CLASSICAL POTENTIAL THEORY AND BROWNIAN MOTION

SIDNEY C. PORT and CHARLES J. STONE
UNIVERSITY OF CALIFORNIA, LOS ANGELES

1. Introduction

Pioneering work of Doob, Kac, and Kakutani showed that there was a beautiful and deep connection between certain problems in the study of Brownian motion and those of classical potential theory. This work stimulated much research on the theory of Markov processes. In spite of all this work, however, there doesn't appear anywhere in the literature any reasonably complete treatment of the connection between potential theory and Brownian motion. In this paper and its companion "Logarithmic Potentials and Planar Brownian Motion" which follows in this volume, we present this connection in a way that is both elementary and essentially selfcontained. Our treatment here is not complete and will be expanded upon in a forthcoming monograph.

This paper, being basically expository in nature, contains essentially nothing new. Its novelty (if any) consists in the treatment given to the topics discussed. In one place, however, we do seem to have a result that is new. This is in finding all bounded solutions of the modified Dirichlet problem for any arbitrary open set G, and in giving a necessary and sufficient condition for there to be a unique such solution.

In this paper, we consider a Brownian motion process X_t in $n \geqq 2$ dimensional Euclidean space R^n. Let B be a Borel set and set

$$V_B = \inf\{t > 0 : X_t \in B\}, \qquad V_B = \infty \text{ if } X_t \notin B \text{ for all } t > 0. \tag{1.1}$$

In Section 2, we present preliminary facts about Brownian motion that are needed to develop classical potential theory from a probabilistic point of view. A set B is called polar if $P_x(V_B < \infty) \equiv 0$. A point x is called regular for B if $P_x(V_B = 0) = 1$. In Section 3, we prove that the set of points in B that are not regular for B is a polar set. In this section, we also show that points are polar and gather together a few more facts of a technical nature that are needed for work in the later sections. The Dirichlet problem for an arbitrary open set G is discussed in Section 4.

Starting with Section 5 and throughout the remainder of this paper, we assume that we are dealing with Brownian motion in $n \geqq 3$ dimensions. (The planar

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case is discussed in the companion paper.) In Section 5, we introduce Newtonian potentials and show how they are related to the Brownian motion process. We also prove that $|X_t| \to \infty$ with probability one as $t \to \infty$ and prove the extended maximum principle for potentials.

The remaining two sections are devoted to developing the notion of capacitary measure and potentials from a probabilistic view point.

2. Preliminaries

A Brownian motion process in $n \geqq 2$ dimensional Euclidean space R^n is a stochastic process X_t, $0 \leqq t < \infty$, having the following properties: (i) for each $t > 0$ and $h \geqq 0$, $X_{t+h} - X_h$ has the normal density $p(t, x) = (2\pi t)^{-n/2} \exp\{-|x|^2/2t\}$; and (ii) for $0 < t_1 < t_2 < \cdots < t_n < \infty$, $X_{t_1} - X_0$, $X_{t_2} - X_{t_1}$, $\cdots$, $X_{t_n} - X_{t_{n-1}}$ are independent random variables. It is well known that a version of this process can be selected so that the sample functions $t \to X_t$ are continuous with probability one. Henceforth, we will always assume that our process has this continuity property.

The distribution of X_t depends on the distribution of X_0. We let $P_x(\cdot)$ denote the probability of $\cdot$ given $X_0 = x$ and we let E_x denote expectation relative to P_x.

Let B be any Borel set. The *first hitting* time V_B of B is defined by $V_B = \inf\{t > 0: X_t \in B\}$ if $X_t \in B$ for some $t > 0$. If $X_t \notin B$ for all $t > 0$, we define $V_B = \infty$. When $V_B < \infty$, the *first hitting place* in B is the random variable X_{V_B}.

In the sequel, we will need to know some continuity properties of $P_x(V_B \leqq t)$. These are given by:

PROPOSITION 2.1. *The function $P_x(V_B \leqq t)$ is a continuous function in t for $t > 0$ for fixed x and a lower semicontinuous function in x for fixed $t > 0$.*

PROOF. Suppose for some $t > 0$, $P_x(V_B = t) > 0$. Then for any h, $0 < h < t$,

$$0 < P_x(V_B = t) \leqq \int_{R^n} p(h, y - x) P_y(V_B = t - h)\, dy, \tag{2.1}$$

so

$$\int_{R^n} P_y(V_B = t - h)\, dy > 0 \tag{2.2}$$

for all h, $0 < h < t$. But this is impossible because for any $r > 0$,

$$\int_{|y| \leqq r} P_y(V_B \leqq t)\, dy \leqq \int_{|y| \leqq r} dy < \infty. \tag{2.3}$$

Thus the measure

$$\int_{|y| \leqq r} P_y(V_B \in dt)\, dy \tag{2.4}$$

can have only countably many atoms.

To establish the lower semicontinuity in x for fixed $t > 0$ note that

$$(2.5) \qquad \int_{R^n} p(h, y - x)P_y(V_B < t - h)\,dy = P_x\{X_s \in B \text{ for some } s \in (h, t)\}$$

is a continuous function in x that increases to $P_x(V_B < t)$ as $h \downarrow 0$. Thus, $P_x(V_B < t) = P_x(V_B \leqq t)$ is lower semicontinuous in x. This establishes the proposition.

The next fact we prove tells us that the mean time to leave any *bounded* set B is finite.

PROPOSITION 2.2. *Let B be relatively compact. Then* $\sup_x E_x V_{B^c} < \infty$.

PROOF. If $x \in (\bar{B})^c$ then $P_x(V_{B^c} = 0) = 1$, so only points $x \in \bar{B}$ need be considered. Let $t > 0$. Then

$$(2.6) \qquad \inf_{x \in \bar{B}} P_x(X_t \in B^c) = \delta > 0,$$

and thus for any $x \in \bar{B}$,

$$(2.7) \qquad P_x(V_{B^c} \leqq t) \geqq P_x(X_t \in B^c) \geqq \delta.$$

Hence, for $x \in \bar{B}$,

$$(2.8) \qquad P_x(V_{B^c} > t) \leqq 1 - \delta.$$

But then

$$(2.9) \qquad P_x(V_{B^c} > 2t) = \int_{\bar{B}} P_x(V_{B^c} > t, X_t \in dz)P_z(V_{B^c} > t) \leqq (1 - \delta)^2,$$

and by induction, for $x \in \bar{B}$,

$$(2.10) \qquad P_x(V_{B^c} > nt) \leqq (1 - \delta)^n.$$

Hence,

$$(2.11) \qquad \sup_{x \in \bar{B}} E_x V_{B^c} \leqq \frac{t}{\delta} < \infty,$$

as desired.

Let A and B be any Borel sets. Then clearly

$$(2.12) \qquad P_x(X_t \in A) = P_x(V_B \leqq t, X_t \in A) + P_x(V_B > t, X_t \in A).$$

Now

$$(2.13) \qquad P_x(V_B \leqq t, X_t \in A) = \int_{0-}^{t} \int_{\bar{B}} P_x(X_t \in A \,|\, V_B = s, X_{V_B} = z)P_x(V_B \in ds, X_{V_B} \in dz).$$

Brownian motion, along with a large variety of Markov processes, possesses the important property that for $s \leqq t$,

$$P_x(X_t \in A \,|\, V_B = s, X_{V_B} = z) = P_z(X_{t-s} \in A). \tag{2.14}$$

This property is a consequence of the *strong Markov property*. It is certainly intuitively plausible that (2.14) should hold and a rigorous proof is not difficult to supply. Since the proof would involve us in a more thorough discussion of the measure theoretic structure of a Brownian motion process than we care to go into in this paper, we will omit the proof. (The interested reader should consult Chapter I of [1] for a complete discussion of the strong Markov property.)

Using (2.14), we obtain from (2.12) that

$$P_x(X_t \in A) - \int_{0-}^{t} \int_{\bar{B}} P_x(V_B \in ds, X_{V_B} \in dz) P_z(X_{t-s} \in A) = P_x(V_B > t, X_t \in A). \tag{2.15}$$

The left side of (2.15) has

$$p(t, y - x) - \int_{0-}^{t} \int_{\bar{B}} P_x(V_B \in ds, X_{V_B} \in dz) p(t - s, y - z) \tag{2.16}$$

as a density. Fatou's lemma shows that

$$\int_{0-}^{t} \int_{\bar{B}} P_x(V_B \in ds, X_{V_B} \in dz) p(t - s, y - z) \tag{2.17}$$

is a lower semicontinuous function in y. As the left side of (2.15) is absolutely continuous, the measure $P_x(V_B > t, X_t \in dy)$ is also absolutely continuous. Let $q(t, x, y)$ be any density of $P_x(V_B > t, X_t \in dy)$. Then

$$p(t, y - x) - \int_{0-}^{t} \int_{\bar{B}} P_x(V_B \in ds, X_{V_B} \in dz) p(t - s, y - z) = q(t, x, y) \quad \text{a.e. } y. \tag{2.18}$$

Consequently, the left side of (2.18) is $\geqq 0$ a.e. y and being upper semicontinuous, it is $\geqq 0$ for all y. We may therefore use the left side of (2.18) to define a density for $P_x(V_B > t, X_t \in dy)$ for all y. Denote this density by $q_B(t, x, y)$. Then for *all* y,

$$p(t, y - x) - \int_{0-}^{t} \int_{\bar{B}} P_x(V_B \in ds, X_{V_B} \in dz) p(t - s, y - z) = q_B(t, x, y). \tag{2.19}$$

Henceforth in this paper, q_B will always denote this density.

The densities $p(t, x)$ have the semigroup property:

$$p(t + s, y - x) = \int_{R^n} p(t, z - x) p(s, y - z)\, dz. \tag{2.20}$$

Define P^t on the bounded or nonnegative measurable functions by

$$P^t f(x) = \int_{R^n} p(t, y - x) f(y)\, dy = E_x f(X_t). \tag{2.21}$$

Then

$$P^{t+s}f = P^t(P^s f). \tag{2.22}$$

Since Brownian motion is a Markov process,

$$P_x(V_B > t + s, X_{t+s} \in A) = \int_{B^c} P_x(V_B > t, X_t \in dz) P_z(V_B > s, X_s \in A), \tag{2.23}$$

and thus for almost all y

$$q_B(t + s, x, y) = \int_{B^c} q_B(t, x, z) q_B(s, z, y)\, dz. \tag{2.24}$$

We will now show that (2.24) holds for *all* $y \in R^n$. To do this note first that by using (2.20) and (2.19), it is easily verified that

$$\lim_{\varepsilon \downarrow 0} \int_{R^n} q_B(t - \varepsilon, x, z) p(\varepsilon, y - z)\, dz = q_B(t, x, y). \tag{2.25}$$

Using this and the fact that (2.24) holds for a.e. y, we see that (2.24) holds for all y.

Let $\lambda > 0$. (Henceforth, λ will always denote a positive real number.) Define

$$\begin{aligned} g^\lambda(x) &= \int_0^\infty p(t, x)\, e^{-\lambda t}\, dt, \\ g_B^\lambda(x, y) &= \int_0^\infty q_B(t, x, y)\, e^{-\lambda t}\, dt, \end{aligned} \tag{2.26}$$

and

$$\Pi_B^\lambda(x, dz) = \int_0^\infty e^{-\lambda t} P_x(V_B \in dt, X_{V_B} \in dz). \tag{2.27}$$

The quantities corresponding to these for $\lambda = 0$ are denoted by the same symbol without the λ (for example, Π_B^λ for $\lambda = 0$ is Π_B). For future reference, we note that $\Pi_B^\lambda(x, dz)$ is supported on $\bar{B}$ and

$$\int_{\bar{B}} \Pi_B^\lambda(x, dz) f(x) = E_x(\exp\{-\lambda V_B\} f(X_{V_B})). \tag{2.28}$$

Also,

$$\int_{R^n} g_B^\lambda(x, y) f(y)\, dy = E_x \int_0^{V_B} f(X_t)\, e^{-\lambda t}\, dt. \tag{2.29}$$

Observe that $g^\lambda(0) = \infty$, but otherwise $g^\lambda(x)$ is a continuous function that vanishes as $x \to \infty$. Since $p(t, y - x) \geqq q_B(t, x, y)$, we see that $g^\lambda(y - x) \geqq g_B^\lambda(x, y)$. It follows from (2.19) that

$$g^\lambda(y - x) = \int_{\bar{B}} \Pi_B^\lambda(x, dz) g^\lambda(y - z) + g_B^\lambda(x, y). \tag{2.30}$$

This equation will play a major role in our development.

Now $g^\lambda(x) = g^\lambda(-x)$, so $g^\lambda(y - x) = g^\lambda(x - y)$. It can be shown that $g_B^\lambda(x, y)$ *is a symmetric function in* x *and* y. A proof of this fact can be obtained by a fairly simple probabilistic argument. We will not prove this fact here but refer the reader to Chapter VI of [1] for a complete discussion.

Using (2.30), we see that the symmetry of g_B^λ implies that

$$\int_{\bar{B}} \Pi_B^\lambda(x, dz) g^\lambda(y - z) = \int_{\bar{B}} \Pi_B^\lambda(y, dz) g^\lambda(x - z). \tag{2.31}$$

This relation, as well as the symmetry of g_B^λ, will be of constant use in our development of potential theory.

Let τ be an orthogonal transformation. Then $p(t, \tau x) = p(t, x)$. It follows from this fact that if τ is a rotation about a point a and S is a sphere of center a, then $P_a(X_{V_S} \in A) = P_a(X_{V_S} \in \tau A)$ for any Borel subset $A \subset S$. Hence, $P_a(X_{V_S} \in dy)$ must be the uniform distribution on S. Let $\sigma_r(a, dy)$ denote the uniform distribution on the sphere of center a and radius r. Then $P_a(X_{V_S} \in dy) = \sigma_r(a, dy)$ when S is the sphere of center a and radius r. Using this fact, we will now prove a fact that will be used several times in the sequel.

PROPOSITION 2.3. *Let* f *be a bounded function such that* $f = P^t f$ *for all* $t > 0$. *Then* f *is a constant.*

PROOF. Since $f = P^t f$ for all $t > 0$, we see that f is continuous and that $f(x) = \lambda \int_0^\infty g^\lambda(y - x) f(y)\, dy$. Let S be the complement of the ball of center a and radius r. Using (2.30), we see that

$$f(x) = \lambda \int_0^\infty g_S^\lambda(x, y) f(y)\, dy + \Pi_S^\lambda f(x). \tag{2.32}$$

Now

$$\int_0^\infty g_S^\lambda(x, y) |f(y)|\, dy \leqq \|f\|_\infty \int_0^\infty e^{-\lambda t} P_x(V_S > t)\, dt. \tag{2.33}$$

Since S^c is bounded, $E_x V_S < \infty$, so

$$\lim_{\lambda \downarrow 0} \int_0^\infty e^{-\lambda t} P_x(V_S > t)\, dt = \int_0^\infty P_x(V_S > t)\, dt = E_x V_S < \infty. \tag{2.34}$$

Letting $\lambda \downarrow 0$ in (2.32), we see that $f(x) = \Pi_S f(x)$. In particular, for $x = a$, we see that

$$f(a) = E_a f(X_{V_S}) = \int_{\partial S} f(y) \sigma_r(a, dy). \tag{2.35}$$

Thus, f is its average over every sphere. Hence, f is harmonic, and as it is bounded it must be a constant. This establishes the proposition.

3. Regular points

Let B be a Borel set. It can be shown by a simple measure theoretic argument that $P_x(V_B = 0)$ is either 0 or 1. (See Chapter I of [1].) A point x is called *regular for B if* this probability is 1. Otherwise x is said to be *irregular for B*. Let B^r denote the set of regular points for B. Clearly, $\mathring{B} \subset B^r \subset \bar{B}$. A simple sufficient condition for regularity is the following.

PROPOSITION 3.1 (Poincaré's test). *Let k be a truncated cone of vertex 0, radius r_0 and angle opening α. Then $x \in B^r$ if $x + k \subset B$.*

PROOF. Clearly,

$$(3.1)\quad P_x(V_B \leqq t) \geqq P_x(X_t \in B) \geqq P_x(X_t \in k + x) = P_0(X_t \in k)$$
$$= (2\pi t)^{-n/2} \int_k \exp\left\{-\frac{|x|^2}{2t}\right\} dx = \alpha(2\pi t)^{-n/2} \int_0^{r_0} \exp\left\{-\frac{r^2}{2t}\right\} r^{n-1}\, dr$$
$$\geqq \delta > 0$$

for all $1 \geqq t > 0$. Thus, $P_x(V_B = 0) > 0$ and hence $P_x(V_B = 0) = 1$, as desired.

Using (2.19), we see that if $x \in B^r$, then $g_B^{\lambda}(x, y) = 0$ for all y. By symmetry, $g_B^{\lambda}(x, y) = 0$ for $y \in B^r$.

A simple but important device will be employed in many proofs. Let B be a closed set and let B_n, $n \geqq 1$, be a family of closed sets such that $B_1 \supset \mathring{B}_1 \supset B_2 \supset \cdots$, $\bigcap_n B_n = \bigcap_n \mathring{B}_n = B$.

PROPOSITION 3.2. *Let B and B_n be as described above. Then $P_x(V_{B_n} \uparrow V_B) = 1$ for $x \in B^c \cup B^r$.*

PROOF. Clearly, the V_{B_n} are nondecreasing, and thus $V_{B_n} \uparrow V \leqq V_B$. If $V = \infty$, then $V_B = \infty$. On the other hand if $V < \infty$, then $X_{V_{B_n}} \to X_V \in \bigcap_n B_n = B$. Thus, $V \geqq V_B$ whenever $V > 0$. Clearly, $P_x(V > 0) = 1$ when $x \in B^c$. If $x \in B^r$ then $P_x(V = V_B = 0) = 1$. This establishes the proposition.

For our later work, we will need the following simple corollary of Propositions 3.1 and 3.2.

COROLLARY 3.1. *Let G be a nonempty open set. Then there is an increasing sequence G_n of open sets with compact closures contained in G such that $G_1 \subset \bar{G}_1 \subset G_2 \subset \cdots$, $\bigcup_n G_n = G$ and such that each point of ∂G_n is regular for G_n^c. The times $V_{\partial G_n}$, $n \geqq 1$, are such that $P_x(V_{\partial G_n} \uparrow V_{\partial G} = 1$ for all $x \in G$.*

PROOF. Let k_n be compact sets such that $k_1 \subset k_2 \subset$ and $\bigcup_n k_n = G$. Cover k_1 by a finite number of open balls whose union D is such that $\bar{D}$ is contained in G. By increasing the radii of some balls if necessary, we can conclude from Poincaré's test that each point of ∂D is regular for $\bar{D}^c$. Let $G_1 = D$. Thus, each point of ∂G_1 is regular for G_1^c and $\bar{G}_1 \subset G$. Apply the same procedure to $\bar{G}_1 \cup k_2$, and so forth. Clearly, $G_1 \subset G_2 \subset \cdots$, $\bigcup_n G_n = G$, so $G_1^c \supset (G_1^c)^\circ \supset \cdots$, and $\bigcap_n G_n^c = G^c$. Using Proposition 3.2, $P_x(V_{G_n^c} \uparrow V_{G^c}) = 1$ for all $x \in G$. But $P_x(V_{\partial G} = V_{G^c}) = 1$, $x \in G$ and $P_x(V_{\partial G_n} = V_{G_n^c}) = 1$ for $x \in G_n$, so $P_x(V_{\partial G_n} \uparrow V_{\partial G}) = 1$, $x \in G$, as desired.

A set B is called *polar* if $P_x(V_B < \infty) \equiv 0$. Clearly, such sets are negligible since no Brownian motion process can ever hit such a set in positive time. Later when we introduce the notion of capacity as given in classical potential theory we will see that a polar set and a set of capacity 0 are equivalent.

Let B be a Borel set. If $x \in \mathring{B}$ then x is a regular point of B, while $x \in (\bar{B})^c$ is irregular for B. Thus, only points on ∂B are in question. An important fact is that the points in B that are irregular for B constitute a polar set, that is, $(B^r)^c \cap B$ is a polar set. We will prove this fact here for B a closed set. To carry out the proof of this fact we will need some preliminary facts, some of which are of interest in their own right.

We first show that $(B^r)^c \cap B$ is a Borel set having measure 0.

LEMMA 3.1. *Let B be a Borel set. Then B^r is a G_δ set and $D = (B^r)^c \cap B$ has measure* 0.

PROOF. A point x is regular if and only if $P_x(V_B = 0) = 1$. Thus,

$$B^r = \{x: P_x(V_B = 0) = 1\} = \bigcap_{n=1}^{\infty} \left\{x: P_x\left(V_B \leqq \frac{1}{n}\right) > 1 - \frac{1}{n}\right\}. \tag{3.2}$$

By Proposition 2.1, $P_x(V_B \leqq 1/n)$ is a lower semicontinuous function so $\{x: P_x(V_B \leqq 1/n) > 1 - 1/n\}$ is open. To see that D has measure 0, we can proceed as follows. Let $A \subset D$ be relatively compact and note that

$$P_x(X_t \in A) \leqq P_x(V_A \leqq t) \leqq P_x(V_B \leqq t), \tag{3.3}$$

so for $x \in A$,

$$\limsup_{t \downarrow 0} P_x(X_t \in A) \leqq \lim_{t \downarrow 0} P_x(V_B \leqq t) = 0. \tag{3.4}$$

Thus,

$$\lim_{t \downarrow 0} \int_A P_x(X_t \in A)\, dx = 0. \tag{3.5}$$

The function

$$f(x) = \int_{R^n} 1_A(z + x) 1_A(z)\, dz \tag{3.6}$$

is a continuous function and

$$\int_A P_x(X_t \in A)\, dx = E_0 f(X_t). \tag{3.7}$$

Hence,

$$f(0) = \lim_{t \downarrow 0} E_x f(X_t) = \lim_{t \downarrow 0} \int_A P_x(X_t \in A) = 0. \tag{3.8}$$

But $f(0) = |A|$, so $|A| = 0$.

Our next result shows that $E_x(\exp\{-\lambda V_B\})$ is the "λ potential" of the measure μ_B^λ defined by

$$\mu_B^\lambda(A) = \lambda \int_{R^n} \Pi_B^\lambda(x, A)\, dx. \tag{3.9}$$

LEMMA 3.2. *Let B be any Borel set. Then*

$$E_x(\exp\{-\lambda V_B\}) = \int_{R^n} g^\lambda(y - x)\mu_B^\lambda(dy). \tag{3.10}$$

PROOF. Integrating both sides of (2.30) on x over R^n, we see that

$$1 = \int_{\bar{B}} \mu_B^\lambda(dz) g^\lambda(y - z) + \lambda \int_{R^n} g_B^\lambda(x, y)\, dx. \tag{3.11}$$

But by the symmetry of g_B^λ, we see that

$$\int_{R^n} g_B^\lambda(x, y)\, dx = \int_{R^n} g_B^\lambda(y, x)\, dx = [1 - E_y(\exp\{-\lambda V_B\})]\lambda^{-1}. \tag{3.12}$$

This establishes the lemma.

REMARK. Since $g^\lambda(x)$ is bounded away from 0 on compacts, it follows from (3.10) that $\mu_B^\lambda(\bar{B}) < \infty$ whenever $\bar{B}$ is compact.

LEMMA 3.3. *Let B be a compact set and let B_n, $n \geqq 1$, be compacts such that $B_1 \supset \mathring{B}_1 \supset B_2 \supset \cdots$, $\bigcap_n \mathring{B}_n = \bigcap_n B_n = B$. Then the total mass $C^\lambda(B_n)$ of the measure $\mu_{B_n}^\lambda$ converges to the total mass $C^\lambda(B)$ of μ_B^λ as $n \to \infty$.*

PROOF. By Proposition 3.2, for $x \in B^r \cup B^c$,

$$E_x(\exp\{-\lambda V_{B_n}\}) \downarrow E_x(\exp\{-\lambda V_B\}). \tag{3.13}$$

By Lemma 3.1, $(B^r)^c \cap B = (B^r \cup B^c)^c$ has measure 0, so (3.13) holds for a.e. x. Thus, by monotone convergence and (3.10),

$$C^\lambda(B_n) = \lambda \int_{R^n} E_x(\exp\{-\lambda V_{B_n}\})\, dx \downarrow \lambda \int_{R^n} E_x(\exp\{-\lambda V_B\})\, dx = C^\lambda(B). \tag{3.14}$$

This establishes the lemma.

LEMMA 3.4. *Suppose k is a compact set such that $\sup_{x \in R^n} E_x(\exp\{-\lambda V_k\}) = \beta < 1$. Then k is polar.*

PROOF. Let k_n be compacts containing k such that $k_1 \supset \mathring{k}_1 \supset k_2 \supset \cdots$, $\bigcap_n k_n = \bigcap_n \mathring{k}_n = k$. Then each point in k is a regular point of k_n for all n. On the one hand (3.10) shows that

$$\int_{R^n} \int_{R^n} g^\lambda(y - x)\mu_k^\lambda(dy)\mu_{k_n}^\lambda(dx) = \int_{R^n} E_x(\exp\{-\lambda V_k\})\mu_{k_n}^\lambda(dx) \leqq \beta C^\lambda(k_n). \tag{3.15}$$

On the other hand,

$$\int_{R^n}\int_{R^n} g^\lambda(y-x)\mu_k^\lambda(dy)\mu_{k_n}^\lambda(dx) = \int_{R^n} E_y(\exp\{-\lambda V_{k_n}\})\mu_k^\lambda(dy) = C^\lambda(k). \tag{3.16}$$

Thus, $C^\lambda(k) \leqq \beta C^\lambda(k_n)$. By Lemma 3.3, $C^\lambda(k_n) \downarrow C^\lambda(k)$, and thus $C^\lambda(k) \leqq \beta C^\lambda(k)$. Hence, $C^\lambda(k) = 0$. But using (3.10), we then see that $E_x(\exp\{-\lambda V_k\}) \equiv 0$, and thus k is polar, as desired.

Let μ be a finite measure having compact support k. Let $G^\lambda\mu(x) = \int_k g^\lambda(y-x)\,\mu(dy)$. Our next lemma shows that $G^\lambda\mu$ satisfies a maximum principle.

LEMMA 3.5. *Let μ be a finite measure having compact support k and let $M = \sup_{x\in k} G^\lambda\mu(x)$. Then $G^\lambda\mu(x) \leqq M$ for all $x \in R^n$.*

PROOF. If $M = \infty$ there is nothing to prove so assume $M < \infty$. Let $\varepsilon > 0$ be given and let $A = \{x: G^\lambda\mu(x) < M + \varepsilon\}$. Each point of A must be a regular point of A. To see this, suppose $x_0 \in A$ and x_0 is irregular for A. Then $P_{x_0}(V_A = 0) = 0$, and thus

$$\lim_{t\downarrow 0} P_{x_0}(X_t \in A) \leqq \lim_{t\downarrow 0} P_{x_0}(V_A \leqq t) = 0. \tag{3.17}$$

However,

$$G^\lambda\mu(x_0) \geqq e^{-\lambda t}P^tG^\lambda\mu(x_0) \geqq e^{-\lambda t}\int_{A^c} p(t, y-x_0)G^\lambda\mu(y)\,dy \geqq (M+\varepsilon)\,e^{-\lambda t}P_{x_0}(X_t \in A^c). \tag{3.18}$$

By (3.17), $\lim_{t\downarrow 0} P_{x_0}(X_t \in A^c)\,e^{-\lambda t} = 1$, and thus $G^\lambda\mu(x_0) \geqq M + \varepsilon$, a contradiction. Hence, $A \subset A^r$. Since $k \subset A \subset A^r$ and $g_A^\lambda(x, y) = 0$ for all $y \in A^r$, we see that $\int_k g_A^\lambda(x, y)\mu(dy) = 0$. By (2.30), we then see that

$$G^\lambda\mu(x) = \int_{\bar{A}} \Pi_A^\lambda(x, dz)G^\lambda\mu(z), \qquad x \in R^n. \tag{3.19}$$

The function $G^\lambda\mu(z)$ is lower semicontinuous in z (since it is the limit of the increasing sequence $e^{-\lambda/n}P^{1/n}G^\lambda\mu(z)$, $n \geqq 1$, of continuous functions). Consequently, $\{z: G^\lambda\mu(z) \leqq M + \varepsilon\}$ is a closed set. Clearly $\bar{A} \subset \{z: G^\lambda\mu(z) \leqq M + \varepsilon\}$, and thus by (3.19), $G^\lambda\mu(x) \leqq M + \varepsilon$ for all $x \in R^n$. As ε is arbitrary, $G^\lambda\mu(x) \leqq M$ for all x. This establishes the lemma.

COROLLARY 3.2. *Let A be a closed set such that $A^r = \varnothing$. Then A is polar.*

PROOF. It suffices to consider the case when A is compact. The function $E_x(\exp\{-\lambda V_A\})$ is lower semicontinuous. Indeed, $P_x(V_A \leqq t)$ is lower semi-

continuous in x and thus by Fatou's lemma,

$$\begin{aligned}(3.20)\quad \liminf_{x\to x_0} E_x(\exp\{-\lambda V_A\}) &= \liminf_{x\to x_0} \lambda \int_0^\infty P_x(V_A \leqq t)\, e^{-\lambda t}\, dt \\ &\geqq \lambda \int_0^\infty \liminf_{x\to x_0} P_x(V_A \leqq t)\, e^{-\lambda t}\, dt \\ &\geqq \lambda \int_0^\infty P_{x_0}(V_A \leqq t)\, e^{-\lambda t}\, dt = E_{x_0}(\exp\{-\lambda V_A\}).\end{aligned}$$

Let

$$(3.21)\qquad A_n = \left\{x: E_x(\exp\{-\lambda V_A\}) \leqq 1 - \frac{1}{n}\right\} \cap A.$$

Then $A_n \subset A$ is compact and by Lemma 3.2, for $x \in A_n$,

$$(3.22)\qquad G^\lambda \mu^\lambda_{A_n}(x) = E_x(\exp\{-\lambda V_{A_n}\}) \leqq E_x(\exp\{-\lambda V_A\}) \leqq 1 - \frac{1}{n}.$$

By Lemmas 3.2 and 3.5, we then see that $E_x(\exp\{-\lambda V_{A_n}\}) \leqq 1 - 1/n$ for all $x \in R^n$, and thus by Lemma 3.4, A_n is polar. Since $A = \cup_n A_n$, A is polar.

We may now prove the following basic theorem.

THEOREM 3.1. *Let B be any closed set. Then $(B^r)^c \cap B$ is polar.*

PROOF. It suffices to consider B compact. Let

$$(3.23)\qquad D_n = \left\{x: E_x(\exp\{-\lambda V_B\}) \leqq 1 - \frac{1}{n}\right\},$$

and let $B_n = B \cap D_n$. Then

$$(3.24)\qquad (B^r)^c \cap B = \bigcup_{n=1}^{\infty} B_n.$$

Since $B_n \subset B$, $E_x(\exp\{-\lambda V_{B_n}\}) \leqq E_x(\exp\{-\lambda V_B\})$. If x is irregular for B then $E_x(\exp\{-\lambda V_{B_n}\}) < 1$ so x is irregular for B_n. Suppose x is regular for B. Then $x \in D_n^c$, and as D_n is closed, x is irregular for D_n and hence also for B_n. Thus, $B_n^r = \varnothing$. Corollary 3.2 then implies that each B_n is polar, and thus $(B^r)^c \cap B$ is polar. This establishes the theorem.

We conclude this section by pointing out a simple corollary of Lemma 3.2. Though its proof is trivial it is important enough to be stated as

THEOREM 3.2. *A one point set is a polar set.*

PROOF. Using (3.10) for the set $B = \{a\}$, we see that

$$(3.25)\qquad E_x(\exp\{-\lambda V_{\{a\}}\}) = g^\lambda(a - x)\, \Pi^\lambda_{\{a\}}(a).$$

Since $E_a(\exp\{-\lambda V_{\{a\}}\}) \leqq 1$ and $g^\lambda(0) = \infty$, it must be that $\Pi^\lambda_{\{a\}}(a) = 0$. But then $E_x(\exp\{-\lambda V_{\{a\}}\}) \equiv 0$ so $P_x(V_{\{a\}} < \infty) \equiv 0$.

4. Dirichlet problem

Let G be an open set. The *classical Dirichlet problem* for G with boundary function φ is as follows. Given φ on ∂G find f harmonic in G and continuous on $\bar{G}$ such that $f = \varphi$ on ∂G. In general, even when φ is restricted to be a bounded continuous function at each point of ∂G, this problem may have no solution. If solutions do exist, then unless G is bounded, they may not be unique. The *modified Dirichlet problem* eases the continuity requirements on $\bar{G}$ by allowing the function f (which must still be harmonic in G) to be discontinuous at an exceptional set of points on ∂G. One of the nicest connections between Brownian motion and potential theory is the elegant and simple treatment it allows for the modified Dirichlet problem.

DEFINITION 4.1. *Let G be an open set. A point $x_0 \in \partial G$ is said to be non-singular if $\lim_{x \in G, x \to x_0} \Pi_{\partial G}\, \varphi(x) = \varphi(x_0)$ for all bounded functions that are continuous on ∂G. Otherwise a point $x_0 \in \partial G$ is called singular.*

THEOREM 4.1. *Let φ be a bounded measurable function defined on ∂G. Then $\Pi_{\partial G}\varphi(x)$ is harmonic in G. Moreover, if $x_0 \in \partial G$ is a point of continuity of φ and is also a regular point of G^c, then $\lim_{x \in G, x \to x_0} \Pi_{\partial G}\varphi(x) = \varphi(x_0)$.*

PROOF. It is easily seen that $P^t \Pi_{\partial G}\, \varphi(x)$ is continuous in x for all $t > 0$ and that $\lim_{t \to 0} P^t \Pi_{\partial G}\, \varphi(x) = \Pi_{\partial G}\, \varphi(x)$ uniformly on compact subsets of G. We conclude that $\Pi_{\partial G}\, \varphi(x)$, $x \in G$, is continuous. Let $x \in G$ and let S_r be a ball of center x and radius r such that $\bar{S}_r \subset G$. Then clearly the process starting from x must first hit ∂S_r in positive time before it can hit ∂G. Thus,

$$\Pi_{\partial G}\, \varphi(x) = \int_{\partial S_r} \Pi_{\partial S_r}(x, dz)\, \Pi_{\partial G}\, \varphi(z). \tag{4.1}$$

But as argued in Section 2, $\Pi_{\partial S_r}(x, dz) = \sigma_r(x, dz)$. Thus,

$$\Pi_{\partial G}\, \varphi(x) = \int_{\partial S_r} \Pi_{\partial G}\, \varphi(z) \sigma_r(x, dz), \tag{4.2}$$

so $\Pi_{\partial G}\, \varphi$ is indeed harmonic at x.

Suppose now that $x_0 \in \partial G$ is both a point of continuity of φ and a regular point of G^c. Given $\delta > 0$ and $\varepsilon > 0$, we can find a t_0 such that $P(\sup_{t \leqq t_0} |X_t - X_0| \leqq \frac{1}{2}\delta) > 1 - \varepsilon$. Let N be any neighborhood of x_0, and let $D(\delta)$ be a closed ball of center x_0 and radius $\delta \subset N$. Let $B = G^c$. Then for $x \in D(\frac{1}{2}\delta)$,

$$\begin{aligned} &P_x(V_B \leqq t_0, X_{V_B} \notin N) \\ &\qquad \leqq P(|X_t - X_0| > \tfrac{1}{2}\delta \text{ for some } t \leqq t_0) \leqq \varepsilon. \end{aligned} \tag{4.3}$$

Thus, for $x \in D(\frac{1}{2}\delta)$,

$$\begin{aligned} P_x(V_B \leqq t_0, X_{V_B} \in N) &= P_x(V_B \leqq t_0) - P_x(V_B \leqq t_0, X_{V_B} \notin N) \\ &\geqq P_x(V_B \leqq t_0) - \varepsilon. \end{aligned} \tag{4.4}$$

Since x_0 is regular for B and $P_x(V_B \leqq t)$ is a lower semicontinuous function in x, we see that

$$1 = P_{x_0}(V_B \leqq t_0) \leqq \liminf_{x \to x_0} P_x(V_B \leqq t_0) \leqq 1. \tag{4.5}$$

Thus, by (4.4), $1 \geqq \liminf_{x \to x_0} P_x(V_B \leqq t_0, X_{V_B} \in N) \geqq 1 - \varepsilon$ and as ε is arbitrary we see that

$$\lim_{x \to x_0} P_x(V_B \leqq t_0, X_{V_B} \in N) = 1. \tag{4.6}$$

But

$$1 \geqq \Pi_B(x, N) \geqq P_x(V_B \leqq t_0, X_{V_B} \in N), \tag{4.7}$$

so for any neighborhood N of x_0

$$\lim_{x \to x_0} \Pi_B(x, N^c) = 0. \tag{4.8}$$

Moreover, as $P_x(V_B \leqq t_0) \leqq P_x(V_B < \infty)$, equation (4.4) shows that

$$\lim_{x \to x_0} P_x(V_B < \infty) = 1. \tag{4.9}$$

Now as φ is continuous at x_0, we can choose a neighborhood N of x_0 such that $|\varphi(x) - \varphi(x_0)| \leqq \varepsilon$, $x \in N$. But

$$\begin{aligned} |\Pi_B \varphi(x) - \varphi(x_0)| &\leqq \int_N \Pi_B(x, dz)|\varphi(z) - \varphi(x_0)| \\ &\quad + \int_{N^c} \Pi_B(x, dz)|\varphi(z) - \varphi(x_0)| + \varphi(x_0)P_x(V_B = \infty) \\ &\leqq \varepsilon + 2\|\varphi\| \Pi_B(x, N^c) + \varphi(x_0)P_x(V_B = \infty). \end{aligned} \tag{4.10}$$

Using (4.8) and (4.9), we see that

$$\lim_{x \to x_0} \Pi_B \varphi(x) = \varphi(x_0). \tag{4.11}$$

Finally, if $x \in G$, then $P_x(V_{\partial G} = V_B) = 1$. Thus (4.11) shows that

$$\lim_{x \to x_0, x \in G} \Pi_{\partial G} \varphi(x) = \varphi(x_0), \tag{4.12}$$

as desired. This establishes the theorem.

COROLLARY 4.1. *A point $x_0 \in \partial G$ is nonsingular if and only if it is a regular point for G^c. The set of singular points of ∂G is thus a polar F^σ set.*

PROOF. If $x_0 \in (G^c)^r \cap \partial G$, then Theorem 4.1 shows that x_0 is a nonsingular point. Suppose now that x_0 is a nonsingular point. Then $\Pi_{\partial G}(x, dy)$ converges weakly to the unit mass $\varepsilon_{x_0}(dy)$ at x_0. Thus, given any $\varepsilon > 0$ and any neighborhood N of x_0, we can find a closed ball S of center $x_0 \subset N$ such that $P_x(X_{V_{\partial G}} \in N) \geqq 1 - \varepsilon$, $x \in S \cap G$. But

$$P_{x_0}(X_{V_{\partial G}} \in N) = P_{x_0}(X_{V_{\partial G}} \in N; V_{\partial G} \leqq V_{\partial S}) + P_{x_0}(X_{V_{\partial G}} \in N; V_{\partial G} > V_{\partial S}). \tag{4.13}$$

Since $S \subset N$, $P_{x_0}(X_{V_{\partial G}} \in N \mid V_{\partial G} \leqq V_{\partial S}) = 1$, so

$$P_{x_0}(X_{V_{\partial G}} \in N;\ V_{\partial G} \leqq V_{\partial S}) = P_{x_0}(V_{\partial G} \leqq V_{\partial S}). \tag{4.14}$$

Also,

$$\begin{aligned} P_{x_0}(X_{V_{\partial G}} \in N \mid V_{\partial G} > V_{\partial S}) &= \int_{\partial S} P_{x_0}(V_{\partial G} > V_{\partial S};\ X_{V_{\partial S}} \in dz) P_z(X_{V_{\partial G}} \in N) \\ &\geqq (1 - \varepsilon) P_{x_0}(V_{\partial G} > V_{\partial S}). \end{aligned} \tag{4.15}$$

Thus, $P_{x_0}(X_{V_{\partial G}} \in N) \geqq 1 - \varepsilon$.

Hence, $\Pi_{\partial G}(x_0, dy)$ is the unit mass at x_0 so $P_{x_0}(X_{V_{\partial G}} = x_0) = 1$. But Theorem 3.2 shows that $\{x_0\}$ is a polar set. Thus, $P_{x_0}\{X_{V_{\partial G}} = x_0\} = 1$ can only be true if $P_{x_0}(V_{\partial G} = 0) = 1$, so x_0 is a regular point of ∂G. Since $\partial G \subset G^c$, x_0 is then also a regular point for G^c. This establishes the theorem.

Theorem 4.1 and its corollary show at once that for φ a bounded continuous function on ∂G, the function $\Pi_{\partial G}\varphi$ is a solution to the Dirichlet problem for G with boundary function φ provided each point of ∂G is nonsingular.

From Theorem 3.1 and Corollary 4.1, we know that the set of singular points of ∂G is a polar set. Let N be any polar set that contains all singular points of ∂G. Let φ be a bounded function on ∂G that is continuous at each point of $N^c \cap \partial G$. The *modified Dirichlet problem* consists in finding a function f harmonic on G and continuous on $G \cup (\partial G \cap N^c)$ such that $f = \varphi$ on ∂G. We know that $\Pi_{\partial G}\varphi$ is a solution to the modified problem. If we choose $\varphi \equiv 1$, we see that $P_x(V_{\partial G} < \infty)$ is a solution. Thus, $P_x(V_{\partial G} = \infty) = 1 - P_x(V_{\partial G} < \infty)$ is a solution to the modified problem with boundary function 0. Our principal goal in the remainder of this section is to show that $\Pi_{\partial G}\varphi(x) + \alpha P_x(V_{\partial G} = \infty)$ are the only bounded solutions to the modified problem.

We will start our investigation with a bounded G.

THEOREM 4.2. *Let G be a bounded open set and let N be a polar set that contains the singular points of ∂G. Suppose φ is a bounded function on ∂G that is continuous on $\partial G \cap N^c$. Then $\Pi_{\partial G}\varphi$ is the unique bounded solution to the modified Dirichlet problem for G with boundary function φ.*

PROOF. Suppose first that all points on ∂G are nonsingular and that the exceptional set N is empty. Then the modified problem becomes just the classical Dirichlet problem. Suppose f is any solution. Then $f - \Pi_{\partial G}\varphi = h$ vanishes on ∂G, is harmonic on G, and continuous on $\bar{G}$. The maximum principle then tells us that h vanishes on G so $f = \Pi_{\partial G}\varphi$ on G.

Consider now an arbitrary bounded open set G and allow an exceptional polar set N containing the singular points of ∂G. By Corollaries 3.1 and 4.1, we can exhaust G by an increasing sequence of open sets $G_1 \subset \bar{G}_1 \subset G_2 \subset \cdots$, $\cup_n G_n = G$, such that all points of ∂G_n are nonsingular, and such that $P_x(V_{\partial G_n} \uparrow V_{\partial G}) = 1$ for all $x \in G$. Let f be a bounded solution to the modified problem on G. Then f is continuous on $\bar{G}_n$ and harmonic on G_n so it is a

solution to the classical Dirichlet problem on G_n with boundary function f on ∂G_n. By what was proved above, $H_{\partial G_n} f$ is the unique solution to this problem, so

$$f(x) = H_{\partial G_n} f(x), \qquad x \in G_n. \tag{4.16}$$

Now as $\bar{G}$ is compact, Proposition 2.2 shows that $E_x V_{\bar{G}^c} < \infty$, so certainly $P_x(V_{G^c} < \infty) = 1$. But for any $x \in G$, $P_x(V_{G^c} = V_{\partial G}) = 1$, so $P_x(V_{\partial G} < \infty) = 1$ for all $x \in G$. Since $\bar{G}_n \subset G$, it must also be that $P_x(V_{\partial G_n} < \infty) = 1$ for all $x \in G_n$. Fix $x \in G$. Then

$$H_{\partial G_n} f(x) = E_x[f(X_{V_{\partial G_n}})]. \tag{4.17}$$

Now $P_x(V_{\partial G_n} \uparrow V_{\partial G}) = 1$ and as $P_x(V_{\partial G} < \infty) = 1$, we see that $P_x(\lim_{n\to\infty} X_{V_{\partial G_n}} = X_{V_{\partial G}}) = 1$. But $P_x(X_{V_{\partial G}} \in N) = 0$ and f is continuous on $(\partial G) \cap N^c$ so

$$P_x\Big(\lim_{n\to\infty} f(X_{V_{\partial G_n}}) = \varphi(X_{V_{\partial G}})\Big) = 1. \tag{4.18}$$

Since f is bounded,

$$\lim_{n\to\infty} E_x f(X_{V_{\partial G_n}}) = E_x\Big[\lim_{n\to\infty} f(X_{V_{\partial G_n}})\Big] = E_x \varphi(X_{V_{\partial G}}). \tag{4.19}$$

Using (4.16) and (4.19), we see that for any $x \in G$, $f(x) = H_{\partial G} \varphi(x)$ as desired.

COROLLARY 4.2. *Let G be a bounded open set. The classical Dirichlet problem has a solution for all continuous boundary functions φ if and only if ∂G has no singular points. In that case $H_{\partial G} \varphi$ is the unique solution with boundary function φ.*

PROOF. Since $\bar{G}$ is compact, a solution f of the classical Dirichlet problem for G for φ continuous on ∂G is automatically a bounded solution for the modified problem with N being the set of all singular points of ∂G. But then $f(x) = H_{\partial G} \varphi(x)$, $x \in G$. As f is a classical solution, $\lim_{x\to x_0} H_{\partial G} \varphi(x_0) = \varphi(x_0)$. Since this is true for all φ continuous on ∂G, we see that all points in ∂G are nonsingular. This establishes the corollary.

To handle the case when G is unbounded, we will require some preliminary information on the process X_t stopped when it hits G^c (for typographical simplicity we shall put $G^c = F$). Let $Y_t = X_{t \wedge V_F}$ and set ${}_F P^t f(x) = E_x f(Y_t)$, and

$$Q_F^t f(x) = E_x[f(X_t),\ V_F > t] = \int_G q_F(t, x, y) f(y)\, dy. \tag{4.20}$$

Note that for $x \in G$, $P_x(V_{\partial G} = V_F) = 1$, so that for $x \in G$,

$${}_F P^t f(x) = Q_F^t f(x) + E_x[f(X_{V_{\partial G}});\ V_{\partial G} \leqq t]. \tag{4.21}$$

We say that a function defined on $\bar{G}$ is invariant for ${}_F P^t$ on G if

$${}_F P^t f(x) = f(x), \qquad x \in G. \tag{4.22}$$

Similarly, a function f defined on G is Q_F^t invariant if $f(x) = Q_F^t f(x)$, $x \in G$.

LEMMA 4.1. *For any bounded function φ on ∂G, $\Pi_{\partial G}\,\varphi(x) + \alpha P_x(V_{\partial G} = \infty)$ is a bounded ${}_FP^t$ invariant function on G. Conversely, every bounded ${}_FP^t$ invariant function on G is of this form.*

PROOF. Let φ be bounded on ∂G and let $x \in G$. Then by (4.21),

$$(4.23)\qquad {}_FP^t\,\Pi_{\partial G}\,\varphi(x) = Q_F^t\,\Pi_{\partial G}\,\varphi(x) + E_x[\Pi_{\partial G}\,\varphi(X_{V_{\partial G}});\, V_{\partial G} \leqq t].$$

By Theorem 3.1, $P_x(X_{V_{\partial G}} \in (\partial G)^r) = 1$. Since $\Pi_{\partial G}\,(x, dy)$ is the unit mass at x if $x \in (\partial G)^r$, we see that $\Pi_{\partial G}\,\varphi(X_{V_{\partial G}}) = \varphi(X_{V_{\partial G}})$ with probability one, and thus the second term on the right in (4.23) is just $E_x[\varphi(X_{V_{\partial G}})\colon V_{\partial G} \leqq t]$. The first term is just

$$(4.24)\qquad Q_F^t\,\Pi_{\partial G}\,\varphi(x) = E_x[\varphi(X_{V_{\partial G}});\, t < V_{\partial G} < \infty].$$

Hence, the right side of (4.23) is just $\Pi_{\partial G}\,\varphi(x)$. Thus, for any φ, $\Pi_{\partial G}\,\varphi$ is ${}_FP^t$ invariant. In particular, for $\varphi \equiv 1$, we see that $P_x(V_{\partial G} < \infty)$ is ${}_FP^t$ invariant, and thus as 1 is clearly ${}_FP^t$ invariant so is $1 - P_x(V_{\partial G} < \infty) = P_x(V_{\partial G} = \infty)$. This shows that $\Pi_{\partial G}\,\varphi + \alpha P_x(V_{\partial G} = \infty)$ is ${}_FP^t$ invariant. It is clearly bounded if φ is bounded.

Suppose now that f is any bounded ${}_FP^t$ invariant function on G. Since constants are ${}_FP^t$ invariant, we can assume that $f \geqq 0$. Then

$$(4.25)\qquad E_x[f(X_{V_{\partial G}});\, V_{\partial G} \leqq t] \uparrow \Pi_{\partial G}\,f(x), \qquad t \to \infty.$$

By (4.21), we then see that $Q_F^t f$ is decreasing as $t \to \infty$. Let h denote its limit. Then for $x \in G$, $f(x) = h(x) + \Pi_{\partial G}\,f(x)$.

By dominated convergence and the semigroup property of Q_F^t, we see that for $x \in G$

$$(4.26)\qquad \begin{aligned} Q_F^t h(x) &= Q_F^t[\lim_{s\to\infty} Q_F^s f](x) = \lim_{s\to\infty} Q_F^t[Q_F^s f](x) \\ &= \lim_{s\to\infty} Q_F^{t+s} f(x) = h(x), \end{aligned}$$

so h is Q_F^t invariant. Note that $Q_F^t h(x) = 0$ for a.e. $x \in F$. Thus, if we define $h(x)$ to be 0 in $G^c(F)$, we see that

$$(4.27)\qquad P^{t+s}h(x) = P^t(P^s h)(x) \geqq P^t(Q_F^s h)(x) = P^t h(x),$$

so $P^t h(x)$ is increasing in t. As $P^t h(x) \leqq \sup_x h(x)$, $\lim_{t\to\infty} P^t h(x) = h_1(x)$ exists. The monotone convergence theorem then shows that $P^t h_1(x) = h_1(x)$ for all $x \in R^n$ and all $t > 0$. Thus, h_1 must be a constant. (See Proposition 2.3.) Denote this constant by α.

Now

$$(4.28)\qquad \begin{aligned} P^t h(x) &= Q_F^t h(x) + E_x[h(X_t);\, V_F \leqq t] \\ &= h(x) + E_x[h(X_t);\, V_F \leqq t], \end{aligned}$$

so taking the limit as $t \to \infty$ we see that

$$\alpha - h(x) = \lim_{t \to \infty} E_x[h(X_t);\ V_F \leqq t] = h_2(x). \tag{4.29}$$

Now $h_2(x) \leqq [\sup_x h(x)]P_x(V_F < \infty)$, so

$$Q_F^s h_2(x) \leqq [\sup_x h(x)]P_x(s < V_F < \infty). \tag{4.30}$$

Thus, $\lim_{s \to \infty} Q_F^s h_2(x) = 0$. But then, as $Q_F^s \alpha = \alpha P_x(V_F > s)$, we see from (4.29) that

$$\alpha P_x(V_F > s) = h(x) + Q_F^s h_2(x). \tag{4.31}$$

Letting $s \to \infty$, we see that $\alpha P_x(V_F = \infty) = h(x)$. Since for $x \in G$, $P_x(V_F = \infty) = P_x(V_{\partial G} = \infty)$ the lemma is proved.

We can now establish the following theorem.

THEOREM 4.3. *Let G be any open set and let N be a polar set that contains the singular points of ∂G. Suppose φ is bounded on ∂G and continuous on $(\partial G) \cap N^c$. Then the only bounded solutions f to the modified Dirichlet problem for G with boundary function φ are*

$$f(x) = \Pi_{\partial G}\, \varphi(x) + \alpha P_x(V_{\partial G} = \infty), \tag{4.32}$$

for α an arbitrary constant. Conversely, every such function is a solution of the modified problem.

PROOF. By Theorem 4.1 and Corollary 4.1, we already know that $\Pi_{\partial G}\, \varphi + \alpha P_x(V_{\partial G} = \infty)$ is a bounded solution. Suppose f is any other bounded solution. Let S_r be the open ball of center 0 and radius r and let $G_r = G \cap S_r$. Consider the modified Dirichlet problem on G_r with boundary function f. Then clearly f as a function on G_r is a solution. Since G_r is bounded, Theorem 4.2 tells us that

$$f(x) = \Pi_{\partial G_r} f(x), \qquad x \in G_r. \tag{4.33}$$

By Lemma 4.1, $\Pi_{\partial G_r} f$ is a bounded ${}_{F_r}P^t$ invariant function on G_r, where we use $(G_r)^c = F_r$ for typographical reasons. Then for any $t > 0$ and $x \in G_r$,

$$\begin{aligned} f(x) = \Pi_{\partial G_r} f(x) &= Q_{F_r}^t\, \Pi_{\partial G_r} f(x) + E_x[\Pi_{\partial G_r} f(x_{V_{\partial G_r}});\ V_{\partial G_r} \leqq t] \\ &= \int_{G_r} P_x(V_{F_r} > t,\ X_t \in dy) f(y) + E_x[f(X_{V_{\partial G_r}});\ V_{\partial G_r} \leqq t]. \end{aligned} \tag{4.34}$$

Now $V_{\partial S_r} \uparrow \infty$ with probability one, and thus for $x \in G$

$$P_x(V_{\partial G_r} = V_{\partial G} \text{ for all sufficiently large } r \text{ whenever } V_{\partial G} < \infty) = 1. \tag{4.35}$$

If $V_{\partial G} = \infty$ then $V_{\partial G_r} \uparrow \infty$. Hence, in every case when $x \in G$, $P_x(\lim_r V_{\partial G_r} = V_{\partial G}) = 1$. Then (4.21) and (4.34) show that as $r \to \infty$, for $x \in G$,

$$(4.36) \qquad f(x) = \int_G P_x(V_F > t, X_t \in dy) f(y) + E_x[f(X_{V_{\partial G}}); V_{\partial G} \leqq t]$$
$$= {}_F P^t f(x).$$

Thus, f is ${}_F P^t$ invariant on G. Lemma 4.1 then shows that $f(x) = \Pi_{\partial G} f + \alpha P_x(V_{\partial G} = \infty)$, $x \in G$. Since f is defined to be φ on ∂G, we see that $f(x) = \Pi_{\partial G} \varphi(x) + \alpha P_x(V_B = \infty)$ as desired. This establishes the theorem.

As a simple application of this theorem we prove the following extension property of harmonic functions relative to polar sets.

COROLLARY 4.3. *Let G be an open set. Suppose f is locally bounded and harmonic on G except perhaps on a relatively closed polar subset N. Then f extends uniquely to a harmonic function on G.*

PROOF. Let S be an open ball such that $\bar{S} \subset G$ and let $G_1 = S \cap N^c$. Then each point on ∂G_1 not in N is regular for G_1^c and f is continuous at each point of ∂G_1 not in N. Thus, as f is bounded on G_1 it is the unique bounded solution to the modified Dirichlet problem for ∂G_1 with boundary function f. But $\Pi_{\partial S} f$ is harmonic on S and assumes boundary value $f(x_0)$ at each point of ∂S not in N. Thus, it too is a bounded solution to the modified Dirichlet problem for G_1. Therefore, $f(x) = \Pi_{\partial S} f(x)$, $x \in G_1$. This shows that f can be extended to be a harmonic function on S and thus $\lim_{x \to x_0} f(x) = f(x_0)$ must exist for each $x_0 \in S$. Since S can be any open ball $\subset G$, f extends everywhere in G as a harmonic function. Define $f^*(x)$ to be $f(x)$ for $x \in G \cap N^c$ and define $f^*(x_0) = \lim_{x \to x_0} f(x)$ for $x_0 \in N$. Then $f^*(x)$ is harmonic on G and agrees with f on $G \cap N^c$. Since N has measure 0, f^* is the unique such function.

5. Newtonian potentials

Throughout the remainder of this paper we will consider Brownian motion in R^n, $n \geqq 3$. The planar case will be treated in our companion paper in this volume.

An easy computation shows that when $n \geqq 3$,

$$(5.1) \qquad \lim_{\lambda \downarrow 0} g^\lambda(x) = \int_0^\infty p(t, x)\, dt$$
$$= \int_0^\infty (2\pi t)^{-n/2} \exp\left\{\frac{-|x|^2}{2t}\right\} dt = \frac{\Gamma\left(\frac{n}{2} - 1\right)}{2\pi^{n/2}} |x|^{2-n}.$$

where the convergence is uniform in compacts not containing 0. For a function f or measure μ define Gf and $G\mu$ by

$$(5.2) \qquad Gf(x) = \int_{R^n} g(y - x) f(y)\, dy,$$

and

$$G\mu(x) = \int_{R^n} g(y - x)\mu(dy), \tag{5.3}$$

respectively. Gf is called *the potential of* f and $G\mu$ the potential of μ. One easily checks that Gf is a continuous function vanishing at ∞ whenever f is a bounded measurable function with compact support and that $G\mu(x)$ is lower semicontinuous and superharmonic whenever μ is a finite measure. It is useful to know that the potential of μ determines μ whenever $G\mu$ is sufficiently finite.

THEOREM 5.1. *If μ is a measure such that $G\mu < \infty$ a.e. then $G\mu$ determines μ.*

PROOF. Suppose μ and ν are two measures such that $G\mu$ and $G\nu < \infty$ a.e. and $G\mu = G\nu$ a.e. Then $P^tG\mu = P^tG\nu$, and for any point x for which $G\mu < \infty$ we have

$$G\mu(x) - P^tG\mu(x) = \int_0^t P^s\mu(x)\, ds, \tag{5.4}$$

where $P^s\mu(x) = \int_{R^n} p(s, y - x)\mu(dy)$. Thus, if x is such that $G\mu(x) < \infty$ and $G\nu(x) < \infty$, then

$$\int_0^t P^s\mu(x)\, ds = \int_0^t P^s\nu(x)\, ds, \tag{5.5}$$

so (5.5) holds for a.e. x. Let h be a bounded, nonnegative function having compact support such that $0 < \int_{R^n} G\mu(x)h(x)\, dx < \infty$ and set $g = Gh$. Then g is a bounded strictly positive continuous function and

$$\begin{aligned}\int_{R^n} g(x)\mu(dx) = \int_{R^n} G\mu(x)h(x)\, dx &= \int_{R^n} G\nu(x)h(x)\, dx \\ &= \int_{R^n} g(x)\nu(dx) < \infty.\end{aligned} \tag{5.6}$$

Let f be any continuous function, $0 \leqq f \leqq 1$. Observe that

$$\int_{R^n} f(x)g(x)P^s\mu(x)\, dx = \int_{R^n} \mu(dx)P^s(fg)(x) \tag{5.7}$$

and

$$\int_{R^n} f(x)g(x)P^s\nu(x)\, dx = \int_{R^n} \nu(dx)P^s(fg)(x). \tag{5.8}$$

Using (5.5), we then see that

$$\frac{1}{t}\int_{R^n} \mu(dx) \int_0^t P^s(fg)(x)\, ds = \frac{1}{t}\int_{R^n} \nu(dx) \int_0^t P^s(fg)(x)\, ds. \tag{5.9}$$

Since fg is a bounded continuous function, $P^sfg \to fg$ as $s \downarrow 0$, and $P^sfg \leqq P^sg = P^sGh \leqq Gh = g$, and by (5.6) g is both μ and ν integrable. Hence, using (5.9) and

dominated convergence, we see upon letting $t \downarrow 0$ in (5.9) that

$$\int_{R^n} \mu(dx)g(x)f(x) = \int_{R^n} \nu(dx)g(x)f(x). \tag{5.10}$$

Since f can be any bounded continuous function, (5.10) shows that $\mu(dx)g(x) = \nu(dx)g(x)$ and as $g > 0$ for all x it must be that $\mu(dx) = \nu(dx)$. This establishes the theorem.

A useful fact about Brownian motion in R^n, $n \geqq 3$, is the following proposition.

PROPOSITION 5.1. *Let B be any bounded set. Then*

$$\lim_{t \to \infty} P_x(X_s \in B \text{ for some } s > t) = 0, \tag{5.11}$$

or equivalently, $P_x(\lim_{t \to \infty} |X_t| = \infty) = 1$.

PROOF. Let k be a compact set of positive measure such that

$$\inf_{0 \leqq s \leqq 1} \inf_{z \in \bar{B}} P_z(X_s \in k) = \delta > 0. \tag{5.12}$$

By integrating from 0 to $t + 1$ on both sides of (2.15) and then integrating by parts, it follows that

$$\begin{aligned} \int_0^{t+1} P_x(X_s \in k)\, ds &\geqq \int_{\bar{B}} \int_t^{t+1} P_x(V_B \leqq s, X_{V_B} \in dz) P_z(X_{t+1-s} \in k)\, ds \\ &\geqq \delta P_x(V_B \leqq t). \end{aligned} \tag{5.13}$$

Letting $t \uparrow \infty$, we see that $\int_k g(y - x)\, dy \geqq \delta P_x(V_B < \infty)$. But

$$\begin{aligned} P_x(X_s \in B \text{ for some } s > t) &= \int_{R^n} p(t, z - x) P_z(V_B < \infty)\, dz \\ &\leqq \delta^{-1} \int_{R^n} p(t, z - x) \left[\int_k g(y - z)\, dy \right] dz \\ &= \delta^{-1} \int_t^\infty P^s 1_k(x)\, ds, \end{aligned} \tag{5.14}$$

where 1_k is the indicator function of k. Since

$$\lim_{t \uparrow \infty} \int_t^\infty P^s 1_k(x) = 0, \tag{5.15}$$

Proposition 5.1 holds.

Using (2.30) and monotone convergence, we see that for any Borel set B,

$$g(y - x) = \int_{\bar{B}} \Pi_B(x, dz) g(y - z) + g_B(x, y). \tag{5.16}$$

It is quite easy to prove the following theorem.

THEOREM 5.2. *The function g_B has the following properties:*

(i) $g_B \geqq 0$;

(ii) $g_B(x, y) = g_B(y, x)$;

(iii) $g_B(x, y) < \infty$ *for* $x \neq y$ *and* $g_B(x, x) = \infty$ *for* $x \in (\bar{B})^c$;

(iv) *for fixed* x, $g_B(x, \cdot)$ *is upper semicontinuous and subharmonic on* $R^n - \{x\}$;

(v) *for fixed* x, $g_B(x, y)$ *is harmonic in* $y \in (\bar{B})^c - \{x\}$;

(vi) *for fixed* x, $g_B(x, y) - g(y - x)$ *is harmonic in* $y \in (\bar{B})^c$;

(vii) $\lim_{y \to y_0} g_B(x, y) = g_B(x, y_0) = 0$ *if* $y_0 \in B^r$.

PROOF. Properties (i) and (ii) follow from the fact that they are true for $g_B^\lambda \uparrow g_B$, $\lambda \downarrow 0$. Properties (iii) to (vi) follow at once from (5.16) and the fact that

$$\int_{\bar{B}} \Pi_B(x, dz) g(y - z), \tag{5.17}$$

as a function of y is lower semicontinuous, superharmonic on R^n, and harmonic on $(\bar{B})^c$. Finally, to see that (vii) is true note that if $y_0 \in B^r$, then $g_B^\lambda(x, y_0) = g_B^\lambda(y_0, x) \equiv 0$, so $g_B(x, y_0) = 0$. But by (iv)

$$0 \leqq \limsup_{y \to y_0} g_B(x, y) \leqq g_B(x, y_0) = 0. \tag{5.18}$$

Let G be an open set. *The Green function* g_G^* of G is the smallest nonnegative function h defined on $G \times G$ such that $h(x, y) - g(y - x)$ is harmonic in y.

An important connection between potential theory and Brownian motion is that g_{G^c} as a function on $G \times G$ is the Green function.

THEOREM 5.3. *The Green function of the open set G is the function g_{G^c} restricted to $G \times G$.*

PROOF. The proof of Theorem 5.2 tells us that g_{G^c} has the required properties so it is only necessary to show that g_{G^c} restricted to $G \times G$ is the smallest such function. Suppose g^* is another function having the required properties. Consider first the case when every point of ∂G is regular for G^c and G is bounded. Then for any $y_0 \in \partial G$ we see by Theorem 5.2 (vii) that

$$\liminf_{y \to y_0} [g^*(x, y) - g_{G^c}(x, y)] = \liminf_{y \to y_0} g^*(x, y) \geqq 0. \tag{5.19}$$

Since $g^*(x, y) - g_{G^c}(x, y)$ is harmonic in G, the minimum principle tells us that

$$g^*(x, y) - g_{G^c}(x, y) \geqq 0 \qquad \text{for all } y \in G. \tag{5.20}$$

Suppose now that G is any open set. By Corollary 3.1, we can find open sets $G_n \subset G$, $G_1 \subset \bar{G}_1 \subset G_2 \subset \cdots$, $\cup_n G_n = G$, such that each point of ∂G_n is regular for G_n^c and such that $P_x(V_{\partial G_n} \uparrow V_{\partial G}) = 1$ for all $x \in G$. The function g^* viewed as a function on $G_n \times G_n$ has the required properties, and thus by what has just been proven

$$g^*(x, y) \geqq g_{G_n}(x, y). \qquad x, y \in G_n. \tag{5.21}$$

Now the functions $g_{G_n^c}$, $n \geqq 1$, are increasing for $x, y \in G$. Indeed, since each point of G_n^c is regular for G_n^c, $0 = g_{G_n^c}(x, y) \leqq g_{G_{n+1}^c}(x, y)$, if either $x \in G_n^c$ or $y \in G_n^c$. On the other hand if x and y are both in G_n, then as $g_{G_{n+1}^c}(x, y)$ has the required properties on $G_{n+1} \supset G_n$ it does so on G_n. Thus, by what was proved above, $g_{G_n^c}(x, y) \leqq g_{G_{n+1}^c}(x, y)$, $x, y \in G_n$. We will finish the proof by showing that the limit, $\lim_n g_{G_n^c}$, agrees with g_{G^c} on $G \times G$. If $x = y$, then $g_{G_n^c}(x, x) = \infty$ for all sufficiently large n and so does $g_{G^c}(x, x)$ so the desired result holds in this case. Suppose $x \neq y$. Then (5.16) shows that

$$g_{G_n^c}(x, y) = g(y - x) - \int_{G_n^c} \Pi_{G_n^c}(x, dz) g(y - z). \tag{5.22}$$

If x and $y \in G$, then x and $y \in G_n$ for all $n \geqq n_0$ for some n_0. Assume $x, y \in G_n$. Then

$$\begin{aligned} \int_{G_n^c} \Pi_{G_n^c}(x, dz) g(y - z) &= E_x g(y - X_{V_{G_n^c}}) = E_x g(y - X_{V_{\partial G_n}}) \\ &= E_x[g(y - X_{V_{\partial G_n}}); V_{\partial G} < \infty] \\ &\quad + E_x[g(y - X_{V_{\partial G_n}}); V_{\partial G} = \infty]. \end{aligned} \tag{5.23}$$

Since $g(y - x)$ is a bounded continuous function in x on G_n^c for $y \in G_n$ and $P_x(X_{V_{\partial G_n}} \to X_{V_{\partial G}} \mid V_{\partial G} < \infty) = 1$, we see that

$$\begin{aligned} \lim_{n \to \infty} E_x[g(y - X_{V_{\partial G_n}}); V_{\partial G} < \infty] &= E_x[g(y - X_{V_{\partial G}}); V_{\partial G} < \infty] \\ &= E_x[g(y - X_{V_{G^c}}); V_{G^c} < \infty] \\ &= \int_{G^c} \Pi_{G^c}(x, dz) g(y - z). \end{aligned} \tag{5.24}$$

Moreover, by Proposition 5.1, $P_x(\lim_{t \to \infty} |X_t| = \infty) = 1$. Since $P_x(V_{\partial G_n} \uparrow \infty \mid V_{\partial G} = \infty) = 1$ and $\lim_{|x| \to \infty} g(y - x) = 0$, it follows by dominated convergence that

$$\lim_{n \to \infty} E_x[g(y - X_{V_{\partial G_n}}); V_{\partial G} = \infty] = 0. \tag{5.25}$$

Thus, from (5.22) and (5.23), we see that for any $x, y \in G$,

$$\lim_{n \to \infty} g_{G_n^c}(x, y) = g(y - x) - \int_{G^c} \Pi_{G^c}(x, dz) g(y - z). \tag{5.26}$$

But by (5.16) the right side of (5.26) is just $g_{G^c}(x, y)$ as desired. This completes the proof.

THEOREM 5.4. *Let μ be a finite measure having support B. Let N be a subset of B such that $\mu(N) = 0$. If $G\mu(x) \leqq M < \infty$ for all $x \in B \cap N^c$, then* $\sup_x G\mu(x) \leqq M$.

PROOF. Choose $\varepsilon > 0$ and let

$$A = \{x: G\mu(x) < M + \varepsilon\}. \tag{5.27}$$

Suppose for some $x_0 \in A$, x_0 is irregular for A. Then $P_{x_0}(V_A = 0) = 0$, and thus

$$\lim_{t \to 0} P_{x_0}(X_t \in A) \leqq \lim_{t \to 0} P_{x_0}(V_A \leqq t) = 0. \tag{5.28}$$

Observe that

$$G\mu(x_0) \geqq P^t G\mu(x_0) \geqq \int_{A^c} p_t(y - x) G\mu(y)\, dy \geqq (M + \varepsilon) P_{x_0}(X_t \in A^c). \tag{5.29}$$

Thus, $G\mu(x_0) \geqq M + \varepsilon$, a contradiction. Therefore, each point of A is regular for A. Using (5.16), we see that

$$G\mu(x) = \int_{\bar{A}} \Pi_A(x, dz) G\mu(z) + \int_B g_A(x, y)\mu(dy). \tag{5.30}$$

Now as $\mu(N) = 0$,

$$\int_B g_A(x, y)\mu(dy) = \int_{B \cap N^c} g_A(x, y)\mu(dy). \tag{5.31}$$

But $B \cap N^c \subset A$, and each point of A is regular for A, so $g_A(x, y) = 0$ for all $y \in B \cap N^c$. Thus, we see that

$$G\mu(x) = \int_{\bar{A}} \Pi_A(x, dz) G\mu(z). \tag{5.32}$$

Now $G\mu(x)$ is a lower semicontinuous function, and thus $\{x: G\mu(x) \leqq M + \varepsilon\}$ is closed. Hence, $\bar{A} \subset \{x: G\mu(x) \leqq M + \varepsilon\}$, and thus (5.32) shows that $G\mu(x) \leqq M + \varepsilon$. As ε is arbitrary, $G\mu(x) \leqq M$ as desired.

6. Equilibrium measure

Let S_r be the closed ball of center 0 and radius r and let $G = S_r^c$. The hitting distribution $\Pi_{S_r}(x, dy)$ of S_r is easily found. Indeed, for $x \in S_r$, $\Pi_{S_r}(x, dy)$ is just the unit mass at x while for $x \in G$, $\Pi_{S_r}(x, dy) = \Pi_{\partial G}(x, dy)$. Since S_r is compact

$$\lim_{|x| \to \infty} P_x(V_{S_r} = \infty) = \lim_{|x| \to \infty} P_x(V_{\partial G} = \infty) = 1. \tag{6.1}$$

Thus, for any continuous function φ on ∂G, $\Pi_{\partial G}\varphi$ is the unique bounded solution to the Dirichlet problem for G with boundary function φ that vanishes at ∞. It is easily checked that

$$h(x) = \int_{\partial G} r^{n-2} \left| |x|^2 - r^2 \right| |\xi - x|^{-n} \varphi(\xi) \sigma_r(d\xi), \tag{6.2}$$

is a bounded harmonic function on G taking values φ on ∂G and vanishes at ∞. Thus, for $x \in G$,

$$\Pi_{S_r}(x, d\xi) = r^{n-2} \left| |x|^2 - r^2 \right| |\xi - x|^{-n}\, d\sigma_r(\xi). \tag{6.3}$$

It follows at once from (6.3) that $\Pi_{S_r}(x, d\xi)g(x)^{-1}$ converges strongly as $|x| \to \infty$ to $k^{-1}r^{n-2}\sigma_r$, where $k = \Gamma\big((n/2) - 1\big)/2\pi^{n/2}$.

Let B be any relatively compact set and let S_r be a closed ball of center 0 and radius r that contains $\bar{B}$ in its interior. Then for $|x| > r$

$$\Pi_B(x, A) = \int_{\partial S_r} \Pi_{\partial S_r}(x, d\xi)\, \Pi_B(\xi, A), \tag{6.4}$$

and thus for any Borel set A,

$$\lim_{|x|\to\infty} \frac{\Pi_B(x, A)}{g(x)} = \int_{\partial S_r} k^{-1}\sigma_r(d\xi)\, \Pi_B(\xi, A). \tag{6.5}$$

We have thus established the following important theorem.

THEOREM 6.1. *Let B be any relatively compact set. Then the measure*

$$\mu_B(dy) = \lim_{|x|\to\infty} \frac{\Pi_B(x, dy)}{g(x)} \tag{6.6}$$

exists in the sense of strong convergence of measures. For any ball S_r of center 0 and radius r containing $\bar{B}$ in its interior

$$\mu_B(dy) = \int_{\partial S_r} \Pi_B(\xi, dy)k^{-1}\sigma_r(d\xi). \tag{6.7}$$

DEFINITION 6.1. *The measure μ_B is called the equilibrium measure of B and its total mass $C(B)$ is called the capacity of B.*

Since $\Pi_B(x, N) \equiv 0$ whenever N is a polar set, we see that $\mu_B(N) = 0$ for any polar set. It is also clear that μ_B is concentrated in the outer boundary of B.

By use of probability theory, we have directly defined an equilibrium measure and capacity for any relatively compact Borel set. We must now show that this is consistent with the definitions usually given in potential theory. The equilibrium measure (also called the capacitory measure) is usually defined only for compact sets and in the following manner.

Let $\mathcal{M}(B)$ denote all *nonzero* bounded measures having compact support contained in B whose potentials are bounded above by 1. When B is compact it is then shown that there is a unique measure γ_B supported on B such that $G\gamma_B = \sup_{\mu\in\mathcal{M}(B)} G\mu$. The measure γ_B is what is usually called the capacitory measure of B and its total mass the capacity. We will now show that $\gamma_B = \mu_B$. (In the classical theory of potentials capacitory measures are only defined for compact sets B.)

As a first step towards this goal we will show the following important theorem.

THEOREM 6.2. *Let B be a relatively compact set. Then $P_x(V_B < \infty) = G\mu_B(x)$.*

PROOF. By (5.16),

$$g(y - x) = \int_{\bar{B}} \Pi_B(x, dz)g(y - z) + g_B(x, y). \tag{6.8}$$

Since $g(y-x)/g(y) \to 1$ as $|y| \to \infty$ uniformly on compacts, it follows from (6.8) that $\lim_{|y|\to\infty} g_B(x,y)g(y)^{-1} = P_x(V_B = \infty)$, the convergence being uniform on compacts. By symmetry,

$$\lim_{|x|\to\infty} \frac{g_B(x,y)}{g(x)} = P_y(V_B = \infty), \tag{6.9}$$

uniformly in y on compacts. Let f be any nonnegative bounded measurable function having compact support. Then $Gf(z)$ is a bounded function. From (6.9), we see that

$$\lim_{|x|\to\infty} \int_{R^n} \frac{g_B(x,y)}{g(x)} f(y)\,dy = \int_{R^n} P_y(V_B = \infty) f(y)\,dy, \tag{6.10}$$

while by Theorem 6.1,

$$\lim_{|x|\to\infty} \int_{\bar{B}} \frac{\Pi_B(x,dz)Gf(z)}{g(x)} = \int_{\bar{B}} \mu_B(dz)Gf(z). \tag{6.11}$$

Thus, using (6.8), we see that

$$\int_{\bar{B}} \mu_B(dz)Gf(z) = \int_{R^n} P_y(V_B < \infty) f(y)\,dy. \tag{6.12}$$

As f is an arbitrary bounded function having compact support, it follows from (6.12) that for a.e. y,

$$G\mu_B(y) = P_y(V_B < \infty). \tag{6.13}$$

Let $\varphi_B(y) = P_y(V_B < \infty)$. Then $P^t\varphi_B(y) = P_y(X_s \in B \text{ for some } s > t)$ increases to $\varphi_B(y)$ as $t \downarrow 0$. Also

$$P^tG\mu_B = G\mu_B - \int_0^t P^s\mu_B\,ds \tag{6.14}$$

increases to $G\mu_B$ as $t \downarrow 0$. From (6.13), we see that $P^tG\mu_B = P^t\varphi_B$, and thus letting $t \downarrow 0$, we see that (6.13) holds for all y. This establishes the theorem.

It follows from Theorem 6.2 that $C(B) = 0$ if and only if B is a polar set. We can now easily show that $\gamma_B = \mu_B$.

THEOREM 6.3. *Let B be a compact set. Then $P_x(V_B < \infty) = \sup_{\mu\in\mathcal{M}(B)} G\mu(x)$.*

PROOF. Since B is compact, we can find compact sets B_n such that $B \subset \mathring{B}_n$ for all n and $B_1 \supset \mathring{B}_1 \supset B_2 \supset \cdots$, $\cap_n B_n = \cap_n \mathring{B}_n = B$. By Proposition 3.2, $P_x(V_{B_n} \uparrow V_B) = 1$ for $x \in B^c \cup B^r$. Thus, for $x \in B^c \cup B^r$ and f a continuous function

$$\begin{aligned} &\lim_n \Pi_{B_n} f(x) \\ &\quad = \lim_n E_x[f(X_{V_{B_n}}); V_B < \infty] + \lim_n E_x[f(X_{V_{B_n}}); V_{B_n} < \infty, V_B = \infty] \\ &\quad = \Pi_B f(x). \end{aligned} \tag{6.15}$$

In particular, by taking f continuous with compact support and equal to 1 on B_1, we see that $P_x(V_{B_n} < \infty) \downarrow P_x(V_B < \infty)$ for $x \in B^r \cup B^c$. Since $(B^c \cup B^r)^c$ has measure 0, $P_x(V_{B_n} < \infty) \downarrow P_x(V_B < \infty)$ a.e. Let $\mu \in \mathcal{M}(B)$. Then as each point of B is a regular point of B_n (since $B \subset B_n^0$), $\int_B g_{B_n}(x, y)\mu(dy) = 0$, and thus by (6.8),

$$G\mu(x) = \int_{B_n} \Pi_{B_n}(x, dz) G\mu(z) \leqq \Pi_{B_n}(x, B_n) = P_x(V_{B_n} < \infty). \tag{6.16}$$

Thus, for each $x \in B^c \cup B^r$ we see upon letting $n \to \infty$ that

$$G\mu(x) \leqq P_x(V_B < \infty) = \varphi_B(x), \tag{6.17}$$

so that (6.17) is valid for a.e. x. Hence, for all x, $P^t G\mu(x) \leqq P^t \varphi_B(x)$, and thus letting $t \downarrow 0$ we see that (6.17) holds for *all* x. Using this and the fact that $\mu_B \in \mathcal{M}(B)$, we see that Theorem 6.3 holds.

Theorems 6.2 and 6.3 and our uniqueness theorem (Theorem 5.1) show that $\gamma_B = \mu_B$ when B is compact.

An immediate consequence of Theorems 6.2 and 6.3 is the following corollary.

COROLLARY 6.1. *Let B be relatively compact. Then for any $\mu \in \mathcal{M}(B)$, $\mu(R^n) \leqq C(B)$.*

PROOF. By Theorems 6.2 and 6.3, we know that

$$\int_{\bar{B}} \frac{g(y - x)}{g(x)} \mu(dy) \leqq \int_{\bar{B}} \frac{g(y - x)}{g(x)} \mu_B(dy). \tag{6.18}$$

Since $g(y - x)g(x)^{-1} \to 1$ as $|x| \to \infty$ uniformly on compacts, we see by letting $|x| \to \infty$ that $\mu(R^n) = \mu(\bar{B}) \leqq \mu_B(\bar{B}) = C(B)$, as desired.

Let U be any open set. We can then find compact sets $k_n \subset U$ such that $k_1 \subset k_2 \subset \cdots$, $\cup_n k_n = U$. Since $X_t \in U$ if and only if $X_t \in k_n$ for all sufficiently large n, $P_x(V_{k_n} \downarrow V_U) = 1$ for all x, so $P_x(V_{k_n} < \infty) \uparrow P_x(V_U < \infty)$. By Theorem 6.2, $P_x(V_{k_n} < \infty) = G\mu_{k_n}(x)$. Thus, $G\mu_{k_n}(x) \uparrow P_x(V_U < \infty)$. But as $\mu_{k_n} \in \mathcal{M}(U)$,

$$P_x(V_U < \infty) \leqq \sup_{\mu \in \mathcal{M}(U)} G\mu(x). \tag{6.19}$$

On the other hand if $\mu \in \mathcal{M}(U)$ has compact support k, then Theorems 6.2 and 6.3 show that

$$G\mu(x) \leqq G\mu_k(x) = P_x(V_k < \infty) \leqq P_x(V_U < \infty). \tag{6.20}$$

Thus, $P_x(V_U < \infty)$ is the *smallest majorant* of potentials of measures in $\mathcal{M}(U)$. This characterization of $P_x(V_U < \infty)$ together with the one given for compact sets by Theorem 6.3 shows that $P_x(V_B < \infty)$ for B an open set or a compact set is the electrostatic potential of B for such sets.

The capacity function $C(\cdot)$ defined for all relatively compact sets has the following properties.

THEOREM 6.4. *Let A and B be relatively compact. Then:*

(i) $C(A) \leqq C(B)$, if $A \subset B$;

(ii) $C(A \cup B) \leqq C(A) + C(B) - C(A \cap B)$;

(iii) $C(A + a) = C(A)$;

(iv) $C(-A) = C(A)$;

(v) $C(rA) = r^{n-2}C(A)$;

(vi) *if B is open and $\bar{B}$ compact,* $C(B) = \sup\{C(k): k \subset B, k \text{ compact}\}$;

(vii) *if B is compact,* $C(B) = \inf\{C(U): U \supset B, U \text{ open}, \bar{U} \text{ compact}\}$.

PROOF. Using Theorem 6.2 and the fact that $g(y - x)/g(x) \to 1$ uniformly on compacts as $|x| \to \infty$, we see that for any relatively compact set B

$$C(B) = \lim_{|x|\to\infty} \frac{P_x(V_B < \infty)}{g(x)} = \int_{\partial S} P_\xi(V_B < \infty)k^{-1}\sigma_r(d\xi), \tag{6.21}$$

for any ball of center 0 and radius r containing $\bar{B}$ in its interior. Thus, to establish (i) to (v), it is only necessary to establish the appropriate inequalities for $P_x(V_B < \infty)$. Hence, (i) to (v) follow from:

(a) $P_x(V_A < \infty) \leqq P_x(V_B < \infty), A \subset B$;

(b) $P_x(V_{A\cap B} < \infty) \leqq P_x(V_A < \infty, V_B < \infty) = P_x(V_A < \infty) + P_x(V_B < \infty) - P_x(V_{A\cup B} < \infty)$;

(c) $P_x(V_A < \infty) = P_{x+a}(V_{A+a} < \infty)$;

(d) $P_x(V_A < \infty) = P_x(V_{-A} < \infty)$;

(e) $P_x(V_A < \infty) = P_{rx}(V_{rA} < \infty)$.

To prove (vi), let k_n be compact sets $\subset B$ such that $k_1 \subset k_2 \subset \cdots$ and $\cup_n k_n = B$. Then $P_x(V_{k_n} < \infty) \uparrow P_x(V_B < \infty)$. Let D be an open relatively compact set containing $\bar{B}$. Then

$$\begin{aligned} C(k_n) &= \int_{R^n} P_x(V_D < \infty)\mu_{k_n}(dx) \\ &= \int_{\bar{D}} G\mu_{k_n}(x)\mu_D(dx) \uparrow \int_{\bar{D}} P_x(V_B < \infty)\mu_D(dx) \\ &= \int_{\bar{B}} G\mu_D(x)\mu_B(dx) = \int_{\bar{B}} P_x(V_D < \infty)\mu_B(dx) = C(B). \end{aligned} \tag{6.22}$$

To establish (vii), let U_n be open, relatively compact, and such that $U_1 \supset \bar{U}_2 \supset U_2 \supset \cdots$, $\cap_n U_n = \cap_n \bar{U}_n = B$. Choose S_r to be an open ball of center 0 and radius r that contains $\bar{U}_1$ in its interior. Then for $\xi \in \partial S_r$, $P_\xi(V_{U_n} < \infty) \downarrow P_\xi(V_B < \infty)$, and thus

$$C(U_n) = \int_{\partial S_r} P_\xi(V_{U_n} < \infty)k^{-1}\sigma_r(d\xi) \downarrow \int_{\partial S_r} P_\xi(V_B < \infty)k^{-1}\sigma_r(d\xi) = C(B). \tag{6.23}$$

This establishes the theorem.

Let $C^*(B) = C(B)$ if B is a compact set. For any open set U define $C^*(U)$ as $\sup \{C(k): k \subset U, k \text{ compact}\}$. We say that a set B is capacitable if $\sup \{C^*(k): k \subset B, k \text{ compact}\} = \inf \{C^*(U): U \supset B, U \text{ open}\}$. For a capacitable set define $C^*(B)$ as $\sup \{C(k): k \subset B, k \text{ compact}\}$. Property (vi) shows that if U is relatively compact then $C^*(U) = C(U)$. This fact, together with (i) and (ii) shows that $C(\cdot)$ is a Choquet capacity (see, for example, [1]) on the compact sets. By Choquet's capacity theorem, C^* is its unique extension to the Borel sets and every Borel set is capacitable.

Now for a relatively compact set B, we have already defined its capacity by $C(B)$. To see that $C^*(B) = C(B)$ note that if $k \subset B, k$ compact then $C(k) \leqq C(B)$, and thus

$$C^*(B) = \sup \{C(k): k \subset B. k \text{ compact}\} \leqq C(B). \tag{6.24}$$

Also if U is open and relatively compact then $C(B) \leqq C(U)$, and thus

$$C^*(B) = \inf \{C(U); U \supset B, U \text{ open}\} \geqq C(B). \tag{6.25}$$

Thus, $C^*(B) = C(B)$.

Now that we have the capacity defined for all Borel sets, let us point out that B has capacity 0 if and only if every compact subset of B has capacity 0. (We could have used this property to define sets of capacity 0 directly.) Our next two results charactize polar sets.

THEOREM 6.5. *Let B be any Borel set. Then*

(i) *B is polar if and only if every compact subset of B is polar;*

(ii) *B is polar if and only if $C^*(B) = 0$.*

PROOF. Clearly, if B is polar, then so is every compact subset of B. On the other hand, if every compact subset of B is polar, then for any relatively compact $A \subset B$, $C(A) = \sup \{C(k): k \subset A, k \text{ compact}\} = 0$, so A is polar. Thus, B must be polar. Therefore (i) holds. Similarly, if B is polar, then every compact subset is polar so $C^*(B) = \sup \{C(k): k \subset B, k \text{ compact}\} = 0$. Conversely, if $C^*(B) = 0$, then $C(k) = 0$ for all compact sets $k \subset B$, and thus by (i) B is polar. This establishes the theorem.

COROLLARY 6.2. *Let B be any Borel set. Then B is polar if and only if $\mathcal{M}(B) = \varnothing$. Equivalently, B is polar if and only if $\sup_x G\mu(x) = \infty$ for any bounded nonzero measure having compact support $\subset B$.*

PROOF. Suppose there is a nonzero measure μ having compact support $k \subset B$ such that $\sup_x G\mu(x) \leqq M < \infty$. The measure μ/M then belongs to $\mathcal{M}(k) \subset \mathcal{M}(B)$ and clearly

$$P_x(V_B < \infty) \geqq P_x(V_k < \infty) \geqq G\mu(x). \tag{6.26}$$

Since μ is nonzero, $G\mu(x) > 0$ for some x, and thus $P_x(V_B < \infty) > 0$ for some x. Hence, B is not polar. On the other hand if B is not polar then some compact subset k of B must also be nonpolar. Hence, $P_x(V_k < \infty) > 0$ for some x. Since $\mu_k \in \mathcal{M}(k)$, $\mathcal{M}(k) \neq \varnothing$, and $G\mu_k(x) \leqq 1$. This establishes the corollary.

7. Equilibrium sets

So far we have discussed equilibrium measures only for relatively compact sets. In this section, we will examine to what extent these notions go over to unbounded sets. We start our discussion with the following theorem.

THEOREM 7.1. *Let B be a Borel set. Then either $P_x(V_B < \infty) \equiv 1$ or $P_x(X_s \in B$ for some $s > t) \to 0$ as $t \to \infty$.*

PROOF. Let $\varphi(x) = P_x(V_B < \infty)$. Then $P^t\varphi$ is a decreasing function in t as $t \to \infty$. Let $r(x) = \lim_{t\uparrow\infty} P^t\varphi(x)$ and set $h = \varphi - r$. Clearly, $P^t h(x) \downarrow 0$ as $t \to \infty$. Using dominated convergence and the semigroup property of P^t, it easily follows that $r = P^t r$ for all $t > 0$. Thus, $r(x) \equiv \alpha$ for some constant α (see Proposition 2.3). Hence, $\varphi = \alpha + h$. Now

$$P_x(t < V_B < \infty) = \int_{R^n} q_B(t, x, y)\varphi(y)\, dy \geqq \alpha P_x(V_B > t). \tag{7.1}$$

Letting $t \uparrow \infty$, we see that $\alpha P_x(V_B = \infty) \leqq 0$. Thus, either $\alpha = 0$ or $P_x(V_B = \infty) \equiv 0$. In the first case $\varphi = h$ and in the second case $P_x(V_B < \infty) \equiv 1$ as desired.

DEFINITION 7.1. *A Borel set B is called recurrent if $P_x(V_B < \infty) \equiv 1$; it is called transient if $P_x(X_s \in B$ for some $s > t) \downarrow 0$.*

By Theorem 7.1, we know that every Borel set is either transient or recurrent.

Our next result extends Theorem 6.2 from relatively compact sets to transient sets.

THEOREM 7.2. *Let B be a transient set. Then there is a unique Radon measure μ_B such that $P_x(V_B < \infty) = G\mu_B(x)$. The measure μ_B is concentrated on ∂B. Moreover, if B_m, $m \geqq 1$, is any sequence of relatively compact sets such that $B_1 \subset B_2 \subset \cdots$, $\cup_m B_m = B$, then $G\mu_{B_m} \uparrow G\mu_B$ as $m \to \infty$ and the measures μ_{B_m} converge vaguely to μ_B.*

PROOF. Let B_m, $m \geqq 1$, be as in the statement of the theorem. Then $\cup_m [V_{B_m} < \infty] = [V_B < \infty]$, and thus $P_x(V_{B_m} < \infty) \uparrow P_x(V_B < \infty)$. By Theorem 6.2,

$$G\mu_{B_m}(x) = P_x(V_{B_m} < \infty) \leqq P_x(V_B < \infty) = \varphi_B(x). \tag{7.2}$$

Let k be any compact set. Since $\inf_{y\in k} g(y - x) = \delta(x) > 0$, it follows from (7.2) that

$$\delta(x)\mu_{B_m}(k) \leqq \int_k g(y - x)\mu_{B_m}(dy) \leqq G\mu_{B_m}(x) \leqq \varphi_B(x). \tag{7.3}$$

Thus, $\sup_m \mu_{B_m}(k) < \infty$. Consequently, there is a subsequence, $\mu_{B'_m}$, of the measures μ_{B_m} that converge vaguely to some measure μ_B.

Let $f \geqq 0$ be continuous with compact support, and let S_r be the closed ball of center 0 and radius r. Then, as $g_{S_f}(x, y) \equiv 0$ for each $x \in S_r^c$, we see that

$$\int_{S_f} \mu_{B'_m}(dx) Gf(x) = \int_{S_f} \mu_{B'_m}(dx)\ \Pi_{S_f}\ Gf(x) \leqq \int_{R^n} \mu_{B'_m}(dx)\ \Pi_{S_f}\ Gf(x). \tag{7.4}$$

By letting $\lambda \downarrow 0$ in (2.31), we see that

$$\int_{R^n} \Pi_{S_f}(x, dz) g(y - z) = \int_{R^n} \Pi_{S_f}(y, dz) g(x - z). \tag{7.5}$$

Using this fact, we compute

$$\begin{aligned} \int_{R^n} \mu_{B'_m}(dx)\ \Pi_{S_f}\ Gf(x) &= \int_{R^n}\int_{R^n}\int_{R^n} \mu_{B'_m}(dx)\ \Pi_{S_f}(x, dz) g(y - z) f(y)\, dy \\ &= \int_{R^n}\int_{R^n}\int_{R^n} \mu_{B'_m}(dx) f(y)\, dy\ \Pi_{S_f}(y, dz) g(x - z) \\ &= \int_{R^n} \Pi_{S_f}\ G\mu_{B'_m}(y) f(y)\, dy. \end{aligned} \tag{7.6}$$

Using (7.2), (7.4), and (7.6), we see that

$$\int_{S_f} Gf(x) \mu_{B'_m}(dx) \leqq \int_{R^n} \Pi_{S_f}\ \varphi_B(y) f(y)\, dy. \tag{7.7}$$

Now for any $t > 0$.

$$\Pi_{S_f}\ \varphi_B(y) = E_y[\varphi_B(X_{VS_f})] \leqq P_y(V_{S_f} \leqq t) + P^t \varphi_B(y). \tag{7.8}$$

Since B is a transient set, $P^t \varphi_B \downarrow 0$ as $t \uparrow \infty$. In addition, $P_y(V_{S_f} \leqq t) \downarrow 0$ as $r \uparrow \infty$ because $P_y(\lim_{r \to \infty} V_{S_f} = \infty) = 1$. It follows from these two facts that

$$\lim_{r \to \infty} \Pi_{S_f}\ \varphi_B(y) = 0. \tag{7.9}$$

Hence, by (7.7) and (7.9), we see that given any $\varepsilon > 0$ there is an $r_0 < \infty$ such that for $r \geqq r_0$,

$$\sup_m \int_{S_f} \mu_{B'_m}(dx) Gf(x) \leqq \varepsilon. \tag{7.10}$$

Using the fact that Gf is a bounded continuous function, we see that

$$\lim_{r\to\infty}\left[\lim_{m\to\infty}\int_{S_r} Gf(x)\mu_{B'_m}(dx)\right] = \lim_{r\to\infty}\int_{S_r} Gf(x)\mu_B(dx) \tag{7.11}$$
$$= \int_{R^n} Gf(x)\mu_B(dx)$$
$$= \int_{R^n} G\mu_B(x)f(x)\,dx.$$

Moreover, monotone convergence shows that

$$\lim_{m\to\infty}\int_{R^n} Gf(x)\mu_{B'_m}(dx) = \lim_{m\to\infty}\int_{R^n} G\mu_{B'_m}(x)f(x)\,dx = \int_{R^n}\varphi_B(x)f(x)\,dx. \tag{7.12}$$

It follows easily from (7.10) through (7.12) that

$$\int_{R^n} G\mu_B(x)f(x)\,dx = \int_{R^n}\varphi_B(x)f(x)\,dx. \tag{7.13}$$

Since f can be any nonnegative continuous function with compact support, (7.13) implies that for a.e. x,

$$G\mu_B(x) = \varphi_B(x). \tag{7.14}$$

From (7.14), we see that $P^tG\mu_B(x) = P^t\varphi_B(x)$ for all x. Since $P^tG\mu_B \uparrow G\mu_B$ and $P^t\varphi_B \uparrow \varphi_B$ as $t\downarrow 0$, it follows that (7.14) holds for *all* x.

Suppose $\mu_{B''_m}$ is another subsequence of the measures μ_{B_m} that converge vaguely to a measure μ'_B. The same argument as used above will again show that $G\mu'_B = \varphi_B$. By Theorem 5.1, it must then be that $\mu'_B = \mu_B$. Thus, the measures μ_{B_m} converge vaguely to μ_B. Theorem 5.1 tells us that μ_B is the unique measure whose potential is φ_B. To see that μ_B is concentrated on ∂B, we can proceed as follows. Let S_m be the closed ball of center 0 and radius m, and let $B_m = B \cap S_m$. Then $B_1 \subset B_2 \subset \cdots$, and $\cup_m B_m = B$. Thus, the measures μ_{B_m} converge vaguely to μ_B. Since μ_{B_m} is concentrated on ∂B_m and each interior point of B is an interior point of B_m for m sufficiently large, μ_B must be concentrated on ∂B. This completes the proof.

We will now show that the total mass of the measure μ_B in Theorem 7.2 is $C^*(B)$. To do this, we will need the following

PROPOSITION 7.1. *Let B be a transient set. Suppose $A \subset B$. Then $\mu_A(R^n) \leqq \mu_B(R^n)$.*

PROOF. Let D_m be a family of relatively compact sets that increase to R^n. Using Theorem 7.2, we then see that

$$\begin{aligned}(7.15)\qquad \int_{R^n} \mu_A(dx) G\mu_{D_m}(x) &= \int_{R^n} \mu_{D_m}(dx) G\mu_A(x) = \int_{R^n} \mu_{D_m}(dx) P_x(V_A < \infty) \\ &\leqq \int_{R^n} \mu_{D_m}(dx) P_x(V_B < \infty) = \int_{R^n} \mu_{D_m}(dx) G\mu_B(x) \\ &= \int_{R^n} \mu_B(dx) G\mu_{D_m}(x).\end{aligned}$$

Since

$$(7.16)\qquad G\mu_{D_m}(x) = P_x(V_{D_m} < \infty) \uparrow 1$$

as $m \uparrow \infty$, it follows from (7.15) (by monotone convergence) that $\mu_A(R^n) \leqq \mu_B(R^n)$, as desired.

THEOREM 7.3. *Let B be a transient set. Then $\mu_B(R^n) = C^*(B)$. Moreover, if B_m, $m \geqq 1$, is a sequence of relatively compact subsets of B such that $B_1 \subset B_2 \subset \cdots$, $\cup_m B_m = B$, then $C(B_m) \uparrow C^*(B)$.*

PROOF. Let B_m, $m \geqq 1$, be as in the statement of the theorem. Then by Proposition 7.1,

$$(7.17)\qquad C(B_1) \leqq C(B_2) \leqq \cdots \leqq \mu_B(R^n).$$

On the other hand, let f_r, $r \geqq 1$, be continuous functions with compact support such that $0 \leqq f_r \leqq 1$ and such that $f_r \uparrow 1$ as $r \uparrow \infty$. Then

$$(7.18)\qquad C(B_m) \geqq \int_{R^n} f_r(x) \mu_{B_m}(dx),$$

and thus by Theorem 7.2,

$$(7.19)\qquad \liminf_{m \to \infty} C(B_m) \geqq \int_{R^n} \mu_B(dx) f_r(x).$$

Letting $r \uparrow \infty$, we see that

$$(7.20)\qquad \liminf_{m \to \infty} C(B_m) \geqq \mu_B(R^n).$$

Hence, $C(B_m) \uparrow \mu_B(R^n)$.

From our results in Section 6, we know that

$$(7.21)\qquad C(B_m) = \sup \{C(k): k \subset B_m, k \text{ compact}\}.$$

Suppose $\mu_B(R^n) = \infty$. Given any $N > 0$, we can then find an m such that $C(B_m) \geqq 2N$. From (7.21), we see that we can find a compact set $k \subset B_m$ such that $C(k) \geqq C(B_m) - N$. Thus, $C(k) \geqq N$. Hence, $\sup \{C(k): k \subset B, k \text{ compact}\} = C^*(B) = \infty$. Suppose now that $\mu_B(R^n) < \infty$, and let $\varepsilon > 0$ be given. We can then choose m such that $C(B_m) \geqq \mu(R^n) - \varepsilon$. From (7.21), we see that

there is a compact set $k \subset B_m$ such that $C(k) \geqq C(B_m) - \varepsilon$, and thus $C(k) \geqq \mu(R^n) - 2\varepsilon$. Hence,

$$C^*(B) = \sup \{C(k): k \subset B, k \text{ compact}\} = \mu_B(R^n). \tag{7.22}$$

This establishes the theorem.

Let B be a closed set. As usual, let $\mathscr{M}(B)$ denote all nonzero bounded measures having compact support $\subset B$ whose potentials are bounded above by 1.

An important link between probability theory and potential theory is the following

THEOREM 7.4. *Let B be a closed set. Then*

$$\sup \{G\mu(x): \mu \in \mathscr{M}(B)\} = P_x(V_B < \infty). \tag{7.23}$$

PROOF. Since B is closed, we can find compact sets $B_n \subset B$, $B_1 \subset B_2 \subset \cdots$, $\cup_n B_n = B$. Then clearly, $P_x(V_{B_n} < \infty) \uparrow P_x(V_B < \infty)$, $n \to \infty$. If $\mu \in \mathscr{M}(B)$ has compact support $k \subset B_n$, then by Theorems 6.2 and 6.3,

$$G\mu(x) \leqq G\mu_{B_n}(x) \leqq P_x(V_{B_n} < \infty) \leqq P_x(V_B < \infty), \tag{7.24}$$

and thus

$$\sup \{G\mu(x): \mu \in \mathscr{M}(B)\} \leqq P_x(V_B < \infty). \tag{7.25}$$

But $G\mu_{B_n}(x) = P_x(V_{B_n} < \infty) \uparrow P_x(V_B < \infty)$, so

$$\sup \{G\mu(x): \mu \in \mathscr{M}(B)\} = P_x(V_B < \infty). \tag{7.26}$$

This establishes the theorem.

COROLLARY 7.1. *A closed set B is transient if and only if there is a Radon measure μ_B supported on ∂B such that*

$$G\mu_B(x) = \sup \{G\mu(x): \mu \in \mathscr{M}(B)\}. \tag{7.27}$$

PROOF. This follows at once from Theorems 7.2 and 7.4.

If B is a transient set the measure μ_B is called the *equilibrium measure* of B and its potential is called the *equilibrium potential*, just as in the case of a compact set. The total mass of μ_B is the capacity of B. Theorem 7.3 shows this is consistent with the extension of $C(\cdot)$ from the relatively compact sets.

We are now in a position to state our results on the Dirichlet problem for G in analytical terms. Note that for $x \in G$, $P_x(V_{\partial G} = \infty) = P_x(V_{G^c} = \infty)$. We want to know when $P_x(V_{\partial G} = \infty) = 0$ for all $x \in G$. Suppose this is the case. Then G^c must be a recurrent set. For suppose $P_x(V_{G^c} < \infty) = 1$ for all $x \in G$. It is always true that $P_x(V_{G^c} < \infty) = 1$ for all $x \in G^c$ except perhaps at the points in G^c that are irregular. But these exceptional points form a set of measure 0, and thus $P_x(V_{G^c} < \infty) = 1$ a.e. Since $P^t\varphi_{G^c} \uparrow \varphi_{G^c}$ as $t \downarrow 0$, we then see that $P_x(V_{G^c} < \infty) = 1$ for all x, so G^c is recurrent. Conversely, if G^c is recurrent then $P_x(V_{G^c} < \infty) \equiv 1$, so $P_x(V_{\partial G} = \infty) = 0$ for all $x \in G$. Thus we have the following.

THEOREM 7.5. *Let G be an open set. The modified Dirichlet problem for ∂G with boundary function φ has $\Pi_{\partial G}\,\varphi$ as its unique bounded solution if and only if G^c is a recurrent set. If G^c is a transient set then a constant multiple of $P_x(V_{G^c} = \infty)$ can be added to the solution $\Pi_{\partial G}\,\varphi$. These constitute the only bounded solutions to the problem.*

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[1] R. BLUMENTHAL and R. GETOOR, *Markov Processes and Potential Theory*, New York, Academic Press, 1968.