APPROXIMATION OF CONTINUOUS ADDITIVE FUNCTIONALS

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1. Introduction

The purpose of this exposition is to give correct proofs of two well known and reasonably important propositions concerning continuous additive functionals. We adopt the terminology and notation of [1] throughout. We fix once and for all a standard process $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ with state space E. (See (I–9.2); all such references are to [1].)

The following two theorems are important facts about continuous additive functionals (CAF's) of such a process. (See (IV–2.21) or [2].)

THEOREM 1. *Let A be a* CAF *of X. Then $A = \Sigma_{n=1}^{\infty} A^n$ where each A^n is a* CAF *of X having a bounded one potential.*

Making use of Theorem 1, one can establish the following result. (See (V–2.1) or [2].)

THEOREM 2. *Suppose that X has a reference measure (that is, satisfies the hypothesis of absolute continuity). Then every* CAF *of X is equivalent to a perfect* CAF.

Unfortunately, the proofs known to me of Theorem 1 are not convincing. For example, the "proof" in [1] goes as follows. Let A be a CAF of X. Define

$$\varphi(x) = E^x \int_0^\infty e^{-t} e^{-A_t}\, dt. \tag{1.1}$$

Clearly, $0 < \varphi \leqq 1$ and φ is universally measurable; actually it is not difficult to see that φ is nearly Borel, but this is not required. Let $R = \inf\{t: A_t = \infty\}$. Then it is easy to check that R is a terminal time and that $P^x(R > 0) = 1$ for all x. Obviously, $\varphi(x) = E^x \int_0^R e^{-t} e^{-A_t}\, dt$. Now if T is any stopping time,

$$\begin{aligned} E^x\{e^{-T}\varphi(X_T); T < R\} &= E^x\left\{e^{-T} \int_0^{R\circ\theta_T} e^{-t} e^{-A_t\circ\theta_T}\, dt; T < R\right\} \\ &= E^x\left\{e^{A_T} \int_T^R e^{-u} e^{-A_u}\, du; R < T\right\}, \end{aligned} \tag{1.2}$$

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and so using T15, Chapter VII, [3], one finds

$$(1.3)\qquad \begin{aligned} U_A^1\varphi(x) &= E^x \int_0^\infty e^{-t}\varphi(X_t)\,dA_t \\ &= E^x \int_0^\infty e^{-t}\varphi(X_t) I_{[0,R)}(t)\,dA_t \\ &= E^x \int_0^R e^{A_t} \int_t^R e^{-u}e^{-A_u}\,du\,dA_t \\ &= E^x \int_0^R e^{-u}e^{-A_u}\left(\int_0^u e^{A_t}\,dA_t\right) du \\ &= E^x \int_0^R e^{-u}(1 - e^{-A_u})\,du \leqq 1. \end{aligned}$$

Next let f_n be the indicator function of $\{1/(n+1) < \varphi \leqq 1/n\}$ for $n \geqq 1$. Clearly, $\Sigma f_n = 1$ and so if we define $A_t^n = \int_0^t f_n(X_s)\,dA_s$, then $\Sigma A^n = A$. Also,

$$(1.4)\qquad \begin{aligned} E^x \int_0^\infty e^{-t}\,dA_t^n &= E^x \int_0 e^{-t} f_n(X_t)\,dA_t \\ &\leqq (n+1)E^x \int_0^\infty e^{-t}\varphi(X_t)\,dA_t \leqq n+1. \end{aligned}$$

Consequently, each A^n is a CAF of X with a bounded one potential.

The *joker*, of course, comes in this last sentence; namely, although $t \to A_t^n$ is continuous almost surely, A^n need not be an additive functional. To see the issue fix n and let $B = A^n$ and $f = f_n$. Then

$$(1.5)\qquad B_{t+s} = B_t + \int_0^s f(X_u)\circ\theta_t\,d_u A_{u+t}.$$

Now $A_{u+t} = A_t + A_u \circ \theta_t$ and so if $A_t < \infty$, $dA_{u+t} = d(A_u \circ \theta_t)$ which yields

$$(1.6)\qquad B_{t+s} = B_t + B_s \circ \theta_t$$

if $A_t < \infty$. Obviously, (1.6) holds if $B_t = \infty$, but there is no reason for (1.6) to hold on $\{A_t = \infty; B_t < \infty\}$. If $A_t = \infty$, then $A_{u+t} = \infty$ for all u and so $dA_{u+t} = 0$. Therefore, although (1.6) need not hold, at least

$$(1.7)\qquad B_{t+s} \leqq B_t + B_s \circ \theta_t.$$

However, something of value can be salvaged from this discussion. Let f_n and A^n be as above. Note that

$$(1.8)\qquad A_t^n = \int_0^t f_n(X_s)\,dA_s = \int_0^{t\wedge R} f_n(X_s)\,dA_s$$

since dA_s puts no mass on the interval $[R, \infty]$. In particular, each A^n is a CAF of (X, R) with a bounded one potential; recall that for B to be an additive functional of (X, R) we only require that $B_{t+s} = B_t + B_s \circ \theta_t$ almost surely on $\{R > t\}$ and that B is continuous at R and constant on $[R, \infty]$. Thus, we have proved the following lemma.

LEMMA 1.1. *Let A be a* CAF *of X. Then $A = \sum_{n=1}^{\infty} A^n$, where each A^n is a* CAF *of (X, R) having a bounded one potential.*

Most likely Lemma 1.1 would suffice in many situations. Still it is of interest to know that Theorem 1 is valid. The main purpose of this note is to present a proof of Theorem 1. It is not at all surprising that Lemma 1.1 will be used in our argument. Once Theorem 1 is established Theorem 2 follows as in [1]. However, because our proof of Theorem 1 is rather long, there is some interest in giving a direct proof of Theorem 2 which avoids an appeal to Theorem 1. We present such a proof in Section 2.

Although our proof of Theorem 1 is rather involved, all of the ideas and techniques that we will need are contained in Section V–5 of [1]. Since these techniques are of some interest in themselves and not particularly well known, it is perhaps worthwhile to present them here in a situation that is substantially simpler than that of Section V–5 of [1]. Consequently, we will give complete details even though this necessitates repeating certain arguments given in [1].

The key fact that we need is the following interesting result which is essentially (V–5.12).

THEOREM 3. *Let T be the hitting time of a finely open nearly Borel set and let A be a* CAF *of (X, T) with a bounded one potential. Let $\eta < 1$ and let $K = \{x : E^x(e^{-T}) < \eta\}$. Then there is a* CAF, *B of X with a bounded one potential such that for every x and $f \in \mathscr{E}_+^*$ which vanishes off K, we have*

$$E^x \int_0^T e^{-t} f(X_t)\, dA_t = E^x \int_0^T e^{-t} f(X_t)\, dB_t. \tag{1.9}$$

Most likely this theorem is true for an arbitrary exact terminal time T, but our proof makes use of the fact that T is the hitting time of a finely open set. Of course, one could easily abstract the property of T needed for the proof to go through, but this would be of very little interest.

As mentioned before, Section 2 is devoted to a proof of Theorem 2. In Section 3 we prove Theorem 1 assuming Theorem 3, while in Section 4 we prove Theorem 3.

2. Proof of Theorem 2

We begin with some preliminary facts that will also be used in Section 3. We fix an additive functional A of X and for the moment we assume only that A has no infinite discontinuity. We assume without loss of generality that $t \to A_t(\omega)$ is right continuous and nondecreasing for all ω. Recall that $A_0 = 0$ and $t \to A_t$ is continuous at ζ. We will usually omit the phrase "almost surely" in our discussions. Let $R = \inf\{t: A_t = \infty\}$. By right continuity $A_R = \infty$ if $R < \infty$ and since A has no infinite discontinuity, A is continuous at R if $R < \infty$. Of course, A is continuous at R if $R = \infty$ because $A_\infty = \lim_{t\uparrow\infty} A_t$ by convention. It is easy to see that R is a terminal time and that $P^x(R > 0) = 1$ for all x. Therefore, R is an exact terminal time. Let $R_n = \inf\{t: A_t \geqq n\}$. Then each R_n is a stopping time and $R_n < R$ when $R < \infty$ because A has no infinite discontinuity. Clearly, $\{R_n\}$ is increasing. Let $T = \lim R_n \leqq R$. Since $A(R_n) \geqq n$ on $\{R_n < \infty\}$, it is clear that $A(T) = \infty$ on $\{T < \infty\}$. Consequently, $T = R$. Thus, $\{R_n\}$ is an increasing sequence of stopping times with limit R and $R_n < R$ for all n if $R < \infty$. Let $\psi(x) = E^x(e^{-R})$. Because R is an exact terminal time ψ is 1-excessive and $0 \leqq \psi < 1$. Let $E_n = \{\psi > 1 - 1/n\}$. Then each E_n is a finely open nearly Borel set, and the E_n decrease to the empty set. Finally, let T_n be the hitting time of E_n. The following lemma is well known. Since a more general and considerably more complicated version is given in [1], we will give the proof here even though only very standard techniques are involved.

LEMMA 2.1. *Using the above notation* $T_n \leqq R$, $\lim T_n = R$, *and* $T_n < R$ *if* $R < \infty$.

PROOF. By the usual supermartingale considerations $e^{-R_n}\psi(X_{R_n}) \to e^{-R} L$ where $0 \leqq L \leqq 1$ and since R is a strong terminal time, we have, for any $\Gamma \in \mathscr{F}_{R_k}$ and $n \geqq k$,

$$E^x\{e^{-R_n}\psi(X_{R_n}); \Gamma; R_n < R\} = E^x\{e^{-R}; \Gamma; R_n < R\}. \tag{2.1}$$

Letting $n \to \infty$, we obtain

$$E^x\{e^{-R}L; \Gamma; R_n < R, \forall\, n\} = E^x\{e^{-R}; \Gamma; R_n < R, \forall\, n\} \tag{2.2}$$

for all $\Gamma \in \vee \mathscr{F}_{R_k}$. Let $\Gamma = \{R < \infty\} \in \vee \mathscr{F}_{R_k}$. Since $R_n < R$ if $R < \infty$, we see that $L = \lim \psi(X_{R_n}) = 1$ if $R < \infty$ and since ψ is 1-excessive, this yields $\lim_{t\uparrow R} \psi(X_t) = 1$ if $R < \infty$.

Now if $0 \leqq t < T_n$, $\psi(X_t) \leqq 1 - 1/n$, and consequently $T_n < R$ if $R < \infty$ because $\lim_{t\uparrow R} \psi(X_t) = 1$. Hence, $T_n \leqq R$ and $T_n < R$ if $R < \infty$. Also, $\psi(X_{T_n}) \geqq 1 - 1/n$ if $T_n < \infty$ and so

$$E^x\{e^{-(R-T_n)}; T_n < R\} = E^x\{\psi(X_{T_n}); T_n < R\} \geqq \left(1 - \frac{1}{n}\right) P^x(T_n < R). \tag{2.3}$$

Letting $n \to \infty$, we see that $\lim T_n = R$ on $\{T_n < R; \forall\, n\}$. But $\lim T_n = R$ on $\{T_n = R$ for some $n\}$, and so Lemma 2.1 is established.

The importance of Lemma 2.1 is that the T_n are *hitting* times of finely open sets and hence are *perfect exact* terminal times.

We now are ready to prove Theorem 2. We assume that A is a CAF of X and we will use the notation developed above. Define $B_t^n = A(t \wedge T_n)$. Then each B^n is a CAF of (X, T_n) and B^n is finite on $[0, T_n)$: this is clear if $R < \infty$ because then $T_n < R$ and it is true *a priori* if $R = \infty$. But $I_{[0,T_n)}(t)$ is a perfect multiplicative functional of X and so it follows from (V-2.1) that each B^n is perfect. (The proof of (V-2.1) is valid for all CAF's of (X, M) which are finite on $[0, S)$ where

$S = \inf\{t: M_t = 0\}$.) As a result for each n there exists $\Lambda_n \in \mathscr{F}$ with $P^x(\Lambda_n) = 0$ for all x such that if $\omega \notin \Lambda_n$, $B^n_{t+s} = B^n_t + B^n_s \circ \theta_t I_{[0,T_n)}(t)$ identically in t and s. Let $\Lambda_0 = \{\lim T_n \neq R\}$ and $\Lambda = \bigcup_{n \geqq 0} \Lambda_n$. The proof of Theorem 2 is completed by observing that

$$\{A_{u+t} \neq A_t + A_u \circ \theta_t \text{ for some } t \text{ and } u\} \subset \Lambda. \tag{2.4}$$

3. Proof of Theorem 1

Let A be a CAF of X. Then by Lemma 1.1 we can write $A = \Sigma A^n$ where each A^n is a CAF of (X, R) with a bounded one potential.

LEMMA 3.1. *Let B be a* CAF *of (X, R) with a bounded one potential. Then there exist* CAF's B^n *of X, each having a bounded one potential such that $B_t = \Sigma B^n_t$ if $t < R$.*

Before coming to the proof of this lemma, let us use it to prove Theorem 1. Applying Lemma 3.1 to each A^n, we have

$$A_t = \sum_n A^n_t = \sum_n \sum_k A^{n,k}_t \qquad \text{if} \quad t < R, \tag{3.1}$$

where each $A^{n,k}$ is a CAF of X with a bounded one potential. But if $t \geqq R$, $A_t = \infty$, and since the double sum in (3.1) is monotone in t, it also must be infinite if $t \geqq R$. Thus, (3.1) holds for all t establishing Theorem 1.

It remains to prove Lemma 3.1. We do this assuming Theorem 3 which will be proved in Section 4. As in Section 2 let $\psi(x) = E^x(e^{-R})$ and let T_n be the hitting time of the finely open set $E_n = \{\psi > 1 - 1/n\}$. Then $T_n \uparrow R$ according to Lemma 2.1. Let $G_n = \{\psi \leqq 1 - 1/n\}$ and let $\varphi_n(x) = E^x(e^{-T_n})$. Next define $K^{n,k} = \{\varphi_n < 1 - 1/k\}$. It is immediate that $K^{n,k}$ increases with both n and k, and so if we let $K_n = K^{n,n}$ then $K_n \subset G_n$ for each n and $\bigcup K_n = E$. Now $t \to B(t \wedge T_n)$ is a CAF of (X, T_n) with a bounded one potential and so by Theorem 3 there exists a CAF, C^n, of X with a bounded one potential such that if $f \in \mathscr{E}^*_+$ and vanishes off K_n then

$$E^x \int_0^{T_n} e^{-t} f(X_t)\, dB_t = E^x \int_0^{T_n} e^{-t} f(X_t)\, dC^n_t. \tag{3.2}$$

We need the following compatibility relationship: if $f \geqq 0$ vanishes off K_n, then for all m

$$E^x \int_0^{T_m} e^{-t} f(X_t)\, dC^n_t = E^x \int_0^{T_m} e^{-t} f(X_t)\, dB_t. \tag{3.3}$$

Suppose firstly that $m < n$. It follows from (3.2) that

$$\bar{B}_t = \int_0^{t \wedge T_n} I_{K_n}(X_u)\, dB_u, \qquad \bar{C}_t = \int_0^{t \wedge T_n} I_{K_n}(X_u)\, dC^n_u \tag{3.4}$$

define CAF's of (X, T_n) with the same bounded one potential. Consequently, by the uniqueness theorem for CAF'S, $\bar{B} = \bar{C}$ (that is, $\bar{B}$ and $\bar{C}$ are equivalent). But $T_m \leqq T_n$ and hence (3.3) holds if $m < n$.

Next suppose that $m > n$. Then $K_n \subset G_n \subset G_m$. Recall that $E_m = E - G_m$ and T_m is the hitting time of E_m. Let S be the hitting time $K_n \cup E_m$ and define stopping times as follows: $S_0 = 0$,

$$S_{2k+1} = S_{2k} + T_n \circ \theta_{S_{2k}}, S_{2k+2} = S_{2k+1} + S \circ \theta_{S_{2k+2}}, \tag{3.5}$$

for $k \geqq 0$. Then $\{S_k\}$ forms an increasing sequence of stopping times and since E_m is finely open, $S_k \leqq T_m$ for all k. Also, $X(S_{2k}) \in K_n$ if $S_{2k} < T_m$ and using the definition of K_n this yields

$$\begin{aligned} E^x\{e^{-S_{2k+1}}; S_{2k+1} < T_m\} &\leqq E^x\{\exp\{-(S_{2k} + T_n \circ \theta_{S_{2k}})\}; S_{2k} < T_m\} \\ &\leqq (1 - 1/n) E^x\{e^{-S_{2k}}; S_{2k} < T_m\} \\ &\leqq (1 - 1/n) E^x\{e^{-S_{2k-1}}; S_{2k-1} < T_m\}. \end{aligned} \tag{3.6}$$

Consequently, $\lim S_k = T_m$. But f vanishes off K_n and $X_t \notin K_n$ if $S_{2k+1} \leqq t < S_{2k+2}$. As a result using (3.2), we obtain

$$\begin{aligned} E^x \int_0^{T_m} e^{-t} f(X_t)\, dB_t &= \sum_{k=0}^{\infty} E^x \int_{S_{2k}}^{S_{2k+1}} e^{-t} f(X_t)\, dB_t \\ &= \sum_{k=0}^{\infty} E^x \left\{ e^{-S_{2k}} E^{X(S_{2k})} \int_0^{T_n} e^{-t} f(X_t)\, dB_t \right\} \\ &= \sum_{k=0}^{\infty} E^x \left\{ e^{-S_{2k}} E^{X(S_{2k})} \int_0^{T_n} e^{-t} f(X_t)\, dC_t^n \right\} \\ &= E^x \int_0^{T_m} e^{-t} f(X_t)\, dC_t^n. \end{aligned} \tag{3.7}$$

Thus, (3.3) is established since it reduces to (3.2) when $m = n$.

Now disjoint the K_n: $J_1 = K_1, \cdots, J_n = K_n - \bigcup_{j<n} K_j$. Thus, $\{J_n\}$ is a disjoint sequence of nearly Borel sets such that $\bigcup J_n = E$ and $J_n \subset K_n$ for each n. Define

$$B_t^n = \int_0^t I_{J_n}(X_s)\, dC_s^n. \tag{3.8}$$

Each B^n is a CAF of X with a bounded one potential. Let $C_t = \Sigma B_t^n$ and let

$f \in \mathscr{E}_+^*$. Then for each n

$$(3.9)\qquad E^x \int_0^{T_n} e^{-t} f(X_t)\, dC_t = \sum_k E^x \int_0^{T_n} e^{-t} (f I_{J_k})(X_t)\, dC_t^k$$

$$= \sum_k E^x \int_0^{T_n} e^{-t} (f I_{J_k})(X_t)\, dB_t$$

$$= E^x \int_0^{T_n} e^{-t} f(X_t)\, dB_t,$$

and letting $n \to \infty$, we obtain

$$(3.10)\qquad E^x \int_0^R e^{-t} f(X_t)\, dC_t = E^x \int_0^R e^{-t} f(X_t)\, dB_t.$$

Since $R > 0$ almost surely, this implies that $t \to C_t$ is finite on $[0, R)$ and it is then easy to see that C is a CAF of X. Once again the uniqueness theorem for CAF's tells us that $B_t = C_t$ if $t < R$. But $C = \Sigma B^n$ where each B^n is a CAF of X with a bounded one potential, and so Lemma 3.1 is established.

4. Proof of Theorem 3

The proof of Theorem 3 is rather long and so we will break it up into several lemmas. We refer the reader to Section 1 for the statement of Theorem 3. We begin with some notation that will be used throughout the proof. Let G be the finely open set such that $T = T_G$. Let $\psi(x) = E^x(e^{-T})$. Then $K = \{\psi < \eta\}$ where $\eta < 1$ and $K \subset \{\psi \leqq \eta\} \subset E - G$. Define $T_0 = 0$ and for $n \geqq 0$

$$(4.1)\qquad T_{2n+1} = T_{2n} + T \circ \theta_{T_{2n}}, \qquad T_{2n+2} = T_{2n+1} + T_K \circ \theta_{T_{2+1}}.$$

Thus, $\{T_n\}$ is an increasing sequence of stopping times, and for any x and $n \geqq 1$

$$(4.2)\qquad \begin{aligned} E^x\{e^{-T_{2n+1}}; T_{2n} < \infty\} &= E^x\{e^{-T_{2n}} \psi(X_{T_{2n}}); T_{2n} < \infty\} \\ &\leqq \eta E^x\{e^{-T_{2n}}; T_{2n} < \infty\} \\ &\leqq \eta E^x\{e^{-T_{2n-1}}; T_{2n-2} < \infty\} \end{aligned}$$

because $\psi(X_{T_{2n}}) \leqq \eta$ if $T_{2n} < \infty$ and $n \geqq 1$. As a result $\lim T_n = \infty$.

Suppose for the moment that there is a CAF, B of X for which the conclusion of Theorem 3 holds. If we define

$$(4.3)\qquad u(x) = E^x \int_0^T e^{-t} I_K(X_t)\, dA_t = U_A^1 I_K(x),$$

then because $X_t \notin K$ if $T_{2n-1} \leqq t < T_{2n}$ we can compute $U_B^1 I_K(x)$ as follows

$$(4.4)\qquad \begin{aligned} U_B^1 I_K(x) &= E^x \int_0^\infty e^{-t} I_K(X_t)\, dB_t \\ &= \sum_{n=0}^\infty E^x \int_{T_{2n}}^{T_{2n+1}} e^{-t} I_K(X_t)\, dB_t \\ &= \sum_{n=0}^\infty E^x \left\{ e^{-T_{2n}} E^{X(T_{2n})} \int_0^T e^{-t} I_K(X_t)\, dB_t \right\} \\ &= \sum_{n=0}^\infty E^x \{ e^{-T_{2n}} u(X_{T_{2n}}) \}. \end{aligned}$$

The main part of the proof of Theorem 3 consists in showing that if we *define*

$$(4.5)\qquad w(x) = \sum_{n=0}^\infty E^x \{ e^{-T_{2n}} u(X_{T_{2n}}) \},$$

then w is a regular one potential of X, and hence the one potential of CAF of X. By hypothesis, u is bounded and since

$$(4.6)\qquad w(x) \leqq \|u\| \sum_{n=0}^\infty E^x(e^{-T_{2n}}) \leqq \|u\| \sum_{n=0}^\infty \eta^n < \infty,$$

w is also bounded.

LEMMA 4.1. *Let K be as above. Then $w = P_K^1 w$.*

PROOF. For typographical simplicity let $Q = T_K$. Then

$$(4.7)\qquad \begin{aligned} P_K^1 w(x) &= E^x \{ e^{-Q} w(X_Q) \} \\ &= \sum_{n=0}^\infty E^x \{ \exp\{-(Q + T_{2n} \circ \theta_Q)\} u(X_{Q + T_{2n} \circ \theta_Q}) \}. \end{aligned}$$

Break each summand into an integral over $\{Q < T_1\}$ and over $\{Q \geqq T_1\}$. A straightforward induction argument shows that if $k \geqq 1$, $Q + T_k \circ \theta_Q = T_k$ on $\{Q < T_1\}$. On the other hand if $Q \geqq T_1$, then $Q = T_2$. But then $Q + T_1 \circ \theta_Q = T_2 + T \circ \theta_{T_2} = T_3$ and again one sees by induction that for $k \geqq 0$, $Q + T_k \circ \theta_Q = T_{k+2}$ if $Q \geqq T_1$. Consequently,

$$(4.8)\qquad P_K^1 w(x) = E^x \{ e^{-Q} u(X_Q); Q < T_1 \} + \sum_{n=1}^\infty E^x \{ e^{-T_{2n}} u(X_{T_{2n}}) \}.$$

Therefore,

$$(4.9)\qquad w(x) - P_K^1 w(x) = u(x) - E^x \{ e^{-Q} u(X_Q); Q < T_1 \}.$$

But $T_1 = T$, $Q = T_K$, and using the definition of u (see (4.3)), we obtain

$$(4.10)\qquad E^x \{ e^{-Q} u(X_Q); Q < T_1 \} = E^x \int_0^T e^{-t} I_K(X_t)\, dA_t = u(x).$$

Therefore, $w = P_K^1 w$, completing the proof of Lemma 4.1.

LEMMA 4.2. *If J is any compact set, then $P_J^1 w \leqq w$.*

PROOF. Let $S = T_J + Q \circ \theta_{T_J}$ where $Q = T_K$ as before. Now $X_S \in K \cup K^r$ if $S < \infty$. But $X_t \notin K \cup K^r$ if $T_{2n+1} \leqq t < T_{2n+2}$, and so $\{S < \infty\} = \cup_n \{T_{2n} \leqq S < T_{2n+1}\}$. Also, it is easy to check by induction that for $k \geqq 0$, $T_{k+2} = T_2 + T_k \circ \theta_{T_2}$. Hence,

$$w(x) = u(x) + \sum_{n=1}^{\infty} E^x\{e^{-T_{2n}} u(X_{T_{2n}})\} \tag{4.11}$$
$$= u(x) + E^x\{e^{-T_2} w(X_{T_2})\}.$$

Again one checks that for $k \geqq 1$, $S + T_k \circ \theta_S = T_k$ if $S < T_1$. Now $\{S < T_1\} \in \mathscr{F}_{T_1} \subset \mathscr{F}_{T_2}$ and so

$$E^x\{e^{-S} w(X_S); S < T_1\} \tag{4.12}$$
$$= E^x\{e^{-S} u(X_S); S < T_1\} + E^x\{e^{-T_2} w(X_{T_2}); S < T_1\}.$$

Using (4.11) and the fact that u is one (X, T) excessive, we obtain

$$E^x\{e^{-S} w(X_S); S < T_1\} + E^x\{e^{-T_2} w(X_{T_2}); S \geqq T_1\} \leqq w(x). \tag{4.13}$$

We next prove by induction that for all $n \geqq 1$

$$w(x) \geqq E^x\{e^{-S} w(X_S); S < T_{2n}\} + E^x\{e^{-T_{2n}} w(X_{T_{2n}}); S \geqq T_{2n}\}. \tag{4.14}$$

If $n = 1$, this reduces to (4.13) because S lies in some interval $[T_{2k}, T_{2k+1})$ when S is finite. Assume (4.14) for a fixed value of n. The second summand on the right side of (4.14) may be written

$$E^x\{e^{-S} w(X_S): S = T_{2n}\} + E^x\{e^{-T_{2n}} w(X_{T_{2n}}); S > T_{2n}\}. \tag{4.15}$$

It is immediate that if $T_{2n} < S$ then $T_{2n-1} < T_J$. Recall that $S = T_J + Q \circ \theta_{T_J}$ and $T_{2n} = T_{2n-1} + Q \circ \theta_{T2n-1}$. But this together with the fact that K is finely open implies that $T_{2n} < T_J$ if $T_{2n} < S$. Consequently, $T_{2n} + S \circ \theta_{T_{2n}} = S$ if $T_{2n} < S$. Combining these observations with (4.13), we obtain

$$\begin{aligned} &E^x\{e^{-T_{2n}} w(X_{T_{2n}}); S > T_{2n}\} \\ &\quad \geqq E^x\{e^{-T_{2n}} E^{X(T_{2n})}[e^{-S} w(X_S); S < T_1]; S > T_{2n}\} \\ &\quad\quad + E^x\{e^{-T_{2n}} E^{X(T_{2n})}[e^{-T_2} w(X_{T_2}); S \geqq T_1]; S > T_{2n}\} \\ &\quad = E^x\{e^{-S} w(X_S); T_{2n} < S < T_{2n+1}\} \\ &\quad\quad + E^x\{e^{-T_{2n+2}} w(X_{T_{2n+2}}); S \geqq T_{2n+1}\}. \end{aligned} \tag{4.16}$$

But $\{T_{2n} < S < T_{2n+1}\} = \{T_{2n} < S < T_{2n+2}\}$ and $\{S \geqq T_{2n+1}\} = \{S \geqq T_{2n+2}\}$. As a result (4.14) holds with n replaced by $n + 1$, and hence it holds for all $n \geqq 1$. Now $\lim T_n = \infty$ and so letting $n \to \infty$ in (4.14), we obtain $w \geqq P_S^1 w$. But $P_S^1 w = P_J^1 P_K^1 w = P_J^1 w$ since $w = P_K^1 w$ by Lemma 4.1, completing the proof of Lemma 4.2.

LEMMA 4.3. *The function w is 1-excessive.*

PROOF. In light of Lemma 4.2 and Dynkin's theorem (II–5.3), it will suffice to show that $\liminf_{t \downarrow 0} P_t^1 w(x) \geqq w(x)$ for all x. Suppose first of all that x is not regular for K. Then almost surely P^x, $t + Q \circ \theta_t = Q$ for t sufficiently small, and since $w = P_K^1 w$ this yields

$$\begin{aligned}(4.17) \qquad \lim_{t \to 0} P_t^1 w(x) &= \lim_{t \to 0} P_t^1 P_K^1 w(x) \\ &= \lim_{t \to 0} E^x\{\exp\{-(t + Q \circ \theta_t)\} w(X_{t + Q \circ \theta_t})\} \\ &= P_K^1 w(x) = w(x).\end{aligned}$$

Suppose on the other hand that x is regular for K. Then $P^x(t < T) \to 1$ as $t \to 0$ and so using (4.11) with $T = T_1$,

$$\begin{aligned}(4.18) \qquad P_t^1 w(x) &\geqq E^x\{e^{-t} w(X_t); t < T\} \\ &= E^x\{e^{-t} u(X_t); t < T\} + E^x\{e^{-T_2} w(X_{T_2}); t < T\}.\end{aligned}$$

Because u is $1 - (X, T)$ excessive this approaches $u(x) + E^x\{e^{-T_2} w(X_{T_2})\} = w(x)$ as $t \to 0$, completing the proof of Lemma 4.3.

LEMMA 4.4. *The function w is a regular one potential.*

PROOF. We must show that if $\{S_n\}$ is an increasing sequence of stopping times with limit S, then $P_{S_n}^1 w \to P_S^1 w$. It follows from (IV–3.6) and (IV–3.8) that we need consider only the case $S_n = T_{B_n}$ where $\{B_n\}$ is a decreasing sequence of nearly Borel sets. In particular each S_n is a strong terminal time and consequently so is their limit S. In checking that $P_{S_n}^1 w(x) \to P_S^1 w(x)$, we may assume that $P^x(S_n > 0) = 1$ since if $S_n = 0$ for all n the conclusion is obvious. Now fix x and let

$$(4.19) \qquad a_{n,k} = E^x\{e^{-S_n} w(X_{S_n}); T_k < S_n \leqq T_{k+1}\}$$

and

$$(4.20) \qquad a_k = E^x\{e^{-S} w(X_S); T_k < S \leqq T_{k+1}\}.$$

Then $P_{S_n}^1 w(x) = \Sigma_k a_{n,k}$ and $P_S^1 w(x) = \Sigma_k a_k$. It will suffice to show that for each k, $a_{n,k} \to a_k$ as $n \to \infty$ because $\Sigma_{k \geqq N} a_{n,k} \leqq \|w\| E^x(e^{-T_N}) \to 0$ as $N \to \infty$. Suppose first of all that k is even, say $k = 2j$. If R is any strong terminal time then on $\{T_{2j} < R \leqq T_{2j+1}\}$ we have $R = T_{2j} + R \circ \theta_{T_{2j}}$, and also because T is the hitting time of a finely open set $R + T_2 \circ \theta_R = T_{2j+2}$. Now using (4.11), we obtain for any strong terminal time R

$$\begin{aligned}(4.21) \qquad & E^x\{e^{-R} w(X_R); T_{2j} < R \leqq T_{2j+1}\} \\ &= E^x\{e^{-R} u(X_R); T_{2j} < R; R \circ \theta_{T_{2j}} \leqq T \circ \theta_{T_{2j}}\} \\ &\quad + E^x\{e^{-R} E^{X(R)}[e^{-T_2} w(X_{T_2})]; T_{2j} < R \leqq T_{2j+1}\} \\ &= E^x\{e^{-T_{2j}} E^{X(T_{2j})}[e^{-R} u(X_R); R \leqq T]; T_{2j} < R\} \\ &\quad + E^x\{e^{-T_{2j+2}} w(X_{T_{2j+2}}); T_{2j} < R \leqq T_{2j+1}\}.\end{aligned}$$

In (4.21), we may replace R by either S_n or S. Observe that the set $\{T_{2j} < S_n\}$ approaches the set $\{T_{2j} < S\}$ as $n \to \infty$ and that $\{T_{2j} < S_n \leqq T_{2j+1}\}$ approaches $\{T_{2j} < S \leqq T_{2j+1}\}$ as $n \to \infty$. Now u is a regular one potential of (X, T) since it is the one potential of a CAF of (X, T), and $u(X_T) = 0$ because X_T is regular for G; recall $T = T_G$ with G finely open. As a result for any y

$$
\begin{aligned}
(4.22) \qquad E^y\{e^{-S_n}u(X_{S_n}); S_n \leqq T\} &= E^y\{e^{-S_n}u(X_{S_n}); S_n < T\} \\
&\to E^y\{e^{-S}u(X_S); S < T\} \\
&= E^y\{e^{-S}u(X_S); S \leqq T\},
\end{aligned}
$$

as $n \to \infty$. Consequently, $a_{n,2j} \to a_{2j}$ as $n \to \infty$. Next consider the case in which k is odd, say $k = 2j + 1$. Using the fact that $w = P_K^1 w$, we obtain

$$
(4.23) \quad a_{n,2j+1} = E^x\{\exp\{-S_n + T_K \circ \theta_{S_n}\}\, w(X_{S_n + T_K \circ \theta_{S_n}}); T_{2j+1} < S_n \leqq T_{2j+2}\}
$$

and a similar expression for a_{2j+1} with S_n replaced by S. But on $\{T_{2j+1} < S_n \leqq T_{2j+2}\}$ we have $S_n + T_K \circ \theta_{S_n} = T_{2j+2}$ while on $\{T_{2j+1} < S \leqq T_{2j+2}\}$, $S + T_K \circ \theta_S = T_{2j+2}$ because K is finely open. From this and the fact that $S_n \uparrow S$, it is immediate that $a_{n,2j+1} \to a_{2j+1}$ as $n \to \infty$. This completes the proof of Lemma 4.4.

We are now prepared to complete the proof of Theorem 3. Since w is a regular one potential there is a CAF, B of X such that $w = U_B^1 1$, that is, w is the one potential of B. Now $D_t = B_{t \wedge T}$ is a CAF of (X, T) and

$$
(4.24) \qquad U_D^1 1(x) = E^x \int_0^T e^{-t}\, dB_t = w(x) - E^x\{e^{-T_1} w(X_{T_1})\}.
$$

From Lemma 4.1

$$
(4.25) \qquad E^x\{e^{-T_1} w(X_{T_1})\} = E^x\{e^{-T_2} w(X_{T_2})\},
$$

and so by (4.11), $U_D^1 1 = u$. Hence, D and $t \to \int_0^t I_K(X_u)\, dA_u$ are equivalent CAF's of (X, T). Therefore, $E^x \int_0^T e^{-t} f(X_t)\, dA_t = E^x \int_0^T e^{-t} f(X_t)\, dB_t$ if f vanishes off K, completing the proof of Theorem 3.

REFERENCES

[1] R. M. BLUMENTHAL and R. K. GETOOR, *Markov Processes and Potential Theory*. New York, Academic Press, 1968.

[2] C. DOLÉANS-DADE, "Fonctionnelles additives parfaites," *Séminaire de Probabilités II Université de Strasbourg. Lecture Notes in Mathematics*. Vol. 51. Berlin-Heidelberg-New York, Springer-Verlag, 1968.

[3] P. A. MEYER, *Probability and Potentials*, Waltham, Blaisdell, 1966.