MARTIN BOUNDARIES OF RANDOM WALKS ON LOCALLY COMPACT GROUPS

R. AZENCOTT
UNIVERSITY OF CALIFORNIA, BERKELEY
and
P. CARTIER
I.H.E.S., BURES-SUR-YVETTE

Introduction

Let G be a separable locally compact space and let (X_t), t in T, be a transient Markov process with values in G, where T is either the set of positive integers (discrete time) or the set of positive real numbers (continuous time). Let (Q^t) be the semigroup of transition kernels of (X_t). Let f and λ be, respectively, a positive Borel function on G and a positive measure on the Borel σ-field of G. Call f (respectively, λ) excessive if $Q^t f \leqq f$ and $\lim_{t \to 0} Q^t f = f$ (respectively, $\lambda Q^t \leqq \lambda$), and invariant if $Q^t f = f$ (respectively, $\lambda Q^t = \lambda$).

Around 1955, the early studies of excessive functions of a Markov process centered around two problems: the relations between Brownian motion and Newtonian potential theory, and the behavior of the trajectories of the process (X_t) as $t \to +\infty$. The latter approach can be traced back to D. Blackwell ([4], 1955) who noticed the link between bounded invariant functions and the subsets of G in which (X_t) stays, from some finite time on, with positive probability. The importance of these 'sojourn' sets became clear after W. Feller's magistral article ([24], 1956), where they are used to construct (discrete T and G) a compactification $G \cup F$ of G such that each bounded invariant function f extends continuously to $G \cup F$ and is uniquely determined by its values on the Feller boundary F.

The other approach was initiated by two papers of J. L. Doob: a study of the behavior of subharmonic functions along Brownian paths ([16], 1954), and a probabilistic approach to the potential theory of the heat equation ([17], 1955). The relation between potential theory and general (transient) Markov processes was completely clarified by G. Hunt soon after ([32], 1957–1958).

These two trends of thought each found their expression in Doob's work ([18], 1959) which revived the methods used by R. Martin ([39], 1941) in his classical study of harmonic functions. In this article, Doob constructed (for

The work of R. Azencott was partially supported by NSF Grant GP-24490.

discrete T and G) a compactification of G by essentially adjoining to G a set of extreme invariant functions, obtained the integral representation of excessive functions in terms of extreme excessive functions, and proved the basic results about the almost sure convergence of (X_t) to the Martin boundary, as $t \to +\infty$. Hunt ([33], 1960) introduced new methods of achieving Doob's results, in particular the reversal of the sense of time for Markov processes.

It soon became clear that Feller's boundary is almost always too large (D. Kendall [34], 1960, J. Feldman [26], 1962), and in later studies the approach of Doob and Hunt has prevailed. The extension of potential theoretic results to the recurrent case began around 1960–1961, first for the case of random walks (K. Itô and H. J. McKean, and F. Spitzer) and then for the case of Markov chains (J. G. Kemeny and J. L. Snell). A very lucid exposition of the main ideas of boundary and potential theory for Markov chains (for discrete T and G) was given by J. Neveu ([46], 1964), using G. Choquet's results on convex cones ([11], 1956) to obtain the integral representation of excessive functions. In [36] (1965), H. Kunita and T. Watanabe treated the case of continuous time and general state space. They considered two processes in duality with respect to a measure on the state space, a situation whose importance had been recognized earlier by G. A. Hunt [32] and P. A. Meyer (thesis). The construction of the (exit) boundary uses, then, the compact caps of the cone of coexcessive measures (that is, measures which are excessive with respect to the dual process).

Meyer ([42], 1968) gave a new presentation of their results using Choquet's integral representation theorem.

When the state space G is a topological group, a class of Markov processes is naturally linked to the group structure, the random walks on G. We now restrict our attention to this situation. In the abelian case, the first significant result was reached by G. Choquet and J. Deny ([12], 1960), who showed that the extreme invariant functions are essentially characters of the group. For the particular case when $G = Z$, group of integers, this was noticed simultaneously by J. L. Doob, J. L. Snell, and R. Williamson ([22], 1960).

For the case $G = Z^n$, the boundaries of random walks were then carefully described by P. Hennequin ([31], thesis 1962) and the asymptotic behavior of the Green function at the boundary was obtained by P. Ney and F. Spitzer ([47], 1966). The basic results of the potential theory of random walks on Z^n, exposed by Spitzer ([49], 1964) were soon extended to countable abelian groups in a joint paper with H. Kesten ([35], 1966).

For the nonabelian case, the extreme invariant functions were obtained first for finitely generated groups by E. Dynkin and M. Maljutov ([23], 1961). A major advance was made by H. Furstenberg ([27], 1963) giving an integral representation of bounded invariant functions for random walks on semisimple connected Lie groups, containing as a particular case the classical Poisson formula relative to harmonic functions on a disc. He obtained partial results ([29], 1965) about the cone of all nonnegative invariant functions for the same class of groups. The main results of [27] were extended by R. Azencott ([3], 1970) to a larger class of groups.

We have tried to outline briefly the evolution of the main currents of ideas relevant to our work. Lack of space has forced us to make a number of sizeable omissions: the essential influences of "abstract" potential theory, the important applications of the theory of excessive functions, and the rich material concerning the recurrent case have been barely alluded to.

The "Poisson formula" obtained by Furstenberg [27] involves a family of explicitly described, compact homogeneous spaces of G, the "Poisson spaces" of G. One question arises naturally: what are the relations between the Poisson spaces of G and the Martin boundaries of random walks of G? This problem is solved in the last part of the present work. We study the case of a transient random walk on a locally compact separable group; there is then a natural random walk in duality with the first one with respect to any right invariant Haar measure on G. Let $\bar{U}$ be the potential kernel of the dual random walk. To any function $r \geqq 0$ on G such that $0 < \bar{U}r < \infty$ ("reference" function), we associate as in [36] a continuous one to one map from G into a compact cap of the cone of coexcessive measures. As in Neveu [46] and Meyer [42], Choquet's theorem gives, then, the integral representation of coexcessive measures. The classical, Martin type compactifications of G have been abandoned here, mainly because G does not in general act continuously on these spaces. Even a strong restriction on the type of reference function used ("adapted" reference function, see Sections 16 and 17) only insures an action of G on part of the boundary. The only favorable case seems to be the one when the closed semigroup generated in G by the support of the law of the random walk is large enough (see Section 8).

We have preferred to imbed G in the sets of *rays* of the cone of coexcessive measures obtaining thus a Hausdorff space $G \cup B$ with countable base, and an "intrinsic boundary" B. This space is neither metrizable nor compact, in general, but the slight measure theoretic technicalities required by this situation are balanced by the fact that G acts continuously on $G \cup B$. We also obtain intrinsic formulations (independent of the reference function r) for the main classical results: integral representation of coexcessive measures and convergence to the boundary.

We then prove a "Poisson formula" for bounded invariant functions, also in intrinsic form, and use it to essentially identify the Poisson space and the "active part" of the intrinsic boundary.

Although reference functions have been eliminated in the formulation of the main results, they have been used in many proofs, and it is, in fact, possible to present the whole question in the more classical setting of Martin compactifications, provided only "adapted" reference functions are used (see Section 17). We also point out that our proofs of the basic (nonintrinsic) results on integral representation and convergence to the boundary follow classical patterns, essentially those outlined by Neveu [46].

Part A. Preliminaries

In this part, the main conventions are stated and a description of the random walks on a group is given. A derivation of the recurrence criterion for such random walks is proposed, after K. L. Chung and W. H. J. Fuchs [15]. The boundary theory described is nontrivial only in the case of transient random walks (see Section 8) which will be considered exclusively in later parts.

1. Notations and conventions

1.1. *Measure theory.* Let E be a Hausdorff space. The smallest σ-algebra containing the class of open subsets of E is denoted by $\mathscr{B}(E)$ and its elements are called the *Borel subsets* of E. We shall consider exclusively real valued functions on E, with nonnegative infinite values included. For instance, $b^+(E)$ denotes the set of all such functions which are Borel measurable (that is, $\mathscr{B}(E)$ measurable) and $C_c(E)$ denotes the functions which are finite at every point, continuous, and have compact support.

A *measure* μ on E is a σ-additive mapping from $\mathscr{B}(E)$ into $[0, +\infty]$ and μ is a *probability* measure if, moreover, $\mu(E) = 1$. The unit point mass at a point x of E is a measure denoted ε_x. We use the notations $\langle \mu, f \rangle$ and $\int_E f(x)\mu(dx)$ for the integral of a function f in $b^+(E)$ with respect to the measure μ; such an integral may be infinite. By definition of ε_x, one gets $f(x) = \langle \varepsilon_x, f \rangle$ for any f in $b^+(E)$.

The measure μ on E is called a *Radon measure* if it enjoys the following properties:

(a) (*local finiteness*) every point of E has an open neighborhood V such that $\mu(V)$ is finite;

(b) (*inner regularity*) for every Borel subset A of E, the number $\mu(A)$ is the L.U.B. of the numbers $\mu(K)$ where K runs over the class of compact subsets of A.

When μ is a Radon measure, $\mu(K)$ is finite whenever K is compact and among the closed subsets of E whose complement is μ null there is a smallest one called the *support* of μ. When E is a separable locally compact space, local finiteness means that $\mu(K)$ is finite for K compact, and it implies inner regularity [8].

Let E and E' be Hausdorff spaces. A *kernel* Q from E into E' is a mapping from $b^+(E')$ into $b^+(E)$ such that $Q(\Sigma_{n=1}^{\infty} f'_n) = \Sigma_{n=1}^{\infty} Qf'_n$ for f'_n in $b^+(E')$, $n \geqq 1$. The kernel Q is called *markovian* if and only if $Q1 = 1$. Let μ be a measure on E; then there exists a unique measure μQ on E' such that $\langle \mu Q, f' \rangle = \langle \mu, Qf' \rangle$ for each f' in $b^+(E')$. In particular, to Q there corresponds a map q from E into the set of measures on E' given by $q(x) = \varepsilon_x Q$. Then $Qf'(x) = \langle q(x), f' \rangle$ for f' in $b^+(E')$. If μ is a measure on E, one gets

$$\mu Q(A') = \int_E q(x)(A')\mu(dx) \tag{1.1}$$

for each Borel subset A' of E'; we shall abbreviate this relation as $\mu Q = \int_E q(x)\mu(dx)$. Finally, let Q' be a kernel from E' into another Hausdorff space E''.

The *composite kernel* QQ' from E into E'' is defined by $(QQ')f'' = Q(Q'f'')$ for f'' in $b^+(E'')$. By duality, one gets $\mu(QQ') = (\mu Q)Q'$ for each measure μ on E.

1.2. *Topological groups.* Let G be a separable locally compact group. The *convolution* $\mu * \mu'$ of two measures μ and μ' on G is the image of the product measure $\mu \otimes \mu'$ by the multiplication map $(g, g') \mapsto gg'$ from $G \times G$ into G. Note the following integral formula

$$(1.2) \qquad \langle \mu * \mu', f \rangle = \int_G \int_G f(xy)\mu(dx)\mu'(dy), \qquad f \text{ in } b^+(G).$$

The n-fold convolution $\mu * \cdots * \mu$ shall be abbreviated as μ^n. For any measure μ on G, the *opposite measure* $\bar{\mu}$ is defined by $\bar{\mu}(A) = \mu(A^{-1})$ for each Borel subset A of G.

A *right invariant Haar measure* m on G is a nonzero Radon measure such that $m(Ag) = m(A)$ for A in $\mathscr{B}(G)$ and g in G. It is unique up to multiplication by a positive real number and there exists a continuous function Δ on G, the *module function of* G, such that $m(g^{-1}A) = \Delta(g)m(A)$ for g in G and A in $\mathscr{B}(G)$.

By a G *space*, we mean a Hausdorff space E upon which G acts continuously from the left. The group G acts on measures on E by $(g\mu)(A) = \mu(g^{-1}A)$ for g in G and A in $\mathscr{B}(E)$. In particular, G acts on itself by left translations, and hence on the measures on G. One gets from the definitions the relations $g\mu = \varepsilon_g * \mu$ and $gm = \Delta(g)m$ for g in G, μ a measure on G, and m a right invariant Haar measure on G.

A probability measure μ on G is *spread out* if it satisfies the following equivalent conditions:

(a) there is an integer $n \geqq 0$ such that the n-fold convolution μ^n is nonsingular with respect to a right invariant Haar measure;

(b) there is an integer $n \geqq 0$, a right invariant Haar measure m, and a nonempty open set V in G such that $\mu^n(A) \geqq m(A)$ for any Borel subset A of V.

This is the case for instance if μ is absolutely continuous with respect to a Haar measure (see [3] for a study of this notion).

2. Sums of independent random variables

In this and the next section, G denotes a separable locally compact group and μ a probability measure on G.

Let us denote by $(Z_n)_{n \geqq 1}$ an independent sequence of G valued random variables with the common probability law μ, and define $S_0 = e$ (the unit element of G) and $S_n = Z_1 \cdots Z_n$ for $n \geqq 1$. The canonical sample space $(\Omega, \mathscr{B}(\Omega), \mathbf{P})$ for the process $(Z_n)_{n \geqq 1}$ is described as follows: Ω is the topological product space $\Pi_{n=1}^{\infty} G_n$, where $G_n = G$ for each n, $\mathscr{B}(\Omega)$ is the class of Borel subsets of Ω, and $\mathbf{P} = \otimes_{n=1}^{\infty} \mu_n$ where $\mu_n = \mu$ for each n. Moreover, Z_n is the projection of Ω onto its nth factor. Since the topology of G is countably generated, $\mathscr{B}(\Omega)$ is the smallest σ-algebra for which the projections Z_n are measurable (as functions with values in $(G, \mathscr{B}(G))$).

Let G_μ be the smallest closed subgroup of G containing the support of μ. For every $n \geqq 0$, the support of μ^n is contained in G_μ; if μ is spread out on G, the support of μ^n has an inner point for some $n \geqq 0$, and hence G_μ has some inner point, that is, G_μ is open. Consequently, one gets $G_\mu = G$ if G is connected and μ spread out on G (for instance μ absolutely continuous with respect to a Haar measure). In any case, the random variables Z_n and S_n take almost surely their values in G_μ, and we can consider $(Z_n)_{n \geqq 1}$ and $(S_n)_{n \geqq 0}$ as random processes carried by the separable locally compact group G_μ.

We shall say that an element g of G is *μ recurrent* if and only if each neighborhood of g is hit infinitely often by almost every path of the process $(S_n)_{n \geqq 0}$. The following theorem is an easy generalization of the results of Chung and Fuchs [15]. According to the previous remarks, there is no real loss of generality in assuming $G_\mu = G$ and this hypothesis simplifies the enunciation.

THEOREM 2.1. *Assume that there is no proper closed subgroup of G containing the support of μ. Let us define the measure $\pi = \Sigma_{n=0}^{\infty} \mu^n$ on G. There is the following dichotomy:*

(i) (*transient case*) *no element of G is μ recurrent and π is a Radon measure;*

(ii) (*recurrent case*) *every element of G is μ recurrent and $\pi(V)$ is infinite for every nonempty open set V in G.*

PROOF. Let R be the set of μ recurrent elements. We shall prove that R is equal to $\varnothing$ or to G. For that purpose, we introduce the set S of all elements g of G enjoying the following property:

for every open neighborhood V of g there exists an integer $n \geqq 0$ such that $\mathbf{P}[S_n \in V] > 0$.

The complement of S in G is the largest open set U such that $\mathbf{P}[S_n \in U] = 0$ for all $n \geqq 0$, hence contains no μ recurrent point. It follows that S is closed and contains R.

We prove next the inclusion $S^{-1}R \subset R$. Indeed, let g be in S and h be in R, and let U be an open neighborhood of $g^{-1}h$. By continuity of the operation in G, we can find open neighborhoods V and W, of g and h, respectively, such that $V^{-1}W \subset U$. By definition of S, there exists an integer $k \geqq 0$ such that the event $A = [S_k \in V]$ has positive probability. Let A' be the set of all ω in A such that there exist infinitely many integers $n \geqq 0$ such that $S_{n+k}(\omega) \in W$. One gets $A' \in \mathscr{B}(\Omega)$, and since W is an open neighborhood of the μ recurrent point h, one has $\mathbf{P}[A'] = \mathbf{P}[A] > 0$. Put $S'_n = S_k^{-1}S_{n+k}$ for $n \geqq 0$; the process $(S'_n)_{n \geqq 0}$ is then independent from $Z_1, \cdots, Z_k$, hence from A', and for every ω in A' the relation $S_{n+k}(\omega) \in W$ entails $S'_n(\omega) = S_k(\omega)^{-1}S_{n+k}(\omega) \in V^{-1}W \subset U$. It follows that almost every path of the process $(S'_n)_{n \geqq 0}$ hits U infinitely often, and since the process $(S'_n)_{n \geqq 0}$ has the same law as $(S_n)_{n \geqq 0}$ and U is an arbitrary open neighborhood of $g^{-1}h$, it follows that $g^{-1}h$ is μ recurrent and we are through.

Assume now R nonempty. From $R \subset S$ and $S^{-1}R \subset R$, one gets $R^{-1}R \subset R$, that is, R is a subgroup of G. Consequently, e belongs to R; hence, $S^{-1} = S^{-1}e \subset S^{-1}R \subset R = R^{-1}$, that is, $S \subset R$. Finally, $S = R$ is a closed subgroup of G and since μ is the probability law of $S_1 = Z_1$, its support is contained in

$S = R$. Hence, $R = G$.

Assume that $\pi(V)$ is finite for some nonempty open set V in G. Since $\pi(V) = \sum_{n=0}^{\infty} \mathbf{P}[S_n \in V]$, it follows from the Borel-Cantelli lemma that almost every path of the process $(S_n)_{n \geqq 0}$ hits V only finitely many times. Consequently, no point of V is μ recurrent and from above there is no μ recurrent point at all.

Therefore, in the case $R = G$, one gets $\pi(V) = +\infty$ for every nonempty open set V in G. When $R = \varnothing$, one can use the reasoning of Chung and Fuchs ([15], p. 4) to show that π is a Radon measure; one needs only to note that there exists a left invariant metric defining the topology of G. *Q.E.D.*

The previous proof gives a useful criterion for transient processes. Indeed, call a subset Γ of G a semigroup, if it contains the unit element e of G and is closed under multiplication. Denote by Γ_μ the smallest closed semigroup containing the support of μ. It is easy to see that the support of μ^n is the closure of the set of products $g_1 \cdots g_n$ for $g_1, \cdots, g_n$ running over the support of μ. Since μ^n is the probability law of S_n, it is easy to see that Γ_μ is the set denoted S in the proof of Theorem 2.1. We have seen that R nonempty entails $S = R = G$. We see therefore that *the inequality* $\Gamma_\mu \neq G_\mu$ *can occur in the transient case only.* When G is the additive real group, the inequality $\Gamma_\mu \neq G_\mu$ means that the probability law μ of the elementary step is supported by either one of the two half lines bounded by 0, and such a one sided process is necessarily transient. There are obvious geometric generalizations of this case.

3. Description of the random walk of law μ

We shall keep the previous notation. For every g in G, the *random walk of law* μ *starting at* g is the process $(gS_n)_{n \geqq 0}$. More generally, let α be a probability measure on G. The *random walk of law* μ *and initial distribution* α is the process $(X_n)_{n \geqq 0}$, of the form $X_n = X_0 S_n$, where X_0 is any G valued random variable with probability law α independent of the process $(Z_n)_{n \geqq 0}$.

The canonical sample space for these processes is $(W, \mathscr{B}(W))$, where W is the topological product space $\prod_{n=0}^{\infty} G_n$ with $G_n = G$ for every $n \geqq 0$, and X_n is the projection $W \mapsto G$ on the nth factor. We denote by $\mathbf{P}^g$ the probability law of the random walk of law μ starting at g, that is, the image of $\mathbf{P}$ by the continuous mapping $(g_1, g_2, \cdots, g_n, \cdots) \mapsto (g, gg_1, gg_1g_2, \cdots, gg_1 \cdots g_n, \cdots)$ from Ω to W. Similarly, one denotes by $\mathbf{P}^\alpha$ the probability law of the random walk of law μ and initial distribution α. It is easily shown that for any A in $\mathscr{B}(W)$, the function $g \mapsto \mathbf{P}^g[A]$ is Borel measurable on G and that

$$\mathbf{P}^\alpha[A] = \int_G \mathbf{P}^g[A]\alpha(dg). \tag{3.1}$$

We shall use this formula as a definition of the measure $\mathbf{P}^\alpha$ on W whenever α is a Radon measure on G. We denote by $\mathbf{E}^g$ and $\mathbf{E}^\alpha$ the expectation functionals corresponding respectively to $\mathbf{P}^g$ and $\mathbf{P}^\alpha$. If the function f on W is nonnegative and Borel measurable, the function $g \mapsto \mathbf{E}^g[f]$ is Borel measurable on G and one gets the formula

$$\mathbf{E}^\alpha[f] = \int_G \mathbf{E}^g[f]\alpha(dg). \tag{3.2}$$

This formula reduces to (3.1) when f is the indicator function I_A of the Borel set A.

The *transition kernel* of the random walk is the kernel Q on G defined by

$$Qf(g) = \int_G f(gh)\mu(dh), \qquad f \text{ in } b^+(G), g \text{ in } G, \tag{3.3}$$

and the *shift* is the kernel θ on W defined by

$$\theta F(g_0, g_1, \cdots, g_n, \cdots) = F(g_1, g_2, \cdots, g_{n+1}, \cdots), \qquad F \text{ in } b^+(W). \tag{3.4}$$

The *Markov property* of the random walk is expressed by the relation $\mathbf{E}^\alpha[F \cdot f(X_{n+1})] = \mathbf{E}^\alpha[F \cdot Qf(X_n)]$, where F depends only on $X_0, \cdots, X_n$. By induction on n, one gets

$$\mathbf{E}^\alpha[f_0(X_0) \cdots f_n(X_n)] = \langle \alpha, f_0 Q f_1 Q \cdots f_{n-1} Q f_n \rangle \tag{3.5}$$

for $f_0, \cdots, f_n$ in $b^+(G)$. Specializing $f_0, \cdots, f_n$ to indicator functions in (3.5) gives

$$\mathbf{P}^\alpha[X_0 \in A_0, \cdots, X_n \in A_n] = \langle \alpha, I_{A_0} Q I_{A_1} Q \cdots I_{A_{n-1}} Q I_{A_n} \rangle \tag{3.6}$$

for any finite sequence of Borel subsets $A_0, \cdots, A_n$ of G.

Part B. Construction of the intrinsic boundary

Here are our main assumptions: *G is a separable locally compact group and μ a probability measure on G*; we assume that $\pi = \Sigma_{n=0}^\infty \mu^n$ is a *Radon measure* (transient case). This part is devoted to an elementary study of the potential theory associated with π and to the construction of the intrinsic boundary corresponding to μ. Finally, we shall prove a certain number of theorems asserting the existence of integral representations.

4. Excessive measures and excessive functions

Let $\bar{\mu}$ and $\bar{\pi}$ be the opposite measures of μ and π. Define the kernels $\bar{Q}$ and $\bar{U}$ on G by the formulas

$$\bar{Q}f(g) = \int_G f(gh)\bar{\mu}(dh), \qquad \bar{U}f(g) = \int_G f(gh)\bar{\pi}(dh) \tag{4.1}$$

for f in $b^+(G)$ and g in G. For a measure λ, one gets dually

$$\lambda\bar{Q} = \lambda * \bar{\mu}, \qquad \lambda\bar{U} = \lambda * \bar{\pi}. \tag{4.2}$$

The *potential kernel* $\bar{U}$ is defined in terms of $\bar{Q}$ by $\bar{U} = \Sigma_{n=0}^{\infty} \bar{Q}^n$, or more precisely

$$(4.3) \qquad \bar{U}f = \sum_{n=0}^{\infty} \bar{Q}^n f, \qquad \lambda\bar{U} = \sum_{n=0}^{\infty} \lambda\bar{Q}^n$$

for f in $b^+(G)$ and any measure λ on G.

The transition kernels Q and $\bar{Q}$ of the random walks of law μ and $\bar{\mu}$ are *in duality* with respect to any right invariant Haar measure m, that is, they satisfy the (easily checked) identity

$$(4.4) \qquad \langle m, f\cdot Qf'\rangle = \langle m, f'\cdot \bar{Q}f\rangle$$

for any f, f' in $b^+(G)$.

DEFINITION 4.1. *A function f on G is called excessive (respectively, invariant), if $f \in b^+(G)$ and $Qf \leqq f$ (respectively, $Qf = f$). Any bounded Borel function such that $Qf = f$ will also be called invariant. Let λ be a measure on G; one calls λ excessive (invariant), if it is a Radon measure and $\lambda\bar{Q} \leqq \lambda (\lambda\bar{Q} = \lambda)$. One calls λ a potential, if there exists a measure α such that $\lambda = \alpha\bar{U}$.*

According to the classical terminology, our excessive measures (respectively, potentials) should be called coexcessive (respectively, copotentials), since the kernels Q and $\bar{Q}$ are in duality (see [36]). Since the construction of the boundary involves only the excessive measures in our sense, there is little inconvenience if we delete the prefix co. We shall come back to the study of excessive functions in Part D. For the moment, we simply note that if an excessive function f is m locally integrable, the measure $f\cdot m$ is excessive (a direct consequence of (4.4)). We also remark that m is an invariant measure.

We shall denote by $\mathscr{E}$ the class of excessive measures and by $\mathscr{I}$ the class of invariant measures. Both are convex cones, that is, closed under addition and multiplication by a nonnegative real number. Any invariant measure is excessive; if a Radon measure is a potential, it is excessive according to the following consequence of (4.3),

$$(4.5) \qquad \alpha\bar{U} = \alpha + (\alpha\bar{U})\bar{Q}.$$

In the following, we shall denote by $\mathscr{M}$ the space of Radon measures on G endowed with the vague topology, that is, the coarsest topology making continuous the real valued functionals $\lambda \mapsto \langle\lambda, f\rangle$ for f in $C_c^+(G)$. We now gather the main algebraic and topological properties of the cone $\mathscr{E}$ of excessive measures. The proofs follow well-known patterns (see, for instance, [46]) and have been included here for the sake of completeness only.

THEOREM 4.1. (i) *Let λ be an excessive measure. There exist two Radon measures α and β on G such that $\lambda = \alpha\bar{U} + \beta$ and $\beta\bar{Q} = \beta$ (Riesz decomposition). The measures α and β are uniquely determined by λ; indeed, $\alpha = \lambda - \lambda\bar{Q}$ and the decreasing sequence $(\lambda\bar{Q}^n)_{n\geq 0}$ tends to β. Moreover, β is the largest among the invariant measures majorized by λ.*

(ii) *The convex cone $\mathscr{E}$ is a lattice for its intrinsic order.*

PROOF. (i) Since λ is a Radon measure and $\lambda\bar{Q} \leqq \lambda$, there exists a Radon measure α such that $\lambda = \alpha + \lambda\bar{Q}$. By induction, one gets $\lambda\bar{Q}^{n+1} \leqq \lambda\bar{Q}^n$ for $n \geqq 0$, and thus there exists the limit $\beta = \lim_{n\to\infty} \lambda\bar{Q}^n$. From the definition of α, one gets

$$\lambda = \alpha + \alpha\bar{Q} + \cdots + \alpha\bar{Q}^{n-1} + \lambda\bar{Q}^n \tag{4.6}$$

by induction on $n \geqq 1$. By going to the limit in (4.6), one gets $\lambda = \alpha\bar{U} + \beta$ as required.

Let us show that β is invariant. Substituting $\lambda = \alpha\bar{U} + \beta$ into the relation $\lambda = \alpha + \lambda\bar{Q}$ gives

$$\alpha\bar{U} + \beta = \alpha + (\alpha\bar{U} + \beta)\bar{Q} = \alpha\bar{U} + \beta\bar{Q} \tag{4.7}$$

by (4.5). Cancelling out $\alpha\bar{U}$ gives $\beta = \beta\bar{Q}$. It is clear that the invariant measure β is majorized by λ. Furthermore, if β' is invariant and $\beta' \leqq \lambda$, one gets $\beta' = \beta'\bar{Q}^n \leqq \lambda\bar{Q}^n$ for any integer $n \geqq 0$, hence, $\beta' \leqq \beta$ by going to the limit.

Let α' and β' be Radon measures such that $\lambda = \alpha'\bar{U} + \beta'$ and $\beta'\bar{Q} = \beta'$. From (4.5), one gets $\lambda = \alpha' + (\alpha'\bar{U})\bar{Q} + \beta'\bar{Q} = \alpha' + \lambda\bar{Q}$: hence $\alpha' = \alpha$, and therefore $\beta' = \beta$.

(ii) We denote by $\lambda_1 \succ \lambda_2$ the intrinsic order in the convex cone $\mathscr{E}$. By definition, this relation means the existence of a measure λ_3 in $\mathscr{E}$ such that $\lambda_1 = \lambda_2 + \lambda_3$. According to (i), write $\lambda_i = \alpha_i\bar{U} + \beta_i$ with β_i invariant for $i = 1, 2$. It is immediate that $\lambda_1 \succ \lambda_2$ is equivalent to $\alpha_1 \geqq \alpha_2$ and $\beta_1 \geqq \beta_2$ (note that $\beta_1 \geqq \beta_2$ implies that $\beta_1 - \beta_2$ is an invariant measure).

With the previous notations, denote by α (respectively, β) the largest among the Radon measures that are majorized in the usual sense by α_1 and α_2 (respectively, β_1 and β_2). The existence of α and β is well known ([7], p. 53). For $i = 1, 2$ one gets $\beta \leqq \beta_i$; hence $\beta\bar{Q} \leqq \beta_i\bar{Q} = \beta_i$. By definition of β, we have $\beta\bar{Q} \leqq \beta$. By (i), there is a largest invariant measure γ majorized in the usual sense by the excessive measure β (that is, by β_1 and β_2) namely, $\gamma = \lim_{n\to\infty} \beta\bar{Q}^n$. It is then immediate that $\lambda_1 \wedge \lambda_2 = \alpha\bar{U} + \gamma$ is the G.L.B. of λ_1 and λ_2 in $(\mathscr{E}, \succ)$.

Finally, from $\lambda_1 \wedge \lambda_2 \prec \lambda_1 \prec \lambda_1 + \lambda_2$, one deduces the existence of an excessive measure $\lambda_1 \vee \lambda_2$ such that $\lambda_1 + \lambda_2 = (\lambda_1 \wedge \lambda_2) + (\lambda_1 \vee \lambda_2)$. The proof that $\lambda_1 \vee \lambda_2$ is the L.U.B. of λ_1 and λ_2 in $(\mathscr{E}, \succ)$ is then straightforward. *Q.E.D.*

THEOREM 4.2. *The convex cone $\mathscr{E}$ of excessive measures is closed in the space $\mathscr{M}$ of all Radon measures on G. Moreover, any excessive measure is the limit of an increasing sequence of potentials.*

PROOF. Let f in $C_c^+(G)$. It is well known that $\bar{Q}f$ is a continuous function on G; hence, $\langle \alpha\bar{Q}, f\rangle = \langle \alpha, \bar{Q}f\rangle$ is the L.U.B. of the numbers $\langle \alpha, g\rangle$ for g in $C_c^+(G)$ and $g \leqq \bar{Q}f$, whatever be the Radon measure α. Hence, $\mathscr{E}$ is singled out from $\mathscr{M}$ by the set of inequalities $\langle \alpha, f\rangle \geqq \langle \alpha, g\rangle$ for f and g in $C_c^+(G)$ such that $g \leqq \bar{Q}f$. Each of these inequalities defines a vaguely closed set in $\mathscr{M}$; thus, $\mathscr{E}$ is vaguely closed in $\mathscr{M}$.

Let λ be any excessive measure with Riesz decomposition $\lambda = \alpha\bar{U} + \beta$. For any compact subset K of G, the reduite β_K of β on K is a potential such that $I_K \cdot \beta \leqq \beta_K \leqq \beta$ and $\beta_K \leqq \beta_L$ for K contained in L (the properties of the reduites needed here are derived again in Part C).

Since G is a separable locally compact space, we can find an increasing sequence $(K_n)_{n \geqq 1}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$ and the sequence of potentials $\alpha\bar{U} + \beta_{K_n}$ is increasing and clearly tends to λ. *Q.E.D.*

5. Intrinsic boundary of G

As before, let $\mathscr{E}$ stand for the space of excessive measures on G with the vague topology. The *ray* generated by a measure $\lambda \neq 0$ in $\mathscr{E}$ is as usual the set of all measures $t \cdot \lambda$, where t runs over the positive real numbers. The rays form a partition of the open subspace $\mathscr{E} - \{0\}$ of $\mathscr{E}$. We shall denote by $\mathscr{R}$ the set of all rays endowed with the topology obtained by considering it as a quotient space of $\mathscr{E} - \{0\}$. The ray D is called *extreme* if and only if the relations $\lambda \prec \lambda'$ and $\lambda' \in D$ imply $\lambda \in D$ for any measure $\lambda \neq 0$ in $\mathscr{E}'$. Let $\mathscr{S}$ be the set of all extreme rays and B the subset of $\mathscr{S}$ consisting of the extreme rays, all of whose elements are invariant measures. Finally, for every g in G, let $i(g)$ be the ray generated by the potential $\varepsilon_g \bar{U} = g\bar{\pi}$.

LEMMA 5.1. *The mapping i is injective and $\mathscr{S}$ is the disjoint union of $i(G)$ and B.*

PROOF. Let g and g' in G be such that $i(g) = i(g')$. There exists therefore a real number $t > 0$ such that $\varepsilon_{g'}\bar{U} = t \cdot \varepsilon_g \bar{U}$; hence, $\varepsilon_{g'} = t \cdot \varepsilon_g$ by Theorem 4.1, (i). This last relation is possible only if $t = 1$ and $g = g'$; hence, i is injective.

Let λ be a nonzero excessive measure with Riesz decomposition $\lambda = \alpha\bar{U} + \beta$. Since $\alpha\bar{U} \prec \lambda$ and $\beta \prec \lambda$, the ray generated by λ can be extreme only if $\alpha\bar{U}$ or β vanishes; that is, if λ is a potential or an invariant measure. Finally, the potential $\alpha\bar{U}$ generates an extreme ray if and only if every measure α' with $\alpha' \leqq \alpha$ is proportional to α; it is well known that this means that α is a point measure. *Q.E.D.*

DEFINITION 5.1. *The intrinsic boundary of G (with respect to μ) is the set B of extreme rays in $\mathscr{E}$ consisting of invariant measures. The intrinsic completion of G (with respect to μ) is the disjoint union $\hat{G}$ of G and B.*

We extend the map $i \colon G \mapsto \mathscr{R}$ to a map $j \colon \hat{G} \mapsto \mathscr{R}$ by $j(x) = x$ for x in B. By definition a set U in $\hat{G}$ is called open if there exist open sets V in G and V' in $\mathscr{R}$ such that $U = V \cup j^{-1}(V')$. The axioms for a topology are easily checked (use the continuity of $i \colon G \mapsto \mathscr{R}$); hence, $\hat{G}$ becomes a topological space. Moreover, by Lemma 5.1, j is a continuous bijection from $\hat{G}$ onto $\mathscr{S}$ (but not necessarily a homeomorphism): furthermore, G with its given topology and B with the topology induced from $\mathscr{R}$ are subspaces of $\hat{G}$ with G open and B closed.

Now we let G operate on $\hat{G}$. For g in G one gets $g(\lambda\bar{Q}) = (g\lambda)\bar{Q}$ (λ in $\mathscr{M}$); hence, the map $\lambda \mapsto g\lambda$ leaves both $\mathscr{E}$ and $\mathscr{I}$ invariant. The group G operates therefore by automorphisms of the convex cone $\mathscr{E}$; hence, it operates on the set $\mathscr{R}$ of rays in $\mathscr{E}$. It is clear that $\mathscr{S}$ and B are invariant under G and that $g \cdot i(g') =$

$i(gg')$ for g, g' in G. The action of G on $\hat{G}$ is given by the left translations on G and the previous action on B, in such a way that the bijection $j: \hat{G} \mapsto \mathscr{S}$ is compatible with the operations of G.

LEMMA 5.2. *The intrinsic completion $\hat{G}$ is a Hausdorff space having a countable base of open sets, and G acts continuously on $\hat{G}$.*

PROOF. By construction, one has a continuous injective map $j: G \mapsto \mathscr{R}$; hence, to show that $\hat{G}$ is Hausdorff, it suffices to show that $\mathscr{R}$ is Hausdorff. The equivalence relation defined in $\mathscr{E} - \{0\}$ by the partition in rays is clearly open, and its graph is closed. Hence, the quotient space $\mathscr{R}$ is Hausdorff. The natural projection $q: \mathscr{E} - \{0\} \mapsto \mathscr{R}$ is open, and $\mathscr{E}$ being a subspace of the separable metrizable space $\mathscr{M}$, has a countable base of open sets: hence, the topology of $\mathscr{R}$ has a countable base. The definition of the topology of $\hat{G}$ implies then immediately that $\hat{G}$ has a countable base of open sets.

Let us prove now that G acts continuously upon the convex cone $\mathscr{E}$ (upon $\mathscr{M}$, indeed!). We have to show that for any f in $C_c^+(G)$, the numerical function F defined on $G \times \mathscr{E}$ by

$$F(g, \lambda) = \langle g \cdot \lambda, f \rangle = \int_G f(gx)\lambda(dx) \tag{5.1}$$

is continuous. Let $\varepsilon > 0$ and (g_0, λ_0) in $G \times \mathscr{E}$ be fixed. Let U be a compact neighborhood of g_0 and S be the (compact) support of f; the set $L = U^{-1}S$ is then compact in G and one can choose a function f' in $C_c^+(G)$ taking the constant value 1 on L. Also, let c be a real number such that $c > \langle \lambda_0, f' \rangle$. Since f is left uniformly continuous, there exists a compact neighborhood V of g_0 contained in U such that

$$|f(gx) - f(g_0x)| \leqq \frac{\varepsilon}{2c} \tag{5.2}$$

for g in V and x in G. The left side of this inequality vanishes for x off L (for g fixed in V). Hence, we can strengthen (5.2) as follows:

$$|f(gx) - f(g_0x)| \leqq \frac{\varepsilon \cdot f'(x)}{2c}, \qquad g \text{ in } V,\ x \text{ in } G. \tag{5.3}$$

The function f'' defined by $f''(x) = f(g_0x)$, x in G, is in $C_c^+(G)$. Hence, the set of measures λ in $\mathscr{E}$ satisfying the inequalities

$$\langle \lambda, f' \rangle < c, \qquad |\langle \lambda, f'' \rangle - \langle \lambda_0, f'' \rangle| < \frac{\varepsilon}{2} \tag{5.4}$$

is an open neighborhood W of λ_0 in $\mathscr{E}$. For g in V and λ in W, one gets

$$\begin{aligned} |F(g, \lambda) - F(g_0, \lambda)| &\leqq |F(g, \lambda) - F(g_0, \lambda)| + |F(g_0, \lambda) - F(g_0, \lambda_0)| \\ &\leqq \int_G |f(gx) - f(g_0x)|\lambda(dx) + |\langle \lambda, f'' \rangle - \langle \lambda_0, f'' \rangle| \\ &\leqq \frac{\varepsilon\langle \lambda, f' \rangle}{2c} + \frac{\varepsilon}{2} < \varepsilon \end{aligned} \tag{5.5}$$

by using (5.1), (5.3), and (5.4). The continuity of F is therefore established.

If q is the canonical mapping from $\mathscr{E} - \{0\}$ onto $\mathscr{R}$, one has a commutative diagram

$$\begin{array}{ccc} G \times \mathscr{E} & \xrightarrow{\ m\ } & \mathscr{E} \\ {\scriptstyle Id_G \times q}\ \big\downarrow & & \big\downarrow\ {\scriptstyle q} \\ G \times \mathscr{R} & \xrightarrow{\ m'\ } & \mathscr{R} \end{array} \tag{5.6}$$

with $m(g, \lambda) = g\cdot\lambda$ and $m'(g, x) = g\cdot x$ for g in G, λ in $\mathscr{E}$ and x in $\mathscr{R}$. We have shown that m is continuous. Clearly, q is surjective and open; hence, $Id_G \times q$ is surjective and open and it follows from (5.6) that m' is continuous. Thus, G acts continuously upon $\mathscr{R}$ and, *a fortiori*, upon the stable subspace $\mathscr{S}$ of extreme rays in $\mathscr{E}$. This fact implies readily that G operates continuously upon $\hat{G}$. *Q.E.D.*

6. Integral representation of excessive measures

We shall need an ancillary notion, that of a reference function.

DEFINITION 6.1. *A reference function on G is a continuous function r on G such that $r(g) > 0$ for each g in G and the potential $\bar{U}r$ is a finite continuous function on G.*

LEMMA 6.1. *For any Radon measure λ on G there exists a bounded reference function r such that $\langle\lambda, r\rangle$ is finite.*

PROOF. Since G is a separable locally compact space, there exists an increasing sequence $(f_n)_{n\geqq 1}$ in $C_c^+(G)$ with limit 1 at any point of G.

The potential $\bar{U}f$ of any function f in $C_c^+(G)$ is a continuous function on G, and hence bounded on the support of f; by the maximum principle ([40], p. 228) the function $\bar{U}f$ is therefore bounded on G.

Then let c_n be the maximum among the numbers 1, $\langle\lambda, f_n\rangle$ and $\sup_{g\in G} \bar{U}f_n(g)$. It is an easy matter to check that $r = \sum_{n=1}^{\infty} c_n^{-1}\, 2^{-n} f_n$ is the required reference function. *Q.E.D.*

We shall now apply Choquet's theory of integral representations to the convex cone of excessive measures. Let r be any continuous function on G with positive values and let $\mathscr{E}_r$ be the set of λ in $\mathscr{E}$ such that $\langle\lambda, r\rangle \leqq 1$. It is immediately verified that $\mathscr{E}_r$ is a *cap* in $\mathscr{E}$, that is, a convex subset containing 0 with convex complement in $\mathscr{E}$. Furthermore, let Σ_r denote the set of nonzero extreme points in $\mathscr{E}_r$, that is, the set of excessive measures λ such that $\langle\lambda, r\rangle = 1$ which generate extreme rays. We claim that the cap $\mathscr{E}_r$ *is vaguely compact*: indeed, the inequality $\langle\lambda, r\rangle \leqq 1$ is equivalent to the set of inequalities $\langle\lambda, f\rangle \leqq 1$ for f in $C_c^+(G)$ majorized by r; hence $\mathscr{E}_r$ is vaguely closed. Furthermore, for f in $C_c^+(G)$, the positive continuous function r has a positive minimum on the compact support of f; hence, there exists a constant $c > 0$ such that $f \leqq c\cdot r$. This last inequality implies $\langle\lambda, f\rangle \leqq c$ for any λ in $\mathscr{E}_r$ and the compactness of $\mathscr{E}_r$ follows from Tychonov's theorem. Finally, from Lemma 6.1, it follows that *$\mathscr{E}$ is the union of its compact caps $\mathscr{E}_r$, where r runs over the reference functions.*

Let r be a reference function and λ be an excessive measure such that $\langle\lambda, r\rangle$ be finite. By Choquet's theorem ([7], [40]) *there exists a bounded measure* δ_r *on* Σ_r *such that* $\lambda = \int_{\Sigma_r} \sigma\cdot\delta_r(d\sigma)$ *and such a measure is unique* since the convex cone $\mathscr{E}$ is a lattice for its intrinsic order. Let g be in G. By assumption, $\bar{U}r(g) = \langle g\bar{\pi}, r\rangle$ is finite; hence, there exists in Σ_r a unique measure generating the extreme ray $i(g)$, namely, $k_r(g) = \bar{U}r(g)^{-1}\cdot g\bar{\pi}$. For f in $C_c^+(G)$, one gets $\langle k_r(g), f\rangle = \bar{U}f(g)/\bar{U}r(g)$ for any g in G; since the functions $\bar{U}f$ and $\bar{U}r$ are continuous on G, it follows that k_r is a vaguely continuous map from G into Σ_r. Furthermore, k_r is injective since i is injective (Lemma 5.1) and Lusin's theorem ([5], p. 135) applies, since G is a separable locally compact space: k_r is a Borel isomorphism of G onto a Borel subset $k_r(G)$ of Σ_r. Therefore, any measure on $k_r(G)$ lifts uniquely to a measure on G and, for instance, the restriction of δ_r to $k_r(G)$ lifts to a bounded measure γ on G. Let $\alpha = (\bar{U}r)^{-1}\cdot\gamma$; by an easy calculation, one gets

$$\lambda = \alpha\bar{U} + \int_{B_r} \sigma\cdot\delta_r(d\sigma) \tag{6.1}$$

where $B_r = \Sigma_r - k_r(G)$. By Lemma 5.1, B_r consists of invariant measures, and therefore the integral in (6.1) represents an invariant measure. Thus, in (6.1) we have the Riesz decomposition of λ. This decomposition corresponds to the decomposition of Σ_r into $k_r(G)$ and B_r and this supports the heuristic view that an invariant measure is the potential of a charge located at the boundary (here B_r is the boundary).

We summarize our discussion in the following theorem.

THEOREM 6.1. *Let* r *be a reference function and* λ *be an invariant measure such that* $\langle\lambda, r\rangle$ *is finite. Let* B_r *be the set of the invariant measures* σ, *such that* $\langle\sigma, r\rangle = 1$, *which generates an extreme ray in* $\mathscr{E}$. *Then there exists a unique bounded measure* δ_r *on* B_r *such that* $\lambda = \int_{B_r} \sigma\cdot\delta_r(d\sigma)$.

7. Intrinsic integral representations

The results derived in the previous section depend on the choice of a reference function r. We now show how to switch to the intrinsic boundary and get intrinsic formulations.

LEMMA 7.1. *Let* λ *be an invariant measure on* G *and* r *a reference function such that* $\langle\lambda, r\rangle$ *is finite. Define* B_r *and* δ_r *as in Theorem* 6.1. *There exists on* $G \times B_r$ *a unique Radon measure* $\Theta_{\lambda,r}$ *taking the following values on the rectangle sets:*

$$\Theta_{\lambda,r}(A \times A') = \int_{A'} \sigma(A)\delta_r(d\sigma), \qquad A \text{ in } \mathscr{B}(G), A' \text{ in } \mathscr{B}(B_r). \tag{7.1}$$

Moreover, $\Theta_{\lambda,r}$ *projects onto the measure* λ *on* G.

PROOF. Since G is a separable locally compact space, each vaguely compact set of Radon measures on G is metrizable. In particular, $\mathscr{E}_r$ is metrizable. Since k_r is a continuous map from G into $\mathscr{E}_r$ and G is a countable union of compact subsets, $k_r(G)$ is the union of a sequence of compact subsets of $\mathscr{E}_r$. It is known ([40], p. 282) that the set Σ_r of extreme points of $\mathscr{E}_r$ is a countable intersection of open subsets of $\mathscr{E}_r$; hence, $B_r = \Sigma_r - k_r(G)$ has the same property. It follows ([5], p. 123) that G and B_r are Polish spaces, that is, homeomorphic to complete separable metric spaces. Hence, the topologies of G and B_r are countably generated and the σ-algebra $\mathscr{B}(G \times B_r)$ is generated by the rectangle sets $A \times A'$, where A is in $\mathscr{B}(G)$ and A' in $\mathscr{B}(B_r)$.

Let f be in $C_c^+(G)$; by definition of the vague topology, the function $\sigma \mapsto \langle \sigma, f \rangle$ on B_r is continuous. By a familiar argument of monotone classes, it follows that the mapping $\sigma \mapsto \langle \sigma, f \rangle$ is Borel measurable on B_r for any f in $b^+(G)$. One defines therefore a Markovian kernel K_r from B_r into G by

$$K_r f(\sigma) = \langle \sigma, rf \rangle, \qquad \sigma \text{ in } B_r, f \text{ in } b^+(G). \tag{7.2}$$

We can now use a construction familiar from the theory of Markov processes. From the bounded measure δ_r on B_r and the Markovian kernel K_r from B_r into G, one derives a bounded measure Θ on $G \times B_r$ characterized by the following relation:

$$\Theta(A \times A') = \langle \delta_r, I_{A'} \cdot K_r I_A \rangle, \qquad A \text{ in } \mathscr{B}(G), A' \text{ in } \mathscr{B}(B_r). \tag{7.3}$$

Since $G \times B_r$ is a Polish space, the bounded measure Θ on it is a Radon measure by Prokhorov's theorem ([8], p. 49). The measure $\Theta_{\lambda,r}$ on $G \times B_r$, product of Θ by the locally bounded continuous function $(g, \sigma) \mapsto r(g)^{-1}$ is therefore a Radon measure.

Equation (7.1) is readily checked. Moreover, one deduces the relation

$$\Theta_{\lambda,r}(A \times B_r) = \int_{B_r} \sigma(A)\delta_r(d\sigma) = \lambda(A) \tag{7.4}$$

as a particular case of (7.1); hence, $\Theta_{\lambda,r}$ projects onto the measure λ on G. Finally, since the σ-algebra $\mathscr{B}(G \times B_r)$ is generated by the rectangle sets, there is at most one measure taking given values on the rectangle sets, hence the uniqueness of $\Theta_{\lambda,r}$. *Q.E.D.*

LEMMA 7.2. *Let λ be an invariant measure on G. For each reference function r, let q_r be the continuous mapping of $G \times B_r$ into $G \times B$ which sends (g, σ) into (g, x), where x is the ray generated by σ. There exists a Radon measure Θ_λ on $G \times B$ with the following property: for each reference function r such that $\langle \lambda, r \rangle$ is finite, the image by q_r of the measure $\Theta_{\lambda,r}$ on $G \times B_r$ defined in Lemma 7.1 is equal to Θ_λ. Moreover, Θ_λ projects onto the measure λ on G.*

PROOF. Let r be a reference function such that $\langle \lambda, r \rangle$ is finite. We denote by Λ_r the image of $\Theta_{\lambda,r}$ by q_r. For any compact subset K of G one gets

$$\Lambda_r(K \times B) = \Theta_{\lambda,r}(K \times B_r) = \lambda(K) < \infty. \tag{7.5}$$

Hence, Λ_r is locally finite and projects onto λ. The inner regularity of $\Theta_{\lambda,r}$ and the continuity of q_r imply inner regularity for Λ_r. Thus, Λ_r is a Radon measure on $G \times B$. If s is any reference function such that $\langle \lambda, s \rangle$ is finite, then $r + s$ is a reference function and $\langle \lambda, r + s \rangle$ is finite. Therefore the proof of the lemma will be achieved if one establishes the equality $\Lambda_r = \Lambda_s$ in the case $r \leqq s$.

From now on, fix two reference functions r and s such that $r \leqq s$ and $\langle \lambda, s \rangle$ is finite. Again, using Lusin's theorem, one sees that the set B'_r of extreme rays generated by the measures belonging to B_r is a Borel subset of B and that B'_r is Borel isomorphic to B_r under the natural map. The construction of Λ_r can be rephrased as follows: for each x in B'_r let $k_r(x)$ be the unique measure σ in the ray x such that $\langle \sigma, r \rangle = 1$; there exists a unique measure δ'_r on B'_r such that $\lambda = \int_{B'_r} k_r(x)\delta'_r(dx)$ and then Λ_r is given by

$$\Lambda_r = \int_{B'_r} (k_r(x) \otimes \varepsilon_x)\delta'_r(dx). \tag{7.6}$$

Since $r \leqq s$, one gets $B'_r \supset B'_s$ and there exists a function f in $b^+(B'_s)$ such that

$$k_s(x) = f(x) \cdot k_r(x), \qquad x \text{ in } B'_s, \tag{7.7}$$

namely, $f(x) = \langle k_s(x), r \rangle$ for x in B'_s. We have then

$$\lambda = \int_{B'_s} k_s(x)\delta'_s(dx) = \int_{B'_s} k_r(x) \cdot f(x)\delta'_s(dx), \tag{7.8}$$

and by the uniqueness of δ'_r one concludes that δ'_r is carried by B'_s and that $\delta'_r(dx) = f(x) \cdot \delta'_s(dx)$ on B'_s. The proof of $\Lambda_r = \Lambda_s$ follows then by a trivial calculation from the definition (7.6) of Λ_r and the corresponding relation for Λ_s. *Q.E.D.*

To summarize, we have attached to any invariant measure λ on G a Radon measure Θ_λ on $G \times B$ with projection λ onto the first factor space. The projection of Θ_λ onto the second factor space is not σ-finite in general, and before disintegrating Θ_λ with respect to the second projection, we have to replace it by an equivalent bounded measure. This is achieved with the help of a reference function r such that $\langle \lambda, r \rangle$ is finite, the result being given by (7.6), namely, $\Theta_\lambda = \int_{B'_r} (k_r(x) \otimes \varepsilon_x)\delta'_r(dx)$. On the other hand, the first projection of Θ_λ being the Radon measure λ, we could appeal to general results ([8], p. 39) to get a disintegration of Θ_λ with respect to the first projection. Such a disintegration is unique up to null sets only, but fortunately we can achieve a very smooth result in an important particular case. The probabilistic significance of the measures Θ_λ and γ will appear in the next part (Theorem 12.2 and 12.3).

LEMMA 7.3. *Let m be a right invariant Haar measure on G. There exists a unique Radon probability measure γ on B such that*

$$\Theta_m = \int_G (\varepsilon_g \otimes g \cdot \gamma)m(dg). \tag{7.9}$$

PROOF. The group G acts upon $G \times B$ by $g \cdot (g', x) = (gg', g \cdot x)$. We shall first establish the relation

$$g \cdot \Theta_m = \Delta(g) \cdot \Theta_m, \qquad g \text{ in } G, \tag{7.10}$$

where Δ is the module function of G. Indeed, let r be a reference function such that $\langle m, r\rangle = 1$ and let B'_r and $k_r(x)$ be as in the proof of Lemma 7.2. There exists a unique probability measure μ_r on B'_r such that

$$m = \int_{B'_r} k_r(x)\mu_r(dx). \tag{7.11}$$

Then one gets

$$\Theta_m = \int_{B'_r} (k_r(x) \otimes \varepsilon_x)\mu_r(dx). \tag{7.12}$$

Let g be in G. It is immediate that the relation $s(x) = \Delta(g)\cdot r(g^{-1}x)$ (for x in G) defines a reference function s such that $\langle m, s\rangle = 1$. One gets easily

$$gk_r(x) = \Delta(g)\cdot k_s(gx), \qquad x \text{ in } B. \tag{7.13}$$

Transforming (7.11) by g one finds

$$\Delta(g)\cdot m = \int_{B'_r} \Delta(g)\cdot k_s(gx)\mu_r(dx), \tag{7.14}$$

since $gm = \Delta(g)\cdot m$. From the uniqueness of the integral representation of an invariant measure and from $gB'_r = B'_s$, one concludes that g transforms the probability measure μ_r on B'_r into the probability measure μ_s on B'_s. We act now upon (7.12) with g and get

$$\begin{aligned} g\Theta_m &= \int_{B'_r} (gk_r(x) \otimes g\varepsilon_x)\mu_r(dx) = \int_{B'_r} \Delta(g)\cdot(k_s(gx) \otimes \varepsilon_{gx})\mu_r(dx) \\ &= \Delta(g)\int_{B'_s} (k_s(y) \otimes \varepsilon_y)\mu_s(dy) = \Delta(g)\cdot\Theta_m. \end{aligned} \tag{7.15}$$

Hence, the sought after relation (7.10) follows.

The function Δ_1 on $G \times B$ defined by $\Delta_1(h, x) = \Delta(h)$ is continuous and locally bounded. Therefore, $\Delta_1\cdot\Theta_m$ is a Radon measure on $G \times B$. We denote by α the image of $\Delta_1\cdot\Theta_m$ by the homeomorphism $(h, x) \mapsto (h, h^{-1}x)$ of $G \times B$ with itself. For any function F in $b^+(G \times B)$, one gets

$$\int_{G\times B} F(h, x)\alpha(dh, dx) = \int_{G\times B} \Delta(h)\cdot F(h, h^{-1}x)\Theta_m(dh, dx). \tag{7.16}$$

In the same manner, (7.10) is made explicit by the following transformation formula

$$\int_{G\times B} F(gh, gx)\Theta_m(dh, dx) = \Delta(g)\int_{G\times B} F(h, x)\Theta_m(dh, dx) \tag{7.17}$$

for any g in G. By an easy calculation, one deduces from (7.16) and (7.17) the following transformation formula for α

$$\int_{G\times B} F(gh, x)\alpha(dh, dx) = \int_{G\times B} F(h, x)\alpha(dh, dx), \tag{7.18}$$

where F is in $b^+(G \times B)$ and g in G.

From (7.16), one deduces that the projection of α onto the first factor of $G \times B$ is equal to $\Delta\cdot m$, and from (7.18), one recovers the well known fact that $\Delta\cdot m$ is a *left invariant* Haar measure. By specializing (7.18), one gets

$$\alpha(gA \times A') = \alpha(A \times A'), \qquad A \text{ in } \mathscr{B}(G), A' \text{ in } \mathscr{B}(B), g \text{ in } G. \tag{7.19}$$

For fixed A' in $\mathscr{B}(B)$, the mapping $A \mapsto \alpha(A \times A')$ of $\mathscr{B}(G)$ into $[0, +\infty]$ is therefore a *left invariant* Radon measure on G. From the uniqueness of Haar measure, one gets the existence of a functional γ on $\mathscr{B}(B)$ such that

$$\alpha(A \times A') = (\Delta\cdot m)(A)\cdot\gamma(A'), \qquad A \text{ in } \mathscr{B}(G), A' \text{ in } \mathscr{B}(B). \tag{7.20}$$

It then follows easily that γ is a Radon probability measure on B and that (7.20) is equivalent to the relation $\alpha = (\Delta\cdot m) \otimes \gamma$. Using (7.16) and Fubini's theorem, one gets finally the following integration formula

$$\langle\Theta_m, F\rangle = \int_G m(dg) \int_B F(g, gx)\gamma(dx), \qquad F \text{ in } b^+(G \times B), \tag{7.21}$$

which is nothing other than the sought after formula (7.9).

It remains to prove that (7.9) characterizes γ uniquely. Let γ' be any Radon probability measure on B such that $\Theta_m = \int_G (\varepsilon_g \otimes g\cdot\gamma')m(dg)$. Making this relation more explicit, one gets

$$\langle\Theta_m, F\rangle = \int_G m(dg) \int_B F(g, gx)\gamma'(dx), \qquad F \text{ in } b^+(G \times B) \tag{7.22}$$

by analogy with (7.21). Using (7.16), one gets $\alpha = (\Delta\cdot m) \otimes \gamma'$; hence, finally $\gamma' = \gamma$. *Q.E.D.*

8. Additional remarks

8.1. *Smoothness of the intrinsic boundary.* The intrinsic boundary B of G (with respect to μ) may seem very large. Since the space of rays in the cone $\mathscr{M} - \{0\}$ of *all positive Radon measures* on G is regular if and only if G is compact, it is highly plausible that B is not always a metrizable space, although we have no nontrivial counter examples (that is, such that $G_\mu = G$). Nevertheless, since the topology of B has a countable base, each compact subset of B is metrizable. It follows that any bounded Radon measure on B is carried by a countable union T of metrizable compact subsets of B. Since G is also a countable union of metrizable compact subsets and G acts continuously upon B, one can even assume that T is stable under G. This applies, for instance, to the measures δ_r' and γ defined above: it follows that the measure Θ_λ is carried by $G \times T$, where T is a subset of B with the previous properties. In summary, the measures we have to work with have all the desirable smoothness.

8.2. *Martin compactification.* Call any continuous nonnegative function r on G (not necessarily positive) such that $\bar{U}r$ is a positive continuous function on G a *generalized reference function.* Classically (*see* [46], [36], for instance), to each generalized reference function r is associated the Martin compactification G_r of G, which is characterized up to homeomorphism by the following properties:

(a) the space G_r is compact and metrizable;

(b) G, with its topology, is an open dense subset of G_r;

(c) for f in $C_c^+(G)$ the function $\bar{U}f/\bar{U}r$ on G extends uniquely to a continuous function L_f on G_r and these functions L_f separate the points of $G_r - G$.

The theorems of existence of an integral representation can be described in terms of G_r. But the main disadvantage of the space G_r is that the action of G on G does not in general extend to a continuous action of G on G_r. The best that can be achieved in general is to obtain a continuous action of G on a Borel subset of G_r, large enough to permit the integral representation of excessive measures; this necessitates the use of reference functions of a special type (see Section 16). We are nevertheless going to describe one case where the Martin compactification seems preferable to the intrinsic boundary of G; let Γ be the support of π; this is also the smallest closed semigroup in G containing the support of μ. One shows easily the equivalence of the two following properties:

(a′) there exists a compact subset K of G such that $G = K \cdot \Gamma$;

(b′) there exists a function r in $C_c^+(G)$ such that $\bar{U}r(g) > 0$ for any g in G, that is, there exists a generalized reference function r having compact support.

Let us assume that (a′) and (b′) hold. Then, the Martin compactification G_r associated with the generalized reference functions r with compact support are all homeomorphic to a metrizable compact space G^* on which G acts continuously. We sketch the construction of G^*. For any r in $C_c^+(G)$, let N_r be the set of all excessive measures λ such that $\langle \lambda, r\rangle = 1$. One first shows that $\bar{U}r > 0$ implies $\langle \lambda, r\rangle > 0$ for each excessive measure $\lambda \neq 0$ and that N_r is vaguely compact. Hence, if $\bar{U}r > 0$, any ray contains one and only one point in the vaguely compact set N_r; this implies that the space $\mathscr{R}$ of rays is compact and metrizable. Call $G_\infty = G \cup \{\infty\}$ the Alexandrov compactification of G and define a map q from G into $G_\infty \times \mathscr{R}$ by $q(g) = (g, i(g))$. There is then a closed subset B^* of $\mathscr{R}$ such that $\overline{q(G)} - q(G) = \{\infty\} \times B^*$. One defines G^* as the disjoint union of G and B^*, one extends q to a bijection q' of G^* onto $\overline{q(G)}$ by mapping any x in B^* into (∞, x), and one gives G^* the topology that makes q' a homeomorphism. It is straightforward to check (a) and (b). If f and r' are in $C_c^+(G)$ and if $\bar{U}r' > 0$, the boundary value of $\bar{U}f/\bar{U}r'$ at x, for x in B^*, is defined as the number $\langle \lambda, f\rangle/\langle \lambda, r'\rangle$, where λ is any representative of the ray x, and the extended function $\bar{U}f/\bar{U}r'$ is continuous on G^*, which proves (c). Hence, G^* is homeomorphic to $G_{r'}$ for any $r' \in C_c^+(G)$ such that $\bar{U}r'$ be continuous and > 0.

The previous construction of G^* shows that the action of G upon itself by left translations extends to a continuous action of G upon G^*. Moreover, using the fact that any excessive measure is the limit of an increasing sequence of potentials, one shows that B is contained in B^* and this fact allows one to consider the intrinsic completion $\hat{G}$ of G as a dense subspace of G^*, namely, a countable intersection of open subsets.

8.3. *Recurrent case.* Let us assume that there exists no proper closed subgroup of G containing the support of μ and that $\Sigma_{n=0}^{\infty} \mu^n(V)$ is infinite for every nonempty open subset V of G (see Theorem 2.1). We shall show that any Radon measure λ such that $\lambda\bar{Q} \leqq \lambda$ is right invariant, hence that the convex cone $\mathscr{E}$ has just one ray. Indeed, let f be in $C_c^+(G)$ and F the nonnegative continuous function on G defined by $F(g) = \int_G f(yg^{-1})\lambda(dy)$. The following calculation shows that $QF \leqq F$:

$$
\begin{aligned}
(8.1)\qquad QF(g) &= \int_G F(gx)\mu(dx) = \int_G F(gx^{-1})\bar{\mu}(dx) \\
&= \int_G \bar{\mu}(dx) \int_G f(yxg^{-1})\lambda(dy) \\
&= \int_G f(zg^{-1})(\lambda * \bar{\mu})(dz) \leqq \int_G f(zg^{-1})\lambda(dz) = F(g).
\end{aligned}
$$

From $QF \leqq F$, it follows that F is constant ([1]; [46], p. 64) hence that λ is right invariant.

It is now clear why the methods used in this part cannot provide nontrivial boundaries in the recurrent case.

Part C. Convergence to the Boundary

Our assumptions are the same as for Part B. The scope of this part is primarily probabilistic. We shall devote ourselves to the proof of several limit theorems giving the asymptotic behavior of the random walk of law μ on G.

9. Relativization

Let λ be an invariant measure. Probabilistically, the relativized process associated with λ is defined as follows. From $\lambda * \bar{\mu} = \lambda$, one gets the existence of a bilateral random walk (Y_n) with n running over the integers of both signs, where each random variable Y_n has λ as distribution and the elementary steps $Y_{n-1}^{-1} Y_n$ are independent with the same probability law $\bar{\mu}$. Then the relativized process is $(Y_{-n})_{n \geqq 0}$ by definition. In the sequel, we shall need only the distribution Π^λ of this process in the path space W and we proceed to give a direct construction of Π^λ.

PROPOSITION 9.1. *Let λ be an invariant measure. There exists on the path space W with projections X_n, $n \geqq 0$, a unique Radon measure Π^λ such that*

$$\Pi^\lambda[X_0 \in A_0, \cdots, X_n \in A_n] = \langle \lambda, I_{A_n}\bar{Q}I_{A_{n-1}} \cdots I_{A_1}\bar{Q}I_{A_0}\rangle \tag{9.1}$$

holds whatever be the integer $n \geqq 0$ and the Borel subsets $A_0, \cdots, A_n$ of G. Moreover, if m is a right invariant Haar measure, Π^m is equal to $\mathbf{P}^m$.

PROOF. It is well known that two measures on W which agree on the cylinder sets $A_0 \times A_1 \times \cdots \times A_n \times G \times G \times \cdots$ are equal; hence, there can be at most one measure Π^λ for which (9.1) obtains.

For each integer $n \geqq 0$, let $W_n = G \times \cdots \times G$ ($n + 1$ factors) and let Π_n be the image of the Radon measure $\lambda \otimes \bar{\mu} \otimes \cdots \otimes \bar{\mu}$ (n factors $\bar{\mu}$) by the homeomorphism of W_n with itself which maps a point $(g_0, g_1, \cdots, g_n)$ onto the point with ith coordinate equal to $g_0 g_1 \cdots g_{n-i}$ for $0 \leqq i \leqq n$. Now let $f_0, f_1, \cdots, f_n$ in $b^+(G)$; the integral of $f_0 \otimes f_1 \otimes \cdots \otimes f_n$ with respect to Π_n is then equal to

$$J = \int_G \cdots \int_G f_0(g_0 g_1 \cdots g_{n-1} g_n) f_1(g_0 g_1 \cdots g_{n-1}), \tag{9.2}$$
$$\cdots f_{n-1}(g_0 g_1) f_n(g_0) \lambda(dg_0) \bar{\mu}(dg_1) \cdots \bar{\mu}(dg_n).$$

Assume $n \geqq 1$. In the previous integral only the first factor contains g_n. Hence, integrating first with respect to g_n and using formula (4.1) defining $\bar{Q}f$, we get a similar integral with the sequence of $n + 1$ functions $f_0, f_1, \cdots, f_n$ replaced by the sequence of n functions $f_1(\bar{Q}f_0), f_2, \cdots, f_n$. By induction on n, one gets

$$\langle \Pi_n, f_0 \otimes \cdots \otimes f_n\rangle = \langle \lambda, f_n \bar{Q} f_{n-1} \bar{Q} \cdots f_1 \bar{Q} f_0\rangle. \tag{9.3}$$

Let $r > 0$ be a continuous function on G such that $\langle \lambda, r\rangle = 1$ and let Π_n^r be the product of the measure Π_n on W_n by the continuous function $(g_0, \cdots, g_n) \mapsto r(g_0)$. Then Π_0^r is the probability measure $r \cdot \lambda$ on $W_0 = G$. For $f_0, \cdots, f_n$ in $b^+(G)$, one gets

$$\langle \Pi_n^r, f_0 \otimes \cdots \otimes f_n\rangle = \langle \lambda, f_n \bar{Q} f_{n-1} \bar{Q} \cdots f_1 \bar{Q}(f_0 r)\rangle \tag{9.4}$$

from (9.3) and $\lambda \bar{Q} = \lambda$ implies

$$\langle \Pi_n^r, f_0 \otimes \cdots \otimes f_{n-1} \otimes 1\rangle = \langle \Pi_{n-1}^r, f_0 \otimes \cdots \otimes f_{n-1}\rangle \tag{9.5}$$

whenever $n \geqq 1$. Otherwise stated, the projection of Π_n^r onto the first n factors of W_n is equal to Π_{n-1}^r, and since Π_0^r is a probability measure, it follows that Π_n^r is a probability Radon measure for each $n \geqq 0$. By Kolmogorov's theorem ([8], p. 54), there exists a unique Radon probability measure $\Pi^{\lambda,r}$ on W whose projection onto the product W_n of the first $n + 1$ factors is equal to Π_n^r for each $n \geqq 0$. As a final step, define Π^λ as the Radon measure on W product of the Radon measure $\Pi^{\lambda,r}$ with the continuous locally bounded function $r(X_0)^{-1}$. From (9.4), one gets

$$\langle \Pi^{\lambda,r}, f_0(X_0) \cdots f_n(X_n)\rangle = \langle \lambda, f_n \bar{Q} f_{n-1} \bar{Q} \cdots f_1 \bar{Q}(f_0 r)\rangle. \tag{9.6}$$

Hence,

$$(9.7)\qquad \langle \Pi^\lambda, f_0(X_0)\cdots f_n(X_n)\rangle = \langle \lambda, f_n\bar{Q}f_{n-1}\bar{Q}\cdots f_1\bar{Q}f_0\rangle$$

whatever the integer $n \geqq 0$ and the functions $f_0, \cdots, f_n$ in $b^+(G)$. The sought after relation (9.1) is the particular case of (9.7), where $f_0, \cdots, f_n$ are indicator functions.

A glance at (3.5) and (9.7) shows that, using the duality between Q and $\bar{Q}$ (relation (4.4)), the proof of $\Pi^m = \mathbf{P}^m$ is reduced to a straightforward induction on n. *Q.E.D.*

In the following, we shall use without further comment the notation Π^λ and the symbol $\mathbf{H}^\lambda$ for the integral defined by Π^λ. For r in $b^+(G)$, we shall denote by $\Pi^{\lambda,r}$ the product of the measure Π^λ on W by the function $r(X_0)$; the integral corresponding to $\Pi^{\lambda,r}$ will be denoted by $\mathbf{H}^{\lambda,r}$.

REMARK. The customary definition of relativized processes works for invariant measures of the form $f\cdot m$ only, where $Qf = f$. Such a process is defined as the Markov process with initial distribution $f\cdot m$ and transition kernel Q^f given by

$$(9.8)\qquad Q^f u = \begin{cases} f^{-1}Q(fu) & \text{on the set } [f > 0], \\ 0 & \text{elsewhere}, \end{cases}$$

for u in $b^+(G)$. The following calculation using (4.4) and the readily verified relation $f\cdot Q^f u = Q(fu)$, shows that our definition agrees with the previous description:

$$\begin{aligned}(9.9)\qquad & \mathbf{H}^{f\cdot m}[f_0(X_0)\cdots f_n(X_n)] \\ &= \langle m, ff_n\bar{Q}f_{n-1}\bar{Q}\cdots f_1\bar{Q}f_0\rangle = \langle m, f_0Qf_1Q\cdots f_{n-1}Q(ff_n)\rangle \\ &= \langle m, ff_0Q^ff_1Q^f\cdots f_{n-1}Q^ff_n\rangle = \langle f\cdot m, f_0Q^ff_1Q^f\cdots f_{n-1}Q^ff_n\rangle.\end{aligned}$$

Similarly, for any invariant measure λ, it is easy to show the existence of a transition kernel Q_λ in duality with $\bar{Q}$ with respect to λ (that is, $\langle \lambda, f\cdot\bar{Q}f'\rangle = \langle \lambda, f'\cdot Q_\lambda f\rangle$) such that the relativized process associated with λ is the Markov process with initial distribution λ and transition kernel Q_λ. We point out that Q_λ is not necessarily unique.

10. Reduites of measures

Let λ be an invariant measure and K a compact subset of G. First, we shall prove the transient character of the relativized process. Indeed, let r be a reference function such that $\langle \lambda, r\rangle = 1$. From (9.6) one gets

$$(10.1)\qquad \sum_{n=0}^{\infty} \Pi^{\lambda,r}[X_n \in K] = \sum_{n=0}^{\infty} \langle \lambda, I_K\cdot\bar{Q}^n r\rangle = \int_K \bar{U}r(g)\lambda(dg).$$

The last integral is finite because the continuous function $\bar{U}r$ is bounded on the compact set K and $\lambda(K)$ is finite. Since $r > 0$, the measures Π^λ and $\Pi^{\lambda,r}$ have the same null sets. From the Borel-Cantelli lemma, one concludes that Π^λ *almost no path in W hits K infinitely often.*

Define W_K as the set of all w in W such that the set of integers $n \geqq 0$ for which $X_n(w)$ belongs to K is finite and nonempty. For w in W_K, one denotes $t_K(w)$ the largest among the integers n such that $X_n(w) \in K$. That is, t_K is the last time that the process is in K.

The *reduite* of λ on K is the measure on G defined by $\lambda_K(A) = \Pi^\lambda[X_0 \in A, W_K]$ for A in $\mathscr{B}(G)$. The following lemma states some elementary properties of the reduites.

LEMMA 10.1. *The reduite λ_K is a potential and $I_K \cdot \lambda \leqq \lambda_K \leqq \lambda$.*

PROOF. By definition, one has

$$\lambda_K(A) = \Pi^\lambda[X_0 \in A, W_K] \leqq \Pi^\lambda[X_0 \in A] = \lambda(A), \qquad A \text{ in } \mathscr{B}(G). \tag{10.2}$$

Hence, $\lambda_K \leqq \lambda$. Moreover, whenever A is contained in K the event $[X_0 \in A]$ is contained up to a Π^λ null set in W_K because of the transient character proved above. We therefore have equality everywhere in the previous calculation, and hence $I_K\lambda \leqq \lambda_K$.

To prove that λ_K is a potential, we need the following formula

$$\mathbf{H}^\lambda[f(X_0)\cdot\theta^n F] = \mathbf{H}^\lambda[\bar{Q}^n f(X_0)\cdot F] \tag{10.3}$$

for f in $b^+(G)$ and F in $b^+(W)$. An easy induction reduces the proof of (10.3) to the proof of the particular case $n = 1$; in this case, we can content ourselves with taking F of the form $f_0(X_0)\cdots f_n(X_n)$, where $f_0, \cdots, f_n$ are in $b^+(G)$ and the sought after relation follows immediately from (9.7).

Define the measure α on G by $\alpha(A) = \Pi^\lambda[X_0 \in A, t_K = 0]$ for A in $\mathscr{B}(G)$ and let J be the indicator of the event $[t_K = 0]$. It is clear that $\theta^n J$ is the indicator of the event $[t_K = n]$ for each integer $n \geqq 0$. Hence, the indicator Φ of W_K is $\Sigma_{n=0}^\infty \theta^n J$. For f in $b^+(G)$, one therefore gets

$$\begin{aligned}\langle\lambda_K, f\rangle = \mathbf{H}^\lambda[f(X_0)\cdot\Phi] &= \sum_{n=0}^\infty \mathbf{H}^\lambda[f(X_0)\cdot\theta^n J] = \sum_{n=0}^\infty \mathbf{H}^\lambda[\bar{Q}^n f(X_0)\cdot J] \\ &= \mathbf{H}^\lambda[\bar{U}f(X_0)\cdot J] = \langle\alpha, \bar{U}f\rangle\end{aligned} \tag{10.4}$$

by using (10.3). Hence, $\lambda_K = \alpha\bar{U}$ is a potential as promised. *Q.E.D.*

Finally, let us consider a compact subset L of G containing K. It is immediate that W_K is contained in W_L up to a Π^λ null set and that $t_K(w) \leqq t_L(w)$ for w in $W_K \cap W_L$. Moreover, $\lambda_K \leqq \lambda_L$.

11. The basic convergence lemma

The following result is the main ingredient to prove convergence of the random walk to the boundary. It is an extension of a theorem of Doob [18] who treated the case of Markov chains with discrete state space. The arrangement of our proof follows rather closely Hunt [33] and Neveu [46].

THEOREM 11.1. *Let λ be an invariant measure, r a reference function such that $\langle \lambda, r\rangle = 1$ and f in $C_c^+(G)$. For every $n \geqq 0$ define the real valued random variable F_n by $F_n = \bar{U}f(X_n)/\bar{U}r(X_n)$. Then the sequence $(F_n)_{n \geqq 0}$ ends Π^λ almost surely to a random variable F_∞ such that $\mathbf{H}^{\lambda,r}[F_\infty] = \langle \lambda, f\rangle$.*

We shall subdivide the proof into several parts.

(A) Let K be a compact subset of G and $t = t_K$; for each integer $n \geqq 0$, we denote by T_n the set of paths w in W_K such that $t(w) \geqq n$, and define the real valued random variable F_n^* by

$$F_n^*(w) = \begin{cases} F_{t(w)-n}(w) & \text{if } w \in T_n, \\ 0 & \text{otherwise.} \end{cases} \tag{11.1}$$

Also, the G valued random variables X_{t-i} are defined on $T_i \supset T_n$ for $0 \leqq i \leqq n$; let $\mathscr{A}_n$ be the smallest σ-algebra of subsets of T_n containing the sets $[X_{t-i} \in A] \cap T_n$ for $0 \leqq i \leqq n$ and A in $\mathscr{B}(G)$. Furthermore, we let $\mathscr{A}_n^*$ be the σ-algebra consisting of the Borel subsets A of W such that $A \cap T_n$ belongs to $\mathscr{A}_n$.

LEMMA 11.1. *The sequence $(F_n^*)_{n\geqq 0}$ is a supermartingale with respect to the increasing family $(\mathscr{A}_n^*)_{n \geqq 0}$ of σ-algebras and the probability measure $\Pi^{\lambda,r}$.*

We fix an integer $n \geqq 0$. Let L be any $\mathscr{A}_n^*$ measurable function on W with values in $[0, +\infty]$. By definition of $\mathscr{A}_n^*$, there exists a function L' in $b^+(G \times \cdots \times G)$ ($n + 1$ factors G) such that $L = L'(X_{t-n}, \cdots, X_{t-1}, X_t)$ on T_n. Let J be the indicator of the event $[t = 0]$ and h be the continuous nonnegative function $\bar{U}f/\bar{U}r$ on G and let $L'' = L'(X_0, \cdots, X_n)\cdot\theta^n J$. On the set T_n, the function F_n^* coincides with $h(X_{t-n})$; hence, the function $F_n^*\cdot L$ coincides with $h(X_{p-n})\cdot L'(X_{p-n}, \cdots, X_{p-1}, X_p)$ on the set $[t = p]$ for any integer $p \geqq n$ and vanishes outside $T_n = \bigcup_{p \geqq n} [t = p]$. Otherwise stated, one has

$$\begin{aligned} F_n^*\cdot L &= \sum_{p=n}^{\infty} h(X_{p-n})L'(X_{p-n}, \cdots, X_{p-1}, X_p)\cdot\theta^p J \\ &= \sum_{q=0}^{\infty} \theta^q[h(X_0)\cdot L'']. \end{aligned} \tag{11.2}$$

Since F_{n+1}^* is zero outside T_{n+1}, one gets by a similar reasoning the relation

$$\begin{aligned} F_{n+1}^*\cdot L &= \sum_{p=n+1}^{\infty} h(X_{p-n-1})\cdot L'(X_{p-n}, \cdots, X_{p-1}, X_p)\cdot\theta^p J \\ &= \sum_{q=0}^{\infty} \theta^q[h(X_0)\cdot\theta L'']. \end{aligned} \tag{11.3}$$

Using (10.3), one derives the formula

$$\mathbf{H}^{\lambda,r}\left[\sum_{q=0}^{\infty} \theta^q R\right] = \mathbf{H}^\lambda[\bar{U}r(X_0)\cdot R], \qquad R \text{ in } b^+(W), \tag{11.4}$$

and from the above representation of $F_n^*\cdot L$, one gets

$$\mathbf{H}^{\lambda,r}[F_n^*\cdot L] = \mathbf{H}^\lambda[\bar{U}r(X_0)\cdot h(X_0)\cdot L''] = \mathbf{H}^\lambda[\bar{U}f(X_0)\cdot L'']. \tag{11.5}$$

In order to get a similar expression for $F_{n+1}^* \cdot L$, we just need to replace L'' by $\theta L''$. Once again using (10.3), we get

$$\text{(11.6)} \qquad \begin{aligned} \mathbf{H}^{\lambda,r}[F_{n+1}^* \cdot L] &= \mathbf{H}^{\lambda}[\bar{U}f(X_0)\cdot \theta L''] = \mathbf{H}^{\lambda}[\bar{Q}\bar{U}f(X_0)\cdot L''] \\ &\leqq \mathbf{H}^{\lambda}[\bar{U}f(X_0)\cdot L''] = \mathbf{H}^{\lambda,r}[F_n^* \cdot L]. \end{aligned}$$

Therefore, we have the inequality $\mathbf{H}^{\lambda,r}[F_{n+1}^* \cdot L] \leqq \mathbf{H}^{\lambda,r}[F_n^* \cdot L]$ for any $\mathscr{A}_n^*$ measurable function L on W with values in $[0, +\infty]$. Since F_n^* is obviously $\mathscr{A}_n^*$ measurable, the lemma is proved.

(B) Let a and b be two rational numbers with $0 < a < b$ and let N be the (random) number of upward crossings of $[a, b]$ by the random sequence $(F_n)_{n \geqq 0}$. Let K and t be as in (A) and define N_K^* as the (random) number of downward crossings of $[a, b]$ by the random sequence $(F_n^*)_{n \geqq 0}$, that is, the number of upward crossings by $(F_n)_{n \geqq 0}$ of $[a, b]$ in the random interval $[0, t]$. According to the classical Doob inequality for a nonnegative supermartingale, one has

$$\text{(11.7)} \qquad (b - a)\cdot \mathbf{H}^{\lambda,r}[N_K^*] \leqq \mathbf{H}^{\lambda,r}[F_0^*].$$

To compute $\mathbf{H}^{\lambda,r}[F_0^*]$, it suffices to let $n = 0$ and $L = 1$ in (11.5), which yields

$$\text{(11.8)} \qquad \mathbf{H}^{\lambda,r}[F_0^*] = \mathbf{H}^{\lambda}[\bar{U}f(X_0)\cdot J].$$

Define the measure α on G by $\alpha(A) = \Pi^{\lambda}[X_0 \in A, t = 0]$; then we have

$$\text{(11.9)} \qquad \mathbf{H}^{\lambda}[\bar{U}f(X_0)\cdot J] = \langle \alpha, \bar{U}f \rangle = \langle \alpha \bar{U}, f \rangle = \langle \lambda_K, f \rangle$$

because $\alpha\bar{U}$ is equal to λ_K by the proof of Lemma 10.1. Since $\lambda_K \leqq \lambda$, we conclude from the relations (11.7) to (11.9) the following inequality

$$\text{(11.10)} \qquad (b - a)\cdot \mathbf{H}^{\lambda,r}[N_K^*] \leqq \langle \lambda, f \rangle.$$

Since G is a separable locally compact space, we can find an increasing sequence $(K_p)_{p \geqq 0}$ of compact subsets of G such that $G = \bigcup_{p=0}^{\infty} K_p$. Because a path is doomed to meet at least one of the compact sets K_p, the transient character shows that up to Π^{λ} null sets $(W_{K_p})_{p \geqq 0}$ is an increasing sequence of Borel subsets of W, whose union exhausts W.

Moreover, for w in W_{K_p}, the sequence $(t_{K_q}(w))_{q \geqq p}$ increases Π^{λ} almost surely without bound. Hence, the sequence of random variables $(N_{K_p}^*)_{p \geqq 0}$ increases to N. Going to the limit in (11.10), we get

$$\text{(11.11)} \qquad (b - a)\cdot \mathbf{H}^{\lambda,r}[N] \leqq \langle \lambda, f \rangle.$$

Hence, whatever be a and b, the number N of upward crossings of $[a, b]$ by $(F_n)_{n \geqq 0}$ is Π^{λ} almost surely finite and the random sequence $(F_n)_{n \geqq 0}$ converges Π^{λ} almost surely (note that Π^{λ} and $\Pi^{\lambda,r}$ have the same null sets).

(C) Define $F_\infty = \lim_{n \to \infty} F_n$. Take the sequence $(K_p)_{p \geqq 0}$ as previously and define the random variables R_p as follows

$$\text{(11.12)} \qquad R_p(w) = \begin{cases} F_{t_p(w)}(w) & \text{for } w \text{ in } W_{K_p}, \\ 0 & \text{otherwise,} \end{cases}$$

where t_p is the exit time associated with the compact set K_p. By the previous remarks about W_{K_p} and the relation $\lim_{p\to\infty} t_p = \infty$ Π^λ almost surely, one gets $F_\infty = \lim_{p\to\infty} R_p$ (Π^λ almost surely). For $K = K_p$, the random variable denoted F_0^* in (B) is nothing else than R_p, and by (11.8) and (11.9), one gets

$$\mathbf{H}^{\lambda,r}[R_p] = \langle \lambda_{K_p}, f\rangle. \tag{11.13}$$

The positive continuous function r on G has a positive minimum on the compact support of f. Hence, there exists a constant $c > 0$ such that $f \leqq c\cdot r$. It follows that $\bar{U}f/\bar{U}r$, and hence F_n and R_p are bounded by c. By the bounded convergence theorem, one gets

$$\mathbf{H}^{\lambda,r}[F_\infty] = \lim_{p\to\infty} \mathbf{H}^{\lambda,r}[R_p] = \lim_{p\to\infty} \langle \lambda_{K_p}, f\rangle \tag{11.14}$$

from (11.13). Since the sequence of measures $(\lambda_{K_p})_{p\geqq 0}$ tends increasingly to λ (see Lemma 10.1 and the proof of Theorem 4.2), the number $\langle \lambda_{K_p}, f\rangle$ tends to $\langle \lambda, f\rangle$ as p tends to infinity. Finally, one gets the desired relation $\mathbf{H}^{\lambda,r}[F_\infty] = \langle \lambda, f\rangle$. *Q.E.D.*

12. Convergence to the boundary

We come to the core of this part and establish three convergence theorems. The first two deal with the relativized process and have an ancillary character.

Recall notation from Section 6. If r is a reference function and $\mathscr{E}_r$ is the set of all excessive measures λ such that $\langle \lambda, r\rangle \leqq 1$, the vaguely continuous map k_r from G into $\mathscr{E}_r$ is defined by $k_r(g) = \bar{U}r(g)^{-1}\cdot g\bar{\pi}$ for g in G. Moreover, Σ_r is the set of nonzero extreme points of the convex set $\mathscr{E}_r$ and $B_r = \Sigma_r - k_r(G)$.

THEOREM 12.1. *Let λ be an invariant measure and r a reference function such that $\langle \lambda, r\rangle = 1$. Then there exists a random element X in B_r such that $k_r(X_n)$ tends Π^λ almost surely to X. Moreover, for each Borel subset A of B_r, one has $\Pi^{\lambda,r}[X \in A] = \delta_r(A)$ where δ_r is the unique probability measure on B_r such that $\lambda = \int_{B_r} \sigma\cdot\delta_r(d\sigma)$. Finally, for λ in B_r, the relation $X = \lambda$ holds Π^λ almost surely.*

PROOF. By definition, one has $\langle k_r(X_n), f\rangle = \bar{U}f(X_n)/\bar{U}r(X_n)$ for f in $C_c^+(G)$ and $n \geqq 0$. Moreover, let D be a countable dense subset of $C_c^+(G)$ (uniform convergence on G). Then a sequence of elements λ_n of $\mathscr{E}_r$ has a limit in $\mathscr{E}_r$ if and only if $\langle \lambda_n, f\rangle$ has a limit for each f in D, and a mapping T from W into $\mathscr{E}_r$ is Borel measurable if and only if the numerical function $\langle T, f\rangle$ is Borel measurable for each f in D.

From these remarks and Theorem 11.1, one gets the existence of a random element X in $\mathscr{E}_r$ defined on the path space W such that $\lim_{n\to\infty} k_r(X_n) = X$ holds Π^λ almost surely and that $\mathbf{H}^{\lambda,r}[\langle X, f\rangle] = \langle \lambda, f\rangle$ holds for each f in D. This last relation can also be written

$$\lambda = \int_{\mathscr{E}_r} \sigma\cdot\nu(d\sigma), \tag{12.1}$$

where ν is the probability measure on $\mathscr{E}_r$ defined by $\nu(A) = \Pi^{\lambda,r}[X \in A]$ for A in $\mathscr{B}(\mathscr{E}_r)$. If λ is in B_r, there can be no nontrivial representation of the form (12.1). Hence, $\nu = \varepsilon_\lambda$, that is, $X = \lambda$ holds Π^λ almost surely.

From (9.6) and the definition of δ_r, one gets immediately

$$(12.2) \qquad \Pi^{\lambda,r}[E] = \int_{B_r} \Pi^{\sigma,r}[E]\delta_r(d\sigma), \qquad E \text{ in } \mathscr{B}(W).$$

We have already shown $\Pi^{\sigma,r}[X = \sigma] = 1$ for σ in B_r. Hence, $\Pi^{\sigma,r}[X \in B_r] = 1$ for each σ in B_r. By (12.2), one therefore gets $\nu(B_r) = \Pi^{\lambda,r}[X \in B_r] = 1$. From (12.1), one gets $\lambda = \int_{B_r} \sigma \cdot \nu(d\sigma)$ and finally $\nu = \delta_r$. *Q.E.D.*

REMARK. Using (9.1) instead of (9.6), one gets the integral formula

$$(12.3) \qquad \Pi^\lambda[E] = \int_{B_r} \Pi^\sigma[E]\delta_r(d\sigma) \qquad E \text{ in } \mathscr{B}(W)$$

instead of (12.2). For σ in B_r, we know that $X = \sigma$ holds Π^σ almost surely. Therefore,

$$(12.4) \qquad \Pi^\sigma[X_0 \in A, X \in A'] = \sigma(A)\varepsilon_\sigma(A')$$

for A in $\mathscr{B}(G)$ and A' in $\mathscr{B}(B_r)$. The last two formulas give

$$(12.5) \qquad \Pi^\lambda[X_0 \in A, X \in A'] = \int_{A'} \sigma(A)\delta_r(d\sigma) = \Theta_{\lambda,r}(A \times A').$$

This gives us the probabilistic meaning of the measure $\Theta_{\lambda,r}$ on $G \times B_r$ defined by Lemma 7.1. Indeed, one gets

$$(12.6) \qquad \Pi^\lambda[(X_0, X) \in C] = \Theta_{\lambda,r}(C), \qquad C \text{ in } \mathscr{B}(G \times B_r).$$

With the previous notations, let p_r be the canonical continuous map from B_r into the intrinsic boundary B of G, namely, $p_r(\lambda)$ is the ray generated by λ. If $(g_n)_{n \geq 0}$ is a sequence of points of G and σ a point in B_r, the relation $\lim_{n\to\infty} k_r(g_n) = \sigma$ in $\mathscr{E}_r$ implies $\lim_{n\to\infty} g_n = p_r(\sigma)$ in $\hat{G}$. Define the random element X_∞ in $\hat{G}$ by $X_\infty = p_r(X)$. Using Theorem 12.1, the previous remark, and Lemma 7.2, one gets the following result immediately.

THEOREM 12.2. *Let λ be an invariant measure. There exists a random element X_∞ in the intrinsic boundary B defined over the sample space W such that the relation $\lim_{n\to\infty} X_n = X_\infty$ holds Π^λ almost surely in the space $\hat{G}$. If the measure λ generates the extreme ray x, then $X_\infty = x$ holds Π^λ almost surely. Furthermore, for any Borel subset C of $G \times B$, one gets*

$$(12.7) \qquad \Pi^\lambda[(X_0, X_\infty) \in C] = \Theta_\lambda(C),$$

where the measure Θ_λ on $G \times B$ has been defined in Lemma 7.2.

We are now ready to prove our main theorem about the convergence to the boundary of the random walk of law μ on G.

THEOREM 12.3. *Let* $(Z_n)_{n \geqq 1}$ *be an independent sequence of* G *valued random elements with the common probability law* μ *and let* $S_0 = e$, $S_n = Z_1 \cdots Z_n$ *for* $n \geqq 1$. *There exists a random element* S_∞ *in the intrinsic boundary* B *of* G (with respect to μ) *such that* S_n *tends to* S_∞ *almost surely in the extended space* $\hat{G} = G \cup B$. *Moreover the probability law of* S_∞ *is the probability measure* γ *on* B *defined by Lemma* 7.3.

PROOF. Roughly speaking, the almost sure convergence of S_n to a point in B is obtained as follows. Take any random element Z in G with distribution a right invariant Haar measure m, independent from the process $(Z_n)_{n \geqq 1}$. Then the process $(ZS_n)_{n \geqq 0}$ has the distribution $\mathbf{P}^m = \Pi^m$ in the path space W. Hence by Theorem 12.2, it converges almost surely to the boundary B. By Fubini's theorem, for m almost any sample value g of Z, the process $(gS_n)_{n \geqq 0}$ converges almost surely to the boundary; since G acts by homeomorphisms upon $\hat{G}$, the process $(S_n)_{n \geqq 0}$ converges almost surely to the boundary B.

The previous argument is marred by some measurability difficulties; indeed, we don't know that $\hat{G}$ is a metrizable space. Hence, the measurability of the limit S_∞ is not ensured *a priori*. One could be tempted to work in $\mathscr{E}_r$ for some reference function r, but there the invariance under G is lost. We shall now repeat the previous reasoning taking more care of the measurability questions.

In the sample space Ω of the process $(Z_n)_{n \geqq 1}$, let us distinguish the part Ω_1 consisting of the sample points ω such that $S_n(\omega)$ converges in $\hat{G}$ to a point in B, to be denoted by $S_\infty(\omega)$. In the same way, W_1 is the set of paths w in W converging in $\hat{G}$ to a point in B, to be denoted by $X_\infty(w)$. There is a homeomorphism Φ of $G \times \Omega$ with W characterized by the following relation

$$X_n(\Phi(g, \omega)) = g \cdot S_n(\omega), \qquad n \geqq 0, g \text{ in } G, \omega \text{ in } \Omega. \tag{12.8}$$

Since G acts by homeomorphisms upon $\hat{G}$, one gets $W_1 = \Phi(G \times \Omega_1)$ and

$$X_\infty(\Phi(g, \omega)) = g \cdot S_\infty(\omega), \qquad g \text{ in } G, \omega \text{ in } \Omega_1. \tag{12.9}$$

By Theorem 12.2 with $\lambda = m$, there exists a Borel subset W_2 of W_1 such that $\Pi^m[W - W_2] = 0$ and that X_∞ induces a Borel measurable map from W_2 into B. Since Φ is a homeomorphism of $G \times \Omega$ with W transforming the measure $m \otimes \mathbf{P}$ into $\mathbf{P}^m = \Pi^m$, by Fubini's theorem, one gets

$$0 = \Pi^m[W - W_2] = \int_G \mathbf{P}[\Omega - \Omega_g] m(dg), \tag{12.10}$$

where Ω_g is the set of ω in Ω such that $\Phi(g, \omega) \in W_2$. Hence, there is at least a point g_0 such that $\mathbf{P}[\Omega - \Omega_{g_0}] = 0$. Thus, $\Omega^* = \Omega_{g_0}$ is a Borel subset of Ω such that $\mathbf{P}[\Omega^*] = 1$ and S_∞ is a Borel measurable map from Ω^* into B such that $\lim_{n \to \infty} S_n(\omega) = S_\infty(\omega)$ for any ω in Ω^*.

It remains to identify the probability law γ of S_∞ in B. Let us modify S_∞ by giving it some fixed value $b \in B$ in $\Omega - \Omega^*$. We modify X_∞ so that (12.9) remains valid. Let F in $b^+(G \times B)$ and $L = F(X_0, X_\infty)$. According to (12.7), one gets

$$\mathbf{E}^m[L] = \mathbf{H}^m[L] = \langle \Theta_m, F \rangle. \tag{12.11}$$

Moreover, from (12.9) and (12.8) one infers $L\Phi(g, \omega) = F(g, g\cdot S_\infty(\omega))$, and since Φ transforms $m \otimes \mathbf{P}$ into $\mathbf{P}^m$, one gets

$$(12.12)\quad \mathbf{E}^m[L] = \int_G \int_\Omega F(g, g\cdot S_\infty(\omega))m(dg)\mathbf{P}(d\omega) = \int_G \int_B F(g, g\cdot x)m(dg)\gamma(dx).$$

Comparing (12.11) and (12.12) gives

$$(12.13)\qquad \langle \Theta_m, F\rangle = \int_G \int_B F(g, g\cdot x)m(dg)\gamma(dx)$$

for an arbitrary F in $b^+(G \times B)$, that is, $\Theta_m = \int_G(\varepsilon_g \otimes g\cdot\gamma)m(dg)$. Hence, γ has the characteristic property stated in Lemma 7.3. *Q.E.D.*

The study of *bounded* invariant functions (Section 15) involves only a part of the boundary B which we now describe.

DEFINITION 12.1. *Let r be a reference function such that $\langle m, r\rangle = 1$. Let μ_r be the image of $\Pi^{m,r} = \mathbf{P}^{r\cdot m}$ by X_∞. We call the active part N of the boundary B the (closed) support in B of the probability measure μ_r. The space N does not depend on r and is invariant by G.*

If r and r' are reference functions such that $\langle m, r\rangle = \langle m, r'\rangle = 1$, we see by (3.1) that the probability measures $\mathbf{P}^{r\cdot m}$ and $\mathbf{P}^{r'\cdot m}$ are equivalent. Hence, μ_r and $\mu_{r'}$ are equivalent, and consequently have the same support. It is obvious, by Theorem 12.1, that μ_r is the measure occuring in (7.11); the proof of Lemma 7.3 shows then that $g\mu_r = \mu_s$, where s is another reference function such that $\langle m, s\rangle = 1$. We then have $\mu_r \sim \mu_s \sim g\mu_r$. The measure μ_r is hence *quasi-invariant* (equivalent to its translates by elements of G) and *a fortiori*, its support N is invariant by G.

To justify the terminology "active part", we note that the limit $X_\infty = \lim_{n\to\infty} X_n$ exists and belongs to the active part N of the boundary, $\mathbf{P}^g$ a.s., for *each* g in G. The proof is completely similar to the proof of Theorem 12.3; we simply have to modify the definitions of Ω_1 and W_1: namely, Ω_1 is the set of ω in Ω such that $S_n(\omega)$ converges in $\hat{G}$ to a point in the active part N of B. A similar definition is used for W_1. Since N is invariant by G, we still have $\Phi(G \times \Omega_1) = W_1$. We thus obtain $\mathbf{P}[S_\infty = \lim_{n\to\infty} S_n$ and S_∞ in $N] = 1$, and using (12.8) and $g\cdot N = N$, we get $\mathbf{P}^g[X_\infty = \lim_{n\to\infty} X_n$ and X_∞ in $N] = 1$, for each g in G.

The interest of the notion of the active part of the boundary lies essentially in the fact that in many cases (see Section 17), N can be determined completely, while the boundary B remains unknown.

13. Additional remarks

13.1. From Theorem 12.2, one deduces that any point in B is the limit in $\hat{G}$ of some sequence of points in G. This could be proved directly by a purely analytical argument. Indeed, from the fact that any excessive measure is the limit of an increasing sequence of potentials (Theorem 4.2), one infers easily that $\mathscr{E}_r$ is the closed convex hull of $k_r(G)$ whatever the reference function r is.

By a classical result, any extreme point of $\mathscr{E}_r$ is of the form $\lambda = \lim_{n\to\infty} k_r(g_n)$. Hence, $x = \lim_{n\to\infty} g_n$ in $\hat{G}$ for the ray x generated by λ. Since any point in B is the ray generated by an extreme point of $\mathscr{E}_r$ for some suitable reference function r, we are through.

13.2. Let r be a *generalized* reference function (see Section 8). One shows easily that $\langle \lambda, r\rangle > 0$ for any excessive measure $\lambda \neq 0$, that the set $\mathscr{E}_r$ of excessive measures λ with $\langle \lambda, r\rangle \leqq 1$ is vaguely compact and that the measures Π^λ and $\Pi^{\lambda, r}$ have the same null sets when $\langle \lambda, r\rangle$ is finite. Using these remarks, one checks that the proofs of Theorem 6.1, Lemmas 7.1 to 7.3, Theorems 11.1 and 12.1 remain valid when r is a generalized reference function.

We have refrained from making this generalization because we feel that the reference functions and the compact spaces $\mathscr{E}_r$ are only auxiliary tools and that the ultimate concern is with the intrinsic boundary B. The most interesting generalized reference functions are those with compact support; but, if they exist at all, the convex cone $\mathscr{E}$ has a compact basis and it is better to work directly with the Martin compactification G^* of Section 8.2 without having recourse to the reference functions.

Part D. Bounded Invariant Functions

In this part, we give an integral representation of the bounded invariant functions analogous to the representation obtained by Furstenberg [27] and we use it to compare the Poisson space of μ to the intrinsic boundary of G.

14. Invariant functions

We have seen in Section 4 that if an excessive function f is locally m integrable, the measure $f\cdot m$ is excessive. This is particularly interesting in the case when μ is spread out since we have the following lemma.

LEMMA 14.1. *Assume that μ is spread out on G. An excessive function f is locally m integrable if and only if f is m almost everywhere finite.*

PROOF. Let f be an excessive function finite on the complement of an m null set A. Since μ is spread out there exists a nonempty open subset V of G, an integer n, and a real number $c > 0$ such that μ^n majorizes $c\cdot m$ on V. Let h be in G; since $m(A) = 0$, there is a g such that $g \in hV^{-1}$ and g is not in A. We have

$$\infty > f(g) \geqq \langle g\mu^n, f\rangle \geqq c\langle gm, I_{gV}\cdot f\rangle \geqq c\Delta(g)\langle m, I_{gV}\cdot f\rangle. \tag{14.1}$$

Since gV is a neighborhood of h and h is arbitrary, f is locally m integrable. The converse is obvious. *Q.E.D.*

When the support of μ is contained in no proper closed subgroup of G, an invariant function is in $L_2(G)$ only if it is m a.e. constant, and hence m a.e. zero when G is not compact (see U. Grenander, *Probabilities of Algebraic Structures*, p. 58).

Now let f be an invariant function in $L_1(G)$. The measure $f\cdot m = \lambda$ is an invariant bounded measure; for any $f' \in C_c^+(G)$, the function f_1 defined by

$f_1(g) = \langle g\lambda, f' \rangle$ is a continuous bounded invariant function. This remark is used (in [3] and [28]) to deduce a representation of the bounded invariant measures from the integral representation of the bounded invariant functions. We recall ([3], Proposition 1.6) that *when μ is spread out, the bounded invariant functions are continuous*. We call H the Banach space of all bounded invariant functions (with the norm of uniform convergence).

15. Integral representation of bounded invariant functions

We assume in this section that μ is spread out. Let r be a reference function such that $\langle m, r \rangle = 1$, let N be the active part of the intrinsic boundary B, and let μ_r be the quasi-invariant measure on N, image of $\mathbf{P}^{r \cdot m}$ by X_∞ (Definition 12.1). Note that the null sets of μ_r are independent of r, so that the Banach space $L_\infty(N, \mu_r)$ does not depend on r.

THEOREM 15.1. *Assume that μ is spread out. There exists an isometry $f \mapsto \hat{f}$ from the Banach space H of bounded invariant functions onto $L_\infty(N, \mu_r)$ such that*

$$\hat{f}(X_\infty) = \lim_{n \to \infty} f(X_n), \qquad \mathbf{P}^{r \cdot m} \text{ a.s.,} \tag{15.1}$$

and

$$f(g) = \langle g\gamma, \hat{f} \rangle, \qquad g \text{ in } G, \tag{15.2}$$

where γ is the probability measure on N occuring in Theorem 12.3 *and Lemma* 7.3.

PROOF. We recall the notation of Sections 6 and 7; B_r is the set of the extreme invariant measures σ such that $\langle \sigma, r \rangle = 1$; B'_r is the corresponding Borel subset of rays in B, and the natural map $p_r: B_r \mapsto B'_r$ is a Borel isomorphism. By Theorem 6.1, there is a unique probability measure δ_r such that

$$m = \int_{B_r} \sigma \cdot \delta_r(d\sigma), \tag{15.3}$$

and, taking account of (7.11), we have $p_r(\delta_r) = \mu_r$.

Let f be a bounded invariant function; let λ be the invariant measure $f \cdot m$; assume first $f \geqq 0$. Since $\langle \lambda, r \rangle$ is finite, by Theorem 6.1 there is a unique bounded measure β_r on B_r such that $\lambda = \int_{B_r} \sigma \cdot \beta_r(d\sigma)$. This result is readily extended to the case when f is not positive by writing $f = (f + \|f\|) - \|f\|$. The measure $\|f\| m - \lambda$ is a positive invariant measure. Hence, by Theorem 6.1, $\|f\| \delta_r - \beta_r$ is a positive measure. There is then a function f_r in $L_\infty(B_r, \delta_r)$ such that $\beta_r = f_r \cdot \delta_r$ and $\|f_r\|_\infty \leqq \|f\|$. On the other hand,

$$\begin{aligned} \lambda = f \cdot m &= \int_{B_r} \sigma f_r(\sigma)\, \delta_r(d\sigma) \\ &\leqq \|f_r\|_\infty \int_{B_r} \sigma \cdot \delta_r(d\sigma) = \|f_r\|_\infty \cdot m. \end{aligned} \tag{15.4}$$

Since f is continuous, (15.4) implies $\|f\| \leqq \|f_r\|_\infty$, and finally $\|f\| = \|f_r\|_\infty$. The map $f \mapsto f_r$ clearly defines an isometry from H onto $L_\infty(B_r, \delta_r)$. Since $L_\infty(B_r, \delta_r)$ is isometric to $L_\infty(N, \mu_r)$ by the map $f_r \mapsto f'_r = f_r \circ p_r^{-1}$, we have an isometry $f \mapsto f'_r$ from H onto $L_\infty(N, \mu_r)$. We shall see, in fact, that the equivalence class of f'_r in $L_\infty(N, \mu_r)$ does not depend on r.

As in Section 12, define $X: W \mapsto B_r$ by $X = \lim_{n\to\infty} k_r(X_n)$ if the limit exists in $\mathscr{E}_r$, and X arbitrary elsewhere. For any function F in $b^+(W)$, we have by (12.2) and (15.3),

$$\mathbf{H}^{m,r}[F] = \int_{B_r} \delta_r(d\sigma)\mathbf{H}^{\sigma,r}[F]. \tag{15.5}$$

Let h be a function in $b^+(G)$; applying (15.5), we get

$$\mathbf{H}^{m,r}[h(X_n)f_r(X)] = \int_{B_r} \delta_r(d\sigma)\mathbf{H}^{\sigma,r}[h(X_n)f_r(X)]. \tag{15.6}$$

Since, by Theorem 12.1, $\Pi^{\sigma,r}[X = \sigma] = 1$ for σ in B_r, (15.6) becomes

$$\mathbf{H}^{m,r}[h(X_n)f_r(X)] = \int_{B_r} \delta_r(d\sigma)f_r(\sigma)\mathbf{H}^{\sigma,r}[h(X_n)]. \tag{15.7}$$

From (9.6), we obtain

$$\mathbf{H}^{\sigma,r}[h(X_n)] = \langle \sigma, h\cdot \bar{Q}^n r\rangle. \tag{15.8}$$

Using (15.8) and (15.4), we transform (15.7) into

$$\mathbf{H}^{m,r}[h(X_n)f_r(X)] = \int_G m(dg)h(g)\bar{Q}^n r(g)f(g). \tag{15.9}$$

In the particular case $f = 1$ (and hence $f_r = 1$), (15.9) yields

$$\mathbf{H}^{m,r}[h(X_n)] = \int_G m(dg)h(g)\bar{Q}^n r(g), \tag{15.10}$$

which shows that the distribution of X_n for the law $\Pi^{m,r}$ is $\bar{Q}^n r\cdot m$. We can then rewrite (15.9) as

$$\mathbf{H}^{m,r}[f_r(X)\,|\,X_n] = f(X_n), \qquad \Pi^{m,r} \text{ a.s.} \tag{15.11}$$

The left side is a bounded martingale and $f_r(X)$ is measurable with respect to the σ-algebra generated by the X_n. Hence, we have

$$\lim_{n\to\infty} \mathbf{H}^{m,r}[f_r(X)\,|\,X_n] = f_r(X), \qquad \Pi^{m,r} \text{ a.s.}, \tag{15.12}$$

which combined with (15.11), implies

$$f_r(X) = \lim_{n\to\infty} f(X_n), \qquad \Pi^{m,r} \text{ a.s.} \tag{15.13}$$

The continuity of p_r and Theorems 12.1 and 12.2 show that

$$p_r(X) = X_\infty, \qquad \Pi^m \text{ a.s.}, \tag{15.14}$$

for any reference function r.

Let s be another reference function such that $\langle m, s\rangle = 1$. Since Π^m, $\Pi^{m,r}$, and $\Pi^{m,s}$ have the same null sets, equations (15.14) and (15.13) imply

$$f'_r(X_\infty) = f'_s(X_\infty), \qquad \Pi^{m,r} \text{ a.s.,} \tag{15.15}$$

(taking account of the definitions $f'_r = f_r \circ p_r^{-1}$ and $f'_s = f_s \circ p_s^{-1}$). The image of $\Pi^{m,r} = \mathbf{P}^{r\cdot m}$ by X_∞ is μ_r. Hence, f'_r and f'_s define the same element of $L_\infty(N, \mu_r)$. We now call $\hat{f}$ the equivalence class (independent of r) of f'_r in $L_\infty(N, \mu_r)$; the equality (15.13) readily implies (15.1), since $f_r(X) = \hat{f}(X_\infty)$, $\Pi^{m,r}$ a.s. Taking account of (3.1), (15.1) implies

$$\hat{f}(X_\infty) = \lim_{n\to\infty} f(X_n), \qquad \mathbf{P}^g \text{ a.s.,} \tag{15.16}$$

for m almost every g in G. Since f is bounded, (15.16) gives

$$\lim_{n\to\infty} \mathbf{E}^g[f(X_n)] = \mathbf{E}^g[\hat{f}(X_\infty)], \qquad m \text{ a.e. } g \text{ in } G. \tag{15.17}$$

We have, since f is invariant,

$$f(g) = Q^n f(g) = \mathbf{E}^g[f(X_n)], \qquad g \text{ in } G. \tag{15.18}$$

From Theorem 12.3, we see that the image of $\mathbf{P}^e$ by X_∞ is the probability measure γ; this shows, by (12.9) and (12.8) that the image of $\mathbf{P}^g$ by X_∞ is $g\gamma$. We can now deduce, from (15.17) and (15.18),

$$f(g) = \langle g\gamma, \hat{f}\rangle, \qquad m \text{ a.e. } g \text{ in } G. \tag{15.19}$$

The definition of X_∞ shows that $F = \hat{f}(X_\infty)$ is shift invariant (see (3.4)), that is, $\theta F = F$. If we define

$$h(g) = \langle g\gamma, \hat{f}\rangle = \mathbf{E}^g[F], \tag{15.20}$$

using the Markov property, we get

$$Qh(g) = \mathbf{E}^g[h(X_1)] = \mathbf{E}^g[\mathbf{E}^{X_1}[F]] = \mathbf{E}^g[\mathbf{E}^g[\theta F \mid X_1]] = \mathbf{E}^g[\theta F] = h(g). \tag{15.21}$$

Since h is bounded and invariant, it is continuous (see Section 14); since f has the same properties, we see that (15.19) implies (15.2). *Q.E.D.*

We now study formula (15.2) in more detail.

PROPOSITION 15.1. *Assume that μ is spread out. There is a Borel positive function u_r on $G \times N$ such that*

$$\frac{d(g\gamma)}{d\mu_r}(x) = u_r(g, x) \qquad g \text{ in } G, x \text{ in } N. \tag{15.22}$$

For μ_r almost every x, the measure $u_r(\cdot, x)\cdot m$ is extreme and invariant. The measure δ_r on B_r such that $m = \int_{B_r} \sigma\cdot\delta_r(d\sigma)$ is carried by the set of extreme invariant measures σ such that $\sigma \ll m$. Any bounded invariant function f has the representation

$$f(g) = \int_N u_r(g, x)\hat{f}(x)\, \mu_r(dx), \qquad g \text{ in } G; \tag{15.23}$$

moreover, if $\mu \ll m$, u_r *can be chosen such that* $u_r(\cdot, x)$ *is an invariant function for* μ_r *almost every* x.

PROOF. Let $\hat{f}$ be any bounded Borel function on N and f the corresponding invariant function. From (15.2), we see that $f = 0$ if and only if for each g in G, $\hat{f} = 0$, $g\gamma$ a.e. By (15.4), $f = 0$ if and only if $f_r = 0$, δ_r a.e., that is to say, if and only if $\hat{f} = 0$, μ_r a.e. Hence, $\hat{f} = 0$, μ_r a.e. if and only if $\hat{f} = 0$, $g\gamma$ a.e. In particular, $g\gamma \ll \mu_r$ for each g in G. Since μ_r is carried by a countable union of metrizable compact sets (see Section 8), there is a positive Borel function u_r on $G \times N$ such that (15.22) is satisfied. Equation (15.23) is then an immediate consequence of (15.2). We have $Qf = f$, which implies, by (15.23) and by the fact that $\hat{f}$ is arbitrary in $L_\infty(N, \mu_r)$,

$$\int \mu(dh)u_r(gh, x) = u_r(g, x), \qquad \mu_r \text{ a.e. } x, \tag{15.24}$$

for each g in G. Applying Fubini's theorem to the set of pairs (g, x) in $G \times N$ for which (15.24) holds, we see that for μ_r a.e. x the function $u_r(\cdot, x)$ satisfies

$$Qu_r(\cdot, x) = u_r(\cdot, x), \qquad m \text{ a.e.} \tag{15.25}$$

Equation (4.4) shows then by duality that the measure $u_r(\cdot, x)\cdot m$ on G is invariant for μ_r a.e. x. Note that if $\mu \ll m$, (15.25) implies $Q^2 u_r(\cdot, x) = Qu_r(\cdot, x)$ everywhere, so that we can replace $u_r(\cdot, x)$ by the invariant function $Qu_r(\cdot, x)$. Coming back to the general case, call $\varphi(x)$ the invariant measure $u_r(\cdot, x)\cdot m$. We get from (15.23)

$$f\cdot m = \int_N \varphi(x)\hat{f}(x)\mu_r(dx). \tag{15.26}$$

Since $\mu_r = p_r(\delta_r)$ and $\hat{f} = f_r \circ p_r^{-1}$, we have

$$f\cdot m = \int_{B_r} \varphi \circ p_r(\sigma) f_r(\sigma)\delta_r(d\sigma). \tag{15.27}$$

Comparing with (15.4), we see that since f_r is arbitrary, we must have $\varphi \circ p_r(\sigma) = \sigma$, for δ_r a.e. σ. Hence, we have $\sigma \ll m$ for δ_r a.e. σ and $\varphi(x)$ is an extreme invariant measure for μ_r a.e. x. *Q.E.D.*

16. A special type of reference function

To prove the main result of this section, we shall have to use compact caps $\mathscr{E}_r$ of the cone of excessive measures, such that the subcone of $\mathscr{E}$ generated by $\mathscr{E}_r$ is stable by G, and on which G acts with some sort of uniformity. The appropriate reference functions are constructed below.

PROPOSITION 16.1. *For any probability measure μ on G (transient case) there is a reference function r such that*

$$\lim_{g \to e} \frac{r(gh) - r(h)}{r(h)} = 0 \tag{16.1}$$

uniformly in h and such that $\langle m, r \rangle = 1$.

We shall need a lemma.

LEMMA 16.1. *Let Δ be the modular function of G. There is a bounded Radon measure η on G, an open subgroup G_0 of G, and a finite positive function ε on G_0 such that*

(i) *$\varepsilon(g)$ tends to 0 when g tends to e,*
(ii) *$\langle \eta, \Delta \rangle = 1$,*
(iii) *$|\langle \eta g, f \rangle - \langle \eta, f \rangle| \leqq \varepsilon(g)\langle \eta, f \rangle$,*

for any g in G_0, f in $b^+(G)$.

PROOF. Assume first that G is a Lie group having a finite number of connected components. Let d be a left invariant distance on G. It is known ([30], p. 75) that there is a positive number k such that for $b > k$ the function $\Delta(g) \exp\{-bd(e, g)\}$ is in $L_1(G, m)$, and such that $\Delta(g) \leqq k \exp\{kd(e, g)\}$ for g in G. Define the measure η on G by $\eta(dg) = C \exp\{-bd(e, g)\} \Delta(g) m(dg)$. For b large enough, η is bounded, and $\langle \eta, \Delta \rangle$ is finite; we then choose C such that $\langle \eta, \Delta \rangle = 1$. The proof of (iii) is readily deduced from the following elementary inequality: for g in G and h in G,

$$|\exp\{-bd(e, hg)\} - \exp\{-bd(e, h)\}| \leqq \varepsilon(g) \exp\{-bd(e, h)\}, \tag{16.2}$$

where $\varepsilon(g) = \exp\{bd(e, g)\} - 1$.

Assume now that the quotient of G by its connected component is compact. There exists ([43], pp. 153 and 175) a normal compact subgroup K of G such that $G_1 = G/K$ is a Lie group with a finite number of connected components. Let m_K be the normed Haar measure on K; for each f in $C_c^+(G)$, the function $\bar{f}(g) = \langle m_K g, f \rangle$ can be considered as a function on G_1. Let η_1 be a measure on G_1 satisfying (i), (ii), (iii). Define η on G by $\langle \eta, f \rangle = \langle \eta_1, \bar{f} \rangle$. It is readily checked that η satisfies (i), (ii), (iii) with $G_0 = G$.

Finally, in the general case, G contains an open subgroup G_1 such that the quotient of G_1 by its connected component is compact ([43], pp. 153 and 175). Write G as a disjoint union $G = \cup_{n \geqq 0} g_n G_1$. Let η_1 be a measure on G_1 satisfying (i), (ii), (iii). Define η on G by $\eta = \Sigma_{n \geqq 0} c_n g_n \eta_1$, where $\Sigma_{n \geqq 0} c_n < \infty$ and $\Sigma_{n \geqq 0} c_n \Delta(g_n) = 1$, which implies (ii). Then, one checks (iii) with $G_0 = G_1$. *Q.E.D.*

PROOF OF PROPOSITION 16.1. The proof of Lemma 6.1 shows the existence of a bounded reference function s such that $\langle m, s \rangle = 1$ and such that $\bar{U}s$ is bounded. Let η be the measure on G obtained in Lemma 16.1. Define r by

$$r(g) = \langle \eta g, s \rangle, \qquad g \text{ in } G. \tag{16.3}$$

We have $\bar{U}r(g) = \langle \eta g, \bar{U}s \rangle$, since $\bar{U}$ commutes with left translations on G. Since s and $\bar{U}s$ are both continuous, bounded and positive, it is clear that r and $\bar{U}r$ have the same properties. Hence, r is a reference function. We have

$$\langle m, r \rangle = \int \eta(dg) \langle gm, s \rangle = \langle \eta, \Delta \rangle = 1. \tag{16.4}$$

For g in G_0, h in G, we have, by (16.3),

$$|r(gh) - r(h)| = |\langle \eta g, R_h s \rangle - \langle \eta, R_h s \rangle|, \tag{16.5}$$

where $R_h s(g) = s(gh)$. Applying inequality (iii) from Lemma 16.1, we get $|r(gh) - r(h)| \leqq \varepsilon(g) r(h)$, for g in G_0 and h in G and the proof is completed.

The following result describes the action of G on B_r when r satisfies (16.1).

PROPOSITION 16.2. *Let r be a reference function satisfying* (16.1) *and such that $\langle m, r \rangle = 1$. Some open subgroup G_0 of G acts then on $\mathscr{E}_r - \{0\}$ by*

$$T_g(\sigma) = \frac{\langle \sigma, r \rangle}{\langle g\sigma, r \rangle} g\sigma, \tag{16.6}$$

and we have

$$\lim_{g \to e, g \in G_0} T_g(\sigma) = \sigma, \tag{16.7}$$

uniformly for σ in B_r. There is a Borel subset Σ of B_r such that the whole group G acts continuously on Σ by (16.6) *and such that $\delta_r(\Sigma) = 1$, where δ_r is the measure on B_r such that $m = \int_{B_r} \sigma \cdot \delta_r(d\sigma)$.*

PROOF. For σ in $\mathscr{E}_r$, define the function F_σ on G by

$$F_\sigma(g) = \langle g\sigma, r \rangle, \qquad g \text{ in } G. \tag{16.8}$$

By the proofs of Proposition 16.1, there exists an open subgroup G_0 of G such that if $F_\sigma(g)$ is finite for some g in G, F_σ is finite on $G_0 g$; moreover, (16.1) shows then that F_σ is continuous at g. We thus see that on each coset $G_0 g$, the function F_σ is either finite and continuous or identically infinite. This has two pleasant consequences used below:

(a) if F_σ is m a.e. finite, F_σ is everywhere finite and continuous;

(b) if we choose a sequence (g_n) such that G is a disjoint union of the sets $G_0 g_n$, we see that F_σ is everywhere finite and continuous if and only if $F_\sigma(g_n)$ is finite for all n.

Also, since σ is in $\mathscr{E}_r$, $F_\sigma(e) = 1$, and all the F_σ are finite and continuous on G_0. It is then easy to check that (16.6) defines a continuous action of G_0 on $\mathscr{E}_r - \{0\}$, which obviously leaves globally invariant the set B_r of extreme invariant points of $\mathscr{E}_r - \{0\}$.

Formula (16.1) implies readily that the (F_σ), σ in $\mathscr{E}_r - \{0\}$, are equicontinuous at e, and it is then trivial to check (16.7)—where the restriction σ in B_r is essential.

Let Σ be the set of all σ in B_r such that F_σ is everywhere finite and continuous. Conclusion (b) above shows clearly that Σ is a Borel set. From $\langle gm, r \rangle = \Delta(g)$

and $gm = \int_{B_r} g\sigma \cdot \delta_r(d\sigma)$ (a consequence of (15.3)), we obtain that, for each g in G, $\langle g\sigma, r\rangle$ is finite δ_r a.e. By Fubini's theorem, this implies that, for δ_r a.e. σ in B_r, the function F_σ is m a.e. finite on G. Using conclusion (a) above, we then see that $\delta_r(\Sigma) = 1$. It is then immediate that (16.6) defines a continuous action of G on Σ. *Q.E.D.*

17. Intrinsic boundary and Poisson space

In this section, *we assume that μ is spread out*. As before, we call H the Banach space of bounded invariant functions. Let H_u be the closed subspace of H consisting of those f in H which are *left uniformly continuous* on G. A construction due to Furstenberg ([27], [3]) shows the existence of a compact G space Π (depending on μ), a probability measure ν on Π, and an isometry $f \mapsto \bar{f}$ from H_u onto $C(\Pi)$ (space of continuous functions on Π) such that

$$f(g) = \langle g\nu, \bar{f}\rangle, \qquad g \text{ in } G; \tag{17.1}$$

Π and ν are called Poisson space and Poisson kernel of μ, respectively. They have been studied extensively in [27] and [3] and in a large number of cases Π is known explicitely. We are going to compare Π and the intrinsic boundary.

We recall briefly the construction of Π (see [3]). For every Borel function f on G, we define a measurable function $F = t(f)$ on the sample space W by

$$F(W) = \begin{cases} \lim_{n\to\infty} f(X_n(w)) & \text{if the limit exists,} \\ 0 & \text{elsewhere.} \end{cases} \tag{17.2}$$

Two functions on W are considered as equivalent if and only if they are $\mathbf{P}^g$ a.s. equal, for each g in G; we call $\bar{t}(f)$ the equivalence class of $t(f)$. The set $\{\bar{t}(f) \mid f \text{ in } H_u\}$ is a C^* algebra A for the norm

$$\|\bar{t}(f)\| = \sup_{g\in G} \|t(f)\|_{L_\infty(W,\, \mathbf{P}^g)} \tag{17.3}$$

The map $\bar{t}$ is an isometry from H_u onto A; the Poisson space Π is the spectrum of A.

THEOREM 17.1. *Let μ be a probability measure on the locally compact separable group G, and let Π and ν be the Poisson space and Poisson kernel of μ. Assume that μ is spread out, and that the random walk of law μ is transient. Let N be the active part of the intrinsic boundary B of G, and γ the measure on N occurring in Theorem* 12.3. *Then there is a Borel subset Π_1 of Π, invariant by G, such that $\nu(\Pi_1) = 1$, and a continuous map ψ from Π_1 to N, commuting with the action of G, such that $\psi(\nu) = \gamma$. If, moreover, Π is a homogeneous space of G, the map ψ is in fact a homeomorphism from Π onto N, commuting with the action of G.*

PROOF. Let r be a reference function *as in Proposition* 16.1. (The notation is that of Section 15.) Let q be the isometry from $L_\infty(N, \mu_r)$ onto H such that

$$f(g) = q(\hat{f})(g) = \langle g\gamma, \hat{f}\rangle, \qquad g \text{ in } G, \hat{f} \text{ in } L_\infty(N, \mu_r). \tag{17.4}$$

Let M be the compact support of δ_r in $\mathscr{E}_r$. Since $\delta_r(B_r) = 1$, the space $C(M)$ of continuous functions on M is naturally identified to a Banach subalgebra of $L_\infty(B_r, \delta_r)$, which we denote by $C_1(M)$. Any function f_r in $C_1(M)$ is clearly uniformly continuous on B_r. Taking account of (16.7), we see that

$$\lim_{g \to e, g \in G_0} \|f_r \circ T_g - f_r\| = 0, \qquad f_r \text{ in } C_1(M). \tag{17.5}$$

The isometry s from $L_\infty(B_r, \delta_r)$ onto $L_\infty(N, \mu_r)$ defined by $\hat{f} = s(f_r) = f_r \circ p_r^{-1}$ maps $C_1(M)$ onto a Banach subalgebra $C_2(M)$ of $L_\infty(N, \mu_r)$, and we rewrite (17.5) as

$$\lim_{g \to e, g \in G_0} \|\hat{f}(g\cdot) - \hat{f}(\cdot)\| = 0, \qquad \hat{f} \text{ in } C_2(M). \tag{17.6}$$

By (17.4), the invariant function corresponding to $\hat{f}(g\cdot)$ is $f(g\cdot)$. Since q is an isometry, $\|f(g\cdot) - f(\cdot)\| = \|\hat{f}(g\cdot) - \hat{f}(\cdot)\|$ and (17.6) shows that if $\hat{f}$ is in $C_2(M)$, f is left uniformly continuous. Hence, q maps $C_2(M)$ into H_u.

From (15.1), we get

$$\hat{f}(X_\infty) = \lim_{n \to \infty} f(X_n) = t \circ q(\hat{f}), \qquad \mathbf{P}^{r \cdot m} \text{ a.s.} \tag{17.7}$$

Let $F_i = \bar{t} \circ q(\hat{f}_i)$ for $i = 1, 2$, with $\hat{f}_i$ in $C_2(M)$. Since $\bar{t}$ is an isometry from H_u onto A and since A is an algebra, there is an f in H_u such that $F_1F_2 = \bar{t}(f)$. We then have in A

$$\bar{t} \circ q(\hat{f}) = \bar{t}(f) = F_1F_2 = \bar{t} \circ q(\hat{f}_1) \cdot \bar{t} \circ q(\hat{f}_2), \tag{17.8}$$

which by definition, is equivalent to

$$t \circ q(\hat{f}) = t \circ q(\hat{f}_1) \cdot t \circ q(\hat{f}_2), \tag{17.9}$$

$\mathbf{P}^g$ a.s., for each g in G.

Combining equations (17.7) and (17.9), we get

$$\hat{f}(X_\infty) = \hat{f}_1(X_\infty)\hat{f}_2(X_\infty), \qquad \mathbf{P}^{r \cdot m} \text{ a.s.} \tag{17.10}$$

The image $\mathbf{P}^{r \cdot m}$ by X_∞ being μ_r, we see that $\hat{f} = \hat{f}_1 \hat{f}_2$ in $L_\infty(N, \mu_r)$. But (17.8) now becomes

$$\bar{t} \circ q(\hat{f}_1 \hat{f}_2) = \bar{t} \circ q(\hat{f}_1) \cdot \bar{t} \circ q(\hat{f}_2), \qquad \hat{f}_1, \hat{f}_2 \text{ in } C_2(M). \tag{17.11}$$

Hence, the isometry $\bar{t} \circ q$ is a homomorphism of algebras from $C_2(M)$ into A, such that $\bar{t} \circ q(1) = 1$. Since $C(M) \mapsto C_1(M)$ and $C_1(M) \mapsto C_2(M)$ are isomorphisms of Banach algebras (preserving 1), we have a homomorphism of $C(M)$ into A (preserving 1). By duality, we obtain a continuous mapping Φ from the spectrum Π of A onto M.

Let $\hat{f}$ be a function in $C_1(M)$ and f the corresponding bounded invariant function (left uniformly continuous); the function f_r on B_r is the restriction to B_r of a continuous function f_r' on M; define $f_r' \circ \Phi = \bar{f}$. For each g in G, we have

$$f(g) = \langle gv, \bar{f} \rangle = \langle \Phi(gv), f_r' \rangle. \tag{17.12}$$

But we also have, by (15.2) and $\hat{f} = f_r \circ p_r^{-1}$,

$$f(g) = \langle g\gamma, \hat{f} \rangle = \langle p_r^{-1}(g\gamma), f_r \rangle. \tag{17.13}$$

Finally, for each g in G,

$$\langle \Phi(g\nu), f_r' \rangle = \langle I_{B_r} \cdot p_r^{-1}(g\gamma), f_r' \rangle. \tag{17.14}$$

Since the function f_r' is arbitrary in $C(M)$, we conclude that

$$\Phi(g\nu) = I_{B_r} \cdot p_r^{-1}(g\gamma) \tag{17.15}$$

for each g in G.

Let Σ be the subset of B_r defined in Proposition 16.2. We have seen in the proof of Proposition 15.1 that $g\gamma \ll \mu_r$ for each g in G. Since $\delta_r(\Sigma) = 1$ and $\delta_r = p_r^{-1}(\mu_r)$, we see that Σ has measure one for $p_r^{-1}(g\gamma)$, for each g in G. Hence, by (17.15), we have

$$\Phi(g\nu) = I_\Sigma \cdot p_r^{-1}(g\gamma) \tag{17.16}$$

for each g in G.

Define $\Pi' = \Phi^{-1}(\Sigma)$. From (17.16), we deduce $g\nu(\Pi') = 1$ for each g in G. The map $\psi = p_r \circ \Phi$ is obviously defined and continuous on Π' and maps Π' into the active part N of the boundary. We then rewrite (17.16) as

$$\psi(g\nu) = g\gamma \tag{17.17}$$

for each g in G.

We now show that ψ commutes with the action of G (which is not obvious since $C(M)$ is not invariant by G *a priori*). Let x in Π' and g in G be such that gx is in Π'. It is known ([3], p. 13) that the measure ν on Π is "contractile," that is, any point mass on Π belongs to the vague closure of the set $(h\nu)_{h\in G}$. Let $(h_i)_{i\in I}$ be a net in G such that $\lim_I (h_i\nu) = \varepsilon_x$ (point mass at x), which implies since Π is a G space, $\lim_I (gh_i\nu) = \varepsilon_{gx}$. Assume that $g\psi(x)$ and $\psi(gx)$ are distinct points of N. Since N is Hausdorff and since G acts continuously on N, we can find in N neighborhoods A of $\psi(x)$ and B of $\psi(gx)$ such that gA and B are disjoint. Then $\psi^{-1}(A)$ is a neighborhood of x in Π'. Hence, $\psi^{-1}(A) = C \cap \Pi'$, where C is a neighborhood of x in Π. The vague convergence of $(h_i\nu)$ to ε_x in the compact space Π implies the existence of i_0 in I such that for $i > i_0$, $h_i\nu(C) > \frac{2}{3}$. Since $h_i\nu(\Pi') = 1$, we then have $h_i\nu[\psi^{-1}(A)] > \frac{2}{3}$ for $i > i_0$, that is, by (17.17), $h_i\gamma(A) > \frac{2}{3}$. Similarly, we find i_1 in I such that for $i > i_1$, $gh_i\gamma(B) > \frac{2}{3}$. Choosing i larger than i_0 and i_1, we have

$$1 \geqq gh_i\gamma(gA) + gh_i\gamma(B) = h_i\gamma(A) + gh_i\gamma(B), \tag{17.18}$$

an obvious contradiction since the last term is larger than $\frac{4}{3}$. We have proved

$$\psi(gx) = g\psi(x) \qquad x, gx \text{ in } \Pi'. \tag{17.19}$$

Let (g_n) be a dense sequence in G, containing the identity of G, and let $\Pi_1 = \bigcap_n g_n^{-1}\,\Pi'$. Since $g\nu(\Pi') = 1$ for any g in G, we have $g\nu(\Pi_1) = 1$ for each g in G, and by (17.19) we have

$$\psi(g_n x) = g_n \psi(x) \tag{17.20}$$

for any n, for x in Π_1. By (16.6), we see that $p_r(T_g\sigma) = gp_r(\sigma)$ for any σ in B_r. Hence, (17.20) implies

$$\Phi(g_n x) = T_{g_n}\Phi(x) \tag{17.21}$$

for any n, for x in Π_1. But x in Π_1 implies $\Phi(x)$ in Σ, which by Proposition 16.2 implies that $T_g\Phi(x)$ is a continuous function of g, for any fixed x in Π_1. Obviously, $\Phi(gx)$ has the same property, and from (17.21) we get

$$\Phi(gx) = T_g\Phi(x) \tag{17.22}$$

for any g, any x in Π_1. Since $T_g(B_r) = B_r$, we see that $G\Pi_1$ is included in Π'. *A fortiori* $G\Pi_1$ is included in $g_n^{-1}\Pi'$ for any g_n, which implies $G\Pi_1 = \Pi_1$. By composition with p_r, from (17.22) we can now deduce

$$\psi(gx) = g\psi(x) \tag{17.23}$$

for any g in G, any x in Π_1.

We now assume that Π is a homogeneous space of G. Then we obviously have $\Pi_1 = \Pi$. Hence, $\psi(\Pi)$ is a compact subset of N; on the other hand, $\psi(\Pi) = p_r(\Phi(\Pi_1)) = p_r(\Sigma)$ so that $\mu_r[\psi(\Pi)] = \delta_r(\Sigma) = 1$. But N is the closed support of μ_r, so that $N = \psi(\Pi)$, and ψ is a continuous map from Π onto N commuting with the action of G. We now show that ψ is an isomorphism. Let $\bar{f}$ be a continuous function on Π and let f be the corresponding left uniformly continuous, bounded invariant function defined by (17.1). By Theorem 15.1, there is an $\hat{f}$ in $L_\infty(N, \mu_r)$ such that $f(g) = \langle g\gamma, \hat{f}\rangle$, for g in G. Using (17.17), we then have

$$\langle g\nu, \bar{f}\rangle = f(g) = \langle g\gamma, \hat{f}\rangle = \langle \psi(g\nu), \hat{f}\rangle = \langle g\nu, \hat{f}\circ\psi\rangle, \quad g \text{ in } G. \tag{17.24}$$

By Theorem 1.3 in [3], this implies $\bar{f} = \hat{f}\circ\psi$, ε a.e. on Π, where ε is any quasi-invariant measure on Π. Define $V(g) = \bar{f}(gx)$ and $F(g) = \hat{f}\circ\psi(gx)$ for g in G, where x is an arbitrary fixed point in Π. Then V is continuous on G and $V = F$ m a.e. on G. Let G' be the stability group of $\psi(x)$ in G and let h be in G'. We have $F(gh) = F(g)$ for each g in G. Hence, $V(gh) = V(g)$ for m a.e. g in G; the continuity of V implies then $V(gh) = V(g)$ for all g in G. In particular, we see that $\bar{f}(hx) = \bar{f}(x)$ for each h in G'. Since $\bar{f}$ is arbitrary in $C(\Pi)$, we obtain $G'x = x$, and G' is included in the stability group of x; the converse inclusion is obvious since ψ commutes with the action of G. Hence, x and $\psi(x)$ have the same stability groups. Then ψ must be a bijection (Π and N are homogeneous spaces), and Π being compact, ψ is a homeomorphism. *Q.E.D.*

REMARKS. The case where Π is a homogeneous space of G has been studied in [27] and [3]. It occurs in particular if G is a semisimple connected Lie group or if G is a compact extension of a solvable group of a certain type (including the nilpotent groups, (see [3] for details)). In both cases, the space Π is known explicitly.

Let us call reference function *adapted* to G any generalized reference function r (see Section 8) such that the functions (F_σ), σ in $\mathscr{E}_r$, defined by (16.8) are finite and equicontinuous at e (as functions on some open subgroup G_0, of G). The reference functions constructed in Proposition 16.1, as well as the generalized reference functions with compact support (see Section 8) are adapted to G. When r is adapted to G, G_0, acts uniformly on B_r (see (16.7) and (16.6)) and it is possible to obtain a result analogous to Theorem 17.1 in terms of the Martin compactification G_r (see Section 8), that is, to identify the Poisson space and the active part of G_r (modulo null sets which are empty in the homogeneous case).

REFERENCES

[1] J. Azema, M. Kaplan-Duflo, and D. Revuz, "Chaînes de Markov récurrentes," unpublished expository paper.

[2] R. Azencott, "Espaces de Poisson et frontière de Martin," *C. R. Acad. Sci. Paris, Ser. A*, Vol. 268 (1969), pp. 1406–1409.

[3] ———, Espaces de Poisson des Groupes Localement Compacts, Lecture Notes in Mathematics, Vol. 148, Berlin-Heidelberg-New York, Springer-Verlag, 1970.

[4] D. Blackwell, "On transient Markov processes with a countable number of states and stationary transition probabilities," *Ann. Math. Statist.*, Vol. 26 (1955), pp. 654–658.

[5] N. Bourbaki, *Topologie Générale*, Chapter 9, Paris, Hermann, 1958 (2nd ed.).

[6] ———, *Espaces Vectoriels Topologiques*, Chapters 1 and 2, Paris, Hermann, 1966 (2*nd* ed.)

[7] ———, *Intégration*, Chapters 1 through 4, Paris, Hermann, 1965 (2nd ed.).

[8] ———, *Intégration*, Chapter 9, Paris, Hermann, 1969.

[9] L. Breiman, "On transient Markov chains with application to the uniqueness problem for Markov processes," *Ann. Math. Statist.*, Vol. 28 (1957), pp. 499–503.

[10] ———, "Transient atomic chains with a denumerable number of states," *Ann. Math. Statist.*, Vol. 29 (1958), pp. 212–218.

[11] G. Choquet, "Existence et unicité des représentations intégrales au moyen des points extrémaux dans les cônes convexes," *Séminaire Bourbaki*, Exp. 139, New York, Benjamin, 1956.

[12] G. Choquet and J. Deny, "Sur l'équation de convolution $\mu = \mu * \sigma$," *C. R. Acad. Sci. Paris, Ser. A*, Vol. 250 (1960), pp. 799–801.

[13] K. L. Chung, "On the boundary theory for Markov Chains, I, II," *Acta Math.*, Vol. 110 (1963), pp. 19–77; Vol. 115 (1966), pp. 111–163.

[14] ———, *Markov Chains with Stationary Transition Probability*, Berlin-New York-Heidelberg, Springer-Verlag, 1960.

[15] K. L. Chung and W. H. J. Fuchs, "On the distribution of values of sums of random variables," *Mem. Amer. Math. Soc.*, Vol. 6 (1951), pp. 1–12.

[16] J. L. Doob, "Semi-martingales and subharmonic functions," *Trans. Amer. Math. Soc.*, Vol. 77 (1954), pp. 86–121.

[17] ———, "A probabilistic approach to the heat equation," *Trans. Amer. Math. Soc.*, Vol. 80 (1955), pp. 216–280.

[18] ———, "Discrete potential theory and boundaries," *J. Math. Mech.*, Vol. 8 (1959), pp. 433–458; correction, p. 993.

[19] ———, "Compactification of the discrete state space of a Markov process," *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, Vol. 10 (1968), pp. 236–251.

[20] ———, "Probability theory and the first boundary value problem," *Illinois J. Math.*, Vol. 2 (1958), pp. 19–36.

[21] ———, "Conditional brownian motion and the boundary limits of harmonic functions," *Bull Soc. Math. France*, Vol. 85 (1957), pp. 431–458.

[22] J. L. DOOB, J. L. SNELL, and R. WILLIAMSON, "Applications of boundary theory to sums of independent random variables," *Contributions to Probability and Statistics*, Stanford, Stanford University Press, 1960, pp. 182–197.

[23] E. DYNKIN and M. MALJUTOV, "Random walks on a group with a finite number of generators," *Soviet Math. Dokl.*, Vol. 2 (1961), pp. 399–402.

[24] W. FELLER, "Boundaries induced by non-negative matrices," *Trans. Amer. Math. Soc.*, Vol. 83 (1956), pp. 19–54.

[25] ———, "On boundaries and lateral conditions for the Kolmogoroff differential equations," *Ann. of Math.*, Vol. 65 (1957), pp. 527–570.

[26] J. FELDMAN, "Feller and Martin boundaries for countable sets," *Illinois J. Math.*, Vol. 6 (1962), pp. 357–366.

[27] H. FURSTENBERG, "A Poisson formula for semisimple Lie groups," *Ann. of Math.*, Vol. 77 (1963), pp. 335–386.

[28] ———, "Noncommuting random products," *Trans. Amer. Math. Soc.*, Vol. 108 (1963), pp. 377–428.

[29] ———, "Translation invariant cones of functions on semisimple Lie groups," *Bull. Amer. Math. Soc.*, Vol. 71 (1965), pp. 271–326.

[30] L. GÅRDING, "Vecteurs analytiques dans les représentations des groupes de Lie," *Bull. Soc. Math. France*, Vol. 88 (1960), pp. 73–93.

[31] P. HENNEQUIN, "Processus de Markov en cascade," *Ann. Inst. H. Poincaré. Sect. B.*, Vol. 19 (1963), pp. 109–196.

[32] G. A. HUNT, "Markoff processes and potentials, I, II, III," *Illinois J. Math.*, Vol. 1 (1957), pp. 44–93; pp. 316–369; Vol. 2 (1958), pp. 151–213.

[33] ———, "Markoff chains and Martin boundaries," *Illinois J. Math.*, Vol. 4 (1960), pp. 313–340.

[34] D. KENDALL, "Hyperstonian spaces associated with Markov chains," *Proc. London Math. Soc.*, Vol. 10 (1960), pp. 67–87.

[35] H. KESTEN and F. SPITZER, "Random walks on countable infinite abelian groups," *Acta Math.*, Vol. 113–114 (1965), pp. 237–265.

[36] H. KUNITA and T. WATANABE, "Markov processes and Martin boundaries, I," *Illinois J. Math.*, Vol. 9 (1965), pp. 485–526.

[37] ———, "On certain reversed processes and their applications to potential theory and boundary theory," *J. Math. Mech.*, Vol. 15 (1966), pp. 393–434.

[38] ———, "Some theorems concerning resolvents over locally compact spaces," *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1967, Vol. 2, Part 2, pp. 131–164.

[39] R. S. MARTIN, "Minimal positive harmonic functions," *Trans. Amer. Math. Soc.*, Vol. 49 (1941), pp. 137–172.

[40] P. A. MEYER, *Probabilités et Potentiel*, Paris, Hermann, 1966.

[41] ———, *Processus de Markov*, Lecture Notes in Mathematics, Vol. 26, Berlin-Heidelberg-New York, Springer-Verlag, 1967.

[42] ———, *Processus de Markov: la Frontière de Martin*, Lecture Notes in Mathematics, Vol. 77, Berlin-Heidelberg-New York, Springer-Verlag, 1968.

[43] D. MONTGOMERY and L. ZIPPIN, *Topological Transformation Groups*," New York, Interscience, 1955.

[44] J. NEVEU, "Lattice methods and submarkovian processes," *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1961, Vol. 1, pp. 347–391.

[45] ———, "Sur les états d'entrée et les états fictifs d'un processus de Markov," *Ann. Inst. H. Poincaré*, Vol. 17 (1962), pp. 324–337.

[46] ———, "Chaînes de Markov et théorie du potentiel," *Ann. Univ. Clermont*, Vol. 24 (1964), pp. 37–89.

[47] P. Ney and F. Spitzer, "The Martin boundary for random walks," *Trans. Amer. Math. Soc.*, Vol. 121 (1966), pp. 116–132.

[48] D. Ray, "Resolvents, transition functions and strongly Markovian processes," *Ann. of Math.*, Vol. 70 (1959), pp. 43–78.

[49] F. Spitzer, *Principles of Random Walk*, New York, Van Nostrand, 1964.

[50] J. B. Walsh, "The Martin boundary and completion of Markov chains," *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, Vol. 14 (1970), pp. 169–188.