t \leqq 1$. Also, for each t,

$$\int_0^t I_{(0,\infty)}[X^n(s-)]\, dX^n(s) \to \int_0^t I_{(0,\infty)}[X(s-)]\, dX(s) \tag{5.24}$$

in L_2. For,

$$\begin{aligned} &\left\| \int_0^t I_{(0,\infty)}[X^n(s-)]\, dX^n(s) - \int_0^t I_{(0,\infty)}[X(s-)]\, dX(s) \right\|_2 \\ &\leqq \left\| \int_0^t I_{(0,\infty)}[X^n(s-)]\, dX^n(s) - \int_0^t I_{(0,\infty)}[X^n(s-)]\, dX(s) \right\|_2 \\ &\qquad + \left\| \int_0^t \{I_{(0,\infty)}[X^n(s-)] - I_{(0,\infty)}[X(s-)]\}\, dX(s) \right\| \\ &= S_1 + S_2. \end{aligned} \tag{5.25}$$

By Property 2.2, $S_2^2 \leqq K \int_0^1 E\{I_{(0,\infty)}[X^n(s-)] - I_{(0,\infty)}[X(s-)]\}^2\, dt$. Moreover, for each ω, $I_{(0,\infty)}[X^n(s-)] \to I_{(0,\infty)}[X(s-)]$ for every fixed s (recall, for every ω, $X^n(s) \to X(s)$ uniformly for $0 \leqq s \leqq 1$), except possibly those s at which $X(s-) = 0$. But if $\nu(R) = \infty$, these s have Lebesgue measure 0, so $I_{(0,\infty)}[X^n(s-)] \to I_{(0,\infty)}[X(s-)]$ for almost all $(\omega, s)(dP \times ds)$. Therefore, by dominated convergence, $S_2 \to 0$. Also

$$S_1 = \left\| \int_0^t I_{(0,\infty)}[X^n(s-)]\, d[X^n(s) - X(s)] \right\|_2 \leqq \|X^n(t) - X(t)\|_2 \to 0 \tag{5.26}$$

as $n \to \infty$. In fact, by using the stronger inequality given in Property 2.2, it is easy to see that we have the stronger result:

$$\sup_{0 \leqq t \leqq 1} \left| \int_0^t I_{(0,\infty)}[X^n(s-)]\, dX^n(s) - \int_0^t I_{(0,\infty)}[X(s-)]\, dX(s) \right| \to 0 \tag{5.27}$$

in L_2, so by choosing a further subsequence if necessary one obtains for each ω

$$(5.28)\qquad \int_0^t I_{(0,\infty)}[X^n(s-)]\,dX^n(s) \to \int_0^t I_{(0,\infty)}[X(s-)]\,dX(s),$$

uniformly in t, $0 \leqq t \leqq 1$. Assume from now on that $n \to \infty$ through this subsequence.

Finally, consider $\lim_{n\to\infty} \Sigma'_{0\leqq s\leqq t} |X^n(s)|$. Let

$$(5.29)\qquad T_{n,1} = \inf\left\{t > 0;\ X \text{ has a zero jump at } t, \text{ and } |j(X, t)| \in \left[\frac{1}{n}, \frac{1}{n-1}\right)\right\},$$

$$(5.30)\qquad T_{n,k+1} = \inf\left\{t > T_{n,k};\ X \text{ has a zero jump at } t \text{ and } |j(X, t)| \in \left[\frac{1}{n}, \frac{1}{n-1}\right)\right\}.$$

Then $\{T_{n,k}; n \geqq 1, k \geqq 1\}$ enumerates all the zero jumps of X; let $\{T_k; k \geqq 1\}$ be a list of the $\{T_{n,k}\}$. By part (b), either $X(T_k) < 0 < X(T_k-)$ or $X(T_k-) < 0 < X(T_k)$ (for a specified ω). It therefore follows from the uniform convergence that if T_k is the time of a zero jump for X, then T_k is also the time of a (strict) zero jump for X^n, for all sufficiently large n (how large will depend on ω). Therefore, for each ω, if $T_k(\omega) \leqq t$, then

$$(5.31)\qquad |X(T_k)| \leqq \liminf_{n\to\infty} \sum_{0\leqq s\leqq t}{}' |X^n(s)| \leqq X(t)^+ - \int_0^t I_{(0,\infty)}[X(s-)]\,dX(s).$$

This clearly continues to hold if we replace the left side by any finite sum $\Sigma_{k=1}^n |X(T_k)|$. The limit on the right is taken through a subsequence for which the limit will exist for every ω and uniformly in t, $0 \leqq t \leqq 1$. The formula of Theorem 5.2 therefore holds for almost all ω, and all t, $0 \leqq t \leqq 1$, the exceptional ω set not depending on t. This completes the proof.

REMARK 5.2. Presumably the inequality of Theorem 5.2 may be replaced by equality, but I have not been able to show this. Notice that the case $\nu(R) = \infty$ could not be treated by taking a limit in the formula (5.2) directly, since the function $F(x) = \int_0^x (\varepsilon \wedge y)^+\,dy$ does not satisfy the hypothesis of Theorem 5.1. The formula of Theorem 5.2 should be compared to the formula attributed to Tanaka (see McKean [13]) for the local time of Brownian motion.

The process $Z(t)$ is not local time (Z does not have continuous paths). However, it is natural to wonder whether, if $Z(t)$ were smoothed out appropriately, the result would be similar to local time (whenever local time exists).

Note added in proof. Equality in Theorem 5.2 can be established by using the theory of Lévy systems developed by S. Watanabe ("On discontinuous additive functionals and Lévy measures of a Markov process," *Japan J. Math.*, Vol. 34 (1964), pp. 53–70). Connections between the process Z and local time are contained in recent work of Getoor and Millar ("Some limit theorems for local time," to appear in *Compositio Math.*).

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