POINT PROCESSES AND FIRST PASSAGE PROBLEMS

YU. K. BELYAYEV
MOSCOW STATE UNIVERSITY, U.S.S.R.

1. Introduction and basic definitions

In many domains of application of probability theory, it becomes necessary to study various properties of random formations of points. Random sequences of arrival times in queuing systems were studied by C. Palm [17], A. Hinčin [9], F. Zitek [23], D. König, K. Matthes and K. Nawrotzki [11] and others. Statistical radiotechnica is also a source of similar problems. Here it became necessary to study the set of times corresponding to the crossing of a fixed level by a random signal (S. Rice [19], V. Tihonov [22]). It is of interest to study random point formations on the plane, on surfaces, and so forth. In this papeı, we describe a general approach which makes it possible to investigate a wide class of random point sets, generated by random processes and fields, from a common point of view.

The theory of random point sets and the random streams which correspond to them can be regarded as a special branch of the theory of random processes. However, this branch can lay claim to a certain degree of independence. Specific concepts and methods of investigation arise in it. If we are considering a random sequence of points on the line, then its definition is easily reduced to the problem of specifying a suitably defined random process. In a general setting, such an approach is frequently insufficient.

We shall consider an adequately general scheme for defining random point sets and random streams (Belyayev [2], [4]). Let $[T, \mathscr{M}_T]$ be a measurable space of values of a parameter $t \in T$, where $\mathscr{M}_T$ is the σ-algebra of measurable sets, and $[\Omega, \mathscr{F}_\Omega, P]$ is the initial probability space of elementary events $\omega \in \Omega$.

DEFINITION 1.1. *By a random stream $\eta(\Delta)$ on $[T, \mathscr{M}_T]$ is meant a random function with domain $\mathscr{M}_T$ (we denote an element of $\mathscr{M}_T$ by Δ), $\eta(\Delta) = 0, 1, \cdots, \infty$, satisfying the relation $\eta(\bigcup_i \Delta_i) = \Sigma_i \eta(\Delta_i)$ for every countable sequence $\Delta_i \in \mathscr{M}_T$ for which $\Delta_i \cap \Delta_j = \varnothing$ whenever $i \neq j$.*

DEFINITION 1.2. *A random point set defined on $[T, \mathscr{M}_T]$ is a function $S = S(\omega)$ defined on Ω, whose values are subsets of T, and such that for every $\Delta \in \mathscr{M}_T$ the number of points in $S \cap \Delta$, denoted by $\eta(\Delta)$, is a random variable. The system of all such random variables $\eta(\Delta)$ is called the random stream generated by the random point set S.*

Interesting examples of random point sets arise in problems involving level crossings. If ξ_t is a real valued random process with continuous sample functions, $T = R^1$, $\mathscr{M}_T$ is the family of Borel sets of R^1, then $S_u = \{t: \xi_t = u\}$, that is, the trace of the process on the level u, is a random point set. The set $S_0 = \{t: \nabla\zeta_t = 0\}$—the set of stationary points of a random field ζ_t on $T = R^m$ whose sample functions are continuously differentiable—is also a random point set. Here

$$\nabla\zeta_t = \left(\frac{\partial\zeta_t}{\partial t_1}, \cdots, \frac{\partial\zeta_t}{\partial t_m}\right)', \qquad t = (t_1, \cdots, t_m)' \in T, \tag{1.1}$$

where ′ denotes transposition of vectors or matrices. One can give many examples of a variety of random point sets defined on the trajectories of random processes and fields.

Let $\mathscr{D}(\Delta)$ be the family of all possible partitions $d(\Delta)$ of a set $\Delta \in \mathscr{M}_T$, that is, all countable collections $d(\Delta) = \{\Delta_\alpha\}$, $\Delta_\alpha \in \mathscr{M}_T$, $\Delta_{\alpha_1} \cap \Delta_{\alpha_2} = \varnothing$, $\alpha_1 \neq \alpha_2$. For a random stream $\eta(\Delta)$, we introduce the set function

$$\lambda(\Delta) = \sup_{d(\Delta)\in\mathscr{D}(\Delta)} \sum_{\Delta_\alpha \in d(\Delta)} P\{\eta(\Delta_\alpha) > 0\}. \tag{1.2}$$

THEOREM 1.1. *The set function defined by* (1.2) *is a measure on* $\mathscr{M}_T$.

PROOF. The property $\lambda(\Delta) \geqq 0$ is obvious. If $\Delta = \cup_i \Delta_i$, $\Delta_{i_1} \cap \Delta_{i_2} = \varnothing$, $i_1 \neq i_2$, $\Delta_i \in \mathscr{M}_T$, then for $d(\Delta) = \cup_i d(\Delta_i)$, $d(\Delta_i) = \{\Delta_{i,\alpha}\}$, we have

$$\lambda(\Delta) \geqq \sum_i \sum_{\Delta_{i,\alpha} \in d(\Delta_i)} P\{\eta(\Delta_{i,\alpha}) > 0\}. \tag{1.3}$$

Now the $d(\Delta_i)$ can be chosen arbitrarily; therefore it follows from (1.3) that $\lambda(\Delta) \geqq \Sigma_i \lambda(\Delta_i)$. It is hardly more complicated to prove the opposite inequality.

Theorem 1.1 makes it possible to introduce a useful numerical characteristic, which generalizes the concept of the parameter of a random stream given by Hinčin [9].

DEFINITION 1.3. *The measure* $\lambda(\Delta)$, *defined by the relation* (1.2), *is called the parametric measure of the random stream* $\eta(\Delta)$. *The principal measure of* $\eta(\Delta)$ *is defined by the relation* $\mu(\Delta) = \mathbf{E}\eta(\Delta)$.

We shall call a system of subsets, $\mathscr{C} = \{\Delta_{n,k}\}$, $n, k = 1, 2, \cdots$, of the space T a fundamental system, if (1) $\Delta_{n,k} \in \mathscr{M}_T$, (2) $\Delta_{n,k} \cap \Delta_{n,\ell} = \varnothing$ for $k \neq \ell$, (3) $\Delta_{n,k} = \cup_{i\in I_{n,k}} \Delta_{n+1,i}$, $I_{n,k} \subset \{1, 2, \cdots\}$, (4) for any $t_1, t_2 \in T$, $t_1 \neq t_2$, there exists $n = n(t_1, t_2)$ and positive integers $i_1 \neq i_2$ such that $t_1 \in \Delta_{n,i_1}$, $t_2 \in \Delta_{n,i_2}$, (5) the σ-algebra generated by the family $\mathscr{C}$ coincides with $\mathscr{M}_T$. The following assertion is a natural generalization of the well-known result of V. S. Koroljuk (see [9]) that the parameter and the intensity of a stationary ordinary random stream coincide.

THEOREM 1.2. *If the random stream* $\eta(\Delta)$ *on* $[T, \mathscr{M}_T]$ *is generated by a random point set and a fundamental system* $\mathscr{C}$ *exists, then* $\lambda(\Delta) \equiv \mu(\Delta)$, $\Delta \in \mathscr{M}_T$.

PROOF. The proof is based upon the construction of a sequence of random functions

$$(1.4) \qquad \tilde{\eta}_n(\Delta_{n,k}) = \begin{cases} 1, & \eta(\Delta_{n,k}) > 0, \\ 0, & \eta(\Delta_{n,k}) = 0, \end{cases}$$

$\tilde{\eta}_n(\bigcup_{i\in I} \Delta_{n,k}) = \Sigma_{i\in I} \tilde{\eta}_n(\Delta_{n,i})$, $I \subset \{1, 2, \cdots\}$, which is nondecreasing and converges to $\eta(\Delta)$ from below.

Let us consider $[T^k, \mathscr{M}_T^k]$, the direct product of k copies of the measure space $[T, \mathscr{M}_T]$, whose points are of the form $(t_1, \cdots, t_k)$, $t_1, \cdots, t_k \in T$. Let S be a random point set on $[T, \mathscr{M}_T]$, and $\eta(\Delta)$ the random stream generated by S. We construct the random set S^{*k} on $[T^k, \mathscr{M}_T^k]$, taking for its points all those of the form $(s_1, \cdots, s_k) \in T^k$, where $s_i \in S$ and $s_i \neq s_j$ for $i \neq j$. We denote by $\eta^{*k}(\Delta^k)$ the random stream generated by S^{*k}.

DEFINITION 1.4. *The k-parametric (k-principal) measure $\lambda_k(\Delta^k)$ $(\mu_k(\Delta^k))$ of the random stream $\eta(\Delta)$ generated by a random point set S is defined as the parametric (principal) measure of the stream $\eta^{*k}(\Delta^k)$ on $[T^k, \mathscr{M}_T^k]$.*

If T has a fundamental system, then we obtain as a corollary of Theorem 1.2 that $\lambda_k(\Delta^k) \equiv \mu_k(\Delta^k)$, that is,

$$(1.5) \qquad \lambda_k(\Delta^k) = \mathbf{E}\eta^{*k}(\Delta^k), \qquad \Delta^k \in \mathscr{M}_T^k.$$

From (1.5) and the definition of S^{*k}, we obtain the possibility of calculating various moments of properties of random point sets. The appropriate result can be stated in the form of a lemma.

LEMMA 1.1. *If $\eta(\Delta)$ is a random stream generated by a random point set on a space $[T, \mathscr{M}_T]$ with a fundamental system $\mathscr{C}$, then the value of the k-parametric measure on a rectangular set $\tilde{\Delta}^k = \Delta_1 \times \cdots \times \Delta_1 \times \Delta_2 \times \cdots \times \Delta_\ell \times \cdots \times \Delta_\ell$, where Δ_1 is repeated k_1 times, $\cdots$, Δ_ℓ is repeated k_ℓ times, $k = k_1 + \cdots + k_\ell$, $\Delta_i \in \mathscr{M}_T$, $i \in [1, \ell]$, and $\Delta_i \cap \Delta_j = \varnothing$ for $i \neq j$, is given by*

$$(1.6) \qquad \lambda_k(\tilde{\Delta}^k) = \mathbf{E}\left\{\prod_{i=1}^{\ell} \prod_{j=1}^{k_i} [\eta(\Delta_i) - j + 1]^+\right\},$$

where $[x]^+ = \max(x, 0)$.

It follows from Lemma 1.1 that for a cubical set $\tilde{\Delta}^k = \Delta \times \cdots \times \Delta$, $\Delta \in \mathscr{M}_T$, the value $\lambda_k(\tilde{\Delta}^k)$ equals the kth factorial moment of the number of points $\eta(\Delta)$. If $k_i = 1$ and $\Delta_i \cap \Delta_j = \varnothing$, $i, j \in [1, \ell]$, $i \neq j$, then it follows from (1.6) that $\lambda_k(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_\ell) = \mathbf{E}\{\prod_{i=1}^{\ell} \eta(\Delta_i)\}$, which for $k = 2$ enables us to compute the covariance of the number of points falling into nonintersecting sets in terms of the 2-parametric measure. We especially single out the possibility of computing the higher moments of characteristics in terms of the parametric measure, due to the fact that in many problems it is possible to compute the value of the parametric measure in explicit analytic form.

If a measure $\nu(\Delta)$ defined on $[T, \mathscr{M}_T]$ is such that $\lambda_k(\Delta^k)$ is absolutely continuous with respect to the product measure $\nu^k = \nu \times \cdots \times \nu$, then

$$(1.7) \qquad \lambda_k(\Delta^k) = \int_{(t_1, \cdots, t_k)\in\Delta^k} \lambda_k(t_1, \cdots, t_k)\nu(dt_1) \cdots \nu(dt_k),$$

where $\lambda_k(t_1, \cdots, t_k)$ is called the k-parameter function of the random stream or random point set with respect to the measure ν.

Thus, with every random point set S on $[T, \mathscr{M}_T]$ we can, generally speaking, associate the infinite sequence of k-parametric measures, which in the presence of a fundamental system $\mathscr{C}$ enables us to find the moments of higher order. One of the important unsolved problems is whether or not the sequence of parametric measures $\lambda_k(\Delta^k)$ determines all probabilistic characteristics of S.

Under very general conditions on random streams which are homogeneous with respect to a group of motions in the space T, the parametric measure is proportional to the corresponding Haar measure, which generalizes the well-known result of Hinčin [9] on the existence of the parameter for a stationary random stream on $T = R^1$ (see Belyayev [4]).

2. Special systems of conditional probabilities

In a large number of applied problems, it becomes necessary to calculate the conditional probabilities of various events, when the conditioning event is the appearance in a random point set of a point with given coordinates. That this problem is not entirely simple is attested to by the fact that a number of papers have been devoted to the question of methods for determining the distribution function of the length of the interval between successive events from a stationary stream on the line (McFadden [15], Ryll-Nardzewski [20], Matthes [14], Cramér and Leadbetter [7], and others).

We shall assume that the random point set S considered below is defined on $[T, \mathscr{M}_T]$, which we shall assume has a fundamental system $\mathscr{C}$, and that the parametric measure is given by $\lambda(\Delta) = \int_\Delta \lambda(t)\nu(dt)$, where $\nu(\Delta)$ is some measure on $\mathscr{M}_T$. We denote the corresponding random stream by $\eta(\Delta, S)$. A random point set $S' \subset S$ generates a substream $\eta(\Delta, S') \leqq \eta(\Delta, S)$. The parametric measure $\lambda'(\Delta)$ of S' is also absolutely continuous with respect to $\nu(\Delta)$, with density $\lambda'(t) \leqq \lambda(t)$ almost everywhere relative to the measure $\nu(\Delta)$. The passage to $S' \subset S$ can be carried out by means of a screening (thinning out) operation, which is defined in the following way. With every point $t \in T$, we associate an event A_t or its complement $\bar{A}_t$, that is, we assume that it is known for each $\omega \in \Omega$ whether the event A_t or $\bar{A}_t$ occurred. We consider (for each ω) the subset $S(\mathscr{A}) \subset S$ consisting of those points $s \in S$ for which the event A_s occurs, assuming that this screening process yields a random point set $S(\mathscr{A})$. Since the parametric measures of the random streams $\eta(\Delta, S(\mathscr{A}))$ and $\eta(\Delta, S)$ satisfy the inequality $\eta(\Delta, S(\mathscr{A})) \leqq \eta(\Delta, S)$, their parameter functions $\lambda(t)$ and $\lambda(t, A_t)$ relative to the measure $\nu(\Delta)$ satisfy almost everywhere the inequality

$$\lambda(t, A_t) \leqq \lambda(t). \tag{2.1}$$

In most of the problems which the author has investigated, the inequality (2.1) can be extended to all values of $t \in T$. Assuming that (2.1) holds for all $t \in T$, we arrive at the following definition.

DEFINITION 2.1. *The conditional probability of the event A_t, the condition being that $t \in S$, is defined by the relation*

$$P^*(A_t | t \in S) = \frac{\lambda(t, A_t)}{\lambda(t)}. \tag{2.2}$$

If the random streams $\eta(\Delta, S(\mathscr{A}))$ and $\eta(\Delta, S)$ are homogeneous with respect to a group of motions $G = \{g\}$ in the space T with invariant measure $\nu(\Delta)$, and $\lambda(t) = \lambda$, $\lambda(t, A_t) = \lambda(gt, A_{gt}) = \lambda(A)$, then

$$P^*(A_t | t \in S) = \frac{\lambda(A)}{\lambda}. \tag{2.3}$$

The probabilities introduced by relations (2.1) and (2.2) turn out to be the natural ones in many concrete problems. In particular for ergodic random processes, the probabilities defined in this way, using the random set of crossings of a fixed level, have a simple statistical interpretation.

As an example, one can consider the case $T = R^1$, $\mathscr{M}_T$ the Borel sets, $\nu(\Delta)$ Lebesgue measure, $G = \{g\}$ the group of translations $t \to t + g$, $A_{k,t}(v) = \{\eta[(t, t + v)]\} = k$; then in accordance with (2.2),

$$\varphi_k(v) = \frac{\lambda(A_{k,t}(v))}{\lambda} \tag{2.4}$$

is the conditional probability that in a time interval of length v exactly k points from S will appear, the condition being that the initial moment of the interval also belongs to S. The $\varphi_k(v)$ are the well-known Palm–Hinčin functions.

Another particular example of this kind would be deducing that the derivative of a stationary Gaussian process at the moments when the trajectory leaves the level u has a Rayleigh distribution. However, it is important for us now to clarify that the method of screening given random streams in a suitable way enables us to define new probability distributions on the space of sample functions of a random process or field. We will therefore consider a more general scheme. This scheme can frequently prove useful in defining conditional probabilities. Suppose that a random point set S is defined on $[T, \mathscr{M}_T]$, where $T = T_1 \times T_2$, $\mathscr{M}_T = \mathscr{M}_{T_1} \times \mathscr{M}_{T_2}$, the $\mathscr{M}_{T_i}$ have fundamental systems $\mathscr{C}_i$, the $\nu_i(\Delta_i)$ are measures on the $\mathscr{M}_{T_i}$, and the parametric measure of S is given by

$$\lambda(\Delta) = \int_\Delta \lambda_2(t_1, t_2)\nu_1(dt_1)\nu_2(dt_2). \tag{2.5}$$

We assume further that with probability 1 all the points $s_\alpha = (s_{1,\alpha}, s_{2,\alpha}) \in S$ have different first coordinate, that is, $\alpha_1 \neq \alpha_2$ implies $s_{1,\alpha_1} \neq s_{1,\alpha_2}$. Considering the random point set $S_1 = \{s_{1,\alpha}\}$, consisting of the coordinates $s_{1,\alpha}$ of all points $s_\alpha = (s_{1,\alpha}, s_{2,\alpha}) \in S$, on the space $[T_1, \mathscr{M}_{T_1}]$, we find that its parametric measure is given by

$$\lambda_1(\Delta_1) = \int_{\Delta_1} \lambda_1(t_1)\nu_1(dt_1), \qquad \Delta_1 \in \mathscr{M}_{T_1}, \tag{2.6}$$

where $\lambda_1(t_1) = \int_{T_2} \lambda_2(t_1, t_2)\nu_2(dt_2)$. Thus, in accordance with (2.2), the conditional probability that for a point $(s_{1,\alpha}, s_{2,\alpha}) \in S$ we will have $s_{2,\alpha} \in \Delta_2 \in \mathscr{M}_{T_2}$, the condition being that $s_{1,\alpha} = t_1$, is given by

$$P^*(s_{2,\alpha} \in \Delta_2 | s_{1,\alpha} = t_1) = \frac{\int_{\Delta_2} \lambda_2(t_1, t_2)\nu_2(dt_2)}{\lambda_1(t_1)}. \tag{2.7}$$

Let us return to the problem of determining the distribution of the derivative, at the moment of leaving level u, of the trajectory of a stationary differentiable Gaussian process ξ_t, $\mathbf{E}\xi_t = 0$, $\mathbf{E}\xi_t^2 = 1$, $\mathbf{E}\dot{\xi}_t^2 = \lambda_2$. Here $T_1 = R^1$, $T_2 = R^+ = [0, \infty)$, and the $\mathscr{M}_{T_i}$ are the algebras of Borel sets. The point $s_{1,\alpha} = \tau$ is the instant of leaving the level u, and $s_{2,\alpha} = \dot{x}$ is the value $\dot{\xi} = \dot{x}$. Here

$$\lambda_2(\tau, \dot{x}) = \frac{\dot{x}}{2\pi\lambda_2^{1/2}} \exp\left\{-\frac{\dot{x}^2}{2\lambda_2} - \frac{u^2}{2}\right\} \tag{2.8}$$

with respect to Lebesgue measure in T. Consequently, in accordance with (2.7),

$$P^*(\dot{\xi}_\tau > v) = \exp\left\{-\frac{v^2}{2\lambda_2}\right\}, \tag{2.9}$$

that is, the conditional distribution of the values of the derivative $\dot{\xi}_\tau$ of a stationary Gaussian process is a Rayleigh distribution ([22]).

Let us consider a pair of correlated random objects (S, ζ_t), where S is a random point set on a space $[T, \mathscr{M}_T]$ having a fundamental system $\mathscr{C}$, and ζ_t is a random process (field) with values from a phase space $[\mathscr{E}, \mathscr{F}_\mathscr{E}]$. Examples of such pairs would be trajectories of random processes ζ_t, with the random point set S consisting of those values of the coordinate t for which the trajectory of the process has a local maximum, a saddle point, and so forth. The operation of screening the points of S described above can be carried out for every elementary event $\omega \in \Omega$ on the basis of the observed trajectory $\zeta_t = \zeta_t(\omega)$. For example, with every point $t \in T$ we can associate a collection of points $t_i = t_i(t) \in T$, $i \in [1, m]$. We form a new set S' by including a point $s \in S$ in S' whenever $\zeta_{t_i} \in A_i \in \mathscr{F}_\mathscr{E}$, and excluding s from S' when the contrary takes place. If $\lambda(t)$ is the parameter function of the random point set S relative to some measure $\nu(\Delta)$, $\Delta \in \mathscr{M}_T$, and S' is a random point set with parameter function $\lambda(t, t_1, \cdots, t_m, A_1, \cdots, A_m) \leqq \lambda(t)$, then one can define the conditional probability

$$P^*(\zeta_{t_i} \in A_i | t \in S) = \frac{\lambda(t, t_1, \cdots, t_m, A_1, \cdots, A_m)}{\lambda(t)}, \tag{2.10}$$

which corresponds to the probability that $\zeta_{t_i} \in A_i$, given that $t \in S$. Regarding (2.10) as a family of finite dimensional probabilities, one can pose the problem of constructing new probability measures in the space of sample functions of ζ_t. Of course there are difficulties here, analogous to those which arise in the construction of conditional probability measures in function spaces. However, the approach itself has definite interest and makes it possible to obtain interesting results.

As an example, which in fact makes use of this approach, we can mention a paper of the author and V. Nosko [5] on the determination of the asymptotic distributions of the duration of excursions of a stationary Gaussian process and of its envelope. In the paper [5], to study the envelope we take $T = R^1 \times \Gamma_r$, $\mathscr{E} = R^2 = \{x = (x_1, x_2)\}$. Let $\zeta_t = (\xi_{1,t}, \xi_{2,t})$ be a two dimensional stationary Gaussian process, $S = \{(\tau, x)\}$, where τ is an exit time and $x \in \Gamma_r$ is the exit point of the trajectory of ζ_t from the circle $\Gamma_r = \{x : x_1^2 + x_2^2 = r^2\}$. The components of the process ζ_t are

$$(2.11)\qquad \begin{aligned} \xi_{1,t} &= \int_0^\infty \cos \lambda t \, du(\lambda) + \int_0^\infty \sin \lambda t \, dv(\lambda), \\ \xi_{2,t} &= \int_0^\infty \sin \lambda t \, du(\lambda) - \int_0^\infty \cos \lambda t \, dv(\lambda), \end{aligned}$$

where $u(\lambda)$ and $v(\lambda)$ are mutually independent stationary Gaussian processes with orthogonal increments;

$$(2.12)\qquad \mathbf{E}[du(\lambda)]^2 = \mathbf{E}[dv(\lambda)]^2 = dF(\lambda),$$

$$\int_0^\infty \lambda^2 [\log (1 + \lambda)]^{1+\varepsilon} \, dF(\lambda) < \infty, \quad \varepsilon > 0.$$

To determine a probability measure on the space of trajectories which exit at time t from the point $x \in \Gamma_r$, we put $t_i(t) = t + t_i$, $A_i \subset R^2$, $i \in [1, m]$; then

$$(2.13)\qquad \begin{aligned} &\lambda(t, x, t_1, \cdots, t_m, A_1, \cdots, A_m) \\ &= \int_{x_i \in A_i, i \in [1,m]} \mathbf{E}\{(n(x)\zeta_t')^+ \mid \zeta_t = x \in \Gamma_r, \zeta_{t_i} = x_i, i \in [1, m]\} \\ &\quad \cdot p_{t, t+t_1, \cdots, t+t_m}(x, x_1, \cdots, x_m) \, dx_1 \cdots dx_m, \qquad x_i \in R^2. \end{aligned}$$

The parameter function of the random point set $S = \{(t, x)\} \subset R^1 \times \Gamma_r$ with respect to the measure $\nu = \nu_1 \times \nu_2$, where ν_1 is Lebesgue measure on R^1 and ν_2 is Lebesgue measure on Γ_r, is given by

$$(2.14)\qquad \lambda(t, x) = \mathbf{E}\{(n(x)\zeta_t')^+ \mid \zeta_t = x \in \Gamma_r\} \, p_t(x).$$

Here $n(x)$ is the normal vector of the circle Γ_r at the point x, while $p_t(x)$ and $p_{t, t+t_1, \cdots, t+t_m}(x, x_1, \cdots, x_m)$ are the probability densities for the values $\zeta_t = x$ and $\zeta_{t+t_i} = x_i$, $i \in [1, m]$. The conditional probabilities are constructed from (2.13) and (2.14) in accordance with (2.10). In [7] the probabilities obtained in this way are called ergodic probabilities. A study of them for $r \uparrow \infty$ leads to the following result ([5]):

THEOREM 2.1. *If ζ_t is a two dimensional stationary Gaussian process which satisfies the conditions enumerated above, then the limit of the distribution of the duration Δ of an excursion of the envelope, that is, $(\xi_{1,t}^2 + \xi_{2,t}^2)^{1/2}$, above the level r, has the following form in terms of ergodic probabilities:*

$$\lim_{r\uparrow\infty} P^*\{r\Delta > v\} = \exp\left\{-\frac{\lambda_2}{8\lambda_0^2}(xv)^2\right\}, \tag{2.15}$$

where

$$\lambda_k = \int_0^\infty \lambda^k\, dF(\lambda), \qquad x^2 = 1 - \frac{\lambda_1^2}{\lambda_0\lambda_2}. \tag{2.16}$$

3. Random point sets generated by random fields

With every trajectory of a random field ζ_t, $t \in R^m$, continuously differentiable with probability one, one can associate various random point sets. The problems which arise here are more diverse than problems concerning the crossing of levels by trajectories of random processes (see Longuet-Higgins [12], [13]). If we proceed along the path of direct generalization of level crossing problems, involving the trajectories of random processes, to the case of random fields, then we arrive at the necessity of studying random point sets $S_u = \{t: \zeta_t = u\}$, which for $m = 2$ form a family of contours (level lines) in the plane. Problems related to the study of the distribution of the number of such contours, their lengths, and so forth, present difficulties. The approach discussed in Sections 1 and 2 can be used to study local objects, while the level lines formed from the curves $\zeta_t = u$ are not local objects. However, here one can study random point sets which are introduced on the level lines in a special way.

A typical example is the study of the random point set S_0 of stationary points of a random field ζ_t, $t \in R^m$, that is, the set of points at which the gradient of the field equals zero. The study of subsets of S_0, such as the sets S_+ of local maxima (bursts), S_- of local minima and S_c of saddle points, deserves attention. If we are studying the reflection from a random surface of a trajectory of ζ_t, then it is natural to study the random set of shines, whose definition is given in [2]. We shall restrict ourselves here to a study of the properties of the random point set S_0.

Let us assume that ζ_t is a random field, $t \in R^m$, which is twice continuously differentiable with probability 1, $S_0 = \{t: \nabla\zeta_t = 0\}$ is the set of stationary points, where

$$\nabla\zeta_t = \left(\frac{\partial\zeta_t}{\partial t_1}, \cdots, \frac{\partial\zeta_t}{\partial t_m}\right)', \qquad t = (t_1, \cdots, t_m)'. \tag{3.1}$$

We denote by $p_t(x)$ the probability density of the values of the vector $\nabla\zeta_t = x$. We will call the joint distribution of the values of $\|\partial^2\zeta_t/\partial t_i\partial t_j\|$, $i, j \in [1, m]$ *determinantly nondegenerate* (det-nondegenerate) if, for every $x \in R^m$,

$$P\left\{\det\left\|\frac{\partial^2\zeta_t}{\partial t_i\partial t_j}\right\| = 0 \,\middle|\, \nabla\zeta_t = x\right\} = 0. \tag{3.2}$$

THEOREM 3.1 ([3]). *Suppose that the homogeneous random field ζ_t is twice continuously differentiable with probability one, $p_t(x) \leqq C < \infty$, and the distribu-*

tion of $\|\partial^2 \zeta_t/\partial t_i \partial t_j\|$ *is det-nondegenerate. Then with probability one we have:* (1) *for* $\tau \in S_0$, $\det \|\partial^2 \zeta_\tau/\partial t_i \partial t_j\| \neq 0$; (2) *every bounded region contains a finite number of points* $\tau \in S_0$.

PROOF. It is sufficient to study the random point set $S_0 \cap K_m$, where $K_m = \{t: 0 \leqq t_i \leqq 1, i \in [1, m]\}$ is the unit cube. We introduce the following notation for the modulus of continuity of a random field and of its first and second derivatives:

$$
\begin{aligned}
\omega_\zeta(h) &= \sup_{t', t'' \in K_m, |t' - t''| < h} |\zeta_{t'} - \zeta_{t''}|, \\
\omega_i(h) &= \sup_{t', t'' \in K_m, |t' - t''| < h} \left| \frac{\partial \zeta_{t'}}{\partial t_i} - \frac{\partial \zeta_{t''}}{\partial t_i} \right|, \\
\omega_{i,j}(h) &= \sup_{t', t'' \in K_m, |t' - t''| < h} \left| \frac{\partial^2 \zeta_{t'}}{\partial t_i \partial t_j} - \frac{\partial^2 \zeta_{t''}}{\partial t_i \partial t_j} \right|, \\
|t' - t''| &= \left(\sum_{i=1}^m (t'_i - t''_i)^2 \right)^{1/2}
\end{aligned}
\tag{3.3}
$$

Since the field is twice continuously differentiable, for any $\varepsilon > 0$ we can choose a continuous function $\omega_\varepsilon(h)$, $\omega_\varepsilon(h) \downarrow 0$ for $h \downarrow 0$, and a constant C_ε, $0 < C_\varepsilon < \infty$, such that for the event

$$
\begin{aligned}
E_\varepsilon = \Bigg\{ &\max_{i,j} \sup_{t \in K_m} \left| \frac{\partial^2 \zeta_t}{\partial t_i \partial t_j} \right| \leqq C_\varepsilon, \\
&\max_{i \in [1, m]} \omega_i(h) \leqq m C_\varepsilon h, \ \max_{i, j \in [1, m]} \omega_{i,j}(h) \leqq \omega_\varepsilon(h), \text{ for all } h \leqq \sqrt{m} \Bigg\}
\end{aligned}
\tag{3.4}
$$

we have $P\{E_\varepsilon\} > 1 - \varepsilon$.

We consider the sequence of decompositions of the cube K_m into 2^{mn} cubes:

$$
K_{n, \bar{k}} = \left\{ t: \frac{k_i}{2^n} \leqq t_i \leqq \frac{k_i + 1}{2^n}, i \in [1, m] \right\},
\tag{3.5}
$$

$$
t_{\bar{k}} = \left(\frac{k_1}{2^n}, \cdots, \frac{k_m}{2^n} \right)', \bar{k} = (k_1, \cdots, k_m).
$$

Let G be the event that there exists $\tau \in S_0 \cap K_m$ for which $\det \|\partial^2 \zeta_t/\partial t_i \partial t_j\| = 0$. In the same way we denote by $G_{n, \bar{k}}$ the similar event when $\tau \in S_0 \cap K_{n, \bar{k}}$. Since $G = \bigcup_{\bar{k}} G_{n, \bar{k}}$, we have

$$
P\{G\} \leqq \sum_{\bar{k}} P\{G_{n, \bar{k}} \cap E_\varepsilon\} + P\{\bar{E}_\varepsilon\},
\tag{3.6}
$$

where $\bar{E}_\varepsilon$ is the complement of E_ε. Using (3.4), we obtain

$$
\{G_{n, \bar{k}} \cap E_\varepsilon\} \subset \left\{ \max_{i \in [1, m]} \left| \frac{\partial \zeta_t}{\partial t_i} \right| \leqq C_\varepsilon \frac{1}{2^n}, \left| \det \left\| \frac{\partial^2 \zeta_t}{\partial t_i \partial t_j} \right\| \right| \leqq (m!)^2 C_\varepsilon^{m-1} \omega_\varepsilon \left(\frac{1}{2^n} \right) \right\}.
\tag{3.7}
$$

In terms of ordinary conditional probabilities, we obtain from (3.7)

$$P\{G_{n,\bar{k}} \cap E_\varepsilon\} \leqq \int_{|x_i| \leqq C_\varepsilon 2^{-n}, i \in [1,m]} P\left\{\left|\det\left\|\frac{\partial^2 \zeta_t}{\partial t_i \partial t_j}\right\|\right| \leqq (m!)^2 C_\varepsilon^{m-1} \omega_\varepsilon\left(\frac{1}{2^n}\right)\middle|\nabla\zeta(t_k) = x\right\} p_t(x)\,dx. \tag{3.8}$$

Since $p_{t_k}(x) \leqq C$, and the conditional probability in the integrand in (3.8) is bounded by unity and tends to zero as $n \to \infty$ for any x, then by Lebesgue's theorem on passing to the limit inside the integral we obtain, for any $\delta > 0$ and $n \geqq n(\delta)$,

$$P\{G_{n,\bar{k}} \cap E_\varepsilon\} \leqq \delta\left(\frac{1}{2^n}\right)^m. \tag{3.9}$$

From (3.6) to (3.9), we obtain

$$P\{G\} \leqq 2^{mn}\delta\left(\frac{1}{2^n}\right)^m + \varepsilon. \tag{3.10}$$

Since δ and ε can be chosen as small as desired, it follows that $P\{G\} = 0$, which corresponds to the first part of the theorem.

We prove the second part of the theorem by assuming that in some cube the number of points from S_0 is infinite. This assumption implies that the event G has occurred, whose probability, as was just proven, is zero.

COROLLARY 3.1. *If ζ_t is a twice continuously differentiable (with probability 1) homogeneous Gaussian field for which the joint distribution of $\partial^2\zeta_t/\partial t_i \partial t_j$, $i, j \in [1, m]$ is nondegenerate, then the assertion of Theorem 3.1 holds.*

The proof follows from the mutual independence of the random variables $\partial\zeta_t/\partial t_k$ and $\partial^2\zeta_t/\partial t_i\partial t_j$ for any values of $i, j, k \in [1, m]$.

It is probable that the assertion of Corollary 3.1 can be strengthened somewhat by replacing the assumption of the twice continuous differentiability with probability 1 of ζ_t with the assumption of twice continuous differentiability of ζ_t in quadratic mean. One can formulate yet another assertion, similar to an assertion of Bulinskaja [6] on the absence of tangency in the problem of level crossings.

THEOREM 3.2 ([3]). *Let ζ_t be a twice continuously differentiable random field which has bounded probability densities for the quantities $\zeta_t = u$, $\nabla\zeta_t = \dot{u}$, and $p_t(u, \dot{u}) < C$. Then with probability 1 there are no stationary points corresponding to a given level r, that is, for $S_{0,r} = \{\tau : \nabla\zeta_\tau = 0, \zeta_\tau = r\}$ we have $P\{S_{0,r} \neq \varnothing\} = 0$.*

The proof can be carried out analogously to the proof of Theorem 3.1.

We remark that for $m = 2$, to transverse crossings of the level line $\zeta_t = r$ there correspond saddle points with height r. Since these are absent, the following alternative must prevail: a connected solution of $\zeta_t = r$, $t \in R^2$, is either a bounded isolated contour, possibly containing within itself another contour, or else it is a curve passing out of any bounded region. Presumably, for homogeneous Gaussian fields the boundedness of the contours satisfying the equation $\zeta_t = r$ is typical.

The k-parameter function which is needed to calculate the moments of functions (the mean, variance and so on) of random point sets generated by random fields, can be obtained in explicit form under weak restrictions. These restrictions, imposed on the field ζ_t, are conveniently formulated in the form of the following complex of conditions.

The complex of conditions $C_{\zeta,k}$. We assume that the real field ζ_t, $t \in R^m$, is with probability 1 twice continuously differentiable, and that the joint distributions of the quantities $\zeta_{t^i} = u_i$, $\nabla\zeta_{t^i} = \dot{u}^i$, $\nabla\zeta_{t^i}\nabla' = \|\ddot{u}^i_{\ell,r}\|$, at the points $t^i \in R^m$ have probability distributions $p_{t^1,\dots,t^k}(u', \dot{U}, \ddot{Z})$ which are jointly continuous in all the variables, where $u' = (u_1, \cdots, u_k)$, $\dot{U} = \|\dot{u}^i_j\|$, $\ddot{Z} = \|\ddot{u}^i_{\ell,r}\|, j, \ell, r \in [1, m]$, $i \in [1, k]$. We also assume that in any bounded region $C \subset R^m$ the moduli of continuity $\omega_{i,j}(h)$ of the fields $\partial^2\zeta_t/\partial t_i \partial t_j$ satisfy the condition

$$P\{\omega_{i,j}(h) > \varepsilon\} = o(h^{km}). \tag{3.11}$$

The following theorem gives conditions which are sufficient for (3.11) to hold.

THEOREM 3.3. *If for a real separable random field* ζ_t, $t, \tilde{h} \in R^m$,

$$\sup_{|\tilde{h}_i| \leqq h, i \in [1,m]} P\{|\zeta_{t+\tilde{h}} - \zeta_t| > \varepsilon(h)\} \leqq g(h), \tag{3.12}$$

where the functions $\varepsilon(h)$ *and* $g(h)$ *satisfy the conditions*

$$\sum_{n=1}^{\infty} 2^{m2^{n+1}} g(2^{-2^n}) < \infty, \qquad \sum_{n=1}^{\infty} \varepsilon(2^{-2^n}) < \infty, \tag{3.13}$$

then for the function

$$\psi(h) = \sum_{n=j(h)}^{\infty} 2^{m2^{n+1}} g(2^{-2^n}), \quad 2^{-2^{j(h)+1}} \leqq h < 2^{-2^{j(h)}}, \tag{3.14}$$

we have for $h \downarrow 0$

$$P\{\omega_\zeta(h) > \varepsilon\} \leqq \psi(h). \tag{3.15}$$

The proof basically follows the method of A. N. Kolmogorov, which is presented in a paper of Slutsky [21], and also the method of Dudley [8]. The partitioning sets have the form

$$\begin{aligned} T_n &= \left\{\left(\frac{k_1}{2^{2^n}}, \cdots, \frac{k_m}{2^{2^n}}\right), 0 \leqq k_i \leqq 2^{2^n}, i \in [1, m]\right\}, \\ T_{n,\bar{k}} &= T_n \cap K_{n,\bar{k}}, \\ K_{n,\bar{k}} &= \left\{t : \frac{k_i}{2^{2^n}} \leqq t_i < \frac{k_i + 1}{2^{2^n}}, i \in [1, m]\right\}, \end{aligned} \tag{3.16}$$

and the basis events are

$$A_{n,\bar{k},t} = \left\{\left|\zeta\left(\frac{k_1}{2^{2^n}}, \cdots, \frac{k_m}{2^{2^n}}\right) - \zeta(t)\right| \leqq \varepsilon(2^{-2^n})\right\}, \tag{3.17}$$

where $\bar{k} = (k_1, \cdots, k_m)$, $\zeta(t) = \zeta_t$, $t \in T_{n+1,\bar{k}}$.

Verifying the hypotheses of Theorem 3.3 for suitably chosen functions $\varepsilon(h)$ and $g(h)$, we easily obtain the following corollary from (3.15).

COROLLARY 3.2. *If ζ_t, $t \in R^m$, is a separable Gaussian field, twice differentiable in quadratic mean, for which*

$$\max_{i,j \in [1,m]} \mathbf{E}\left|\frac{\partial^2 \zeta_{t+\tilde{h}}}{\partial t_i \partial t_j} - \frac{\partial^2 \zeta_t}{\partial t_i \partial t_j}\right|^2 \leqq \frac{C}{|\log |\tilde{h}||^{1+\varepsilon}}, \qquad C > 0, \varepsilon > 0, \tag{3.18}$$

then with probability one the fields $\partial^2 \zeta_t / \partial t_i \partial t_j$ have continuous trajectories for which (3.11) *is satisfied for any $\varepsilon > 0$ and any integer $\ell = km$. The complex of conditions $C_{\zeta,k}$, $k = 1, 2, \cdots$, holds for ζ_t.*

We will denote the positive (negative) definiteness of a matrix A by $A \succ 0$ ($A \prec 0$), and also put $\nabla \zeta_t \nabla' = \|\partial^2 \zeta_t / \partial t_i \partial t_j\|$. We introduce the following functions:

$$I_+(\nabla \zeta_t \nabla') = \begin{cases} 1, & \nabla \zeta_t \nabla' \prec 0, \\ 0, & \nabla \zeta_t \nabla' \not\prec 0, \end{cases} \qquad I_-(\nabla \zeta_t \nabla') = \begin{cases} 1, & \nabla \zeta_t \nabla' \succ 0, \\ 0, & \nabla \zeta_t \nabla' \not\succ 0. \end{cases} \tag{3.19}$$

THEOREM 3.4. *If the complex of conditions $C_{\zeta,k}$ is satisfied, then the k-parameter functions with respect to Lebesgue measure in R^m of the random point sets of stationary points $S_0 = \{\tau\}$, of bursts $S_+ = \{\tau\}$, and of local minima $S_- = \{\tau\}$ generated by the random field ζ_t, $t \in R^m$, at which $\zeta_\tau > u$, are given respectively by*

$$\begin{aligned} &\lambda_u^0(t^1, \cdots, t^k) \\ &= \int_{u_i \geqq u, i \in [1,k]} \mathbf{E}\left\{\prod_{i=1}^k |\det \nabla \zeta_{t^i} \nabla'| \,\middle|\, \zeta_{t^i} = u_i, \nabla \zeta_{t^i} = 0, i \in [1, k]\right\} \\ &\qquad p_{t^1 \cdots t^k}(u_1, \cdots, u_k, \dot{0})\, du_1 \cdots du_k, \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\lambda_u^+(t^1, \cdots, t^k) \\ &= \int_{u_i \geqq u, i \in [1,k]} \mathbf{E}\left\{\prod_{i=1}^k |\det \nabla \zeta_{t^i} \nabla'| I_+(\nabla \zeta_{t^i} \nabla') \,\middle|\, \zeta_{t^i} = u, \nabla \zeta_{t^i} = 0, i \in [1, k]\right\} \\ &\qquad p_{t^1 \cdots t^k}(u_1, \cdots, u_k, \dot{0})\, du_1 \cdots du_k, \end{aligned} \tag{3.21}$$

$$\begin{aligned} &\lambda_u^-(t^1, \cdots, t^k) \\ &= \int_{u_i \geqq u, i \in [1,k]} \mathbf{E}\left\{\prod_{i=1}^k |\det \nabla \zeta_{t^i} \nabla'| I_-(\nabla \zeta_{t^i} \nabla') \,\middle|\, \zeta_{t^i} = u, \nabla \zeta_{t^i} = 0, i \in [1, k]\right\} \\ &\qquad p_{t^1 \cdots t^k}(u_1, \cdots, u_k, \dot{0})\, du_1 \cdots du_k. \end{aligned} \tag{3.22}$$

The proof is carried out by constructing upper and lower bounds for the probabilities of the events that there lies in the cube

$$\Delta_h(t) = \left\{s : |t_i - s_i| \leqq \frac{h}{2}, i \in [1, m], t, s \in R^m\right\}, \qquad h \downarrow 0, \tag{3.23}$$

at least one point τ from S_0, S_+, S_-, at which $\zeta_\tau \geqq u$. In this connection, one first considers those subsets of S_0, S_+, S_- for which

$$\max_{i,j}\left|\frac{\partial^2\zeta_\tau}{\partial t_i \partial t_j}\right| \leqq A, \ |\det \nabla\zeta_\tau \nabla'| \geqq \varepsilon. \tag{3.24}$$

Using the properties of the moduli of continuity $\omega_{i,j}(h)$, expressed by (3.11), we obtain the necessary bounds, following which we let $\varepsilon \downarrow 0, A \uparrow \infty$. The proof is not difficult, but because the formulas are cumbersome it would take up a lot of space, and will therefore not be given.

The relations (3.20) through (3.22) could be used as the basis for obtaining numerical results for concrete random fields. In this connection, it seems natural to use the method of statistical modeling in order to compute the integrals in (3.20) through (3.22). The corresponding programs have been tested for $m = 2, 3$, $k = 1, 2$. For $k = 1$ for a homogeneous Gaussian field satisfying $C_{\zeta,1}$, the principal terms in an asymptotic expansion of $\lambda_u^+(t)$ and $\lambda_u^0(t)$ for $u \uparrow \infty$ are found in a paper by Nosko [16]. It turns out that

$$\begin{aligned} \lambda_u^0(t) &= \lambda_u^+(t)\left[1 + O\left(\frac{1}{u}\right)\right] \\ &= \frac{[\det \Lambda^{(1)}]^{1/2}}{(2\pi)^{(m+1)/2}\lambda_0^{(2m-1)/2}}\, u^{m-1} \exp\left\{\frac{-u^2}{2\lambda_0}\right\}\left[1 + O\left(\frac{1}{u}\right)\right], \end{aligned} \tag{3.25}$$

$$\mathbf{E}\zeta_t = 0, \qquad \mathbf{E}\zeta_t^2 = \lambda_0, \qquad \Lambda^{(1)} = \mathbf{E}\nabla\zeta_t(\nabla\zeta_t)'.$$

It follows from (3.25) that for $u \uparrow \infty$ the parameter functions of the random point set S_0 and its subset S_+ are equivalent. This corresponds to the intuitive assumption that flattenings of the field, that is, points τ where $\nabla\zeta_\tau = 0$, at high levels correspond, as a rule, to local maxima. This is related to the fact that high saddle points and high local minima can appear only close to excursions having a complicated structure, for example near two-vertex bursts or vertices having hollows resembling craters, and so forth. Thus, for normal fields high vertices of a complicated structure are encountered infrequently.

Numerical calculations carried out for a homogeneous isotropic Gaussian field ζ_t, $t \in R^2$, with covariance function $R(x) = \exp\{-x_1^2 - x_2^2\}$, have shown that already at the level $u = \sqrt{3}$ the remainder term $O(1/u)$ does not have a practical influence. Thus, it may be hoped that calculations with the formula (3.25) for moderate values of u can be carried out on the basis of the principal term of the asymptotic expansion.

Using the method presented in Section 2 for determining the special conditional probabilities, it is easy to obtain the distribution of high bursts $\zeta_\tau = u + m_u$, given that a burst with height greater than u occurred at $\tau = t$. In particular, for $u \uparrow \infty$ we obtain from (3.25):

COROLLARY 3.3. *The limit of the special conditional distribution of the height of bursts of a homogeneous Gaussian field ζ_t, $t \in R^m$, satisfying $C_{\zeta,1}$, given that the height of the burst is greater than u, is given by*

$$\lim_{u\uparrow\infty} P\{um_u > v\} = \exp\left\{\frac{-v}{\lambda_0}\right\}, \tag{3.26}$$

for $\mathbf{E}\zeta_t = 0$, $\mathbf{E}\zeta_t^2 = \lambda_0$.

Study of the structure of a field ζ_t in the "neighborhood" of a burst also can be carried out by means of the special conditional probabilities. Here with every family of points $t^i \in R^m$ and families of sets $A_i \subset R^1$, $B_i \subset R^m$, $C_i \subset R^{m(m+1)/2}$, and so forth, one can associate the random point subset S'_+ of those bursts of the homogeneous field ζ_t such that if $\tau \in S_+$, then $\tau \in S'_+$ if $\zeta_\tau \in A_0$, $\zeta_{\tau+t^i} \in A_i$, $\nabla\zeta_{\tau+t^i} \in B_i$, $\nabla\zeta_{\tau+t^i}\nabla' \in C_i$, $i \in [1, n]$. If the conditions $C_{\zeta,k}$, $k = 1, 2, \cdots$ are satisfied, then the parameter function of S'_+ is given by

$$\lambda^+_{t^1\cdots t^n}(A_0, A_i, B_i, C_i, i\in[1,n]) = \int_{G_0} |\det \|\ddot{u}^0_{j,\ell}\|| p_{0,t^1,\cdots,t^n}(u, \dot{U}, \ddot{Z})\, du\, d\ddot{Z}, \tag{3.27}$$

where

$$G_0 = \{u_0 \in A_0, u_i \in A_i, (\dot{u}^i_1, \cdots, \dot{u}^i_m) \in B_i, \|\ddot{u}^i_{j,\ell}\| \in C_i, i \in [1, n], \ell, j \in [1, m], \|\ddot{u}^0_{j,\ell}\| \prec 0\}, \tag{3.28}$$

$$du = du_0\, du_1 \cdots du_n, \qquad \dot{U} = \|\dot{u}^i_j\|, \qquad \ddot{Z} = \|\ddot{u}^i_{j,\ell}\|, \qquad \dot{u}^0_j = 0.$$

The special conditional probabilities in the space of trajectories of the field having bursts at the point $\tau = 0$, of height $h \in A_0$, such that $\zeta_{t^i} \in A_i$, and so forth, are introduced as the ratios

$$P^*_{t^1\cdots t^n}(A_i, B_i, C_i, i \in [1, n] | A_0) = \frac{\lambda^+_{t^1\cdots t^n}(A_0, A_i, B_i, C_i, i \in [1, n])}{\lambda^+_0(A_0)}. \tag{3.29}$$

Interesting results were obtained in a paper by Nosko [16] on the structure of excursions of homogeneous Gaussian random fields above unboundedly increasing levels $u \uparrow \infty$. It was shown that when the complex of conditions $C_{\zeta,k}$, $k = 1, 2, \cdots$, is satisfied, then with probability arbitrarily close to unity an excursion of a trajectory of the field ζ_t, $t \in R^m$, containing a burst at the point τ, can be approximated within $o(1/u)$ by that part of the second order surface

$$z = u + \tfrac{1}{2}(t - \tau)'\Lambda_u(t - \tau) \tag{3.30}$$

lying on the plane $z = u$ in the space R^{m+1} of points $(t_1, \cdots, t_m, z)$, where

$$\Lambda_u = u \cdot \|-\lambda_{i,j}/\lambda_0\|, \qquad \lambda_{i,j} = \mathbf{E}\,\frac{\partial\zeta_t}{\partial t_i}\,\frac{\partial\zeta_t}{\partial t_j}. \tag{3.31}$$

This result was obtained by introducing in the space of trajectories special conditional probabilities analogous to (3.29). For the particular case $m = 2$, the excursions are approximated by segments of an elliptical paraboloid. We mention here the following result.

THEOREM 3.5 (Nosko [16]). *Let ζ_t, $t \in R^2$, be a homogeneous Gaussian field satisfying the complex of conditions $C_{\zeta,k}$, $k = 1, 2, \cdots$. Then for $u \uparrow \infty$ the special conditional probabilities of the following functionals of excursions above the level u:*

the maximum m_u of the excursion, its cross sectional area S_u and its volume V_u, satisfy the relations

$$\lim_{u\uparrow\infty} P^*\{um_u > v\} = \lim_{u\uparrow\infty} P^*\{\gamma u S_u^2 > v\} = \lim_{u\uparrow\infty} P^*\{(2\gamma u^3 V_u)^{1/2} > v\} = \exp\left\{-\frac{v}{\lambda_0}\right\}, \tag{3.32}$$

where

$$\gamma = (2\pi\lambda_0)^{-1}\{\lambda_{1,1}\lambda_{2,2} - \lambda_{1,2}^2\}^{1/2}. \tag{3.33}$$

4. Estimation of the distribution of the maximum of a random field

Problems concerning the distribution of the maximum of a random field ζ_t when $t \in G$, where G is a region or surface in R^m, are interesting. An attempt to obtain the exact solution for such problems meets with considerable difficulties. The situation, however, is made easier by the fact that frequently one only needs to find an estimate for the probability of a trajectory of the field going above a high level u. In such cases, one can use an asymptotic approach [1]. For example, if ζ_t, $t \in R^m$, is a real Gaussian random field, and S_+ is the random point set of bursts of ζ_t above the level u, then for a region $G \subset R^m$ with a smooth boundary ∂G we have

$$P\{\sup_{t\in G} \zeta_t > u\} \leqq \mathbf{E}\eta_u^+(G) + \mathbf{E}\eta_u^+(\partial G), \tag{4.1}$$

where $\mathbf{E}\eta_u^+(G)$ is the average number of bursts in $S_+ \cap G$, and $\mathbf{E}\eta_u^+(\partial G)$ is the average number of bursts above the level u of the field ζ_t, $t \in \partial G$. One can show that if the complex of conditions $C_{\zeta,1}$ is satisfied, then the principal term in an asymptotic expression for the right side of (4.1) for $u \uparrow \infty$ has the form $\lambda_u^0 V$, where V is the volume of the region G and λ_u^0 can be found from (3.25). Thus, the level u_α for which $P\{\sup_{t\in G} \zeta_t > u_\alpha\} \approx \alpha$ can be found from the simple transcendental equation

$$VC_m\left(\frac{K_\alpha}{\sqrt{2\lambda_0}}\right)^{m-1} \exp\left\{-\left(\frac{K_\alpha}{\sqrt{2\lambda_0}}\right)^2\right\} = \alpha, \tag{4.2}$$

where

$$C_m = \frac{[\det \Lambda^{(1)}]^{1/2}}{2\pi^{(m+1)/2}\lambda_0^{m/2}}. \tag{4.3}$$

The results of the numerical computations which were mentioned in Section 3 show that, presumably, the u_α obtained from (4.2) gives good results if $u_\alpha/\sqrt{\lambda_0} > 3$. However, the problem of estimating the error which occurs here has not yet been solved.

If one assumes that the random stream of bursts above an unboundedly increasing level u is Poisson, then following the method of Cramér [7] one can obtain that for $G \uparrow R^m$

$$\lim_{V=V(G)\uparrow\infty} P\left\{\sup_{x\in G}\frac{\zeta_x}{\sqrt{\lambda_0}} < \sqrt{2\log V} + \frac{(m-1)\log\log V}{2\sqrt{2\log V}} + \frac{(m-1)\log(\sqrt{2\lambda_0}) + \log C_m + z}{\sqrt{2\log V}}\right\} = \exp\{-\exp\{-z\}\}. \tag{4.4}$$

As G increases, one has to assume that ∂G is regular, that is, the volume V_d of the set of points whose distance from G does not exceed d is such that $V_d/V \to 0$. The relation (4.4) also can be used to calculate the critical level u_α. Denoting by z_β the quantile of the level β, $\exp\{-\exp\{-z_\beta\}\} = \beta$, $0 < \beta < 1$, we obtain from (4.4)

$$\sqrt{\lambda_0}u_\alpha \approx \sqrt{2\log V} + \frac{(m-1)\log\log V}{2\sqrt{2\log V}} + \frac{(m-1)\log(\sqrt{2\lambda_0}) + \log C_m + z_{1-\alpha}}{\sqrt{2\log V}}. \tag{4.5}$$

It should be kept in mind that for small α ($\alpha \leqq 0.05$) and small values of V, calculation of the critical level by means of (4.5) can yield a nonmonotone dependence upon V. The method of calculation by means of (4.2) appears to be preferable.

By methods similar to those mentioned in Section 3, we have derived expressions for the parameter functions of the random set of critical points on the level lines of fields defined on manifolds [18] in R^m [1]. These expressions can be used to estimate the distribution of the maximum of a field defined over a surface.

REFERENCES

[1] Yu. K. Belyayev, "Distribution of the maximum of a random field and its application to problems of reliability," *Izv. Akad. Nauk SSSR, Tehn. Kibernet.*, No. 2 (1970), pp. 77–84.

[2] ———, "New results and generalizations of problems of crossing type," in the Russian edition only of H. Cramér and M. Leadbetter, *Stationary Stochastic Processes*, Moscow, Izdat. "Mir," 1969, pp. 341–378.

[3] ———, "The random sets of bursts and shines of random fields," *Proceedings USSR-Japan Symposium on Probability*, Novosibirsk, 1969, pp. 11–18.

[4] ———, "On the characteristics of a random flow," *Theor. Probability Appl.*, Vol. 13 (1968), pp. 543–544.

[5] Yu. K. Belyayev and V. P. Nosko, "Characteristics of excursions above a high level for a Gaussian process and its envelope," *Theor. Probability Appl.*, Vol. 14 (1969), pp. 296–309.

[6] E. V. Bulinskaya, "On the mean number of crossings of a level by a stationary Gaussian process," *Theor. Probability Appl.*, Vol. 6 (1961), pp. 435–438.

[7] H. Cramér and M. Leadbetter, *Stationary and Related Stochastic Processes; Sample Function Properties and Their Applications*, New York, Wiley, 1967.

[8] R. M. Dudley, "Gaussian processes on several parameters," *Ann. Math. Statist.*, Vol. 36 (1965), pp. 771–788.

[9] A. Ya. Hinčin, *Papers on Queuing Theory*, Moscow, Fizmatgiz, 1963.

[10] M. KAC and D. SLEPIAN, "Large excursions of a Gaussian process," *Ann. Math. Statist.*, Vol. 30 (1959), pp. 1215–1228.

[11] D. KÖNIG, K. MATTHES, and K. NAWROTZKI, "Verallgemeinerungen der Erlangschen und Engsetschen formeln, eine Methode in der Bedienungstheorie," *Angewandte Mathematik und Mechanik*, Vol. 5 (1967), pp. 1–123.

[12] M. S. LONGUET-HIGGINS, "The statistical analysis of a random, moving surface," *Philos. Trans. Roy. Soc. London Ser. A*, Vol. 249 (1957), pp. 321–387.

[13] ———, "The statistical geometry of random surfaces," *Hydraulic Instability*, Symposium in Applied Mathematics, Vol. 13 (1962), pp. 105–143.

[14] K. MATTHES, "Stationäre zufällige Punktfolgen," *Jahresbericht d. Deutsch. Math. Verein*, Vol. 66 (1963/4), pp. 66–79.

[15] J. A. MCFADDEN, "On the lengths of intervals in a stationary point process," *J. Roy. Statist. Soc. Ser. B*, Vol. 24 (1962), pp. 364–382.

[16] V. P. NOSKO, "The characteristics of excursions of Gaussian homogeneous fields above a high level," *Proceedings USSR-Japan Symposium on Probability*, Novosibirsk, 1969.

[17] C. PALM, "Intensitätsschwankungen in Fernsprechverkehr," *Ericsson Technics*, Vol. 44 (1943), pp. 1–189.

[18] P. K. RAŠEVSKIĬ, *Riemannian Geometry and Tensor Analysis*, Moscow, Gostehizdat, 1953.

[19] S. O. RICE, "The mathematical analysis of random noise," *Bell System Tech. J.*, Vol. 23 (1944), pp. 282–332; Vol. 24 (1945), pp. 46–156.

[20] C. RYLL-NARDZEWSKI, "Remarks on processes of calls," *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1961, Vol. 2, pp. 66–79.

[21] E. SLUTSKY, "Qualche proposizioni relativa alla teoria delle funzioni aleatorie," *Giorn. Ist. Ital. Attuari*, Vol. 8 (1937), pp. 183–199.

[22] V. I. TIHONOV, *Excursions of Random Processes*, Moscow, Izdat. "Nauka," 1970.

[23] F. ZÍTEK, "On the theory of ordinary streams," *Czechoslovak Math. J.*, Vol. 8 (1958), pp. 448–459. (In Russian, summary in German.)