CENTRAL LIMIT THEOREM FOR STATIONARY PROCESSES

M. ROSENBLATT
UNIVERSITY OF CALIFORNIA, SAN DIEGO

1. Introduction

A discussion of strong mixing and uniform ergodicity is presented, partly in terms of their relation to the central limit problem. Some of the gaps in one's understanding of the proper domain of validity of the central limit theorem for stationary sequences are pointed out. A definition of strong mixing appropriate for stationary random fields is given. A version of a limit theorem for stationary random fields with asymptotic normality is then derived. The argument for this limit theorem uses martingalelike ideas.

2. Stationary sequences

By this time there is an extensive literature on the central limit theorem for stationary processes, especially with respect to asymptotic normality. However, much of this is still rather unsatisfactory since it leads to effective computational results only under limited circumstances. We shall give a brief sketch of some of the ideas that have been used. For convenience, discrete time stationary processes will be discussed for the most part since the case of continuous time parameter processes can usually be easily reduced to the discrete time case.

Let $\{X_n, n = \cdots, -1, 0, 1, \cdots\}$ be a discrete time parameter stationary process. The Borel fields $\mathscr{B}_n = \mathscr{B}(X_k, k \leqq n)$, $\mathscr{F}_m = \mathscr{B}(X_k, k \geqq m)$ are generated by the random variables up to time n and from time m, respectively. They represent the past relative to n and future relative to m, respectively. A condition called strong mixing was proposed in [12] and amounted to

$$\sup_{B \in \mathscr{B}_0,\, F \in \mathscr{F}_n} |P(BF) - P(B)P(F)| \to 0 \tag{2.1}$$

as $n \to \infty$ where P is the probability measure of the stationary process. The condition has interest on its own but it was originally proposed together with some additional moment conditions to get asymptotic normality for partial sums of the random variables of a process properly normalized. A later version of such a central limit theorem using strong mixing can be found in Ibragimov's paper [6]. However, the condition (2.1) also has an amusing alternative interpretation

This research was supported by the Office of Naval Research, Grant No. N00014-69-A-0200-6016.

in terms of prediction. Suppose we were to predict the indicator function I_F of the event $F \in \mathscr{F}_m$, $m > n$, in terms of the past relative to n. The best predictor in terms of mean square error would be $P(F \mid \mathscr{B}_n)$ with

$$E|P(F \mid \mathscr{B}_m) - I_F|^2 = P(F) - \int_F P(F \mid \mathscr{B}_m)\, dP. \tag{2.2}$$

The predictor making no use of the available information would simply approximate I_F by the constant $P(F)$. The error would then be

$$E|P(F) - I_F|^2 = P(F) - P(F)^2. \tag{2.3}$$

It would then be appropriate to call a process *uniformly purely nondeterministic* if

$$\sup_{F \in \mathscr{F}_m} \left| \int_F [P(F \mid \mathscr{B}_0) - P(F)]\, dP \right| \to 0 \tag{2.4}$$

as $m \to \infty$. A simple argument making use of the basic properties of conditional probabilities leads to the following lemma.

LEMMA 2.1. *A stationary process is strongly mixing if and only if it is uniformly purely nondeterministic.*

A weaker condition than (2.1) was considered by Cogburn [2]. Let τ be the shift transformation of the stationary process. He called the process uniformly ergodic if

$$\sup_{B \in \mathscr{B}_0,\, F \in \mathscr{F}_0} \left| \frac{1}{n} \sum_{k=1}^{n} P(B \cap \tau^k F) - P(B)P(F) \right| \to 0 \tag{2.5}$$

as $n \to \infty$. Let $h_n(\cdot)$ be a sequence of instantaneous functions (measurable). Consider the sequence of partial sums $\Sigma_{k=1}^{n} h_n(X_k)$ already adjusted (say by normalization) so that the typical term converges to zero in probability as $n \to \infty$. If the process is uniformly ergodic, such partial sums can always be approximated in distribution by infinitely divisible laws. In fact, Cogburn has shown that if $\{X_k\}$ is Markov, uniform ergodicity is not only sufficient, but also necessary for this approximation property to always hold. Uniform ergodicity certainly implies strong mixing. It is natural to ask how much stronger it is. If $\{X_k\}$ is uniformly ergodic, it can have a tail field consisting of at most a finite number of cyclically moving sets (see [2]). Suppose we consider the random walk on the circle group generated by the probability measure η, that is, the Markov process with transition function

$$P(x, A) = \eta(A - x) \tag{2.6}$$

where $x \in (0, 1]$ and A is a Borel set. The invariant measure is the uniform measure on $(0, 1]$. Let $\{X_k\}$ be the stationary Markov process generated by (2.6) and the invariant measure. Then if $\{X_k\}$ is uniformly ergodic and has a trivial tail field, it must be strongly mixing. It would be very interesting to find out whether or not this is true generally.

So as to get a better understanding of a condition like (2.1), we look at it in the case of a stationary Markov process. Let $P(\cdot, \cdot)$ be the transition probability function of the process and μ the invariant probability measure. Given a function f, let $\|f\|_p$ denote the $L^p(d\mu)$ norm of f, $1 \leqq p \leqq \infty$. Set

$$(Tf)(x) = \int P(x, dy) f(y). \tag{2.7}$$

A simple argument shows that strong mixing for the stationary Markov process with transition function $P(\cdot, \cdot)$ and invariant measure μ is equivalent to

$$\sup_{f \perp 1} \frac{\|T^n f\|_1}{\|f\|_\infty} \to 0 \tag{2.8}$$

as $n \to \infty$ where by $f \perp 1$ we mean $Ef(X_n) = \int f(x)\mu(dx) = 0$. Let us consider the following condition of Harris (see [5] and [10]). Assume that

$$\sum_{j=1}^{\infty} P_j(x, A) = \infty \tag{2.9}$$

for every x and every measurable set A with $\mu(A) > 0$. Here $P_j(\cdot, \cdot)$ is the jth step transition function generated by $P(\cdot, \cdot)$, that is,

$$P_{j+1}(x, A) = \int P_j(x, dy) P(y, A), \qquad j = 1, 2, \cdots. \tag{2.10}$$

One can then show that if there are no cyclically moving sets (the Markov process is purely nondeterministic) then (2.9) implies that

$$\operatorname{Var}\left(\mu - P_n(x, \cdot)\right) \to 0 \tag{2.11}$$

as $n \to \infty$ for each x where Var (η) is the total variation of the set function η. However, this implies that (2.8) is satisfied; the process is strongly mixing. In particular, this means that any stationary positive recurrent Markov chain with trivial tail field (no nontrivial invariant or cyclically moving sets) is strongly mixing. Similarly one can show that uniform ergodicity is equivalent to

$$\sup_{f \perp 1} \frac{\left\| \frac{1}{n} \sum_{j=1}^{n} T^j f \right\|_1}{\|f\|_\infty} \to 0 \tag{2.12}$$

as $n \to \infty$ for a stationary Markov process. It is clear that if a stationary Markov process satisfies the Harris condition it is uniformly ergodic. However, it is easy to construct many Markov processes which are strongly mixing but do not satisfy the Harris condition.

It is surprising that even with such tools available, in the case of stationary Markov chains there isn't any central limit theorem that naturally and fully generalizes the one available for independent and identically distributed random variables. Let $\{X_k\}$ be a stationary ergodic Markov chain. Assume that $h \equiv h_n$

and set $Y_k = h(X_k)$. There is then an old but very elegant result due to Doeblin [1] that runs as follows. Let $1 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_\ell \leqq n$ be the successive distinct (random) times that the trajectory of the chain $\{X_k\}$ hits a fixed state i between one and n. Let

$$Z_j = \sum_{k=\alpha_j}^{\alpha_{j+1}-1} Y_k. \tag{2.13}$$

Doeblin's result states that $(1/\sqrt{n}) \sum_{k=1}^{n} \{Y_k - E(Y)\}$ is asymptotically normally distributed as $n \to \infty$ if

$$0 < \sigma^2 = E(Z - EZ)^2 < \infty. \tag{2.14}$$

However, conditions directly in terms of the transition probabilities of the process $\{X_k\}$ and the moments of the Y_k would be much more natural. A requirement involving (2.13) unfortunately makes use of information on hitting distributions as well. There is no broad result of the kind one would like. However, one can obtain a very limited result of this type. Assume that the chain has no cyclically moving states. Consider a fixed state i of the chain. The number of times the state is occupied in n steps is asymptotically normal (when appropriately centered and normalized) if the recurrence time distribution for the state has finite second moment. An argument of Feller (see [3]) indicates that the recurrence time distribution for i has finite second moment if and only if $\sum_{n=1}^{\infty} |p_{i,i}^{(n)} - (1/\mu_i)| < \infty$. By a standard comparison argument we obtain the following result.

LEMMA 2.2. *Let $\{X_k\}$ be a stationary Markov chain that is purely nondeterministic. Consider any bounded nonconstant function f on the states of the chain. If for some state i*

$$\sum_{n=1}^{\infty} \left| p_{i,i}^{(n)} - \frac{1}{\mu_i} \right| < \infty \tag{2.15}$$

(this then holds for every i) and $\sigma^2\left(\sum_{j=1}^{n} f(X_j)\right) \to \infty$ as $n \to \infty$, it follows that $(1/\sqrt{n}) \sum_{j=1}^{n} \{f(X_j) - E(X)\}$ is asymptotically normally distributed as $n \to \infty$.

The techniques referred to thus far involve strong specifications such as (2.1) or (2.5) on the asymptotic behavior of the process. One could alternatively try to get a result that involves the specific functional f dealt with more explicitly rather than a global requirement on the underlying process. An interesting result of this sort has been obtained by Gordin [4]. The argument depends on a reduction to a central limit theorem for martingale differences.

THEOREM 2.1. *Let f be a measurable function on the probability space of an ergodic stationary process. If*

$$\sum_{n=1}^{\infty} E^{1/2}|E(f(w)|\mathscr{B}_{-n})|^2 + \sum_{n=1}^{\infty} E^{1/2}|f(w) - E(f(w)|\mathscr{F}_n)|^2 < \infty \tag{2.16}$$

and

$$\sigma^2\left(\sum_{j=1}^{n} f(\tau^j w)\right) \cong \sigma^2 n, \qquad \sigma^2 > 0, \tag{2.17}$$

then $(1/\sqrt{n}\,\sigma)\sum_{j=1}^{n}(f(\tau^j w) - Ef)$ *is asymptotically Gaussian as* $n \to \infty$ *with mean zero and variance one.*

In particular cases it is not clear whether Gordin's theorem would allow one to obtain results obtained by very special techniques more readily. An example of such a result obtained by computations with cumulants is the following theorem of Sun [14].

THEOREM 2.2. *Let* $\{X_n; n = \cdots, -1, 0, 1, \cdots\}$, $EX_n \equiv 0$, *be a stationary Gaussian process with absolutely continuous spectrum and spectral density* $f \in L^2$. *Assume that*

$$\lim_{N\to\infty} \frac{1}{2\pi N}\int_{-\pi}^{\pi} \frac{\sin^2 (N/2)\lambda}{\sin^2(\lambda/2)} f(\lambda)\, d\lambda = a \tag{2.18}$$

exists with $0 < a < \infty$. *Consider the derived process* $Y_n = G(X_n)$ *obtained by a real instantaneous function* G *where* $EY_n \equiv 0$ *and* $EY_n^2 < \infty$. *Then* $N^{-1/2}\sum_{n=1}^{N} Y_n$ *is asymptotically normal with mean zero and finite variance.*

3. Stationary fields

In this section we shall consider getting a central limit theorem for stationary processes with a multidimensional time index, that is, for stationary fields. Since the essence of all the ideas dealt with in this section already arise fully in the two dimensional case, no generality will be lost in dealing with a real-valued stationary random field $\{X_{n,m}(w); n, m = \cdots, -1, 0, 1, \cdots\}$. In the usual Kolmogorov construction, one would have the points $w = (w_{n,m})$ of the probability space real-valued functions on pairs of integers with $X_{n,m}(w) = w_{n,m}$. There are the two commuting shift transformations τ_1, τ_2 with $\tau_1 w = (w_{n+1,m})$ and $\tau_2 \omega = (w_{n,m+1})$. Stationarity of the process amounts to invariance of the probability measure P under these two shifts, that is, for any measurable set A (set in the Borel field $\mathscr{B}$ of the process where $\mathscr{B}$ is the Borel field generated by the random variables $X_{n,m}$)

$$P(\tau_1 A) = P(\tau_2 A) = P(A). \tag{3.1}$$

There are some additional preliminary observations we shall make before discussing the central limit theorem. By *ergodicity* of the process one means that

$$\lim_{n_1, n_2 \to \infty} \frac{1}{n_1 n_2} \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} P(\tau_1^j \tau_2^k A) = P(A) \tag{3.2}$$

for each measurable set A. A set A is said to be *invariant* if $\tau_1 A = \tau_2 A = A$ up to an exceptional set of probability zero. Just as in the case of stationary sequences, one can show that the process $\{X_{n,m}\}$ is ergodic if and only if the only measurable invariant sets are trivial, that is, of probability one or probability zero. The process $\{X_{n,m}\}$ is mixing if for every pair of measurable sets A and B

$$\lim_{|k|\to\infty} P(A \cap \tau_1^{k_1}\tau_2^{k_2}B) = P(A)P(B) \tag{3.3}$$

where $|k| = (k_1^2 + k_2^2)^{1/2}$. It is interesting to see what these conditions amount to in the case of a Gaussian stationary process $\{X_{n,m}\}$ with mean $EX_{n,m} \equiv 0$. The covariances

$$r_{\alpha,\beta} = E(X_{n,m}X_{n+\alpha,m+\beta}) \tag{3.4}$$

have the Fourier representation

$$r_{\alpha,\beta} = \int_{-\pi}^{\pi}\int \exp\{i\alpha\lambda + i\beta\mu\}\, dF(\lambda, \mu) \tag{3.5}$$

in terms of monotone nondecreasing function F of bounded increase commonly called the spectral distribution function of the process. The following results characterize the class of ergodic and mixing Gaussian stationary processes.

LEMMA 3.1. *The Gaussian stationary process* $\{X_{n,m}; n, m = \cdots, -1, 0, 1, \cdots\}$ *is ergodic if and only if the spectral distribution function of the process has no jumps.*

A simple adaptation of the proof given in [9] for a one dimensional time parameter to the present context leads to the desired result.

LEMMA 3.2. *The Gaussian stationary process* $\{X_{n,m}\}$ *is mixing if and only if*

$$\lim_{|k|\to\infty} r_{k_1,k_2} = 0. \tag{3.6}$$

The argument of [9] is again all that is required.

Let us now introduce some notation that will be helpful in stating and deriving the central limit theorem. A derived stationary process

$$Y_{\alpha,\beta} = f(\tau_1^{\alpha}\tau_2^{\beta}w) \tag{3.7}$$

is given by the measurable function f on the probability space of the process $\{X_{n,m}\}$. Actually we can for the most part think of the derived process given by an instantaneous function

$$Y_{\alpha,\beta} = f(X_{\alpha,\beta}). \tag{3.8}$$

First we shall discuss a simple adaptation to the case of multidimensional index of the idea of strong mixing. Consider the following idea of strong mixing. Let S and S' be two sets of indices. The Borel fields $\mathscr{B}(S) = \mathscr{B}(X_\tau, \tau \in S)$ and $\mathscr{B}(S') = \mathscr{B}(X_\tau, \tau \in S')$ are the Borel fields generated by the random variables with subscript τ belonging to S and S', respectively. Let $d(S, S')$ be the distance between the sets of indices S and S'. We shall say that the process X is strongly mixing if

$$\sup_{A\in\mathscr{B}(S),\, B\in\mathscr{B}(S')} |P(AB) - P(A)P(B)| \leqq \varphi(d(S, S')) \tag{3.9}$$

where φ is a function such that $\varphi(d) \to 0$ as $d \to \infty$.

A result like that of Kolmogorov and Rozanov [8] for Gaussian stationary fields will be obtained now by a similar argument.

THEOREM 3.1. *Let $\{X_{n,m}\}$ be a Gaussian stationary field with absolutely continuous spectrum and continuous positive spectral density*

$$f(\lambda, \mu) = \frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu} F(\lambda, \mu) \tag{3.10}$$

(looked at as a function on the compact 2-torus $(-\pi, \pi] \times (-\pi, \pi]$). The process is then strongly mixing.

Just as in the paper of Kolmogorov and Rozanov, one can show that a Gaussian stationary process is strongly mixing if and only if the maximal correlation between $H(S)$ and $H(S')$ (the Hilbert spaces generated by linear forms in random variables with indices in S and S', respectively) is bounded by a function $\varphi(d(S, S'))$ which tends to zero as $d \to \infty$. If

$$\rho(d) = \sup_{\xi \in H(S),\, \xi' \in H(S')} \operatorname{Corr}(\xi, \xi') \tag{3.11}$$

it is clear that

$$\rho(d) = \sup_{p_1, p_2} \int p_1(\eta) p_2(\eta) f(\eta)\, d\eta \tag{3.12}$$

(here $\eta = (\lambda, \mu)$ and $d\eta = d\lambda d\mu$) with

$$\begin{aligned} p_1(\eta) &= \sum_{\tau_j \in S} \alpha_j \exp\{i\tau_j \cdot \eta\} \\ p_2(\eta) &= \sum_{\tau_j \in S'} \beta_j \exp\{-i\tau_j \cdot \eta\} \end{aligned} \tag{3.13}$$

trigonometric polynomials and

$$\int |p_i(\eta)|^2 f(\eta)\, d\eta \leqq 1, \qquad i = 1, 2. \tag{3.14}$$

It then follows that

$$\rho(d) \leqq \sup_p \int p(\eta) f(\eta)\, d\eta \tag{3.15}$$

where the p are trigonometric polynomials of the form

$$p(\eta) = \sum_{|\tau_j| \geqq d} \alpha_j \exp\{i\eta \cdot \tau_j\} \tag{3.16}$$

satisfying $\int |p(\eta)| f(\eta)\, d\eta$. Using the basic lemma in the Kolmogorov-Rozanov paper [8] it follows that

$$\rho(d) \leqq \inf_p \operatorname{ess\,sup}_\lambda |f(\eta) - p(\eta)| \frac{1}{f(\eta)} \tag{3.17}$$

where one lets p run over all polynomials of the type

$$p(\eta) = \sum_{|\tau_j| < d} \alpha_j \exp\{i\eta \cdot \tau_j\}. \tag{3.18}$$

By the Weierstrass approximation theorem for trigonometric polynomials it follows that $\rho(d) \to 0$ as $d \to \infty$ so that the process is strongly mixing. A random field arising in a simplified fluid flow model is given in [13]. Heuristically, arguments like this are occasionally used to justify a Gaussian approximation for the velocity distribution in the final period of decay of turbulence.

Rozanov briefly discusses quasi-Markov Gaussian random fields in [11]. A regular (with respect to interpolation) Gaussian field on the lattice points in the plane is quasi-Markov if and only if it has an absolutely continuous spectrum with spectral density

$$f(\lambda, \mu) = a\big(1 - \alpha \cos(\lambda + \mu) + \beta \cos(\lambda - \mu)\big)^{-1}. \tag{3.19}$$

By inspection one can show that one must have $|\alpha| + |\beta| < 1$ to insure integrability and a simple approximation using (3.17) shows that $\rho(d)$ in this case must decrease at least exponentially as a function of d.

We shall now consider a central limit theorem for a two dimensional stationary random field that is an analogue of the Theorem of Gordin for stationary random sequences. As before $\{X_{n,m}\}$ is ergodic and stationary with $\{Y_{n,m}\}$ as given by (3.7) a derived stationary field. Let $\{H_{n,m}\}$ be the Hilbert space of square integrable functions measurable with respect to

$$\mathscr{B}_{n,m} = \mathscr{B}(X_{j,k}; j < n \quad \text{or} \quad j = n, k \leqq m). \tag{3.20}$$

Set $S_{n,m} = H_{n,m} \ominus H_{n,m-1}$ (the orthogonal complement of $H_{n,m-1}$ in $H_{n,m}$). Assume that

$$EY_{n,m} \equiv 0, \quad EY_{n,m}^2 < \infty. \tag{3.21}$$

The projection operator (orthogonal) on a space H will be denoted by $P(H)$, Assume that the function f generating the Y process is in the direct sum $\oplus_{n,m=-\infty}^{\infty} S_{n,m}$. It then follows that if

$$f_k = P\Bigg(\bigoplus_{n,m=-k}^{k} S_{n,m}\Bigg) f \tag{3.22}$$

then $f_k \to f$ in mean square as $k \to \infty$. For convenience let

$$P_{\ell_1,\ell_2} = P(S_{\ell_1,\ell_2}) \tag{3.23}$$

and

$$\begin{aligned} U_1 f(w) &= f(\tau_1 w) \\ U_2 f(w) &= f(\tau_2 w). \end{aligned} \tag{3.24}$$

The operators U_1 and U_2 are unitary. Now

$$f = f_k + f - f_k = \sum_{\ell_1,\ell_2=-\infty}^{\infty} P_{\ell_1,\ell_2} f_k + f - f_k \tag{3.25}$$

$$
\begin{aligned}
= & \sum_{\ell_1,\ell_2=-\infty}^{\infty} U_1^{-\ell_1}U_2^{-\ell_2}P_{\ell_1,\ell_2}f_k \\
& + \sum_{\ell_1>0}\sum_{\ell_2>0}\Bigg\{-(U_1U_2 - U_1 - U_2 + I)\sum_{m_1=-\ell_1}^{-1}\sum_{m_2=-\ell_2}^{-1} U_1^{m_1}U_2^{m_2}P_{\ell_1,\ell_2}f_k \\
& \qquad + (U_1 - I)\sum_{m_1=-\ell_1}^{-1} U_1^{m_1}P_{\ell_1,\ell_2}f_k + (U_2 - I)\sum_{m_2=-\ell_2}^{-1} U_2^{m_2}P_{\ell_1,\ell_2}f_k\Bigg\} \\
& + \sum_{\ell_1>0}\sum_{\ell_2<0}\Bigg\{(U_1U_2 - U_1 - U_2 + I)\sum_{m_1=-\ell_1}^{-1}\sum_{m_2=0}^{-\ell_2-1} U_1^{m_1}U_2^{m_2}P_{\ell_1,\ell_2}f_k \\
& \qquad + (U_1 - I)\sum_{m_1=-\ell_1}^{-1} U_1^{m_1}P_{\ell_1,\ell_2}f_k - (U_2 - I)\sum_{m_2=0}^{-\ell_2-1} U_2^{m_2}P_{\ell_1,\ell_2}f_k\Bigg\} \\
& + \sum_{\ell_1<0}\sum_{\ell_2>0}\Bigg\{(U_1U_2 - U_1 - U_2 + I)\sum_{m_1=0}^{-\ell_1-1}\sum_{m_2=-\ell_2}^{-1} U_1^{m_1}U_2^{m_2}P_{\ell_1,\ell_2}f_k \\
& \qquad - (U_1 - I)\sum_{m_1=0}^{-\ell_1-1} U_1^{m_1}P_{\ell_1,\ell_2}f_k + (U_2 - I)\sum_{m_2=-\ell_2}^{-1} U_2^{m_2}P_{\ell_1,\ell_2}f_k\Bigg\} \\
& - \sum_{\ell_1<0}\sum_{\ell_2<0}\Bigg\{(U_1U_2 - U_1 - U_2 + I)\sum_{m_1=0}^{-\ell_1-1}\sum_{m_2=0}^{-\ell_2-1} U_1^{m_1}U_2^{m_2}P_{\ell_1,\ell_2}f_k \\
& \qquad - (U_1 - I)\sum_{m_1=0}^{-\ell_1-1} U_1^{m_1}P_{\ell_1,\ell_2}f_k - (U_2 - I)\sum_{m_2=0}^{-\ell_2-1} U_2^{m_2}P_{\ell_1,\ell_2}f_k\Bigg\} \\
& + (U_1 - I)\Bigg\{\sum_{\ell_1>0}\sum_{m_1=-\ell_1}^{-1} - \sum_{\ell_1<0}\sum_{m_1=0}^{-\ell_1-1}\Bigg\} U_1^{m_1}P_{\ell_1,0}f_k \\
& + (U_2 - I)\Bigg\{\sum_{\ell_2>0}\sum_{m_2=-\ell_2}^{-1} - \sum_{\ell_2<0}\sum_{m_2=0}^{-\ell_2-1}\Bigg\} U_2^{m_2}P_{0,\ell_2}f_k \\
= & \; h_k + (U_1U_2 - U_1 - U_2 + I)g_k + (U_1 - I)g_k' + (U_2 - I)g_k'' + f - f_k,
\end{aligned}
$$

where

(3.26)
$$h_k = \sum_n\sum_m U_1^{-n}U_2^{-m}P_{n,m}f_k.$$

Notice that

(3.27)
$$
\begin{aligned}
& \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(Y_{m,n} - U_1^mU_2^nh_k) \\
& \quad = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} U_1^mU_2^n(f - h_k) = \{U_1^MU_2^N - U_1^M - U_2^N + I\}g_k \\
& \qquad + \sum_{n=0}^{N-1}(U_1^M - I)U_2^ng_k' + \sum_{m=0}^{M-1}(U_2^N - I)U_1^mg_k'' \\
& \qquad\qquad + \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} U_1^mU_2^n(f - f_k).
\end{aligned}
$$

Set $a_k = f - f_k$. Consider

$$E[a_k U_1^m U_2^n a_k] = c(m, n; k) \tag{3.28}$$

and

$$E Y_{\alpha,\beta} Y_{\alpha+m,\beta+n} = c_{m,n}. \tag{3.29}$$

Assume that

$$\sum_{m,n} |c(m, n; k)| \to 0, \tag{3.30}$$

as $k \to \infty$ and that

$$\sum_{m,n} |c_{m,n}| < \infty \tag{3.31}$$

with

$$\sum_{m,n} c_{m,n} = \sigma^2 > 0. \tag{3.32}$$

Then $(1/\sqrt{MN}) \Sigma_{m=0}^{M-1} \Sigma_{n=0}^{N-1} (Y_{m,n} - U_1^m U_2^n h_k)$ converges to zero in mean square. The sum of the first three terms on the right of (3.27) normalized by $\sqrt{MN}$ obviously tends to zero in mean square. The last term over $\sqrt{MN}$ is small in mean square because of condition (3.30) if k is large. Thus (3.27) normalized will be small in mean square as $M, N \to \infty$ if k is sufficiently large. However the process $\{U_1^m U_2^n h_k\}$ has the property

$$E\{U_1^m U_2^n h_k \mid \mathscr{B}_{m,n-1}\} \equiv 0 \tag{3.33}$$

for all m, n and is ergodic. Further, conditions (3.30), (3.31), and (3.32) imply that the variance $\sigma^2(h_k)$ is positive for sufficiently large k and that $\sigma^2(h_k) \to \sigma^2$ as $k \to \infty$. A variant of the standard development of a central limit theorem for martingale differences (see [7]) indicates that $(1/\sqrt{MN}) \Sigma_{m=0}^{M-1} \Sigma_{n=0}^{N-1} U_1^m U_2^n h_k$ is asymptotically normally distributed with mean zero and variance $\sigma^2(h_k)$. A standard approximation argument then shows that $(1/\sqrt{MN}) \Sigma_{m=0}^{M-1} \Sigma_{n=0}^{N-1} Y_{m,n}$ is asymptotically normally distributed with mean zero and variance σ^2. We therefore have the following central limit theorem.

THEOREM 3.2. *Let $\{X_{m,n}\}$ be a strictly stationary ergodic random field. Assume that $\{Y_{m,n}\}$ is a derived random field generated by the measurable function f as in formula* (3.7). *Let $Ef = 0, Ef^2 < \infty$ with f in $\oplus_{n,m=-\infty}^{\infty} S_{n,m}$. If conditions* (3.30), (3.31), *and* (3.32) *are satisfied then $(1/\sqrt{MN}) \Sigma_{m=0}^{M-1} \Sigma_{n=0}^{N-1} Y_{m,n}$ is asymptotically normally distributed with mean zero and variance $\sigma^2 > 0$.*

REFERENCES

[1] K. L. CHUNG, *Markov Chains with Stationary Transition Probabilities*, Springer Verlag, Berlin, 1960.

[2] R. COGBURN, "Conditional probability operators," *Ann. Math. Statist.*, Vol. 33 (1962), pp. 634–658.

[3] W. FELLER, "Fluctuation theory of recurrent events," *Trans. Amer. Math. Soc.*, Vol. 67 (1949), pp. 98–119.

[4] M. I. GORDIN, "The central limit theorem for stationary processes," *Soviet. Math. Dokl.*, Vol. 10 (1969), pp. 1174–1176.

[5] T. E. HARRIS, "The existence of stationary measures for certain Markov processes," *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 2 (1956), pp. 113–124.

[6] I. A. IBRAGIMOV, "Some limit theorems for stationary processes," *Theory Prob. Appl.*, Vol. 7 (1962), pp. 349–382.

[7] ———, "A central limit theorem for a class of dependent random variables," *Theory Prob. Appl.*, Vol. 8 (1963), pp. 83–89.

[8] A. N. KOLMOGOROV and YU. A. ROZANOV, "On strong mixing conditions for stationary Gaussian processes," *Theory Prob. Appl.*, Vol. 5 (1960), pp. 204–208.

[9] G. MARUYAMA, "The harmonic analysis of stationary stochastic processes," *Mem. Facult. Sci. Kyusyu Univ. Ser. A*, Vol. 4 (1949), pp. 45–106.

[10] S. OREY, "Recurrent Markov Chains," *Pac. J. Math.*, Vol. 9 (1959), pp. 805–827.

[11] YU. A. ROZANOV, "On Gaussian fields with given conditional distributions," *Theory Prob. Appl.*, Vol. 12 (1967), pp. 381–391.

[12] M. ROSENBLATT, "A central limit theorem and a strong mixing condition," *Proc. Nat. Acad. Sci. U.S.A.*, Vol. 42 (1956), pp. 43–47.

[13] M. ROSENBLATT, "Remarks on the Burgers equation," *J. Math. Phys.*, Vol. 9 (1968), pp. 1129–1136.

[14] T. C. SUN, "Some further results on central limit theorems for non-linear functions of a normal stationary process," *J. Math. Mech.*, Vol. 14 (1965), pp. 71–85.