THE RANGE OF RANDOM WALK

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1. Introduction

Let $\{X_n, n \geqq 1\}$ be a sequence of independent identically distributed random variables, defined on a probability space $(\Omega, \mathscr{F}, P)$, which take values in the d dimensional integer lattice E_d. The sequence $\{S_n, n \geqq 0\}$ defined by $S_0 = 0$ and $S_n = \Sigma_{k=1}^n X_k$ is called a random walk. The range of the random walk, denoted by R_n, is the cardinality of the set $\{S_0, S_1, \cdots, S_n\}$; it is the number of distinct points visited by the random walk up to time n. Our object here is to study the asymptotic behavior of R_n. Two specific problems are considered:

(i) Does $R_n/ER_n \to 1$ a.s.? If so, this will be called the strong law for R_n.

(ii) Does $(R_n - ER_n)(\text{Var } R_n)^{-1/2}$ converge in distribution? If so, this will be called the central limit theorem for R_n.

The random walk may take place on a proper subgroup of E_d. In this case, the subgroup is isomorphic to some E_k for $k \leqq d$; if $k < d$, then the transformation should be made and the problem considered in k dimensions. We will assume throughout the paper that this reduction has been made, if necessary, and that d is the genuine dimension of the random walk.

Dvoretzky and Erdös [2] proved the strong law for the range of simple random walk for $d \geqq 2$. (Simple random walk is one for which the distribution of X_1 assigns probability $(2d)^{-1}$ to each of the $2d$ neighbors of the origin.) Their method was to obtain a somewhat crude estimate of Var R_n and then use the Chebyshev inequality. While this worked fairly easily for $d \geqq 3$, they had to work much harder for $d = 2$. By a rather sophisticated technique, they managed to improve the required probability estimate enough to obtain the proof.

Let $p = P[S_1 \neq 0, S_2 \neq 0, \cdots]$. The random walk is called transient if $p > 0$ and recurrent otherwise. Using a very elegant technique Kesten, Spitzer, and Whitman ([12], p. 38) proved that for *all* random walks $R_n/n \to p$ a.s. For transient random walks $ER_n \sim pn$, so that their result includes the strong law for all transient random walks.

There are recurrent random walks only if the dimension is one or two. In [7], we attempted to prove the strong law for R_n for the general recurrent random walk in two dimensions, but we succeeded only partially. Our method there was to imitate the proof of Dvoretzky and Erdös, that is, to obtain an estimate for Var R_n and then to improve the probability estimate by their methods. In Section 3, we will prove the strong law for R_n for *all* two dimensional recurrent random walks by an essentially different technique. We use a very delicate

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method for estimating Var R_n, and the estimate is good enough so that once it is available the strong law follows in a fairly straightforward manner. In Section 4, we will show that if $EX_1 = 0$, $E|X_1|^2 < \infty$, and $d = 2$, then Var $R_n \sim cn^2/\log^4 n$ for some positive constant c. This should be compared with the Dvoretzky-Erdös bound which was $O(n^2 \log \log n/\log^3 n)$.

The second problem was first considered by Jain and Orey [6] who showed that if the random walk is strongly transient with $p < 1$, then Var $R_n \sim cn$ and the central limit theorem applies with the limit law normal. (The random walk is strongly transient if $\Sigma_{n=1}^{\infty} \Sigma_{j=n}^{\infty} P[S_j = 0] < \infty$.) Note that the case $p = 1$ is not interesting, since then $R_n = n + 1$ a.s. Thus, criteria for strong transience are of interest, since they will also imply the central limit theorem. All random walks with $d \geqq 5$ are strongly transient and if $EX_1 = 0$ and $E|X_1|^2 < \infty$, the random walk is strongly transient if and only if $d \geqq 5$. In Section 5, we give two additional sufficient conditions for strong transience. The first is that the random walk is aperiodic and not irreducible and the second is that $EX_1 \neq 0$ and $E|X_1|^2 < \infty$. These are both valid regardless of the dimension.

In [8], we considered the central limit theorem for random walks which were transient but not strongly transient. We proved that if $d = 4$, then Var $R_n \sim cn$ and the central limit theorem applies with a normal limit law. The growth of the variance should be compared with the Dvoretzky-Erdös bound which was $O(n \log n)$. For $d = 3$, we proved that Var $R_n = O(n \log n)$ in general. Under the additional assumption that $EX_1 = 0$ and $E|X_1|^2 < \infty$, we proved that Var $R_n \sim cn \log n$ and that the central limit theorem still is valid with a normal limit law. The Dvoretzky-Erdös bound in three dimensions was $O(n^{3/2})$.

The case of one dimension is rather unique. If Var $X_1 < \infty$, the random walk is either strongly transient if $EX_1 \neq 0$, or recurrent if $EX_1 = 0$. In the first case, the strong law and central limit theorem are both known for R_n as we have mentioned above. However, when $EX_1 = 0$ the situation is quite different and we shall prove in Section 6 that $n^{-1/2}R_n$ converges in distribution to a proper law. This implies that it is impossible to have a strong law, for $(R_n - \alpha_n)/\beta_n$ cannot converge even in probability to a nonzero constant for any sequences $\{\alpha_n\}$, $\{\beta_n\}$ except in the trivial case that R_n/β_n already converges to zero in probability.

It would be interesting to know whether the central limit theorem is valid for R_n when $d = 2$, at least when $EX_1 = 0$ and $E|X_1|^2 < \infty$, in which case the behavior of Var R_n is known. Another question is whether there is a limit law in general for R_n when $d = 3$. It is known that there is a limit law when $EX_1 = 0$ and $E|X_1|^2 < \infty$, or when the random walk is strongly transient. We have also said nothing about random walk on the line with Var $X_1 = \infty$. Of course, it is known that the strong law holds if the random walk is transient and the central limit theorem holds if it is strongly transient but it would be nice to have more information than this. One can also ask more delicate questions about the growth of R_n in general, that is, to obtain better upper and lower envelopes for R_n than are given by the strong law. Some results of this type are mentioned in [2].

2. Preliminaries

It will be convenient to think of the random walk as a Markov chain and we will use some of the terminology of general Markov chains. For $x \in E_d$, the random walk starting at x will refer to the random walk with $S_0 = x$ and $S_n = x + \Sigma_{k=1}^{n} X_k$. The notation $P_x[\cdot]$ will be used to denote probabilities of events related to this random walk; when $x = 0$, we will simply use $P[\cdot]$. Thus, for $n \geqq 0$ and $x, y \in E_d$, we let

$$P^n(x, y) = P_x[S_n = y] = P[S_n = y - x], \tag{2.1}$$

and note that $P^n(x, y) = P^n(0, y - x)$. For transient random walk the Green function is defined by $G(x, y) = \Sigma_{k=0}^{\infty} P^k(x, y)$. For an arbitrary set H of lattice points, T_H will denote the first hitting time of H, that is,

$$T_H = \min \{k \geqq 1 : S_k \in H\}; \tag{2.2}$$

if there are no positive integers k with $S_k \in H$, then $T_H = \infty$. If H consists of a single point x, we will write T_x instead of $T_{\{x\}}$. The taboo probabilities are defined by

$$P_H^n(x, y) = P_x[S_n = y, T_H \geqq n] \tag{2.3}$$

for $n \geqq 1$. We will use u_n for $P^n(0, 0)$, f_n for $P_0^n(0, 0)$, and

$$r_n = P[T_0 > n] = p + \sum_{k=n+1}^{\infty} f_k. \tag{2.4}$$

Another equation which is satisfied is $\Sigma_{k=0}^{n} u_k r_{n-k} = 1$; since r_n is monotone, it follows that

$$r_n \leqq \left(\sum_{k=0}^{n} u_k \right)^{-1}. \tag{2.5}$$

Kesten and Spitzer [10] proved that for any two dimensional random walk, r_n is slowly varying and this will be quite useful. We will need the following simple observation about slowly varying functions that decrease.

LEMMA 2.1. *Let $\{\ell_n\}$ be slowly varying and decreasing. Then there is a positive constant c such that if $j \leqq n$, then $j\ell_j \leqq cn\ell_n$. In particular, this implies for any two dimensional random walk there is a c such that $jr_j^4 \leqq cnr_n^4$ for $j \leqq n$.*

PROOF. Since ℓ_n is slowly varying, there exists an integer N such that for $n \geqq N$, $\ell_{2^n}/\ell_{2^{n+1}} < 2$. Let $2^N \leqq j \leqq n$; then there are integers β, γ with $\beta \leqq \gamma$ such that $2^\beta \leqq j < 2^{\beta+1}$ and $2^\gamma \leqq n < 2^{\gamma+1}$. Since ℓ_n is decreasing,

$$j\ell_j \leqq 2^{\beta+1}\ell_{2^\beta} \leqq 2 \cdot 2^{\gamma+1}\ell_{2^{\gamma+1}} \leqq 4n\ell_n. \tag{2.6}$$

To cover the cases with $j < 2^N$, we replace the 4 by a possibly larger constant c.

For any random walk in two dimensions with $p < 1$, there is a positive constant A such that

$$P^n(0, x) \leqq An^{-1} \tag{2.7}$$

for all $x \in E_2$ and $n \geqq 1$. This is a standard estimate; it is proved under these conditions in [8]. Another standard result which we shall use is that for any $n \geqq 1$ and $x \in E_d$,

$$P_0^n(0, x) = P_x^n(0, x). \tag{2.8}$$

This is proved by considering the dual or reversed random walk.

Lemma 2.2 is proved in [8] for $\gamma = 0, 1$. Although the general proof is the same, we shall give it since it is short and we will need some of the intermediate steps in Section 4.

LEMMA 2.2. *For* $\gamma \geqq 0$,

$$\sum_{k=1}^{m} P_0^k(0, x) r_{m-k}^{\gamma} \leqq \sum_{k=1}^{m} P^k(0, x) r_{m-k}^{\gamma+1}. \tag{2.9}$$

Equality holds for $\gamma = 0$.

PROOF. By considering the first return to zero,

$$P^k(0, x) = P_0^k(0, x) + \sum_{j=1}^{k-1} f_j P^{k-j}(0, x), \tag{2.10}$$

so that

$$\sum_{k=1}^{m} P_0^k(0, x) r_{m-k}^{\gamma} = \sum_{k=1}^{m} P^k(0, x) \left[r_{m-k}^{\gamma} - \sum_{j=1}^{m-k} f_j r_{m-k-j}^{\gamma} \right]. \tag{2.11}$$

The proof is completed by observing that

$$r_{m-k}^{\gamma} - \sum_{j=1}^{m-k} f_j r_{m-k-j}^{\gamma} = r_{m-k}^{\gamma+1} + \sum_{j=1}^{m-k} f_j [r_{m-k}^{\gamma} - r_{m-k-j}^{\gamma}] \tag{2.12}$$

and then using the monotonicity of r_n.

3. The strong law for R_n in the plane

The result that we will prove in this section is

THEOREM 3.1. *For any recurrent random walk in two dimensions,* $R_n/ER_n \to 1$ a.s. *as* $n \to \infty$.

The main part of the proof is to find an estimate of the form $\operatorname{Var} R_n = O(\varphi(n))$, where φ is a function with the property that for every given $\alpha > 1$ there is a sequence of positive integers $\{n_k\}$ such that $n_{k+1}/n_k \to \alpha$ and $\Sigma_{k=1}^{\infty} \varphi(n_k) n_k^{-2} r_{n_k}^{-2} < \infty$. Once we have such an estimate for the variance the proof of Theorem 3.1 can be finished as in [2] by the following argument. Since $ER_n \sim n r_n$, we have by Chebyshev's inequality, for every $\varepsilon > 0$,

$$P[|R_n - ER_n| \geqq \varepsilon ER_n] = O(\varphi(n) n^{-2} r_n^{-2}). \tag{3.1}$$

Let $\alpha > 1$ and $\{n_k\}$ be as above. Then by the Borel-Cantelli lemma, $R_n/ER_n \to 1$ a.s. along this subsequence. To fill in between, by the monotonicity of R_n, for $n_k \leqq n < n_{k+1}$,

$$(3.2)\qquad \frac{R_{n_k}}{ER_{n_k}}\frac{ER_{n_k}}{ER_{n_{k+1}}} \leqq \frac{R_n}{ER_n} \leqq \frac{R_{n_{k+1}}}{ER_{n_{k+1}}}\frac{ER_{n_{k+1}}}{ER_{n_k}},$$

and we know that the lower and upper bounds converge to α^{-1} and α, respectively, since $ER_n \sim nr_n$ and r_n is slowly varying. But α can be chosen as close to 1 as we please, so this is sufficient.

The remainder of this section will be devoted to obtaining the desired estimate for Var R_n.

THEOREM 3.2. *For any recurrent random walk in two dimensions*, Var $R_n = O(\varphi(8n))$ where

$$(3.3)\qquad \varphi(n) = n + nr_n^4 \sum_{k=1}^{n} ku_k \log\left(\frac{n}{k}\right).$$

REMARK 3.1. Since ku_k is bounded, it follows from the theorem that Var $R_n = O(n^2 r_n^4)$. This is a very good bound for the case of simple random walk, but it is very poor in general.

PROOF. Let $Z_0 = 1$ and for $k \geqq 1$ let

$$(3.4)\qquad Z_k = I[S_k \neq S_{k-1}, \cdots, S_k \neq S_0].$$

Then $R_n = \Sigma_{k=0}^n Z_k$ and so

$$(3.5)\qquad \operatorname{Var} R_n = \sum_{k=1}^{n} \operatorname{Var} Z_k + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \operatorname{Cov}(Z_i, Z_j).$$

The first sum is $O(n)$, so we will concentrate on the second one. By reversing the time parameter, it is easy to see that $EZ_i = r_i$ and that for $i < j$

$$(3.6)\qquad EZ_iZ_j = \sum_{x\neq 0} P_0^{j-i}(0, x)P_x[T_x > i, T_0 > i].$$

Hence, we can write

$$(3.7)\qquad \operatorname{Cov}(Z_i, Z_j) = EZ_iZ_j - r_ir_j = \sum_{x\neq 0} P_0^{j-i}(0, x)b_i(x),$$

where

$$(3.8)\qquad \begin{aligned} b_i(x) &= P_x[T_x > i, T_0 > i] - P_x[T_x > i]P_x[T_0 > i] \\ &= P_x[T_x \leqq i, T_0 \leqq i] - P_x[T_x \leqq i]P_x[T_0 \leqq i]. \end{aligned}$$

Using the last expression for $b_i(x)$, we will find a more useful form for it. We have

$$(3.9)\qquad \begin{aligned} P_x[T_x < T_0 \leqq i] &= \sum_{\nu=1}^{i} P_x[\nu = T_x < T_0 \leqq i] \\ &= \sum_{\nu=1}^{i}\sum_{\xi=1}^{i-\nu} P_{x0}^{\nu}(x, x)P_0^{\xi}(x, 0) \\ &= \sum_{\xi=1}^{i}\sum_{\nu=1}^{i-\xi} P_{x0}^{\nu}(x, x)P_0^{\xi}(x, 0). \end{aligned}$$

Similarly,

$$P_x[T_0 < T_x \leqq i] = \sum_{\nu=1}^{i} \sum_{\xi=1}^{i-\nu} P_{x0}^{\xi}(x, 0) P_x^{\nu}(0, x). \tag{3.10}$$

Considering the first visit to x,

$$\begin{aligned} \sum_{\xi=1}^{i-\nu} P_0^{\xi}(x, 0) &= \sum_{\xi=1}^{i-\nu} P_{x0}^{\xi}(x, 0) + \sum_{\xi=1}^{i-\nu} \sum_{\eta=1}^{\xi-1} P_{x0}^{\eta}(x, x) P_0^{\xi-\eta}(x, 0) \\ &= \sum_{\xi=1}^{i-\nu} P_{x0}^{\xi}(x, 0) + \sum_{\beta=1}^{i-\nu} \sum_{\eta=1}^{i-\nu-\beta} P_{x0}^{\eta}(x, x) P_0^{\beta}(x, 0). \end{aligned} \tag{3.11}$$

Substituting in (3.10) for $\Sigma_{\xi=1}^{i-\nu} P_{x0}^{\xi}(x, 0)$ from this expression, we get

$$P_x[T_0 < T_x \leqq i] = \sum_{\nu=1}^{i} \sum_{\xi=1}^{i-\nu} P_x^{\nu}(0, x) P_0^{\xi}(x, 0) \left[1 - \sum_{\eta=1}^{i-\nu-\xi} P_{x0}^{\eta}(x, x) \right]. \tag{3.12}$$

Changing the order of summation and combining with (3.9), we have

$$\begin{aligned} &P_x[T_x \leqq i, T_0 \leqq i] \\ &\quad = \sum_{\xi=1}^{i} \sum_{\nu=1}^{i-\xi} P_0^{\xi}(x, 0) \left[P_{x0}^{\nu}(x, x) + P_x^{\nu}(0, x) \left\{ 1 - \sum_{\eta=1}^{i-\nu-\xi} P_{x0}^{\eta}(x, x) \right\} \right]. \end{aligned} \tag{3.13}$$

By considering the first visit to zero,

$$\sum_{\nu=1}^{i-\xi} P_x^{\nu}(x, x) = \sum_{\nu=1}^{i-\xi} P_{x0}^{\nu}(x, x) + \sum_{\beta=1}^{i-\xi} \sum_{\eta=1}^{i-\xi-\beta} P_{x0}^{\eta}(x, 0) P_x^{\beta}(0, x). \tag{3.14}$$

We substitute the expression this gives for $\Sigma_{\nu=1}^{i-\xi} P_{x0}^{\nu}(x, x)$ in the last expression. Since for $x \neq 0$,

$$\begin{aligned} 1 - \sum_{\eta=1}^{i-\nu-\xi} \{P_{x0}^{\eta}(x, x) + P_{x0}^{\eta}(x, 0)\} \\ = P_x[T_x > i - \nu - \xi, T_0 > i - \nu - \xi], \end{aligned} \tag{3.15}$$

we have

$$\begin{aligned} &P_x[T_x \leqq i, T_0 \leqq i] \\ &\quad = \sum_{\xi=1}^{i} \sum_{\nu=1}^{i-\xi} P_0^{\xi}(x, 0) \{ f_\nu + P_x^{\nu}(0, x) P_x[T_x > i - \nu - \xi, T_0 > i - \nu - \xi] \}. \end{aligned} \tag{3.16}$$

But since

$$P_x[T_0 \leqq i] P_x[T_x \leqq i] = \sum_{\xi=1}^{i} P_0^{\xi}(x, 0) \sum_{\nu=1}^{i} f_\nu, \tag{3.17}$$

we obtain the cancellation we need in the expression for $b_i(x)$:

$$(3.18) \qquad b_i(x) = \sum_{\xi=1}^{i} P_0^{\xi}(x, 0) \left\{ - \sum_{\nu=i-\xi+1}^{i} f_\nu + \sum_{\nu=1}^{i-\xi} P_x^{\nu}(0, x) P_x[T_x > i - \nu - \xi, T_0 > i - \nu - \xi] \right\}.$$

At this point, we can afford to be somewhat crude and use the bound

$$(3.19) \qquad b_i(x) \leqq \sum_{\xi=1}^{i} \sum_{\nu=1}^{i-\xi} P_0^{\xi}(x, 0) P_x^{\nu}(0, x) r_{i-\nu-\xi}.$$

Using (2.8) and Lemma 2.2, we obtain

$$(3.20) \qquad b_i(x) \leqq \sum_{\xi=1}^{i} \sum_{\nu=1}^{i-\xi} P^{\xi}(x, 0) P^{\nu}(0, x) r_{i-\nu-\xi}^3.$$

Letting $\lambda = j - i$ in (3.7), we see that

$$(3.21) \qquad \begin{aligned} C_n &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathrm{Cov}\,(Z_i, Z_j) \\ &\leqq \sum_{i=1}^{n} \sum_{\lambda=1}^{n-i} \sum_{\xi=1}^{i} \sum_{\nu=1}^{i-\xi} \sum_{x \neq 0} P_0^{\lambda}(0, x) P^{\xi}(x, 0) P^{\nu}(0, x) r_{i-\nu-\xi}^3 \\ &= \sum_{i=1}^{n} \sum_{\lambda=1}^{n-i} \sum_{\xi=1}^{i} \sum_{\nu=1}^{i-\xi} \sum_{x \neq 0} P^{\lambda}(0, x) P^{\xi}(x, 0) P^{\nu}(0, x) r_{i-\nu-\xi}^3 r_{n-i-\lambda}, \end{aligned}$$

where Lemma 2.2 has again been used at the last step. We want to sum first on i; the relevant part is

$$(3.22) \qquad \sum_{i=\xi+\nu}^{n-\lambda} r_{i-\nu-\xi}^3 r_{n-i-\lambda}.$$

This is a convolution and since r_n is slowly varying and decreasing this is dominated by a constant times $(n - \lambda - \xi - \nu) r_{n-\lambda-\xi-\nu}^4$, which in turn is dominated by cnr_n^4 by Lemma 2.1. Thus, we have $C_n \leqq cnr_n^4 D_n$, say, where

$$(3.23) \qquad D_n = \sum_{\lambda=1}^{n} \sum_{\xi=1}^{n-\lambda} \sum_{\nu=1}^{n-\lambda-\xi} \sum_{x \neq 0} P^{\lambda}(0, x) P^{\xi}(x, 0) P^{\nu}(0, x).$$

Since this expression is symmetric in λ and ν, we need only consider $\nu \geqq \lambda$. Using the uniform estimate (2.7) on $P^{\nu}(0, x)$, we see that

$$(3.24)\qquad \begin{aligned} D_n &\leqq 2A \sum_{\xi=1}^{n} \sum_{\lambda=1}^{[(n-\xi)/2]} \sum_{\nu=\lambda}^{n-\xi-\lambda} \sum_{x\neq 0} P^{\lambda}(0, x)P^{\xi}(x, 0)\nu^{-1} \\ &\leqq 2A \sum_{\xi=1}^{n} \sum_{\lambda=1}^{[(n-\xi)/2]} u_{\lambda+\xi}[\log e(n - \xi - \lambda) - \log \lambda] \\ &\leqq 2A \sum_{\xi=1}^{n} \sum_{\lambda=1}^{n-\xi} u_{\lambda+\xi}[\log en - \log \lambda] \\ &\leqq 2A \sum_{\beta=1}^{n} \sum_{\lambda=1}^{\beta} u_{\beta}[\log en - \log \lambda]. \end{aligned}$$

Next we use the inequality

$$(3.25)\qquad \sum_{\lambda=1}^{\beta} \log \lambda \geqq \int_0^{\beta} \log y \, dy = \beta \log\left(\frac{\beta}{e}\right)$$

to obtain

$$(3.26)\qquad D_n \leqq 2A \sum_{\beta=1}^{n} \beta u_{\beta} \log\left(\frac{e^2 n}{\beta}\right) \leqq 2A \sum_{\beta=1}^{8n} \beta u_{\beta} \log\left(\frac{8n}{\beta}\right).$$

Recalling that $C_n \leqq cnr_n^4 D_n$, this completes the proof of the theorem.

In order to complete the proof of Theorem 3.1, we must show that φ has the desired summability property. To do this, let $\psi(n) = \varphi(n)n^{-2}r_n^{-2}$, $n'_k = [\alpha^k]$ for $k \geqq 0$, and $n_k = 8n'_k$. We need only show that $\Sigma_k \psi(n_k) < \infty$. We note first that since $n^{-1}r_n^{-2} \leqq n^{-1/2}$ for large n, this is trivial for the contribution of the term n to $\varphi(n)$, so we need only consider the second term in $\varphi(n)$. From this point on the term n in $\varphi(n)$ will be ignored. Define an increasing sequence of integers by

$$(3.27)\qquad m_j = \min\{k : u_0 + \cdots + u_k \geqq j\}$$

for $j \geqq 1$. Since $\Sigma_{k=0}^{\infty} u_k$ diverges and $u_k \to 0$, we have that the m_j are defined for all j and

$$(3.28)\qquad \sum_{k=m_j+1}^{m_{j+1}} u_k \to 1 \quad \text{as} \quad j \to \infty.$$

Define

$$(3.29)\qquad v_j = \sum_{k=n_j+1}^{n_{j+1}} u_k;$$

by (2.7),

$$(3.30)\qquad v_j \leqq An_j^{-1}(n_{j+1} - n_j) \sim A(\alpha - 1),$$

so that the v_j are bounded. For $m_j < n_i \leqq m_{j+1}$, we have by (2.5),

$$\psi(n_i) \leqq n_i^{-1} j^{-2} \sum_{k=1}^{n_i} k u_k \log\left(\frac{n_i}{k}\right) \tag{3.31}$$

$$= n_i^{-1} j^{-2} \sum_{\beta=0}^{i-1} \sum_{k=n_\beta+1}^{n_{\beta+1}} k u_k \log\left(\frac{n_i}{k}\right) + O(n_i^{-1}\log n_i)$$

$$\leqq j^{-2} \sum_{\beta=0}^{i-1} \sum_{k=n_\beta+1}^{n_{\beta+1}} \left(\frac{n_{\beta+1}}{n_i}\right) u_k \log\left(\frac{n_i}{n_\beta}\right) + O(in_i^{-1})$$

$$\leqq cj^{-2} \sum_{\beta=0}^{i-1} \left(\frac{n_{\beta+1}}{n_i}\right) v_\beta(i-\beta) + O(in_i^{-1}),$$

where the error term is to take care of the sum for $k = 1, \cdots, 8$. Since this error term is summable, we will drop it. Now let

$$I_j = \{i : m_j < n_i \leqq m_{j+1}\}. \tag{3.32}$$

Then

$$\sum_{i \in I_j} \psi(n_i) \leqq cj^{-2} \sum_{i \in I_j} \sum_{\beta=0}^{i-1} \left(\frac{n_{\beta+1}}{n_i}\right) v_\beta(i-\beta) \tag{3.33}$$

and so it will suffice to show that the double sum is bounded uniformly in j. We change the order of summation and write ($\beta < I_j$ means $\beta \in \{i : n_i \leqq m_j\}$)

$$\sum_{\beta < I_j} \sum_{i \in I_j} \left(\frac{n_{\beta+1}}{n_i}\right) v_\beta(i-\beta) + \sum_{\beta \in I_j} \sum_{i \in I_j, i > \beta} \left(\frac{n_{\beta+1}}{n_i}\right) v_\beta(i-\beta). \tag{3.34}$$

Since v_β is bounded and $n_i = 8[\alpha^i]$, this is bounded by

$$c \sum_{i \in I_j} \sum_{\beta < I_j} \alpha^{\beta+1-i}(i-\beta) + c \sum_{\beta \in I_j} v_\beta. \tag{3.35}$$

By (3.28) and (3.30), the second sum is dominated for large j by

$$\sum_{k=m_j+1}^{m_{j+1}} u_k + A(\alpha - 1) \leqq 2 + A(\alpha - 1). \tag{3.36}$$

Hence, it remains to look at the double sum. This is of the form

$$\sum_{i=\lambda}^{\nu} \sum_{\beta=0}^{\lambda-1} (i-\beta)\alpha^{\beta+1-i} \leqq \sum_{k=1}^{\infty} k^2 \alpha^{1-k} < \infty, \tag{3.37}$$

independent of λ and ν and hence of j. This proves that φ has the desired summability property and therefore completes the proof of Theorem 3.1.

4. The variance of R_n in the plane

We start this section with a theorem which gives the asymptotic behavior of f_n for all strongly aperiodic random walks in two dimensions with $EX_1 = 0$

and $E|X_1|^2 < \infty$. The result will be used to establish the asymptotic behavior of Var R_n in this case, but it is also of some independent interest since it establishes for this special case Kesten's conjecture [11] that

$$\lim_{n\to\infty} f_n^{-1} \sum_{k=0}^{n} f_k f_{n-k} = 2 \tag{4.1}$$

for all strongly aperiodic recurrent random walks. In his paper, Kesten found the asymptotic behavior of f_n and thereby verified (4.1) for one dimensional recurrent random walks with X_1 in the domain of attraction of a symmetric stable law. Some of the other results of [11] are also valid for the two dimensional random walks we are considering since they follow from (4.1).

THEOREM 4.1. *For a strongly aperiodic, two dimensional random walk with* $EX_1 = 0$ *and* $E|X_1|^2 < \infty$,

$$f_n \sim \frac{c_1}{n \log^2 n}, \tag{4.2}$$

where $c_1 = 2\pi|Q|^{1/2}$ *and* Q *is the covariance matrix of* X_1.

REMARK 4.1. The asymptotic behavior of r_n and also of $\Sigma_{j=1}^{n} jf_j$ can be obtained directly from Tauberian theorems, but this method does not give information about f_n since this sequence is not known to be monotone.

PROOF. Let $\gamma = [n/\log^3 n]$ and $\beta = n - 2\gamma$, and write

$$f_n = \sum_{x\neq 0} \sum_{y\neq 0} P_0^{\gamma}(0, x)P_0^{\beta}(x, y)P_0^{\gamma}(y, 0). \tag{4.3}$$

First we shall show that the error made by neglecting the taboo on the middle factor is small. Consider

$$\begin{aligned} 0 \leqq P^{\beta}(x, y) - P_0^{\beta}(x, y) &= P_x[T_0 \leqq \beta, S_{\beta} = y] \\ &\leqq \sum_{i=1}^{[\beta/2]} P_0^i(x, 0)P^{\beta-i}(0, y) + \sum_{i=[\beta/2]}^{\beta-1} P^i(x, 0)P_0^{\beta-i}(0, y), \end{aligned} \tag{4.4}$$

where the terms are classified according to the first time zero is hit if that is prior to $\beta/2$, but the last time if it occurs after $\beta/2$. This distinction is important. The estimate $f_n = O(1/n \log^2 n)$ which follows from (4.3) with $\gamma = [n/3]$ will be needed below; this is in Kesten and Spitzer [10]. Then

$$\begin{aligned} &\Big|f_n - \sum_{x\neq 0} \sum_{y\neq 0} P_0^{\gamma}(0, x)P^{\beta}(x, y)P_0^{\gamma}(y, 0)\Big| \\ &\leqq \sum_{i=1}^{[\beta/2]} \sum_{x\neq 0} \sum_{y\neq 0} P_0^{\gamma}(0, x)P_0^{i}(x, 0)P^{\beta-i}(0, y)P_0^{\gamma}(y, 0) \\ &\qquad + \sum_{i=[\beta/2]}^{\beta-1} \sum_{x\neq 0} \sum_{y\neq 0} P_0^{\gamma}(0, x)P^{i}(x, 0)P_0^{\beta-i}(0, y)P_0^{\gamma}(y, 0) \end{aligned} \tag{4.5}$$

$$= \sum_{i=1}^{[\beta/2]} \sum_{y \neq 0} f_{\gamma+i} P^{\beta-i}(0, y) P_0^{\gamma}(y, 0) + \sum_{i=[\beta/2]}^{\beta-1} \sum_{x \neq 0} P_0^{\gamma}(0, x) P^{i}(x, 0) f_{\beta-i+\gamma}$$

$$\leqq A \sum_{i=1}^{[\beta/2]} f_{\gamma+i} (\beta - i)^{-1} r_\gamma + A \sum_{i=[\beta/2]}^{\beta-1} r_\gamma i^{-1} f_{\beta-i+\gamma}$$

$$= O\left(\beta^{-1} r_\gamma \sum_{i=1}^{[\beta/2]} f_{\gamma+i}\right) = O\left(\beta^{-1} r_\gamma (\log \gamma)^{-2} \sum_{i=1}^{\beta} (\gamma + i)^{-1}\right)$$

$$= O\left(\frac{\log \log n}{n \log^3 n}\right).$$

With c_1 as in the statement of the theorem, the local limit theorem ([12], p. 77) gives

$$\left| P^{\beta}(x, y) - \frac{1}{c_1 \beta} \right| \leqq \frac{\varepsilon}{\beta} + O(\beta^{-2} |y - x|^2) \tag{4.6}$$

uniformly in x, y for β sufficiently large. Now

$$\beta^{-2} \sum_{x \neq 0} \sum_{y \neq 0} P_0^{\gamma}(0, x) |y - x|^2 P_0^{\gamma}(y, 0) \tag{4.7}$$

$$\leqq \beta^{-2} \sum_{x \neq 0} P^{\gamma}(0, x) \{\gamma E|X_1|^2 + |x|^2\}$$

$$\leqq 2\gamma \beta^{-2} E|X_1|^2 = O\left(\frac{1}{n \log^3 n}\right)$$

The proof is now complete for

$$\sum_{x \neq 0} \sum_{y \neq 0} P_0^{\gamma}(0, x) \frac{1}{c_1 \beta} P_0^{\gamma}(y, 0) = \frac{1}{c_1 \beta} r_\gamma^2 \sim \frac{c_1}{n \log^2 n} \tag{4.8}$$

the last step being a consequence of the known limit $r_n \sim c_1/\log n$. (See Lemma 2.3 of [7].)

THEOREM 4.2. *For two dimensional random walk with* $EX_1 = 0$ *and* $E|X_1|^2 < \infty$,

$$\operatorname{Var} R_n \sim \frac{c_2 n^2}{\log^4 n}, \tag{4.9}$$

where $c_2 = 8\pi^2 K|Q|$, *the covariance matrix of* X_1 *is* Q, *and*

$$K = -\int_0^1 \frac{\log w}{1 - w + w^2} dw + \frac{1}{2} - \frac{\pi^2}{12} = 0\ 84948659 \cdots \tag{4.10}$$

Note, however, that if the random walk takes place on a proper subgroup of E_2, *then a transformation should be made so that it will take place on all of* E_2 *and* Q *should be the covariance matrix for this transformed random walk.*

PROOF. The first step is to make a transformation, if necessary, so that the random walk does not take place on a proper subgroup. This does not change R_n and the fact that Q may change has been allowed for in the statement of the theorem. We shall assume for now that the random walk is strongly aperiodic and show at the end how to remove this assumption. From (3.5) and (3.7), we have that

$$\text{Var } R_n = 2 \sum_{i=1}^{n} \sum_{\lambda=1}^{n-i} \sum_{x \neq 0} P_0^{\lambda}(0, x) b_i(x) + O(n). \tag{4.11}$$

The expression we will use for $b_i(x)$ is given in (3.18). The contribution to the variance from the negative part of $b_i(x)$ will be considered first. This part will be denoted $-V_n$ and the positive part U_n so that $\text{Var } R_n = U_n - V_n + O(n)$. Now

$$\begin{aligned} V_n &= 2 \sum_{i=1}^{n} \sum_{\lambda=1}^{n-i} \sum_{x \neq 0} \sum_{\xi=1}^{i} \sum_{\nu=i-\xi+1}^{i} P_0^{\lambda}(0, x) P_0^{\xi}(x, 0) f_\nu \\ &= 2 \sum_{i=1}^{n} \sum_{\lambda=1}^{n-i} \sum_{\xi=1}^{i} \sum_{\nu=i-\xi+1}^{i} f_{\lambda+\xi} f_\nu \\ &\sim 2c_1^2 \sum_{i=2}^{n} \sum_{\xi=2}^{i-2} \frac{\log (n - i + \xi/\xi)}{\log (n - i + \xi) \log \xi} \frac{\log (i/i - \xi)}{\log i \log (i - \xi)}, \end{aligned} \tag{4.12}$$

where Theorem 4.1 has been used at the last step. Since the summands are bounded by 1, the contribution for $\xi \leqq n/\log^5 n$ is $O(n^2/\log^5 n)$, while for $\xi > n/\log^5 n$, one can replace $\log \xi$ with $\log n$. The same method applies to the other log terms in the denominator by considering where i, $i - \xi$, and $n - i + \xi$ are respectively $\leqq n/\log^5 n$. Thus,

$$\begin{aligned} V_n &\sim 2c_1^2 (\log n)^{-4} \sum_{i=2}^{n} \sum_{\xi=2}^{i-2} \log \frac{n - i + \xi}{\xi} \log \frac{i}{i - \xi} \\ &\sim 2c_1^2 n^2 (\log n)^{-4} \int_0^1 \int_0^z \log \frac{1 - z + y}{y} \log \frac{z}{z - y} \, dy \, dz \\ &= 2c_1^2 n^2 (\log n)^{-4} \frac{1}{2}\left(\frac{\pi^2}{6} - 1\right), \end{aligned} \tag{4.13}$$

since the double integral can be evaluated. Now we must consider the positive contribution. In (3.18), write

$$\begin{aligned} &P_x[T_x > i - \nu - \xi, T_0 > i - \nu - \xi] \\ &\qquad = r_{i-\nu-\xi} - P_x[T_x > i - \nu - \xi, T_0 \leqq i - \nu - \xi]; \end{aligned} \tag{4.14}$$

the next step will be to show that using the last of these probabilities in (3.18) will lead to a term of smaller order. To obtain a bound, note that by thinking of η as the last time zero is hit prior to $i - \nu - \xi$,

$$P_x[T_x > i - \nu - \xi, T_0 \leqq i - \nu - \xi] \leqq \sum_{\eta=1}^{i-\nu-\xi} P_x^{\eta}(x, 0) r_{i-\nu-\xi-\eta}. \tag{4.15}$$

Substituting this bound in (3.18) and (4.11), we see that this contribution to the variance is of order

$$(4.16)\quad \sum_{i=1}^{n}\sum_{\lambda=1}^{n-i}\sum_{x\neq 0}\sum\sum_{\xi+\nu+\eta\leq i}\sum P_0^\lambda(0,x)P_0^\xi(x,0)P_x^\nu(0,x)P_x^\eta(x,0)r_{i-\nu-\xi-\eta}$$

$$\leqq \sum_{i=1}^{n}\sum_{\lambda=1}^{n-i}\sum_{x\neq 0}\sum\sum_{\xi+\nu+\eta\leq i}\sum P^\lambda(0,x)P^\xi(x,0)P^\nu(0,x)P^\eta(x,0)r_{n-i-\lambda}r_{i-\nu-\xi-\eta}^4$$

$$= O\left(nr_n^5\sum_{\lambda=1}^{n}\sum_{x\neq 0}\sum\sum_{\xi+\nu+\eta\leq n-\lambda}\sum P^\lambda(0,x)P^\xi(x,0)P^\nu(0,x)P^\eta(x,0)\right),$$

where Lemma 2.2 has been applied four times at the second step and the i sum has been moved inside and Lemma 2.1 used at the last step. We will now show that the multiple sum is of order n. Since it is symmetric in λ and ν, in ξ and η, a bound is

$$(4.17)\quad 4A^2\sum_{\lambda=1}^{n}\sum_{\xi=1}^{n}\sum_{x\neq 0}\sum_{\nu=\lambda}^{n}\sum_{\eta=\xi}^{n}P^\lambda(0,x)P^\xi(x,0)\nu^{-1}\eta^{-1}$$

$$\leqq 4A^2\sum_{\lambda=1}^{n}\sum_{\xi=1}^{n}u_{\lambda+\xi}\log\frac{en}{\lambda}\log\frac{en}{\xi}$$

$$\leqq 4A^3\sum_{k=1}^{2n}\sum_{\xi=1}^{k-1}\frac{1}{k}\log\frac{en}{k-\xi}\log\frac{en}{\xi}$$

$$= O\left(\sum_{k=1}^{2n}\left(\log\frac{e^2n}{k}\right)^2\right) = O(n).$$

Thus, we have for the positive contribution to the variance

$$(4.18)\quad U_n = 2\sum_{i=1}^{n}\sum_{\lambda=1}^{n-i}\sum_{x\neq 0}\sum_{\xi=1}^{i}\sum_{\nu=1}^{i-\xi}P_0^\lambda(0,x)P_0^\xi(x,0)P_x^\nu(0,x)r_{i-\nu-\xi} + O(n^2r_n^5).$$

We will now apply Lemma 2.2 three times to the main term here to obtain

$$(4.19)\quad 2\sum_{i=1}^{n}\sum_{\lambda=1}^{n-i}\sum_{x\neq 0}\sum_{\xi=1}^{i}\sum_{\nu=1}^{i-\xi}P^\lambda(0,x)P^\xi(x,0)P^\nu(0,x)r_{n-i-\lambda}r_{i-\nu-\xi}^3.$$

But since this is now the leading term, we need to examine the error made in the application of the lemma. To this end, note that for $\gamma \geqq 1$,

$$(4.20)\quad 0\leqq\sum_{j=1}^{m-k}f_j(r_{m-k-j}^\gamma - r_{m-k}^\gamma)\leqq\gamma\sum_{j=1}^{m-k}f_jr_{m-k-j}^{\gamma-1}(r_{m-k-j}-r_{m-k}).$$

Now, for $1\leqq j\leqq (m-k)/2$, by Theorem 4.1,

$$(4.21)\quad r_{m-k-j}-r_{m-k}=\sum_{\beta=m-k-j+1}^{m-k}f_\beta = O\left(\frac{j}{m-k}r_{m-k}^2\right),$$

and so

$$\sum_{j=1}^{m-k} f_j(r_{m-k-j}^{\gamma} - r_{m-k}^{\gamma}) \tag{4.22}$$
$$= O\left(\sum_{j=1}^{(m-k)/2} jf_j(m-k)^{-1}r_{m-k}^{\gamma+1} + \sum_{j=(m-k)/2}^{m-k} f_j r_{m-k-j}^{\gamma}\right)$$
$$= O(r_{m-k}^{\gamma+3} + r_{m-k}^{\gamma+2}) = O(r_{m-k}^{\gamma+2}).$$

Utilizing this in (2.12) and (2.11) yields

$$\sum_{k=1}^{m} P_0^k(0, x)r_{m-k}^{\gamma} = \sum_{k=1}^{m} P^k(0, x)[r_{m-k}^{\gamma+1} + O(r_{m-k}^{\gamma+2})]. \tag{4.23}$$

Thus, the error introduced by Lemma 2.2 will always lead to an extra factor of an r and this will ultimately give a term of order $n^2/\log^5 n$. We now return to the estimation of the main term (4.19). By summing first on i, we obtain

$$U_n = 2\sum_{k=3}^{n} a_k b_{n-k} + O\left(\frac{n^2}{\log^5 n}\right), \tag{4.24}$$

where

$$a_k = \sum\sum_{\xi+\nu+\lambda=k} \sum_{x\neq 0} P^{\lambda}(0, x)P^{\xi}(x, 0)P^{\nu}(0, x), \tag{4.25}$$

$$b_k = \sum_{j=0}^{k} r_j r_{k-j}^3 \sim kr_k^4. \tag{4.26}$$

We will prove next that $a_k \to a$; this will imply that

$$U_n \sim 2a\sum_{k=3}^{n} b_{n-k} \sim an^2 r_n^4 \sim \frac{ac_1^4 n^2}{\log^4 n}. \tag{4.27}$$

To show that $a_k \to a$, the local limit theorem ([12], pp. 77 and 79) will again be used. Write

$$P^{\beta}(0, x) = Q_{\beta}(x) + o(E_{\beta}(x)), \tag{4.28}$$

uniformly in x as $\beta \to \infty$, where

$$Q_{\beta}(x) = (2\pi\beta)^{-1}|Q|^{-1/2} \exp\left\{\frac{-x\cdot Q^{-1}x}{2\beta}\right\}, \tag{4.29}$$

$$E_{\beta}(x) = \min\{\beta^{-1}, |x|^{-2}\}. \tag{4.30}$$

Since it is clear that $Q_{\beta}(x) = O(E_{\beta}(x))$, the sum of all the error terms will go to zero if

$$\sum\sum_{\xi+\nu+\lambda=k} \sum_{x\neq 0} E_{\lambda}(x)E_{\xi}(x)E_{\nu}(x) \tag{4.31}$$

is bounded. Since this is symmetric in ξ, ν, λ, we may as well assume that $\xi \leqq \nu \leqq \lambda$. This means that $\lambda \geqq k/3$, and so (4.31) is bounded by a constant times

(4.32)
$$k^{-1}\sum_{\xi=1}^{k}\sum_{\nu=1}^{k}\sum_{|x|^2\leqq k}E_\xi(x)E_\nu(x)+k^2\sum_{|x|^2\geqq k}|x|^{-6}$$
$$\leqq k^{-1}\sum_{|x|^2\leqq k}\left(\log\frac{e^2k}{|x|^2}\right)^2+O(1)$$
$$=O\left[k^{-1}\sum_{j=1}^{k^{1/2}}j\left(2\log\frac{ek^{1/2}}{j}\right)^2\right]=O(1).$$

Now we must examine the leading terms. Letting

(4.33)
$$\mu=\xi^{-1}+\nu^{-1}+(k-\xi-\nu)^{-1},\qquad y=x\mu^{1/2},$$

we can write

(4.34)
$$a_k=\sum\sum_{\xi+\nu+\lambda=k}\sum_{x\neq 0}Q_\lambda(x)Q_\xi(x)Q_\nu(x)+o(1)$$
$$=(2\pi)^{-2}|Q|^{-1}\sum_{\xi=1}^{k-2}\sum_{\nu=1}^{k-\xi-1}\sum_y\{\xi\nu(k-\xi-\nu)\}^{-1}Q_1(y)+o(1)$$
$$\sim c_1^{-2}\sum_{\xi=1}^{k-2}\sum_{\nu=1}^{k-\xi-1}\{\xi\nu(k-\xi-\nu)\mu\}^{-1}\sim 2c_1^{-2}K_1,$$

where (see [5], p. 533)

(4.35)
$$K_1=\frac{1}{2}\int_0^1\int_0^{1-z}\frac{1}{z-z^2+y-y^2-yz}\,dydz=-\int_0^1\frac{\log w}{1-w+w^2}\,dw$$
$$=1.17195361935\cdots.$$

Recalling (4.13) and (4.27),

(4.36)
$$\operatorname{Var}R_n\sim 2c_1^2\left(K_1+\frac{1}{2}-\frac{\pi^2}{12}\right)n^2(\log n)^{-4}.$$

Thus, the proof of the theorem is complete for the case when the random walk is strongly aperiodic. To make the transition to the general case, let S_n denote the original random walk which has been made aperiodic by a transformation and let $P(x, y)$ denote its transition function. Then the random walk S_n' with transition function

(4.37)
$$P'(x,y)=\tfrac{1}{2}\delta(x,y)+\tfrac{1}{2}P(x,y)$$

will be strongly aperiodic. One can describe the paths of the new random walk by flipping a coin and each time remaining stationary if the coin falls heads and moving according to the original transition function if it falls tails. The range of the S_n' random walk will be denoted by R_n'. By conditioning on the sequence of heads and tails, it is clear that

(4.38)
$$ER_n'=\sum_{k=0}^{n}\binom{n}{k}2^{-n}ER_k,\qquad E(R_n')^2=\sum_{k=0}^{n}\binom{n}{k}2^{-n}ER_k^2.$$

For even n,

$$(4.39)\qquad |ER_{n/2}^2 - ER_k^2| = \Big|\sum_{0 \leqq i,j \leqq n/2}\sum EZ_iZ_j - \sum_{0 \leqq i,j \leqq k}\sum EZ_iZ_j\Big| \leqq n|n - 2k|.$$

Thus, by (4.38),

$$(4.40)\qquad |E(R_n')^2 - ER_{n/2}^2| \leqq \sum_{|n-2k| \leqq n^{2/3}} \binom{n}{k} 2^{-n} n|n - 2k| + \sum_{|n-2k| > n^{2/3}} \binom{n}{k} 2^{-n} |ER_k^2 - ER_{n/2}^2| \leqq n^{5/3} + n^2 P\left[\left|Y - \frac{n}{2}\right| \geqq \frac{1}{2} n^{2/3}\right],$$

where Y is a binomial random variable. By Chebyshev's inequality this probability is $O(n^{-1/3})$. Thus, we have

$$(4.41)\qquad E(R_n')^2 = ER_{n/2}^2 + O(n^{5/3}).$$

The same estimate can be obtained for $(ER_n')^2$ and $(ER_{n/2})^2$ in essentially the same way. Therefore,

$$(4.42)\qquad \operatorname{Var} R_n \sim \operatorname{Var} R_{2n}' \sim \frac{8\pi^2 K|Q'|4n^2}{\log^4 n}$$

by the strongly aperiodic case. But since $Q' = Q/2$, it follows that $|Q'| = |Q|/4$ and the constant has the right form.

5. Sufficient conditions for strong transience

The following theorems give simple sufficient conditions for a random walk to be strongly transient. As we already mentioned in the introduction, the results of [6] apply to the range of such random walks.

First we need to introduce some terminology. A nonempty set $F \subset E_d$ is said to be closed if $\Sigma_{y \in F} P(x, y) = 1$ for every $x \in F$. The random walk is called irreducible if the only closed set is E_d. This is equivalent to saying that every lattice point can be reached from every other lattice point. The assumption that the random walk is aperiodic (that is, does not take place on a proper subgroup of E_d) which we have made without any real loss of generality is equivalent in the present terminology to the assumption that no two closed sets are disjoint.

THEOREM 5.1. *Every aperiodic random walk that is not irreducible is strongly transient.*

PROOF. Let $F_1 = \{x: G(0, x) > 0\}$ and $F_0 = \{x: G(x, 0) = 0\}$. Since the random walk is not irreducible, F_1 is a proper subset of E_d and hence F_0 is nonempty. The sets F_1 and F_0 are both closed and by the aperiodicity must have a common element x. Then there is a k such that

$$(5.1) \qquad P^k(0, x) = \varepsilon > 0, \qquad G(x, 0) = 0.$$

We may also assume that $\varepsilon < 1$, for if $\varepsilon = 1$, then $u_n = 0$ for $n \geqq k$ by (5.1) and the random walk is clearly strongly transient. Since $P^k(0, y) \leqq 1 - \varepsilon$ for $y \neq x$,

$$(5.2) \qquad P^k(0, y) \leqq \alpha = \max(\varepsilon, 1 - \varepsilon) \qquad \text{for all} \quad y \in E_d.$$

Now suppose that $z \in F_1$. Since $G(x + z, z) = G(x, 0) = 0$ and $G(0, z) > 0$, it follows that $G(x + z, 0) = 0$ or $x + z \in F_0$. Thus, for $z \in F_1$,

$$(5.3) \qquad \sum_{y \notin F_0} P^k(z, y) \leqq 1 - P^k(z, x + z) = 1 - \varepsilon \leqq \alpha.$$

Now suppose for all $z \in F_1 \cap F_0^c$ that $P^{nk}(z, 0) \leqq \alpha^n$; this is true for $n = 1$ by (5.2). For $z \in F_1 \cap F_0^c$, then

$$(5.4) \qquad P^{(n+1)k}(z, 0) = \sum_{y \in F_1 \cap F_0^c} P^k(z, y)P^{nk}(y, 0) \leqq \alpha^n \sum_{y \notin F_0} P^k(z, y) \leqq \alpha^{n+1}$$

by the induction hypothesis and (5.3). If $0 \in F_0$, the random walk is trivially strongly transient so we may assume that $0 \in F_1 \cap F_0^c$. Thus, we have $u_{nk} \leqq \alpha^n$ and for $0 < r < k$

$$(5.5) \qquad u_{nk+r} = \sum_{y \in F_1 \cap F_0^c} P^r(0, y)P^{nk}(y, 0) \leqq \alpha^n;$$

this is clearly sufficient for strong transience.

THEOREM 5.2. *A random walk with $EX_1 \neq 0$ and $E|X_1|^2 < \infty$ is strongly transient regardless of the dimension.*

PROOF. Since $EX_1 \neq 0$, there must be at least one component of X_1 with nonzero expectation. Furthermore, whenever the original random walk visits 0, so will the component ones so that $u_n \leqq u_n'$ where u_n' refers to one of the component random walks. Thus, if the component random walk is strongly transient, the original random walk will be as well. This reduces the problem to the one dimensional case. Let S_n be a random walk with $EX_1 \neq 0, EX_1^2 < \infty$, and $d = 1$. If $\operatorname{Var} X_1 = 0$, the random walk is degenerate and $EX_1 \neq 0$ implies $u_n = 0$ for all $n > 0$. We may then assume that $0 < \operatorname{Var} X_1 < \infty$ and by considering $-S_n$, if necessary, that $\mu = EX_1 > 0$. Now

$$(5.6) \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} u_{n+k} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{x} P^n(0, x)P^k(x, 0) = \sum_{x} G(0, x)G(x, 0)$$
$$= G^2(0, 0) + 2 \sum_{x>0} G(0, x)G(x, 0).$$

Thus, the random walk is strongly transient if and only if the last sum converges. If the random walk is aperiodic, the renewal theorem asserts that $\lim_{x \to \infty} G(0, x) = \mu^{-1} > 0$. If it is not aperiodic, the limit will still be positive if we restrict x to the subgroup. In either case, the random walk is strongly transient if and only if $\Sigma_{x>0}\, G(x, 0)$ converges. But

$$(5.7)\qquad \sum_{x>0} G(x, 0) = \sum_{x<0} G(0, x) = \sum_{x<0}\sum_{n=0}^{\infty} P^n(0, x)$$

$$= \sum_{n=0}^{\infty} P[S_n < 0] \leqq \sum_{n=0}^{\infty} P[|n^{-1}(S_n - n\mu)| \geqq \mu].$$

This series converges provided $\text{Var}\, X_1 < \infty$ by a result of Erdös [3]. (Also, see [9].)

REMARK 5.1. The condition $E|X_1|^2 < \infty$ is essential in Theorem 5.2 even though the random walk is transient once EX_1 exists and is nonzero. To see that it is essential, note that the proof shows that a necessary and sufficient condition for strong transience in one dimension when $EX_1 = \mu > 0$ is the convergence of the series $\Sigma_n P[S_n < 0]$. It is easy to construct examples for which this series diverges once the requirement $EX_1^2 < \infty$ is dropped.

6. The range of one dimensional random walk

Let $\{S_n\}$ be a one dimensional random walk with $EX_1^2 < \infty$. If $EX_1 \neq 0$, then by Theorem 5.2 the random walk is strongly transient. The results of [6] then apply and the central limit theorem is valid provided only that the range R_n does not grow deterministically. The case of $EX_1 = 0$ is somewhat different and we shall deal with it now.

THEOREM 6.1. *Let $\{S_n\}$ be a one dimensional random walk with $EX_1 = 0$ and $0 < \text{Var}\, X_1 = \sigma^2 < \infty$. Then R_n/ER_n converges in distribution to a proper law. The limit law is that of*

$$(6.1)\qquad \left(\frac{\pi}{8}\right)^{1/2} \left\{\max_{0 \leqq t \leqq 1} Y(t) - \min_{0 \leqq t \leqq 1} Y(t)\right\},$$

where $Y(t)$ is standard one dimensional Brownian motion.

PROOF. By making a transformation, if necessary, we may assume that the random walk is aperiodic since the transformation does not change R_n. Now $r_n \sim \sigma(2/\pi n)^{1/2}$ (see [12], p. 381), and

$$(6.2)\qquad ER_n = \sum_{k=0}^{n} r_k \sim \sigma\left(\frac{8}{\pi}\right)^{1/2} n^{1/2}.$$

Let

$$(6.3)\qquad M_n = \max_{0 \leqq j \leqq n} S_j, \qquad m_n = \min_{0 \leqq j \leqq n} S_j.$$

It follows readily from the results in [4] (see also [12], p. 232) that $EM_n \sim \sigma(2/\pi)^{1/2} n^{1/2}$ and so

$$(6.4)\qquad EM_n - Em_n \sim \sigma\left(\frac{8}{\pi}\right)^{1/2} n^{1/2} \sim ER_n.$$

Since $M_n - m_n + 1 - R_n \geqq 0$ and

$$(6.5)\qquad P[M_n - m_n + 1 - R_n \geqq \varepsilon ER_n] \leqq \frac{1 + EM_n - Em_n - ER_n}{\varepsilon ER_n} \to 0$$

as $n \to \infty$, it follows that

$$\frac{M_n - m_n - R_n}{ER_n} \xrightarrow{P} 0. \tag{6.6}$$

Thus, R_n/ER_n and $(M_n - m_n)/ER_n$ have the same asymptotic distribution, and the limit behavior of the latter quantity follows from Donsker's invariance principle [1].

Theorem 6.1 is different from the other limit theorems for the range in that the limit law is not normal. We are also using a different scheme of normalization, but this difference is only apparent as we can now show that the standard deviation of R_n grows at the same rate as ER_n. Thus, using the usual normalization will only effect a scale change and translation on the limit law.

THEOREM 6.2. *Let* $\{S_n\}$ *be a one dimensional random walk with* $EX_1 = 0$ *and* $0 < \operatorname{Var} X_1 < \infty$. *Then there is a positive constant c such that* $\operatorname{Var} R_n \sim cn$.

PROOF. We know from the last theorem that $n^{-1/2}R_n$ converges in distribution and we will show that $\{n^{-1}R_n^2\}$ is uniformly integrable. This will prove the theorem with c being the variance of the limit law for $n^{-1/2}R_n$. Note that if $i \leqq j \leqq k$,

$$\begin{aligned} EZ_iZ_jZ_k \leqq P[S_k \neq S_{k-1}, \cdots, S_k \neq S_j; S_j \neq S_{j-1}, \cdots, \\ S_j \neq S_i; S_i \neq S_{i-1}, \cdots, S_i \neq 0] \\ = r_{k-j}r_{j-i}r_i, \end{aligned} \tag{6.7}$$

and so

$$ER_n^3 \leqq 6 \sum_{i=0}^{n} \sum_{j=i}^{n} \sum_{k=j}^{n} r_{k-j}r_{j-i}r_i = O(n^{3/2}). \tag{6.8}$$

Thus,

$$\int_{[n^{-1}R_n^2 \geqq M]} n^{-1}R_n^2 \, dP \leqq M^{-1/2} \int n^{-3/2}R_n^3 \, dP = O(M^{-1/2}) \tag{6.9}$$

uniformly in n.

REMARK 6.1. In the same way, one can show that $n^{-k/2}ER_n^k$ converges to the kth moment of the limit law, which exists for each $k \geqq 1$.

Added in proof. W. Feller (*Ann. Math. Statist.*, Vol. 22 (1951), pp. 427–432) considered the asymptotic distribution of $(M_n - m_n)n^{-1/2}$ for $d = 1$. He also obtained a series expansion for the density of the random variable in (6.1).

REFERENCES

[1] M. D. DONSKER, "An invariance principle for certain probability limit theorems," *Mem. Amer. Math. Soc.*, No. 6 (1951), pp. 1–12.

[2] A. DVORETZKY and P. ERDÖS, "Some problems on random walk in space," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1951, pp. 353–367.

[3] P. ERDÖS, "On a theorem of Hsu and Robbins," *Ann. Math. Statist.*, Vol. 20 (1949), pp. 286–291.
[4] P. ERDÖS and M. KAC, "On certain limit theorems of the theory of probability," *Bull. Amer. Math. Soc.*, Vol. 52 (1946), pp. 292–302.
[5] I. S. GRADSHTEYN and I. M. RYZHIK, *Table of Integrals, Series, and Products*, New York and London, Academic Press, 1965.
[6] N. C. JAIN and S. OREY, "On the range of random walk," *Israel J. Math.*, Vol. 6 (1968), pp. 373–380.
[7] N. C. JAIN and W. E. PRUITT, "The range of recurrent random walk in the plane," *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, Vol. 16 (1970), pp. 279–292.
[8] ———, "The range of transient random walk," *J. Analyse Math.*, Vol. 24 (1971), pp. 369–393.
[9] M. L. KATZ, "The probability in the tail of a distribution," *Ann. Math. Statist.*, Vol. 34 (1963), pp. 312–318.
[10] H. KESTEN and F. SPITZER, "Ratio theorems for random walks I," *J. Analyse Math.*, Vol. 11 (1963), pp. 285–322.
[11] H. KESTEN, "Ratio theorems for random walks II," *J. Analyse Math.*, Vol. 11 (1963), pp. 323–379.
[12] F. SPITZER, *Principles of Random Walk*, Princeton, Van Nostrand, 1964.