ON THE LAW OF THE ITERATED LOGARITHM FOR MAXIMA AND MINIMA

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1. Introduction and summary

Let $w(t)$, $0 \leqq t \leqq \infty$, denote a standard Wiener process. The general law of the iterated logarithm (see [6], p. 21) says that if g is a positive function such that $g(t)/\sqrt{t}$ is ultimately nondecreasing, then

$$P\{w(t) \geqq g(t) \text{ i.o. } t \uparrow \infty\} \tag{1.1}$$

equals 0 or 1, according as

$$\int_1^\infty \frac{g(t)}{t^{3/2}} \exp\left\{-\frac{1}{2}\frac{g^2(t)}{t}\right\} dt < \infty \text{ or } = \infty. \tag{1.2}$$

(The notation i.o. $t \uparrow \infty$ $(t \downarrow 0)$ means for arbitrarily large (small) t.) In particular, for $k \geqq 3$ and

$$g(t) = \left[2t\left(\log_2 t + \tfrac{3}{2}\log_3 t + \sum_{i=4}^{k} \log_i t + (1+\delta)\log_{k+1} t\right)\right]^{1/2}, \tag{1.3}$$

the probability (1.1) is 0 or 1 according as $\delta > 0$ or $\delta \leqq 0$. (We write $\log_2 = \log\log$, $e_2 = e^e$, and so on.)

For applications in statistics it is of interest to compute as accurately as possible

$$P\{w(t) \geqq g(t) \text{ for some } t \geqq \tau\} \tag{1.4}$$

for functions g for which this probability is < 1; that is, functions for which (1.2) converges (see [3], [10], [12]). In [11], we gave a method for computing (1.4) exactly for a certain class of functions g. A sketch of this method follows. Since $\exp\{\theta w(t) - \frac{1}{2}\theta^2 t\}$, $0 \leqq t < \infty$, is a martingale for each θ, Fubini's theorem shows that $\int_0^\infty \exp\{\theta w(t) - \frac{1}{2}\theta^2 t\}\, dF(\theta)$, $0 \leqq t < \infty$, is also a martingale for any σ-finite measure F on $(0, \infty)$. Let

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$$f(x, t) = \int_0^\infty \exp\left\{\theta x - \frac{\theta^2 t}{2}\right\} dF(\theta), \tag{1.5}$$

and for each $t \geqq 0$ and $\varepsilon > 0$ let $A(t, \varepsilon)$ be the solution of

$$f(x, t) = \varepsilon. \tag{1.6}$$

Then

$$\begin{aligned} P\{w(t) &\geqq A(t, \varepsilon) \text{ for some } t \geqq \tau\} \\ &= P\{f(w(t), t) \geqq \varepsilon \text{ for some } t \geqq \tau\}, \end{aligned} \tag{1.7}$$

and we use an elementary martingale equality to evaluate the right side of (1.7). The relation of $w(t)$ to the sequence of sums of i.i.d. random variables with mean 0 and variance 1 then permits the asymptotic evaluation of boundary crossing probabilities for partial sums.

In view of (1.3) a choice of F of particular interest is, for $\delta > 0$,

$$dF(\theta) = \begin{cases} \left[\theta\left(\log\frac{1}{\theta}\right)\cdots\left(\log_{k-1}\frac{1}{\theta}\right)\left(\log_k\frac{1}{\theta}\right)^{1+\delta}\right]^{-1} d\theta & \text{for } \theta \leqq \frac{1}{e_k}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.8}$$

for which it is shown in [11] that for any $\varepsilon > 1/\delta$,

$$A(t, \varepsilon) = \left[2t\left(\log_2 t + \tfrac{3}{2}\log_3 t + \sum_{i=4}^{k} \log_i t + (1 + \delta)\log_{k+1} t + \log\frac{\varepsilon}{2\sqrt{\pi}} + o(1)\right)\right]^{1/2} \tag{1.9}$$

as t $\to \infty$ and

$$P\{w(t) \geqq A(t, \varepsilon) \text{ for some } t \geqq 0\} = \frac{1}{\delta\varepsilon}. \tag{1.10}$$

The purpose of this paper is to obtain analogous results for maxima and minima of sequences $x_1, x_2, \cdots$ of i.i.d. random variables. We begin in Section 2 by establishing an analogue of the criterion (1.2) for a law of the iterated logarithm for sample minima. In Section 3, we give an application of this result to a conjecture of Darling and Erdös [2]. In Sections 4 and 5, we introduce a continuous time process v_t, $0 < t < \infty$, related to min $(x_1, \cdots, x_n)$ in much the same way that $w(t)$ is related to $x_1 + \cdots + x_n$, and apply the methods of [11] to the study of this process. In spite of the dissimilarity in the behavior of v_t and $w(t)$, the measure F defined by (1.8) plays the same role for v_t as for $w(t)$.

2. The law of the iterated logarithm for minima of uniform variables

THEOREM 1. *Let* $u_1, u_2, \cdots$ *be independent and uniform on* $(0, 1)$, *and let* $V_n = \min(u_1, \cdots, u_n)$. *Let* (c_n) *be any sequence of positive numbers. Then:*

(i) *if* $c_n/n \downarrow$ *for all sufficiently large* n, *then* $P\{nV_n \leqq c_n \text{ i.o.}\} = 0$ *or* 1 *according as*

$$(2.1) \qquad \sum_1^\infty \frac{c_n}{n}$$

converges or diverges;

(ii) *if* $c_n/n \downarrow$ *and* $c_n \uparrow$ *for all sufficiently large* n, *then* $P\{nV_n \geqq c_n \text{ i.o.}\} = 0$ *or* 1 *according as*

$$(2.2) \qquad \sum_1^\infty \frac{c_n}{n} e^{-c_n}$$

converges or diverges.

COROLLARY 1. *For* $k \geqq 3$,

$$(2.3) \qquad P\left\{nV_n \geqq \log_2 n + 2\log_3 n + \sum_{i=4}^{k} \log_i n + (1+\delta)\log_{k+1} n \text{ i.o.}\right\}$$

is equal to 0 *or* 1 *according as* $\delta > 0$ *or* $\delta \leqq 0$.

REMARK 2.1. The proof of (i) is an immediate consequence of the Borel–Cantelli lemma and the fact that if c_n/n is ultimately decreasing, then $V_n \leqq c_n/n$ i.o. if and only if $u_n \leqq c_n/n$ i.o. The proof of (ii) is much harder and will be given below.

REMARK 2.2. If $M_n = \max(u_1, \cdots, u_n)$, then Theorem 1 holds with V_n replaced by $1 - M_n$.

REMARK 2.3. Under different regularity conditions on the sequence (c_n), Ville [13] has shown that if (2.2) converges, then $P\{nV_n \geqq c_n \text{ i.o.}\} = 0$. His approach is similar to the one we take in Section 4. Pickands [8] has also obtained some results in the direction of Theorem 1.

REMARK 2.4. The condition in (ii) that c_n be ultimately increasing is bothersome in some applications (see Remark 2.5 below), but it cannot be dropped completely. For example, if $c_n = 1/n$, then both (2.1) and (2.2) converge. Hence by (i), $P\{nV_n \geqq c_n \text{ for all sufficiently large } n\} = 1$, which is incompatible with the conclusion of (ii) applied to the same sequence c_n.

REMARK 2.5. Let $x_1, x_2, \cdots$ be independent random variables with a common continuous distribution function F. Since $u_n = F(x_n)$ is uniform on $(0, 1)$ and

$$(2.4) \qquad \begin{aligned} F[\min(x_1, \cdots, x_n)] &= \min[F(x_1), \cdots, F(x_n)] \\ &= \min(u_1, \cdots, u_n) = V_n, \end{aligned}$$

Theorem 1 implies a law of the iterated logarithm for $\min(x_1, \cdots, x_n)$. In particular, (ii) implies that if (a_n) is any sequence of numbers such that a_n is ultimately decreasing and $nF(a_n)$ is ultimately increasing, then $P\{\min(x_1, \cdots,$

$x_m) \geqq a_n$ i.o.$\} = 0$ or 1 according as

$$\sum_1^\infty F(a_n) \exp\{-nF(a_n)\} < \infty \text{ or } = \infty. \tag{2.5}$$

The condition that $nF(a_n)$ be ultimately increasing may be difficult to verify for a given F, and hence, it is worth observing (as will become apparent in the proof below) that the condition in (ii) that $c_n(= nF(a_n))$ be ultimately increasing may be replaced by the growth condition

$$\liminf_{n\to\infty} \frac{c_n}{\log_2 n} \geqq 1. \tag{2.6}$$

Moreover, it follows *a fortiori* that if $c_n \leqq (\geqq)\, c_n'$ and $P\{nV_n \geqq c_n' \text{ i.o.}\} = 1(0)$, then $P\{nV_n \geqq c_n \text{ i.o.}\} = 1(0)$. Hence, (ii) may be applied indirectly to some sequences (c_n) which satisfy neither the monotonicity conditions of (ii) nor the growth condition (2.6).

REMARK 2.6. Let $x_1, x_2, \cdots$ be independent $N(0, 1)$ random variables with distribution function $\Phi(x) = \int_{-\infty}^x \varphi(y)\, dy$, where $\varphi(y) = (2\pi)^{-1/2} \exp\{-\frac{1}{2}y^2\}$. For $k \geqq 3$ and δ arbitrary, let

$$a_n = -\left[2 \log \frac{n}{2\sqrt{\pi}} - \log_2 n - 2 \log\left(\log_2 n + 2 \log_3 n + \sum_{i=4}^{k} \log_i n + (1+\delta) \log_{k+1} n\right)\right]^{1/2}. \tag{2.7}$$

From the fact that

$$\Phi(x) = \frac{1}{|x|}\varphi(x)\left[1 + O\left(\frac{1}{x^2}\right)\right], \qquad \text{as } x \to -\infty, \tag{2.8}$$

it can be shown that $c_n = n\Phi(a_n)$ satisfies (2.6), and hence, by (ii) and the preceding remark, that $P\{\min(x_1, \cdots, x_n) \geqq a_n \text{ i.o.}\} = 0$ or 1 according as $\delta > 0$ or $\delta \leqq 0$. Alternatively, it is possible using (2.8) to replace the criterion (2.5) by one involving the normal density φ; the argument of Lemma 8 below (together with (2.5), (2.8) and Remark 2.2) shows that if (a_n) is any ultimately increasing sequence of positive numbers such that $n a_n^{-1} \varphi(a_n)$ is ultimately increasing, then $P\{\max(x_1, \cdots, x_n) \leqq a_n \text{ i.o.}\} = 0$ or 1 according as

$$\sum_1^\infty \frac{\varphi(a_n)}{a_n} \exp\left\{-n \frac{\varphi(a_n)}{a_n}\right\} \tag{2.9}$$

converges or diverges.

The truth of (ii) follows from Theorem 2 and from Lemma 8 below which shows that the conditions of (ii) imply those of Theorem 2.

THEOREM 2. *Let $\alpha > 0$ and $n_k = \exp\{\alpha k/\log k\}$, $k = 2, 3, \cdots$, and assume that c_n/n is ultimately decreasing.*

(i) *If*

$$\sum_k \exp\{-c_{[n_k]}\} \tag{2.10}$$

converges for some α, *then* $P\{nV_n \geqq c_n \text{ i.o.}\} = 0$.

(ii) *If* (2.6) *holds and* (2.10) *diverges for some* α, *then* $P\{nV_n \geqq c_n \text{ i.o.}\} = 1$.

As usual, $[x]$ denotes the largest integer $\leqq x$. To avoid burdensome detail in the proof, we have ignored the difference between n_k and $[n_k]$.

PROOF. For (i), suppose that (2.10) converges for some α. By replacing c_n by $\min(c_n, 2\log_2 n)$, we may assume, without loss of generality, that

$$c_n \leqq 2\log_2 n. \tag{2.11}$$

By the Borel–Cantelli lemma, it suffices to show that

$$\sum_k P\{nV_n \geqq c_n \text{ for some } n_k < n \leqq n_{k+1}\} < \infty, \tag{2.12}$$

and hence, by the monotonicity of V_n and the ultimate monotonicity of c_n/n, to show that

$$\sum_k P\left\{V_{n_k} \geqq \frac{c_{n_{k+1}}}{n_{k+1}}\right\} < \infty. \tag{2.13}$$

But

$$\begin{aligned} \log P\left\{V_{n_k} \geqq \frac{c_{n_{k+1}}}{n_{k+1}}\right\} &= \log\left(1 - \frac{c_{n_{k+1}}}{n_{k+1}}\right)^{n_k} \leqq -\frac{n_k}{n_{k+1}} c_{n_{k+1}} \\ &\leqq -c_{n_{k+1}} \exp\left\{\frac{\alpha k}{\log(k+1)} - \frac{\alpha(k+1)}{\log(k+1)}\right\} \\ &= -c_{n_{k+1}} \exp\left\{\frac{-\alpha}{\log(k+1)}\right\} \\ &\leqq -c_{n_{k+1}}\left(1 - \frac{\alpha}{\log(k+1)}\right) \\ &\leqq -c_{n_{k+1}} + 2\alpha + o(1), \end{aligned} \tag{2.14}$$

where the last inequality follows from (2.11). Inequality 2.13 now follows immediately from the convergence of (2.10).

For (ii), assume that (2.6) holds and that the series (2.10) diverges for some α. Let $c_n' = \min(c_n, 2\log_2 n)$. Then $\Sigma_k \exp\{-c_{n_k}'\} \geqq \Sigma_k \exp\{-c_{n_k}\} = \infty$, and since by the first part of the theorem

$$P\{nV_n \geqq 2\log_2 n \text{ i.o.}\} = 0, \tag{2.15}$$

it follows that with no loss of generality we may again assume that (2.11) holds.

Let $A_k = \{n_k V_{n_k} \geqq c_{n_k}\}$. By Kolmogorov's 0–1 law, $P(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k) = 0$ or 1, and hence, to show that infinitely many of the events A_k occur with probability 1, it suffices to show that for all k_0

$$P\left(\bigcup_{k=k_0}^{\infty} A_k\right) \geqq \frac{1}{8}. \tag{2.16}$$

Let $k_1 > k_0$ and for $k_0 \leqq k \leqq k_1$ let $B_k = A_k \cap A_{k+1}^c \cap \cdots \cap A_{k_1}^c$. Then

$$\bigcup_{k=k_0}^{\infty} A_k \supset \bigcup_{k=k_0}^{k_1} A_k = \bigcup_{k=k_0}^{k_1} B_k, \tag{2.17}$$

and the events $B_{k_0}, B_{k_0+1}, \cdots, B_{k_1}$ are disjoint. Hence, to prove (2.16), it suffices to show that there exists a $k_1 > k_0$, k_1 depending on k_0, such that

$$\sum_{k=k_0}^{k_1} P(B_k) \geqq \frac{1}{8}. \tag{2.18}$$

But for each $k_0 \leqq k \leqq k_1$,

$$B_k = A_k \cap \left\{ V_{n_r}^{n_k} < \frac{c_{n_r}}{n_r} \text{ for all } k < r \leqq k_1 \right\}, \tag{2.19}$$

where we have set $V_j^i = \min_{i<n \leqq j} u_n$. Hence, by the independence of the u_n,

$$\begin{aligned} P(B_k) &= P(A_k) P\left\{ V_{n_r}^{n_k} < \frac{c_{n_r}}{n_r} \text{ for all } k < r \leqq k_1 \right\} \\ &\geqq P(A_k)\left(1 - \sum_{r=k+1}^{k_1} P\left\{ V_{n_r}^{n_k} \geqq \frac{c_{n_r}}{n_r} \right\}\right). \end{aligned} \tag{2.20}$$

It is easy to see from (2.6) that as $k \to \infty$, $P(A_k) \to 0$, and from Lemma 1 below, $\Sigma_k P(A_k) = \infty$. Hence, there exists a number K_0 (to be further specified below) such that for any $k_0 \geqq K_0$ and for some $k_1 > k_0$,

$$\frac{1}{4} \leqq \sum_{k=k_0}^{k_1} P(A_k) \leqq \frac{5}{16}. \tag{2.21}$$

It follows from (2.20) and (2.21) that to prove (2.18) it suffices to show that

$$\sup_{k_0 \leqq k \leqq k_1} \sum_{r=k+1}^{k_1} P\left\{ V_{n_r}^{n_k} \geqq \frac{c_{n_r}}{n_r} \right\} \leqq \frac{1}{2}. \tag{2.22}$$

It will be shown in Lemma 8 below that if (2.6) holds, then (2.10) converges or diverges simultaneously for all values of α, and hence, it suffices to prove (2.22) for one value of α. This will be done in Lemmas 2 through 7 below, completing the proof of the theorem.

LEMMA 1. $\Sigma_k P(A_k) = \infty$.

PROOF. Since $\log(1 - x) \geqq -x - x^2$ for all sufficiently small positive x, we have from (2.11) as $k \to \infty$,

$$\log P(A_k) = n_k \log\left(1 - \frac{c_{n_k}}{n_k}\right) \geqq n_k\left(-\frac{c_{n_k}}{n_k} - \frac{c_{n_k}^2}{n_k^2}\right) \geqq -c_{n_k} + o(1). \tag{2.23}$$

The lemma now follows from the divergence of (2.10).

In Lemmas 2 through 7 below, $\alpha > 1$ and $0 < \lambda < 1$ will be fixed numbers satisfying

$$\lambda > \left(\frac{5}{6}\right)^{1/2}, \tag{2.24}$$

and

$$\exp\{\alpha\lambda^3\} > 17. \tag{2.25}$$

LEMMA 2. *There exists a number K_1 such that for all $k \geqq K_1$ and $r > k$,*

$$\frac{n_k}{n_r} \leqq \exp\left\{\frac{-\lambda\alpha(r - k)}{\log r}\right\}. \tag{2.26}$$

PROOF. Let $\nu = r - k$. Then since $\log(1 + x) \leqq x$,

$$\begin{aligned}\log\frac{n_k}{n_r} &= \frac{\alpha k}{\log k} - \frac{\alpha(k + \nu)}{\log(k + \nu)} = \frac{\alpha k \log(1 + \nu/k)}{\log k \log(k + \nu)} - \frac{\alpha\nu}{\log(k + \nu)} \\ &\leqq -\frac{\alpha\nu}{\log(k + \nu)}\left(1 - \frac{1}{\log k}\right) \leqq -\lambda\alpha\frac{(r - k)}{\log r}\end{aligned} \tag{2.27}$$

for $k \geqq K_1$, provided $\log K_1 \geqq (1 - \lambda)^{-1}$.

LEMMA 3. *For each k let $r_1 = r_1(k)$ be the largest integer r such that $r - k < (\log r)^{1/2}$. There exists a number K_2 such that for all $k \geqq K_2$ and $k < r \leqq r_1$,*

$$P\left\{V_{n_r}^{n_k} \geqq \frac{c_{n_r}}{n_r}\right\} \leqq \exp\{-\alpha\lambda^3(r - k)\}. \tag{2.28}$$

PROOF. From the inequality $1 - x \leqq e^{-x}$ and Lemma 1, for $r > k \geqq K_1$, we obtain

$$\begin{aligned}P\left\{V_{n_r}^{n_k} \geqq \frac{c_{n_r}}{n_r}\right\} &= \left(1 - \frac{c_{n_r}}{n_r}\right)^{n_r - n_k} \leqq \exp\left\{-\left(1 - \frac{n_k}{n_r}\right)c_{n_r}\right\} \\ &\leqq \exp\left\{-\left(1 - \exp\left\{-\lambda\alpha\left(\frac{r - k}{\log r}\right)\right\}\right)c_{n_r}\right\}.\end{aligned} \tag{2.29}$$

For all sufficiently small positive x, $1 - e^{-x} \geqq \lambda x$. Hence, there exists K_3 so large that for all $k \geqq K_3$ and $k < r \leqq r_1$,

$$1 - \exp\left\{-\lambda\alpha\left(\frac{r - k}{\log r}\right)\right\} \geqq \alpha\lambda^2\left(\frac{r - k}{\log r}\right). \tag{2.30}$$

Finally, by (2.6), there exists K_4 so large that for $r > K_4$

$$c_{n_r} \geqq \lambda \log r. \tag{2.31}$$

With $K_2 = \max(K_1, K_3, K_4)$, the lemma follows from (2.29), (2.30), and (2.31).

LEMMA 4. *For each* $k \geqq K_2$ *and* $r > r_1$,

$$P\left\{V_{n_r}^{n_k} \geqq \frac{c_{n_r}}{n_r}\right\} \leqq \exp\{-\lambda^3(\log r)^{1/2}\}. \tag{2.32}$$

PROOF. From (2.29) and (2.31), we have

$$\begin{aligned} P\left\{V_{n_r}^{n_k} \geqq \frac{c_{n_r}}{n_r}\right\} &\leqq \exp\left\{-\left(1 - \exp\left\{-\lambda\frac{r-k}{\log r}\right\}\right)c_{n_r}\right\} \\ &\leqq \exp\{-(1 - \exp\{-\lambda(\log r)^{-1/2}\})\lambda \log r\} \\ &\leqq \exp\{-\lambda^3(\log r)^{1/2}\}. \end{aligned} \tag{2.33}$$

LEMMA 5. *For each* k, *let* $r_2 = r_2(k)$ *be the least integer* $r > k$ *such that* $r \geqq k + (\log r)^2$. *Then for all* $k > K_1$ *and* $r \geqq r_2$, $n_k/n_r \leqq 1/r$.

PROOF. By (2.27), for $k \geqq K_1$ and $r \geqq r_2$,

$$\log \frac{n_k}{n_r} \leqq -\lambda\alpha\left(\frac{r-k}{\log r}\right) \leqq -\lambda\alpha \log r < -\log r \tag{2.34}$$

LEMMA 6. *There exists a number* K_5 *such that for all* $k \geqq K_5$ *and* $r > r_2$,

$$P\left\{V_{n_r}^{n_k} \geqq \frac{c_{n_r}}{n_r}\right\} \leqq \frac{1}{\lambda^2} P(A_r). \tag{2.35}$$

PROOF. From (2.23), we have

$$P(A_r) \geqq \lambda \exp\{-c_{n_r}\} \tag{2.36}$$

for all $r \geqq$ some K_6. Hence, by Lemma 5, (2.11), (2.29), and (2.36), we have, for all $k \geqq K_5 \geqq \max(K_1, K_6)$ and $r \geqq r_2$,

$$\begin{aligned} P\left\{V_{n_r}^{n_k} \geqq \frac{c_{n_r}}{n_r}\right\} &\leqq \exp\left\{-\left(1 - \frac{n_k}{n_r}\right)c_{n_r}\right\} \\ &\leqq \frac{1}{\lambda} P(A_r) \exp\left\{\frac{2\log_2 n_r}{r}\right\} \leqq \frac{1}{\lambda^2} P(A_r). \end{aligned} \tag{2.37}$$

Note that $r_2(k) \sim k$ as $k \to \infty$. Let K_7 be so large that for all $k \geqq K_7$,

$$r_2(k) \leqq 2k \tag{2.38}$$

and

$$8(\log k)^2 \exp\{-\lambda^3(\log k)^{1/2}\} < \tfrac{1}{16}. \tag{2.39}$$

LEMMA 7. *For* $K_0 = \max(K_1, \cdots, K_7)$ *and all* $k_0 \geqq K_0$, (2.22) *holds.*

PROOF. For all $k_0 \leqq k \leqq k_1$,

$$\sum_{r=k+1}^{k_1} P\left\{V_{n_r}^{n_k} \geqq \frac{c_{n_r}}{n_r}\right\} \leqq \sum_{r=k+1}^{r_1} + \sum_{r=r_1+1}^{r_2} + \sum_{r=r_2+1}^{k_1}, \tag{2.40}$$

which by Lemmas 3, 4, and 6, equations (2.21), (2.24), (2.25), (2.38), and (2.39), does not exceed

$$\begin{aligned} (2.41)\quad & \sum_{r=k+1}^{\infty} \exp\{-\alpha\lambda^3(r-k)\} + 2(\log r_2)^2 \exp\{-\lambda^3(\log k)^{1/2}\} + \frac{1}{\lambda^2}\sum_{r=r_2}^{k_1} P(A_r) \\ & \leqq \frac{\exp\{-\alpha\lambda^3\}}{1-\exp\{-\alpha\lambda^3\}} + 2(2\log k)^2 \exp\{-\lambda^3(\log k)^{1/2}\} + \frac{1}{\lambda^2}\left(\frac{5}{16}\right) \\ & \leqq \frac{1}{16} + \frac{1}{16} + \frac{6}{16} = \frac{1}{2}. \end{aligned}$$

The following lemma shows that the conditions of Theorem 1(ii) imply those of Theorem 2. Note that the condition that (c_n) be ultimately increasing is used only to show that (c_n) may without loss of generality be assumed to satisfy (2.6). This substantiates Remark (2.5) above.

LEMMA 8. *Let (c_n) be any sequence of positive numbers such that c_n/n is ultimately decreasing and either (c_n) is ultimately increasing or* (2.6) *holds. Then* (2.2) *converges if and only if* (2.10) *converges for all* $\alpha > 0$.

PROOF. First observe that without loss of generality we may assume that (2.11) holds. In fact, if c_n is ultimately increasing, so is $c_n' = \min(c_n, 2\log_2 n)$, while if (2.6) holds then it also holds for c_n', and it is easy to see that replacing c_n by c_n' does not alter the convergence or divergence of either (2.2) or (2.10).

We next show that with no loss of generality it may be assumed that (2.6) holds. Suppose that $c_n \leqq c_{n+1}$ for all $n \geqq n_0$. If $\lim_{n\to\infty} c_n < \infty$, then (2.2) and (2.10) both diverge and continue to do so if c_n is replaced by $c_n' = \max(c_n, \log_2 n)$. Suppose on the other hand that $c_n \to \infty$. Since xe^{-x} is decreasing for large x, we have

$$\sum_{n_0}^{n} \frac{c_k}{k} e^{-c_k} \geqq c_n e^{-c_n} \sum_{n_0}^{n} k^{-1} \geqq c_n e^{-c_n}\log n - O(1), \tag{2.42}$$

which $\to \infty$ along any subsequence n' for which $c_{n'} \leqq \log_2 n'$. Hence, if (2.2) converges, (2.6) holds. If (2.2) diverges, we see from (2.42) that we may replace c_n by $c_n' = \max(c_n, \log_2 n)$ and maintain divergence, so in this case as well we may assume that (2.6) holds.

It remains to prove the lemma under the assumption

$$\tfrac{1}{2}\log_2 n \leqq c_n \leqq 2\log_2 n. \tag{2.43}$$

Now

$$\left(1 - \frac{n_k}{n_{k+1}}\right) \sim 1 - \exp\left\{-\frac{\alpha}{\log k}\right\} \sim \frac{\alpha}{\log k} \sim \left(\frac{n_{k+1}}{n_k} - 1\right) \quad \text{as } k \to \infty, \tag{2.44}$$

and hence, from (2.43),

$$c_{n_{k+1}}\left(1 - \frac{n_k}{n_{k+1}}\right), \qquad c_{n_k}\left(\frac{n_{k+1}}{n_k} - 1\right) \tag{2.45}$$

are bounded away from 0 and ∞. Since c_n/n is decreasing for large n, if (2.2) diverges, we have

$$\begin{aligned} \infty = \sum_k \sum_{n_k < n \leqq n_{k+1}} \frac{c_n}{n} \exp\left\{\frac{-c_n}{n} n\right\} &\leqq \sum_k \left(\frac{c_{n_k}}{n_k} \exp\left\{-\frac{c_{n_{k+1}}}{n_{k+1}} n_k\right\}\right)(n_{k+1} - n_k) \\ &\leqq \sum_k c_{n_k}\left(\frac{n_{k+1}}{n_k} - 1\right) \exp\left\{-c_{n_{k+1}} + c_{n_{k+1}}\left(1 - \frac{n_k}{n_{k+1}}\right)\right\} \\ &\leqq \text{const.} \sum_k \exp\left\{-c_{n_{k+1}}\right\}. \end{aligned} \tag{2.46}$$

The case in which (2.2) converges is treated similarly.

3. A conjecture of Darling and Erdös

In [2] Darling and Erdös obtained the limiting distribution of

$$y_t = \max_{0 \leqq \tau \leqq t} \frac{w(\tau)}{(\tau + 1)^{1/2}} \qquad \text{as } t \to \infty. \tag{3.1}$$

(This question was suggested by an inequality in [9] concerning the statistical consequences of "optional stopping.") They also conjectured an iterated logarithm law for the process y_t, namely:

(a) there exists a constant $c_1 > 0$ such that

$$P\left\{y_t \geqq (2 \log_2 t)^{1/2} + \frac{\log_3 t}{2(2 \log_2 t)^{1/2}} + \frac{(c_1 + \delta) \log_4 t}{(2 \log_2 t)^{1/2}} \text{ i.o. } t \uparrow \infty\right\} = 0 \text{ or } 1 \quad \text{according as } \delta > 0 \text{ or } \delta < 0; \tag{3.2}$$

and

(b) there exists a constant $c_2 > 0$ such that

$$P\left\{y_t \leqq (2 \log_2 t)^{1/2} + \frac{\log_3 t}{2(2 \log_2 t)^{1/2}} - \frac{(c_2 + \delta) \log_4 t}{(2 \log_2 t)^{1/2}} \text{ i.o. } t \uparrow \infty\right\} = 0 \text{ or } 1 \quad \text{according as } \delta > 0 \text{ or } \delta < 0. \tag{3.3}$$

Since y_t is increasing in t, it follows that for any ultimately increasing function $\psi(t)$, $y_t \geqq \psi(t)$ for arbitrarily large t if and only if $w(t) \geqq (t + 1)^{1/2}\psi(t)$ for arbitrarily large t (see Remark 2.1). Hence, from (1.3) we see that the probability (3.2) is 1 for all c_1 and δ; that is, conjecture (a) is false. A correct version of (a) is

$$P\left\{y_t \geqq (2 \log_2 t)^{1/2} + \frac{3 \log_3 t}{2(2 \log_2 t)^{1/2}} + \frac{(1 + \delta) \log_4 t}{(2 \log_2 t)^{1/2}} \text{ i.o. } t \uparrow \infty\right\} = 0 \text{ or } 1 \quad \text{according as } \delta > 0 \text{ or } \delta < 0. \tag{3.4}$$

Conjecture (b) is correct, but more difficult to prove. In this section, we shall use Theorem 1(ii) to verify (b) and identify the constant c_2 as 1. We shall only sketch the proof, which relies greatly on the method of Motoo [6]. (Motoo's method illuminates the entire paper [2] as well as the relation of Theorem 1(i) to (a) and Theorem 1(ii) to (b).)

Let

$$U(t) = e^{-t}w(e^{2t} - 1). \tag{3.5}$$

It is easy to verify that

$$P\{U(t+s)\varepsilon\, dx \mid U(s) = u\} = \varphi\left(\frac{x - ue^{-t}}{(1-e^{-2t})^{1/2}}\right)\frac{dx}{(1-e^{-2t})^{1/2}}, \tag{3.6}$$

and hence, that $U(t)$, $0 \leqq t < \infty$, is a Markov process with stationary transition probabilities and infinitesimal generator

$$Df(x) = f''(x) - xf'(x). \tag{3.7}$$

(This $U(t)$ is the Ornstein–Uhlenbeck process with $U(0) = 0$.) To prove (b) with $c_2 = 1$, it suffices to show that

$$\begin{aligned} P\{\max_{0 \leqq \tau \leqq t} U(\tau) \leqq (2 \log t + \log_2 t - (2+\delta)\log_3 t)^{1/2} \text{ i.o. } t \uparrow \infty\} \\ = 0 \text{ or } 1 \qquad \text{according as } \delta > 0 \text{ or } \delta < 0. \end{aligned} \tag{3.8}$$

Define $T_0 = 0$ and for each $n = 1, 2, \cdots$,

$$\begin{aligned} T_{2n-1} &= \inf\{t : t > T_{2n-2},\, U(t) = 1\}, \\ T_{2n} &= \inf\{t : t > T_{2n-1},\, U(t) = 0\}. \end{aligned} \tag{3.9}$$

It may be shown that $\gamma = ET_2 < \infty$, and since $T_2 - T_0$, $T_4 - T_2, \cdots$ are independent and identically distributed, it follows from the strong law of large numbers that

$$P\left\{\frac{T_{2n}}{n} \to \gamma\right\} = 1. \tag{3.10}$$

Let $x_n = \max_{T_{2n-2} \leqq t < T_{2n}} U(t)$, $n = 1, 2, \cdots$. Then $x_1, x_2, \cdots$ are i.i.d., and for $a > 1$, $P\{x_n > a\}$ is the probability that the process $U(t)$ starting from 1 reaches the level a before it reaches 0. From (3.7) and standard diffusion theory it follows that $P\{x_n > a\} = g(1)$, where $g(x)$ satisfies $g''(x) - xg(x) = 0$, $0 < x < a$, subject to the boundary conditions $g(0) = 0$, $g(a) = 1$. Hence,

$$P\{x_n > a\} = \frac{\int_0^1 \exp\{\frac{1}{2}y^2\}\, dy}{\int_0^a \exp\{\frac{1}{2}y^2\}\, dy} = \eta a \exp\{-\tfrac{1}{2}a^2\}\left(1 + O\left(\frac{1}{a^2}\right)\right) \tag{3.11}$$

as $a \to \infty$, where we have put $\eta = \int_0^1 \exp\{\frac{1}{2}y^2\}\, dy$. Let $\delta > 0$ and

$$\psi(t) = [2 \log t + \log_2 t - (2 + \delta) \log_3 t]^{1/2}. \tag{3.12}$$

Since ψ is ultimately increasing, we have by (3.10) for any $\varepsilon > 0$,

$$\begin{aligned} P\{\max_{0 \leqq \tau \leqq t} U(t) \leqq \psi(t) \text{ i.o. } t \uparrow \infty\} &\leqq P\{\max_{0 \leqq t \leqq T_{2n}} U(t) \leqq \psi(T_{2n+2}) \text{ i.o.}\} \\ &\leqq P\{\max_{1 \leqq k \leqq n} x_k \leqq \psi(n(\gamma + \varepsilon)) \text{ i.o.}\}. \end{aligned} \tag{3.13}$$

It follows from Remarks 2.2 and 2.5 and some straightforward calculation using (3.11) that

$$P\{\max_{0 \leqq \tau \leqq t} U(t) \leqq \psi(t) \text{ i.o. } t \uparrow \infty\}, \tag{3.14}$$

that is, the probability (3.8), is 0 for $\delta > 0$. A similar argument shows that (3.8) is 1 for $\delta > 0$.

Motoo's method of proof of the criterion (1.2) for the Wiener process [6] is essentially a combination of the preceding argument with Theorem 1 (i) instead of Theorem 1 (ii). It is interesting to note that neither Motoo's nor our argument requires knowledge of the exact value of $\gamma = ET_2$ or of the constant η appearing in (3.11). A more careful analysis shows that under certain regularity conditions on the function ψ,

$$\begin{aligned} &P\{\max_{0 \leqq \tau \leqq t} U(\tau) \leqq \psi(t) \text{ i.o. } t \uparrow \infty\}, \\ &\qquad = 0 \text{ or } 1, \text{ according as } \int_1^\infty f(\psi(t)) \exp\left\{-\frac{t}{\sqrt{2\pi}} f(\psi(t))\right\} dt \end{aligned} \tag{3.15}$$

is convergent or divergent, where we have set $f(x) = x \exp\{-\frac{1}{2}x^2\}$. However, establishing this deeper criterion requires knowledge of the constants γ and η and in particular that $\gamma/\eta = (2\pi)^{1/2}$. (It is interesting to observe that if we had defined the stopping times T_n in our proof in terms of 0 and an arbitrary number $b > 0$, then γ and η would depend on b, but the ratio γ/η would not.) It is also necessary to sharpen (3.10) to, say,

$$P\left\{\frac{T_{2n} - n\gamma}{n^{1/2} \log n} \to 0\right\} = 1, \tag{3.16}$$

which is a consequence of the fact that $ET_2^2 < \infty$ and the usual proof of the strong law of large numbers using Kolmogorov's inequality and Kronecker's lemma (see [7]). We omit the details.

4. A continuous time extremal process

In this section, we introduce a continuous time process v_t which bears more or less the same relation to the process V_n as the Wiener process does to the sequence of partial sums of i.i.d. mean 0, variance 1, random variables, and give boundary crossing probabilities for this process analogous to those of [11] for the Wiener process.

Consider the sequence of processes $nV_{[nt]}, 0 \leqq t < \infty$. For any $0 = t_0 < t_1 < t_2 < \cdots < t_r$ and $a_1 \geqq a_2 \geqq \cdots \geqq a_r > 0$, we have as $n \to \infty$

$$(4.1)\qquad P\{nV_{[nt_1]} \geqq a_1, \cdots, nV_{[nt_r]} \geqq a_r\} = \prod_{i=1}^{r}\left(1 - \frac{a_i}{n}\right)^{[nt_i]-[nt_{i-1}]} \to \exp\left\{-\sum_{i=1}^{n} a_i(t_i - t_{i-1})\right\}.$$

This suggests defining $v_t, 0 < t < \infty$, by the following consistent family of joint distributions: for $0 = t_0 < t_1 < \cdots < t_r$ and $-\infty < a_i < \infty$, $i = 1, 2, \cdots, r$,

$$(4.2)\quad P\{v_{t_1} \geqq a_1, \cdots, v_{t_r} \geqq a_r\} = \exp\left\{-\sum_{i=1}^{r} \max(a_i, \cdots, a_r)^+ (t_i - t_{i-1})\right\}.$$

By Kolmogorov's consistency theorem, there exists a process, say $\tilde{v}_t$, having the finite dimensional joint distributions given by (4.2). Defining $v_t = \lim_{s\downarrow t} \tilde{v}_s$, where s runs through rationals greater than t, we obtain a process having the same finite dimensional joint distributions as $\tilde{v}_t$ and in addition right continuous, decreasing sample paths. We shall call any such process a *standard extremal process*.

It is easy to see from (4.2) that the process $v_t, 0 < t < \infty$, is Markovian with stationary transition probability

$$(4.3)\quad P\{v_t \geqq a_2 | v_\tau = a_1\} = \begin{cases}\exp\{-a_2(t - \tau)\} & \text{for } a_1 \geqq a_2 \geqq 0,\\ 0 & \text{for } 0 \leqq a_1 < a_2.\end{cases}$$

For each $\tau > 0$, let $h = h(\tau) = \inf\{t: t > \tau, v_t < v_\tau\}$. Then $\{h > t\} = \{v_t = v_\tau\}$, and hence, by (4.3),

$$(4.4)\qquad P\{h > t | v_\tau = a\} = \exp\{-a(t - \tau)\}.$$

Also for $a_2 < a_1$,

$$\begin{aligned}(4.5)\qquad P\{v_h \leqq a_2, t < h < t + \delta | v_\tau = a_1\} &= P\{v_{t+\delta} \leqq a_2, t < h < t + \delta | v_\tau = a_1\} + o(\delta)\\ &= P\{v_{t+\delta} \leqq a_2, v_t \geqq a_1 | v_\tau = a_1\} + o(\delta)\\ &= (1 - \exp\{-\delta a_2\})\exp\{-(t - \tau)a_1\} + o(\delta),\end{aligned}$$

so

$$(4.6)\qquad P\{v_h \leqq a_2 | v_\tau = a_1\} = \int_\tau^\infty P\{v_h \leqq a_2, h \in dt | v_\tau = a_1\} = \frac{a_2}{a_1}.$$

It follows that the sample paths of the process $v_t, 0 < t < \infty$, may be described as follows: for any $\tau > 0$, if the process is in the state a at time τ, it remains there for a random length of time having a negative exponential distribution with parameter a and then jumps to a point uniformly distributed on $(0, a)$. By (4.2), $P\{v_{0+} = +\infty\} = 1$, and with probability 1 there are infinitely many jumps in each neighborhood of $t = 0$. Except for the behavior of v_t near $t = 0$, this description is analogous to that of the discrete time process V_n, which holds in each state a a random length of time which is geometrically distributed with parameter a and then moves to a point uniformly distributed on $(0, a)$. If $x_1, x_2, \cdots$ are i.i.d. with $P\{x_i \geqq x\} = e^{-x}$, and $v_n^* = \min(x_1, \cdots, x_n)$, then the process v_t interpolates the process v_n^* in the sense that the two sequences v_n and v_n^*, $n = 1, 2, \cdots$, have the same joint distributions.

Trivial modifications of the proof of Theorem 1(ii) prove:

THEOREM 3. *If $c(t) \geqq 0$ is ultimately increasing and $c(t)/t$ is ultimately decreasing, then $P\{v_t \geqq c(t)/t, \text{ i.o. } t \uparrow \infty\} = 0$ or 1, according as $\int_1^\infty (c(t)/t)e^{-c(t)}\,dt$ converges or diverges.*

Since $P\{v_{0_+} = +\infty\} = 1$, it is of interest to obtain a description of the rate of growth of v_t as $t \downarrow 0$. A law of the iterated logarithm for v_t as $t \downarrow 0$ follows from Theorem 3 and the following inversion theorem.

THEOREM 4. *For each $v > 0$, let $T(v) = \sup\{t: v_t \geqq v\}$. The process $T(v)$, $0 < v < \infty$, is a standard extremal process.*

PROOF. The fact that the sample paths of $T(v)$, $0 < v < \infty$ are decreasing, right continuous step functions follows at once from the corresponding properties of v_t, $0 < t < \infty$. Hence, it suffices to show that $T(v)$ and v_t have the same finite dimensional joint distributions. For $0 < u < v$ and $\tau > t > 0$, except for a set of probability 0,

$$\{T(u) \geqq \tau, T(v) \geqq t\} = \{v_t \geqq v, v_\tau \geqq u\}, \tag{4.7}$$

and hence, by (4.2),

$$\begin{aligned} P\{T(u) \geqq \tau, T(v) \geqq t\} \\ = \exp\{-tv - (\tau - t)u\} = \exp\{-u\tau - (v - u)t\}. \end{aligned} \tag{4.8}$$

The general case of an arbitrary finite number of time points $u, v, \cdots, z$ follows by the same argument.

For any strictly decreasing function ψ defined on $(0, \infty)$, $v_t \geqq \psi(t)$ i.o. $t \downarrow 0$ if and only if $T(v) > \psi^{-1}(v)$ i.o. $v \uparrow \infty$. Hence, by Theorem 4,

$$P\{v_t \geqq \psi(t) \text{ i.o. } t \downarrow 0\} = P\{v_t > \psi^{-1}(t) \text{ i.o. } t \uparrow \infty\}. \tag{4.9}$$

For example, by Theorem 3 and (4.9), we have

$$P\left\{\limsup_{t \to 0} \frac{tv_t}{\log_2 \frac{1}{t}} = 1\right\} = 1. \tag{4.10}$$

We now show that the method of [11] yields boundary crossing probabilities for the process v_t, $0 < t < \infty$, analogous to those obtained there for the Wiener process.

Let F denote a measure on $(0, \infty)$ assigning finite measure to bounded intervals, and define for $x > 0$, $t \geqq 0$, $\varepsilon > 0$,

$$(4.11)\qquad \begin{aligned} f(x, t) &= \int_{\{0<y\leqq x\}} e^{yt}\, dF(y), \\ A(t, \varepsilon) &= \inf\{x : f(x, t) \geqq \varepsilon\} (= \infty \text{ if } f(x, t) < \varepsilon \text{ for all } x). \end{aligned}$$

It is easily seen that for all $t \geqq 0$, $x \geqq A(t, \varepsilon)$ if and only if $f(x, t) \geqq \varepsilon$, and that if $\tau_0 = \inf\{\tau : A(\tau, \varepsilon) < \infty\}$, then $A(\cdot, \varepsilon)$ is continuous and decreasing on (τ_0, ∞) and $A(\tau_0, \varepsilon) = \lim_{t\downarrow\tau_0} A(t, \varepsilon)$. Moreover, if $F\{A(t, \varepsilon)\} = 0$, then $f(A(t, \varepsilon), t) = \varepsilon$.

Let $\mathscr{F}_t = \mathscr{B}(v_\tau, \tau \leqq t)$. Since $\{I[v_t \geqq y]e^{yt}, \mathscr{F}_t, 0 \leqq t < \infty\}$ is a martingale for each $y > 0$, as may be verified by direct computation using (4.3), it follows from Fubini's theorem that $\{f(v_t, t), \mathscr{F}_t, 0 < t < \infty\}$ is also a martingale.

THEOREM 5. *For any* $\varepsilon > F\{(0, \infty)\}$

$$(4.12)\qquad P\{v_t \geqq A(t, \varepsilon) \textit{ for some } t > 0\} = \frac{F\{(0, \infty)\}}{\varepsilon}.$$

For each $\tau > 0$,

$$(4.13)\qquad \begin{aligned} P\{v_t \geqq A(t, \varepsilon) \textit{ for some } t \geqq \tau\} &= \exp\{-\tau A(\tau, \varepsilon)\} \\ &+ \varepsilon^{-1}\left[F\{(0, A(\tau, \varepsilon))\} - \exp\{-\tau A(\tau, \varepsilon)\}\int_{\{0<y<A(\tau,\varepsilon)\}} e^{\tau y}\, dF(y)\right] \\ &\left(= \varepsilon^{-1}F\{(0, A(\tau, \varepsilon))\} \textit{ if } F\{A(\tau, \varepsilon)\} = 0\right). \end{aligned}$$

PROOF. The parenthetical part of (4.13) follows at once from the preceding line and the observation that if $F\{A(\tau, \varepsilon)\} = 0$, then

$$(4.14)\qquad \int_{\{0<y<A(\tau,\varepsilon)\}} e^{\tau y}\, dF(y) = f(A(\tau, \varepsilon), \tau) = \varepsilon;$$

equation (4.12) follows from the parenthetical part of (4.13), by letting $\tau \downarrow \tau_0 = \inf\{t : A(t, \varepsilon) < \infty\}$ through any sequence of values such that $F\{A(\tau, \varepsilon)\} = 0$. The proof of (4.13) follows from Lemma 1 of [11], Remark (d) at the end of Section 3 of [11], and Lemma 9 below (see the proof of Theorem 1 of [11]).

LEMMA 9. *The function* $f(v_t, t)$ *tends to* 0 *in probability*.

PROOF. Let $c > 0$. By the weak convergence of the family $F_t\{\cdot\} = F\{(\cdot) \cap (0, c]/t\}$ to the 0 measure,

$$(4.15)\qquad \int_{\{0<y\leqq c/t\}} e^{yt}\, dF(y) = \int_{\{0<y\leqq c\}} e^{y}\, dF\left(\frac{y}{t}\right) \to 0$$

as $t \to \infty$. Hence, for any $\varepsilon > 0$, for all t sufficiently large,

$$(4.16)\qquad P\{f(v_t, t) \geqq \varepsilon\} \leqq P\left\{v_t \geqq \frac{c}{t}\right\} \doteq e^{-c},$$

which can be made arbitrarily small by taking c sufficiently large.

5. Asymptotic expansions for $A(t, \varepsilon)$

If the measure F of the preceding section is taken to be Lebesgue measure on $(0, \infty)$, it is easily seen from (4.11) that

$$A(t, \varepsilon) = \frac{1}{t}\log(1 + \varepsilon t) = \frac{1}{t}\left[\log t + \log \varepsilon + O\left(\frac{1}{t}\right)\right] \tag{5.1}$$

as $t \to \infty$. By Theorem 3 there exist functions $g(t) \sim \log_2 t/t$ as $t \to \infty$ such that $P\{v_t \geqq g(t) \text{ for some } t > 0\} < 1$, and it is natural to ask whether we can find boundaries with this rate of growth to which Theorem 5 applies.

THEOREM 6. *If F is defined by* (1.8), *then for $k = 2$*

$$A(t, \varepsilon) = \frac{1}{t}[\log_2 t + (2 + \delta)\log_3 t + \log \varepsilon + o(1)] \tag{5.2}$$

as $t \to \infty$, while for $k \geqq 3$,

$$A(t, \varepsilon) = \frac{1}{t}\left[\log_2 t + 2\log_3 t + \sum_{i=4}^{k}\log_i t + (1 + \delta)\log_{k+1} t + \log \varepsilon + o(1)\right] \tag{5.3}$$

as $t \to \infty$.

(See equation (10) of [11] which describes the corresponding result for the Wiener process.)

To prove (5.2), let F be given by (1.8) with $k = 2$ and let $F'(y) = dF/dy$. (The proof of (5.3) when F is given by (1.8) with $k \geqq 3$ is similar and will be omitted.) It follows easily from (4.11) that

$$A = A(t, \varepsilon) \to 0 \tag{5.4}$$

as $t \to \infty$. Similarly,

$$tA \to \infty. \tag{5.5}$$

In fact, if there exists a number C such that $tA < C$ along a sequence of t values, then

$$\varepsilon \leqq \int_0^C e^z \, dF\left(\frac{z}{t}\right). \tag{5.6}$$

But $F(z/t) \to 0$ as $t \to \infty$ for all $z > 0$, and hence, $\int_0^C e^z \, dF(z/t) \to 0$ as $t \to \infty$, contradicting our supposition. Since F' is decreasing in a neighborhood of the origin, we have by (5.4) for all t sufficiently large,

$$\varepsilon = f(A, t) = \int_0^A e^{yt}F'(y)\, dy \geqq \frac{F'(A)(e^{tA} - 1)}{t}, \tag{5.7}$$

and hence, by (1.8) and (5.5),

$$\limsup_{t\to\infty} \frac{e^{tA}}{tA \log \dfrac{1}{A}\left(\log_2 \dfrac{1}{A}\right)^{1+\delta}} \leqq \varepsilon. \tag{5.8}$$

Rewriting (5.8) as

$$A \leqq \frac{1}{t}\left[\log \varepsilon t + \log A + \log_2 \frac{1}{A} + (1+\delta)\log_3 \frac{1}{A} + o(1)\right], \tag{5.9}$$

we see by (5.4) and (5.5) that $A \leqq (\log t/t)[1 + o(1)]$, and hence $\log A \leqq \log_2 t - \log t + o(1)$. Since (5.5) implies *a fortiori* that for all sufficiently large t we have $1/A \leqq t$, and hence

$$\log_k 1/A \leqq \log_k t, \tag{5.10}$$

we have from (5.9),

$$A \leqq \frac{1}{t}[\log \varepsilon + 2\log_2 t + (1+\delta)\log_3 t + o(1)]. \tag{5.11}$$

Thus $\log A \leqq -\log t + \log_3 t + O(1)$, which by substituting once more in (5.9) yields $A \leqq \log_2 t/t\,(1 + o(1))$ and hence

$$\log A \leqq -\log t + \log_3 t + o(1). \tag{5.12}$$

Finally, substituting (5.12) and (5.10) in (5.9), yields one half of (5.2), to wit

$$A \leqq \frac{1}{t}[\log_2 t + (2+\delta)\log_3 t + \log \varepsilon + o(1)]. \tag{5.13}$$

In particular,

$$\limsup_{t\to\infty} \frac{tA}{\log_2 t} \leqq 1. \tag{5.14}$$

To prove (5.13) with the inequality reversed let $0 < b < c < 1$. From (4.11), we have for all t sufficiently large

$$\begin{aligned} \varepsilon &= \left(\int_0^{bA} + \int_{bA}^{cA} + \int_{cA}^{A}\right) e^{yt}F'(y)\,dy \\ &\leqq e^{btA}F(bA) + \frac{e^{ctA}}{t}F'(bA) + \frac{e^{tA}}{t}F'(cA) \\ &\leqq \frac{e^{btA}}{\delta\left(\log \dfrac{1}{2A}\right)^{\delta}} + \frac{e^{ctA}}{btA\log\dfrac{1}{A}\left(\log\dfrac{1}{2A}\right)^{1+\delta}} + \frac{e^{tA}}{ctA\log\dfrac{1}{A}\left(\log\dfrac{1}{2A}\right)^{1+\delta}}. \end{aligned} \tag{5.15}$$

Let $b = 1/\log_2 t$. Then from (5.4), (5.5), (5.14), and (5.15), we obtain for large t

$$\varepsilon \leqq o(1) + \frac{e^{tA}}{\log 1/A} \leqq o(1) + \frac{e^{tA}}{\frac{1}{2}\log t}, \tag{5.16}$$

and hence,

$$\frac{\log_2 t}{t} = O(A). \tag{5.17}$$

From (5.14) and (5.17), it follows that for all η sufficiently small,

$$\eta \leqq \frac{tA}{\log_2 t} \leqq 1 + \eta \tag{5.18}$$

for all sufficiently large t. Using (5.18), we see that the second term on the right side of (5.15) is majorized by $\exp\{c(1+\eta)\log_2 t\}/(\eta/2 \log t)$ for large t, which converges to 0 as $t \to \infty$ for η so small that $c(1 + \eta) < 1$. Hence, letting $t \to \infty$, then $c \to 1$ in (5.15), we have

$$\liminf_{t\to\infty} \frac{e^{tA}}{tA \log \frac{1}{A}\left(\log_2 \frac{1}{A}\right)^{1+\delta}} \geqq \varepsilon. \tag{5.19}$$

The reverse of inequality (5.13) now follows from (5.19) by an argument similar to that which led from (5.8) to (5.13). This completes the proof of (5.2).

6. Remarks

REMARK 6.1. Extremal processes in continuous time have been studied by Dwass [4] and Lamperti [5]. Lamperti proved an invariance theorem which is helpful in the proof of (6.2) below.

REMARK 6.2. Theorem 2(i) of [11] states that if g is a positive continuous function such that $g(t)/t^{1/2}$ is ultimately increasing and (1.2) converges, then for each $\tau > 0$,

$$\begin{aligned} &P\{w(t) \geqq g(t) \text{ for some } t \geqq \tau\} \\ &\qquad = \lim_{m\to\infty} P\{S_n \geqq m^{1/2} g(n/m) \text{ for some } n \geqq \tau m\}, \end{aligned} \tag{6.1}$$

where $S_n = x_1 + \cdots + x_n$, and $x_1, x_2, \cdots$ is any sequence of i.i.d. random variables having $Ex_1 = 0$, $Ex_1^2 = 1$. An analogous limit theorem for minima of uniform random variables is that if g is continuous and decreasing on some interval $(\tau_0, \infty)([\tau_0, \infty))$ and $\equiv \infty$ on $[0, \tau_0]([0, \tau_0))$, and if (2.10) converges with $c_n = ng(n)$, then for each $\tau > 0$,

$$\begin{aligned} &P\{v_t \geqq g(t) \text{ for some } t \geqq \tau\} \\ &\qquad = \lim_{m\to\infty} P\left\{V_n \geqq \frac{1}{m} g\left(\frac{n}{m}\right) \text{ for some } n \geqq \tau m\right\}. \end{aligned} \tag{6.2}$$

(As in Theorem 2(ii) of [11] a similar result holds if in (6.2) we replace $t \geqq \tau$ by $t > 0$ and $n \geqq \tau m$ by $n \geqq 1$.) The proof is similar in spirit to the proof of Theorem 2 of [11], but the details are much simpler.

REMARK 6.3. Using the probability integral transform, one can immediately obtain analogous results for random variables having arbitrary continuous distributions. (See Remark 2.5.) For example, if $x_1, x_2, \cdots$ are i.i.d. with $P\{x_i \geqq x\} = e^{-x}(x > 0)$ and if $g = e^{-h}$ satisfies the conditions of Remark 6.2 above, then the left side of (6.2) equals

$$\lim_{m\to\infty} P\left\{\max_{1\leqq k\leqq n} x_k \leqq \log m + h\left(\frac{n}{m}\right) \quad \text{for some } n \geqq \tau m\right\}. \tag{6.3}$$

If the x are standard normal random variables, the left side of (6.2) equals

$$\lim_{m\to\infty} P\left\{\max_{1\leqq k\leqq n} x_k \leqq 2^{1/2}\left[\log m + h\left(\frac{n}{m}\right) - \tfrac{1}{2}\log\left(\log m + h\left(\frac{n}{m}\right)\right) - \log 2\sqrt{\pi}\right]^{1/2} \quad \text{for some } n \geqq \tau m\right\}. \tag{6.4}$$

REMARK 6.4. It is possible to give a proof of Theorem 3 which is in the spirit of Motoo's proof [6] of the law of the iterated logarithm for the Wiener process. In fact, it may be shown that $\{x_t \equiv e^t v_{e^t}, 0 \leqq t < \infty\}$ is a positive recurrent Markov process, and since the sample paths of this process are continuous and increasing except for jumps in the negative direction, Motoo's method (as sketched in Section 3) applies with minor changes. To complete the argument it is necessary to compute (at least asymptotically as $a \to \infty$)

$$P\{x(T) \geqq a \mid x(0) = 1\}, \tag{6.5}$$

where $T = \inf\{t: x(t) \notin [1, a)\}$. Since the generator A of the process $x(t)$ is given by

$$Af(x) = xf'(x) + \int_0^x (f(u) - f(x))\, du \tag{6.6}$$

and since $p_a(x) \equiv P\{x(T) \geqq a \mid x(0) = x\}$ satisfies $Ap_a(x) = 0, 1 < x < a$, subject to $p_a(a) = 1$ and $p_a(x) = 0, 0 < x < 1$, it may be shown that

$$p_a(x) = \frac{e + \int_1^x \frac{e^u}{u}\, du}{e + \int_1^a \frac{e^u}{u}\, du}, \qquad 1 \leqq x \leqq a, \tag{6.7}$$

and hence $p_a(1) \sim ae^{-a}$ as $a \to \infty$.

Extensions of this approach are being investigated by Mr. J. Frankel.

REMARK 6.5. Minor changes in the method of proof of Theorem 1 applied to $\max_{0\leqq s\leqq t}|w(s)|$ yield Chung's law of the iterated logarithm [1]; to wit: if ultimately $c(t)\uparrow$ and $c(t)/t\downarrow$, then

$$P\left\{\max_{0\leqq s\leqq t}|w(s)| \leqq \left(\frac{t}{c(t)}\right)^{1/2} \text{ for arbitrarily large } t\right\} = 0 \text{ or } 1, \tag{6.8}$$

according as

$$\int^{\infty} \frac{c(t)}{t} \exp\left\{-\frac{\pi^2}{8} c(t)\right\} dt < \infty \text{ or } = \infty. \tag{6.9}$$

The required computations are virtually identical with those of Lemmas 2 through 7 in light of the observation that for $0 < \tau < t$, $0 < y < x$,

$$P\{\max_{0 \leqq s \leqq t} |w(s)| \leqq x \mid \max_{0 \leqq s \leqq \tau} |w(s)| \leqq y\} \leqq P\{\max_{0 \leqq s \leqq t-\tau} |w(s)| \leqq x\}. \tag{6.10}$$

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