SETS OF BOUNDEDNESS AND CONTINUITY FOR THE CANONICAL NORMAL PROCESS

JACOB FELDMAN
UNIVERSITY OF CALIFORNIA, BERKELEY

1. Introduction

Let X be a linear space; by a *linear stochastic process* will be meant a map L from X to the linear space of measurable functions on some probability space $(\Omega, S, \Pr)$ such that, for each real a, b and each x, y in X, $aL_x + bL_y = L_{ax+by}$ with probability one. Two such maps are equivalent if they have the same finite dimensional joint distributions. Such processes have been objects of active investigation in recent years, both for their intrinsic interest and in connection with the study of other stochastic processes. For background, we refer the reader to [2].

A central role is played by the *canonical normal processes* on real Hilbert spaces, which are characterized by the conditions that each L_x is Gaussian with zero mean and $E\{L_xL_y\} = (x, y)$. The restrictions of these processes provide models for all Gaussian processes with zero mean, in the following way. Let $\{\xi_t, t \in T\}$ be a Gaussian process with zero mean. Let H be the closed subspace of $L^2(\Pr)$ spanned by the random variables of the process. Thus, a map $\phi : T \to H$ is defined. Let L be the map $: H \to L^0(\Pr)$ which assigns to each $x \in H$ the selfsame random variable regarded as a member of $L^0(\Pr)$. Then L is a linear map $: H \to L^0(\Pr)$, that is, a linear stochastic process on H. Each L_x is Gaussian with zero mean, and $E(L_xL_y) = (x, y)$, so that L is a version of the canonical normal process on H. The original process ξ is then given by $\xi_t = L_{\phi(t)}$.

Recently this viewpoint has been developed, notably in [2] and [8], in connection with the study of pathwise boundedness and pathwise continuity of Gaussian processes. It is easy to see that ξ will have a pathwise bounded version if and only if L has a version whose restriction to $\phi(T)$ is pathwise bounded. If T has a topology, and $t \overset{\text{stoch}}{\to} \xi_t$ is stochastically continuous, then ϕ is continuous: $T \to H$, so the existence of a version of L whose restriction to $\phi(T)$ is pathwise continuous will imply the existence of a pathwise continuous version for ξ; while if ϕ is also an open map, as is the case when T is compact, then the converse implication also holds.

This research was partially supported by NSF Grant GP-7176.

DEFINITION 1. *A set S in H is called a GB* set *if the restriction to S of the canonical normal process in H has a pathwise bounded version, a GC* set *if it has a pathwise continuous version.*

DEFINITION 2. *For any linear process L on a linear space X, $\bar{L}(S)$ is defined as the essential supremum of the set of random variables $\{|L_x| : x \in S\}$. Thus, $\bar{L}(S)$ is a random variable with values in $[0, +\infty]$.*

It is easy to see that L has a version whose restriction to S is pathwise bounded if and only if $\bar{L}(S) < \infty$ with probability one. Also, if $S \subset R$ then $\bar{L}(S) \leqq \bar{L}(R)$; while if R is the convex symmetric hull of S then $\bar{L}(S) = \bar{L}(R)$. So the property of possessing a pathwise bounded version on S persists through the operations of convexifying, symmetrizing, and, of course, passing to a subset. Furthermore, if X is a topological vector space and L is stochastically continuous, the property is preserved through closure, since $\bar{L}(S) = \bar{L}(\bar{S})$. This is convenient, since convex closed symmetric sets are more amenable to study. It is also fairly clear that the vector sum of two GB sets is again GB.

Continuity is more subtle: it is *not* true that a GC set will have a closed convex symmetric hull which is a GC set; indeed, Example 2 shows that convexification or symmetrization or closure alone can destroy the property. However, surprisingly, if S is a *compact* GC set, then its closed convex symmetric hull is again a GC set (Theorem 2). Also, it remains true, although no longer obvious, that in a fairly general situation the vector sum of two GC sets is GC.

It is shown in [1] that a closed GB set is compact; and it is obvious that every compact GC set is a GB set. But how badly can a GC set be non-GB? Our results in this direction are summarized in Corollary 2 following Theorem 5: every convex symmetric GC set is contained in the sum of a finite dimensional subspace and a compact GC (and therefore also GB) set. As for the question of how non-GC a GB set can be, our Theorem 6 makes more precise Theorem 4.7 of [2].

Finally, we discuss the connection between the GB and GC properties and extendability of the images of the cylinder set measure associated with L to certain Banach spaces arising naturally in the context. This is done both for the Gaussian case and for a general linear process.

2. Persistence of the GC property

DEFINITION 3. *The set $B(S)$ will be the smallest closed convex symmetric set containing S.*

EXAMPLE 1. Let S be an infinite orthonormal set. Then S is a GC set, since it has the discrete topology; S is closed and bounded. However, S is not a GB set, and $B(S)$ is not a GC set (see [1], Proposition 6.7).

EXAMPLE 2. Let $R = \{a_j e_j : j = 2, 3, \cdots\}$, where $\{e_2, e_3, \cdots\}$ is orthonormal and $a_j = (\log j)^{-1/2}$. Then R is a GC set, since discrete; $R \cup \{0\}$ is a GB set, but not a GC set (see [1], Propositions 6.7 and 6.9). Let x_0 be a nonzero point not in the convex hull of R, but in its closure; for example, $\Sigma_{m=1}^{\infty} (1/2m) a_j e_j$.

Let $S = (R - x_0) \cup \{x_0\}$. Then S is a GB set and also a GC set; but neither the convex hull of S, nor the symmetrization of S, nor the closure of S is a GC set.

DEFINITION 4. *For any set Σ and any family $\mathcal{F}$ of real valued functions on Σ, let $\mathcal{M}(\Sigma, \mathcal{F})$ be the smallest σ-algebra of subsets of Σ making all elements of $\mathcal{F}$ measurable.*

THEOREM 1. *Let L be a linear process on the topological vector space X, and let S be a convex symmetric separable subset of X whose closure $\bar{S}$ is metrizable. Suppose L has a version whose restriction to S is pathwise continuous at 0. Then L has a version which is pathwise linear, and whose restriction to $\bar{S}$ is uniformly continuous.*

PROOF. Let d be a compatible metric on $\bar{S}$. Let $\|x\| = d(\frac{1}{2}x, 0)$. Then it is easy to see that $x, y \mapsto \|x - y\|$ is also a compatible metric on $\bar{S}$.

Assume that L has restriction to S which is pathwise continuous at 0. Let S_0 be a countable, rationally convex, symmetric, dense subset of S; such clearly exist. Let X^a be the algebraic dual of X and

$$Y = \{x^a \in X^a : x^a \text{ continuous at } 0 \text{ on } S_0\} \tag{2.1}$$
$$= \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{x \in S_0, \|x\| < 1/k} \left\{x^a \in X^a : |\langle x^a, x\rangle| < \frac{1}{k}\right\}.$$

The set Y is in $\mathcal{M}(X^a, X)$. Now, there is a probability measure P on $\mathcal{M}(X^a, X)$ such that $x \mapsto \langle \cdot, x\rangle$ becomes a version of L (see [2] for details). Thus

$$P(Y) = \Pr \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{x \in S_0, \|x\| < 1/k} \left\{w : |L_x(w)| < \frac{1}{k}\right\}, \tag{2.2}$$

which by assumption is 1. Observe that if $y \in Y$, then $\langle y, \cdot\rangle$ is uniformly continuous on S_0, since $|\langle y, x_1\rangle - \langle y, x_2\rangle| = 2|\langle y, \frac{1}{2}(x_1 - x_2)\rangle|$, which tends to 0 as $\|x_1 - x_2\| \to 0$, since $\frac{1}{2}(x_1 - x_2) \in S_0$.

Let $V = \bigcup_{j=1}^{\infty} j\bar{S}$. This is a linear subspace of X. Let W be a complementary subspace (in the purely algebraic sense; these need not be closed subspaces). For $x = v + w$, define $M_x(y)$, $y \in Y$, as follows: write v as $j\bar{s}$ with $\bar{s} \in \bar{S}$. Then $\bar{s} = \lim_{k\to\infty} s_k$, $s_k \in S_0$. Set

$$M_x(y) = j \lim_{k\to\infty} \langle y, s_k\rangle + \langle y, w\rangle. \tag{2.3}$$

To see that this definition is unique: first, it is independent of which sequence (s_k) is chosen to approximate $\bar{s}$, because of uniform continuity of $\langle y, \cdot\rangle$ on S_0. Next, if $v = j\bar{s} = i\bar{r}$ with $\bar{r} \in \bar{S}$, and $i \leqq j$, then $((i/j)s_k)$ is a sequence in S_0 which converges to $\bar{r}$ as $k \to \infty$, and

$$i \lim_{k\to\infty} \langle y, (i/j)s_k\rangle = j \lim_{k\to\infty} \langle y, s_k\rangle. \tag{2.4}$$

So the definition is independent of the representation of v.

It is clear that M has pathwise uniformly continuous restriction to $\bar{S}$. As for linearity: $M_x(y) = \langle y, x\rangle$ for $x \in S_0$, hence $x \mapsto M_x(y)$ is rationally convex linear on S_0. By continuity, it is convex linear on $\bar{S}$. Then, by construction, it

is linear on V. Since it equals $\langle x, y\rangle$ for $x \in W$, it is also linear there. So $x \twoheadrightarrow M_x(y)$ is linear on X for each y.

To see that M is equivalent to L as a stochastic process, it suffices to check it on $\bar{S}$. If $x \in \bar{S}$ and $x = \lim_{k\to\infty} s_k$, then we need to show that

$$\Pr\big(\{y : \langle y, x\rangle \neq M_x(y)\}\big) = 0. \tag{2.5}$$

But $\langle y, x\rangle = \langle y, x - s_k\rangle + \langle y, s_k\rangle$, and

$$M_x(y) = M_{x-s_k}(y) + M_{s_k}(y) = M_{x-s_k}(y) + \langle y, s_k\rangle. \tag{2.6}$$

So $\langle y, x\rangle - M_x(y) = \langle y, x - s_k\rangle - M_{x-s_k}(y)$. Now, $\langle \cdot, x - s_k\rangle \to 0$ in probability, because of the pathwise continuity at zero of L, while $M_{x-s_k}(y) \to 0$ for each y, by definition. *Q.E.D.*

COROLLARY 1. *If S is a convex symmetric GC set in Hilbert space, so is $\bar{S}$.*

THEOREM 2. *Let S be a compact GC set in Hilbert space. Then $B(S)$ is a GC set.*

PROOF. Let L be a version of the canonical normal process on H which is pathwise continuous on S. First we will show that $\Pr\big(\{\bar{L}(S) < \varepsilon\}\big) > 0$ for each $\varepsilon > 0$. Choose S_0 countable and dense in S. Now,

$$\begin{aligned}\{\omega : x, y \in S \text{ and } \|x - y\| < \delta \Rightarrow |L_x(\omega) - L_y(\omega)| \leqq \tfrac{1}{2}\varepsilon\} \\ = \{\omega : x, y \in S_0 \text{ and } \|x - y\| < \delta \Rightarrow |L_x(\omega) - L_y(\omega)| \leqq \tfrac{1}{2}\varepsilon\}\end{aligned} \tag{2.7}$$

because of continuity of $x \twoheadrightarrow L_x(\omega)$ on S and density of S_0. Thus, since each $x \twoheadrightarrow L_x(\omega)$ is uniformly continuous on S, there is some δ such that this set, call it Λ, has positive probability. Choose $x_1, \cdots, x_N$ such that the open spheres of radius $\delta > 0$ about the x_i cover S. Let $\tau(x)$ be the first x_i such that $\|x - x_i\| < \delta$. Let $T = \{x - \tau(x) : x \in S\}$. Let $T_0 = \{x - \tau(x) : x \in S_0\}$. Then T_0 is dense in T; for if $x^{(k)} \in S_0$ and $x^{(k)} \to x \in S$, then $\tau(x^{(k)})$ eventually $= \tau(x)$. So

$$\begin{aligned}0 < \Pr(\Lambda) &\leqq \Pr\big(\{|L_y| \leqq \tfrac{1}{2}\varepsilon \text{ for all } y \in T_0\}\big) \\ &= \Pr\big(\{|L_y| \leqq \tfrac{1}{2}\varepsilon \text{ for all } y \in T\}\big) \\ &= \Pr\big(\{\bar{L}(T) \leqq \tfrac{1}{2}\varepsilon\}\big).\end{aligned} \tag{2.8}$$

Let F be the subspace spanned by $x_1, \cdots, x_N$. Then, by [1], Proposition 4.1,

$$\Pr\big(\{\bar{L}\big(P_{F^\perp}(T)\big) \leqq \tfrac{1}{3}\varepsilon\}\big) \geqq \Pr\big(\{\bar{L}(T) \leqq \tfrac{1}{2}\varepsilon\}\big). \tag{2.9}$$

For $x \in S$, write

$$x = P_F x + P_{F^\perp} x = P_F x + P_{F^\perp}\big(x - \tau(x)\big), \tag{2.10}$$

since $\tau(x) \in F$. Now, $P_F(S)$ is a compact subset of F and $P_F x \in P_F(S)$, while $P_{F^\perp}\big(x - \tau(x)\big) \in P_{F^\perp}(T)$. Thus,

$$\bar{L}(S) \leqq \bar{L}\big(P_F(S)\big) + \bar{L}\big(P_{F^\perp}(T)\big) \tag{2.11}$$

with probability one. But $\bar{L}\big(P_F(S)\big)$ and $\bar{L}\big(P_{F^\perp}(T)\big)$ are independent, so

$$(2.12)\qquad \Pr\{\{\bar{L}(S) < \varepsilon\}\} \geqq \Pr\{\{\bar{L}(P_F(S)) < \tfrac{1}{2}\varepsilon\} \cap \{\bar{L}(P_{F^\perp}(T)) \leqq \tfrac{1}{2}\varepsilon\}\}$$

$$= \Pr\{\{\bar{L}(P_F(S)) < \tfrac{1}{2}\varepsilon\}\}\Pr\{\{\bar{L}(P_{F^\perp}(T)) \leqq \tfrac{1}{2}\varepsilon\}\} > 0.$$

Now, $\Pr\{\bar{L}(S) < \varepsilon\} = \Pr\{\bar{L}(B(S)) < \varepsilon\}$, so, by [1], Theorem 4.6, $B(S)$ is a GC set. *Q.E.D.*

REMARK. The essence of the above proof is to show that compact GC sets S are characterized by the property $\Pr\{\bar{L}(S) < \varepsilon\} > 0$ for all $\varepsilon > 0$.

EXAMPLE 3. A plausible approach to the above theorem would be to try to show that if a linear functional on H is continuous on S, then it is continuous on $B(S)$. Unfortunately, this is not true, as this example shows. Let R and x_0 be as in Example 2. Let $S = R - x_0$. Choose a linear functional x^a on H which is 1 at x_0 and 0 on the (nonclosed) linear subspace spanned by R. Then x^a is continuous on S, in fact is identically -1 on S. But $0 \in \bar{S}$, and x^a is zero there, of course.

THEOREM 3. *If S and T are compact GC sets, then $S \cup T$ is a GC set.*

PROOF. By Theorem 2, we may assume that S and T are also convex and symmetric. Consider now the seminorms in H having S^0 and T^0 as unit spheres; by Theorem 3 of [4], these are both measurable seminorms, in the terminology of that paper. Then, by Lemma 5 of that same paper, their sum is also a measurable seminorm. But its unit sphere is $S^0 \cap T^0$; so, again by Theorem 3 of [4], $(S^0 \cap T^0)^0 = (S \cup T)^{00}$ is a GC set. *Q.E.D.*

3. Structure of GC sets

If S is a closed convex symmetric bounded set in H, and S^0 is its polar, that is, $S^0 = \{y : \langle (y, x) \rangle \leqq 1 \text{ for all } x \in S\}$, then let $|\cdot|$ be the seminorm on H which has S^0 as its unit sphere; this exists, because S^0 is absorbing, since S is bounded. This seminorm $|\cdot|$ is continuous, because S^0 contains a neighborhood of 0 in H. Then there is a Banach space W and a continuous map $\theta : H \to W$, with $\theta(H)$ dense in W, and $\|\theta(x)\| = |x|$. Furthermore, $\bigcup_{n=1}^{\infty} nS$ may be given a norm for which S is the unit sphere, and in this norm it becomes a Banach space. Furthermore, the map $\theta' : W' \to H$ is a homeomorphism from W' onto this Banach space. For details see [7].

THEOREM 4. *A closed convex symmetric bounded GC set is compact.*

PROOF. Let S be the set. Let $|\cdot|$, θ, W be as above. By Theorem 1, there exists a version of the canonical normal process on H which is pathwise linear and pathwise continuous on S. Let L be such a version. Then $w' \twoheadrightarrow L_{\theta'(w')}(\omega)$ is bounded on the unit sphere of W' and linear on W'. Thus, $w' \twoheadrightarrow L_{\theta'(w')}(\omega)$ is continuous on the unit sphere of W'. So $\bar{L}(\theta'(\text{unit sphere of } W')) = \bar{L}(S) < \infty$ with probability 1. *Q.E.D.*

Thus, a closed convex symmetric GC set satisfies the assumptions of the following lemma.

LEMMA 1. *Suppose S is a closed symmetric convex set, and has compact intersection with every closed ball in H. Then either S is bounded or S contains some line through the origin.*

PROOF. For $c > 0$, let

$$S_c = \frac{1}{c}\{x \in S : \|x\| = c\}. \tag{3.1}$$

This S_c is compact. If S is unbounded, then each S_c is nonempty. Finally, if $c \leqq d$, then $S_c \supset S_d$; for if $x \in S$ and $\|x\| = d$, then

$$\frac{1}{d}x = \frac{1}{c}\left(\frac{c}{d}x\right), \frac{c}{d}x \in S \tag{3.2}$$

because S is convex and contains zero, and $\|(c/d)x\| = c$. So $\cap_{c>0} S_c$ contains some x_0. The elements of each S_c have norm 1, so

$$\|x_0\| = 1. \tag{3.3}$$

Now, $cx_0 \in S$ for all $c > 0$, so, by symmetry of S, also for all $c < 0$, and, by convexity, for $c = 0$. *Q.E.D.*

THEOREM 5. *If S is a closed convex symmetric GC set, then S contains a largest subspace F, which is finite dimensional. Furthermore, $S \cap F^\perp$ is compact and $S \subset F + 2(S \cap F^\perp)$.*

PROOF. Convexity of S implies that it contains a largest subspace F. Since the intersection of F with the unit sphere is convex, symmetric, and GC, so is its closure (by Corollary 1); hence, this closure is compact (by Theorem 4) and so F must be finite dimensional.

I now claim that the image of S in H/F has compact intersection with any closed ball in H/F. For if $x_i \in S$ and $f_i \in F$ and $\|x_i + f_i\| \leqq c$, then

$$\frac{1}{2}(x_i + f_i) \in \left\{x \in S : \|x\| \leqq \frac{c}{2}c\right\}, \tag{3.4}$$

so $x_i + f_i$ has a subsequence converging to some $x_0 \in S$; and $x_i + F$ therefore has a convergent subsequence. Thus, by Lemma 1, the image of S in H/F is bounded. This image is sent to $(S + F) \cap F^\perp$, in the canonical isomorphism between H/F and $F^\perp$; thus, $(S + F) \cap F^\perp$ is bounded, but also GC, since it is contained in $2S$. Therefore it is compact. Finally, given $x \in S$, write

$$x = P_F x + (x - P_F x) \in F + (S + F) \cap F^\perp \subset F + 2(S \cap F^\perp). \tag{3.5}$$

Q.E.D.

Combining Corollary 1, Theorem 4, and Theorem 5, gives the following.

COROLLARY 2. *A convex symmetric GC set is contained in the sum of a finite dimensional subspace and a compact convex symmetric GC set.*

COROLLARY 3. *If S and T are convex symmetric GC sets, then so is their vector sum $S + T$.*

PROOF. Corollary 2 immediately reduces us to the case where $S = C + E$, $T = D + F$, C and D being compact convex symmetric GC sets, E and F being finite dimensional subspaces. Now, $C + D$ is contained in the convex hull of $2C \cup 2D$, which is GC *by* Theorem 3. Since

$$S + T = (C + D) + (E + F), \tag{3.6}$$

it remains only to consider the effect of adding a finite dimensional subspace to a compact convex symmetric GC set. But it is easy to see that the GC property persists through this operation. *Q.E.D.*

Following [2], we call the closed convex symmetric GB set S *maximal*, provided there is no closed convex symmetric GB set S_0 such that S is contained in $\bigcup_j jS_0$ and is compact there in the topology which has S_0 as unit sphere. Theorem 4.7 of [2] asserts that if a closed convex symmetric GB set is *not* GC, then it is maximal.

LEMMA 2. *If $K_1, K_2, \cdots$ are compact convex symmetric sets in H, $d_j = \sup\{\|x\| : x \in K_j\}$, and $\Sigma_j c_j d_j < \infty$, then*

$$\{\sum c_j x_j : x_j \in K_j, \text{ only finitely many } x_j \neq 0\} \tag{3.7}$$

is a convex symmetric set with compact closure.

PROOF. Let

$$x^{(k)} = \sum_j c_j x_j^{(k)}. \tag{3.8}$$

By a diagonal process, choose k_i so that $x_j^{(k_i)}$ converges for each j. Then

$$\|x^{(k_h)} - x^{(k_i)}\| \leqq \sum_{j=1}^{N} c_j \|x_j^{(k_h)} - x_j^{(k_i)}\| + 2 \sum_{j=N+1}^{\infty} c_j d_j. \tag{3.9}$$

First choose N so the second term is less than $\frac{1}{2}\varepsilon$; then choose h, i so large that the first term is less than $\frac{1}{2}\varepsilon$, so

$$\|x^{(k_h)} - x^{(k_i)}\| < \varepsilon, \tag{3.10}$$

which completes the proof.

Now we can show the converse of the result mentioned before the above lemma.

THEOREM 6. *A closed convex symmetric GB set which is actually a GC set cannot be maximal.*

PROOF. Let $|\cdot|$, θ, W be as in the beginning of this section. Then Theorem 2 of [4] says that, if n is the cylinder set measure on H associated with L, (see [2] for details), $n \circ \theta^{-1}$ extends to a regular measure μ on W. Thus, there exist compact sets $K_1, K_2, \cdots$ with $\mu(K_j) \uparrow 1$. Now, θ' maps the unit sphere of W' onto the compact set S, so θ' is a compact operator; therefore, θ is also compact and θ sends the unit sphere of H into a precompact set. Thus we may assume that K_1 contains this image and also that $K_1, K_2, \cdots$ are all convex and symmetric. Then by Lemma 2 there exist constants $c_1, c_2, \cdots > 0$ such that the closure of $\{\Sigma_j c_j x_j : x_j \in K_j\}$ is a compact, convex, symmetric set K in W. Let

W_0 be $\bigcup_{j=1}^{\infty} jK$, made into a Banach space with K as unit sphere. Since each $K_j \subset W_0$, we have

$$\mu(W_0) = 1, \tag{3.11}$$

so let

$$\mu_0 = \mu \big| W_0 \tag{3.12}$$

and $\theta(H) \subset W_0$. The injection $\psi: W_0 \to W$ is continuous, indeed compact, so $\psi': W' \to W_0'$ is likewise.

Let θ_0 be θ regarded as a map from H into W_0. Let S_0 be the closure of the image of the unit sphere of W_0' under θ_0'. Let

$$V = \bigcup_{j=1}^{\infty} jS_0. \tag{3.13}$$

Then V may be regarded as a Banach space with S_0 as unit sphere. Since θ_0' sends the unit sphere of W_0' into that of V, it is continuous, in fact norm reducing: $W_0' \to V$. Now,

$$S = \theta' \text{ (unit sphere of } W') = \theta_0'\psi' \text{ (unit sphere of } W'). \tag{3.14}$$

But ψ' is compact, so ψ' (unit sphere of W') is compact in W_0' and its image S under θ_0' is compact in V.

Finally, we must show that S_0 is a GB set. For each $w_0 \in W_0$,

$$\{|\langle w_0', w_0\rangle| : w_0' \in \text{unit sphere of } W_0'\} \tag{3.15}$$

is bounded. But

$$w_0' \twoheadrightarrow L_{\theta_0'(w_0')} \tag{3.16}$$

is equivalent to the linear process $w_0' \twoheadrightarrow \langle w_0', \cdot\rangle$ on the probability space $(W_0, \mathscr{B}(W)\big|W_0, \mu_0)$, where $\mathscr{B}(W)$ is the Borel sets of W. Thus, $\bar{L}(\theta_0'$ (unit sphere of $W_0'))) < \infty$ with probability one, and therefore $\bar{L}(S_0) < \infty$ with probability one. *Q.E.D.*

4. Connection with countable additivity of cylinder set measures

For the moment we consider an arbitrary locally convex separated topological vector space X. There is a one to one correspondence between equivalence classes of random linear functionals L on X and cylinder set measures m on the cylinder sets $\mathscr{C}(X, X')$ of X induced by the duality with X' (see [3] for details). The correspondence is characterized by the equalities

$$\Pr\{L_{x_j} \in A_j, j = 1, \cdots, n\} = m(\{x' : \langle x_j, x'\rangle \in A_j, j = 1, \cdots, m\}). \tag{4.1}$$

Throughout this section, let S be a convex, symmetric, weakly compact subset of X. This is equivalent to the statement that $S^{00} = S$, where polars of subsets of X are taken in X' and polars of subsets of X' are taken in X. It is also

equivalent to saying that S is the polar of a Mackey neighborhood of zero in X'; recall that the Mackey topology is the finest locally convex topology in X' for which X is the dual.

The set S^0 serves as the unit sphere for a seminorm on X'. Let W be the Banach space constructed for X' by means of this seminorm; thus, there is a continuous linear map $\theta: X' \to W$ having dense range, and if $\Sigma(W)$ is the unit sphere of W, then $\theta^{-1}(\Sigma(W)) = S^0$. Furthermore, there is an adjoint map $\theta': W' \to X$, continuous from the norm topology on W' to the original topology on X, which is one to one (since θ has dense range) and for which $\theta'(\Sigma(W')) = S$. (For details see [4].) Denote by i the canonical injection $W \to W''$. Then for a cylinder set measure m on $\mathscr{C}(X', X)$, $m \circ \theta^{-1}$ is a cylinder set measure on $\mathscr{C}(W, W')$, and $m \circ \theta^{-1} \circ i^{-1}$ is a cylinder set measure on $\mathscr{C}(W'', W')$.

Theorem 7 should perhaps be called a "folk theorem"; slight variants of it are contained in, for example, [5] and [6].

THEOREM 7. *The process L on S has a pathwise bounded version if and only if $m \circ \theta^{-1} \circ i^{-1}$ is concentrated on spheres; in this case $m \circ \theta^{-1} \circ i^{-1}$ is also countably additive.*

PROOF. Existence of a pathwise bounded version on S implies that for each $\varepsilon > 0$ there exists a $t > 0$ such that if $x_1, \cdots, x_m \in S$, then

$$\Pr\{|L_{x_j}| \leqq t, j = 1, \cdots, m\} \geqq 1 - \varepsilon. \tag{4.2}$$

Now,

$$\begin{aligned} \Pr\{|L_{x_j}| &\leqq t, j = 1, \cdots, n\} \\ &= m(\{x' : |\langle x_j, x' \rangle| \leqq t, j = 1, \cdots, n\}) \\ &= m(t\{x_1, \cdots, x_n\}^0). \end{aligned} \tag{4.3}$$

Furthermore, for any $x_1, \cdots, x_n \in X$, we have

$$\{x_1, \cdots, x_n\} \subset S \Leftrightarrow \{x_j\}^0 \supset S^0, \qquad j = 1, \cdots, n, \tag{4.4}$$

since $S^{00} = S$. In particular, let $e'_1, \cdots, e'_n \in W'$ and $e'_j \leqq 1$. Notice that if $i \circ \theta(x) \in \{e'_1, \cdots, e'_n\}^0$, then

$$x \in \theta'(\{e'_1, \cdots, e'_n\}^0), \tag{4.5}$$

since

$$\langle i \circ \theta(x), e' \rangle = \langle x, \theta'(e') \rangle. \tag{4.6}$$

So

$$1 - \varepsilon \leqq m(t\{\theta'(e'_1), \cdots, \theta'(e'_n)\}^0) \leqq m \circ \theta^{-1} \circ i^{-1}(\{e'_1, \cdots, e'_n\}^0). \tag{4.7}$$

Thus, any finite intersection of symmetric closed slabs containing $t\Sigma(W'')$ has $m \circ \theta^{-1} \circ i^{-1}$ measure at least $1 - \varepsilon$. Then the same holds for closed half spaces instead of slabs. Now Lemma 3 of [6] or Theorem 4 of [9] apply, after a little reformulation, to give countable additivity as well.

Conversely, let μ be a countably additive measure on the σ-field generated by $\mathscr{C}(W'', W')$ which agrees with $m \circ \theta^{-1} \circ i^{-1}$ on $\mathscr{C}(W'', W')$ and which is concentrated on spheres. The map $\theta'(e') \twoheadrightarrow \langle e', \cdot \rangle$ gives a linear process on $\theta'(W')$, and if

$$\|e'_j\| \leqq 1, \qquad j = 1, \cdots, n, \tag{4.8}$$

then

$$\begin{aligned} \mu\{|\langle e'_j, \cdot \rangle| \leqq t, j = 1, \cdots, n\} &= m \circ \theta^{-1} \circ i^{-1}(\{e'' : |\langle e'_j, e'' \rangle| \leqq t, j = 1, \cdots, n\}) \\ &= m(\{x' : |\langle \theta'(e'_j), x' \rangle| \leqq t, j = 1, \cdots, n\}) \\ &= \Pr\{|L_{\theta'(e'_j)}| \leqq t, j = 1, \cdots, n\}. \end{aligned} \tag{4.9}$$

Since this can be made arbitrarily small for large t, and since $\theta'(\Sigma(W')) = S$, this shows that L has a pathwise bounded version. *Q.E.D.*

THEOREM 8. *Assume S is compact in the original topology of X; then*

(i) *countable additivity of $m \circ \theta^{-1}$ implies existence of a pathwise continuous version of L on S, and*

(ii) *for separable W, existence of a pathwise continuous version of L on S implies countable additivity of $m \circ \theta^{-1}$.*

PROOF. For (i), let μ be a countably additive extension of $m \circ \theta^{-1}$. Then, as in the previous theorem, the stochastic process

$$\theta'(e') \twoheadrightarrow \langle e', \cdot \rangle\tilde{}, \qquad e' \in \Sigma(W'), \tag{4.10}$$

is equivalent to the restriction of L to $\theta'(\Sigma(W'))$, which is S. Furthermore, θ' is easily seen to be continuous from the weak* topology on $\Sigma(W')$ to the weak topology on S. Since $\Sigma(W')$ is weak* compact, the map is a homeomorphism. But also the identity map on S is a homeomorphism from the original topology on S to the weak topology, since S is assumed compact in the original topology. So

$$\theta'(e') \twoheadrightarrow \langle e', e \rangle \tag{4.11}$$

is continuous on S.

For (ii), assume L has a pathwise continuous version. As noted in (i), θ' is a homeomorphism from $\Sigma(W')$ onto S, when $\Sigma(W')$ is given the weak* topology. Let

$$M_{e'} = L_{\theta'(e')}. \tag{4.12}$$

Then M has a pathwise continuous version on $\Sigma(W')$ for the weak* topology on $\Sigma(W')$. But separability of W implies that $\Sigma(W')$ is separable and metrizable in its weak* topology. Then Theorem 1 shows that M has a version for which every path function is linear and, when restricted to $\Sigma(W')$, is weak* continuous. From this it follows that the probability measure may be transferred to W. *Q.E.D.*

Finally, assume that X is separable Hilbert space with its usual topology and L is the canonical normal process. Weak compactness of S then amounts to boundedness of S. The following theorem combines known results from [1] and [4] with the results of this paper. Recall that we have assumed also convexity and symmetry of S.

THEOREM 9. (i) *If S is a GC set then S is compact and $m \circ \theta^{-1}$ is countably additive.*

(ii) *If $m \circ \theta^{-1}$ is countably additive then S is a compact GC set.*

PROOF. The proof is immediate from [4], Theorem 3 and the present Theorem 4.

The author wishes to express his gratitude to Lucien LeCam for several stimulating conversations connected with the present work. In particular, the proof of Theorem 1 was the outcome of one such conversation. He also wishes to thank Haskell Rosenthal for Example 3. Also, Lemma 2 was noted by Dick Dudley in a letter.

REFERENCES

[1] S. CHEVET, "p-ellipsoïdes de 1^q, exposant d'entropie, mesures cylindriques gaussiennes," *C.R. Acad. Sci. Paris Sér. A*, Vol. 269 (1969), pp. 658–660.

[2] R. M. DUDLEY, "The sizes of compact subsets of Hilbert space and continuity of Gaussian processes," *J. Functional Analysis*, Vol. 1 (1967), pp. 290–330.

[3] ———, "Random linear functionals," *Trans. Amer. Math. Soc.*, Vol 136 (1969), pp. 1–24.

[4] R. M. DUDLEY, J. FELDMAN, and L. LECAM, "On seminorms, probabilities, and abstract Wiener spaces," *Ann. of Math.*, Vol. 93 (1971), pp. 390–408.

[5] L. LECAM, "Remarques sur les variables aléatoires dans les espaces vectoriels non séparables," *Comptes Rendus d'un Colloque International*, Paris, CNRS, 1969, pp. 39–53.

[6] YU. V. PROHOROV, "The method of characteristic functionals," *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1961, Vol. 2, pp. 403–419.

[7] H. H. SCHAEFER, *Topological Vector Spaces*, New York, Macmillan, 1966.

[8] V. N. SUDAKOV, "Gaussian measures, Gauchy measures, and ε-entropy," *Soviet Math. Dokl.*, Vol. 10 (1969), pp. 310–313.

[9] R. K. GETOOR, "On characteristic functions of Banach space valued random variables," *Pacific J. Math.*, Vol. 7 (1957), pp. 885–896.