ISOTONIC TESTS FOR CONVEX ORDERINGS

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1. Introduction

The problem of testing the hypothesis that F is a negative exponential distribution with unknown scale parameter against the alternative that F has monotone increasing nonconstant failure rate (F has Increasing Failure Rate, IFR) has been studied by a number of authors, some of whom are Proschan and Pyke [18], Nadler and Eilbott [17], Barlow [1], Bickel and Doksum [7], and Bickel [6]. Bickel and Doksum show that the test proposed by Proschan and Pyke is asymptotically inadmissible. They then take an essentially parametric approach to the problem. In particular they obtain the studentized asymptotically most powerful linear spacings tests for selected parametric families of distributions which are IFR when the parameter $\theta > 0$ and exponential when $\theta = 0$. Bickel [6] proves that these tests are actually asymptotically equivalent to the level α tests which are most powerful among all tests which are similar and level α (for the associated parametric problems).

Since the problem is essentially nonparametric, we take a nonparametric approach similar to the one taken by Chapman [10] and Doksum [12] in studying the problem of testing for goodness of fit to a specified distribution against stochastically ordered alternatives. In addition, we consider a more general class of problems which includes the problem of testing for monotone failure rate. The setup is similar to that in Barlow and van Zwet [2].

Let $\mathscr{F}$ be the class of absolutely continuous distribution functions F such that $F(0) = 0$ with positive and right (or left) continuous density f on the interval where $0 < F < 1$. It follows that the inverse function F^{-1} is uniquely defined on $(0, 1)$. We take $F^{-1}(1)$ to be equal to the right endpoint of the support of F (possibly $+\infty$) and define $F^{-1}(0) = 0$. For $F, G \in \mathscr{F}$ we say that F is *c-ordered* (convex ordered) with respect to G ($F \underset{c}{<} G$) if and only if $G^{-1}F$ is convex on the

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interval where $0 < F < 1$ (van Zwet [21]). Denoting the densities of F and G by f and g, we find that $F \underset{c}{<} G$ implies that

$$r(x) = \frac{d}{dx} G^{-1}F(x) = \frac{f(x)}{g[G^{-1}F(x)]} \tag{1.1}$$

is nondecreasing in x on the interval where $0 < F < 1$. The problem of estimating $r(x)$ when G is known was considered by Barlow and van Zwet [2], [3]. When $G(x) = 1 - \exp\{-x\}$, it is easy to verify that $r(x) = f(x)/[1 - F(x)]$, the failure rate function of F.

We assume G known, $F \underset{c}{<} G$ and consider the problem of testing

$$H_0 : F \underset{c}{=} G \tag{1.2}$$

(that is, $G^{-1}F$ is linear on the support of F) against the alternative

$$H_1 : F \underset{c}{<} G \quad \text{and} \quad F \underset{c}{\neq} G, \tag{1.3}$$

given a random sample $X = (X_1, X_2, \cdots, X_n)$ from F.

We call (1.2) and (1.3) the problem of testing for c-equivalence versus c-ordering. We study tests based on the "total time on test" statistics for this problem (Section 3). In the cases when G is the uniform or exponential distribution we show that the tests corresponding to the "cumulative total time on test statistics" are asymptotically minimax over a class of alternatives based on the Kolmogorov distance (Sections 6, 7, and 8) and in each of the classes of statistics considered by Bickel and Doksum [7] (in the exponential case).

2. Preliminaries

We can simplify our problem by introducing the following transformation

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} g[G^{-1}F(u)]\, du, \qquad 0 \leqq t \leqq 1. \tag{2.1}$$

Recall that G is always fixed in this discussion. Note that H is a distribution since H^{-1} (the inverse of H) is strictly increasing on $[0, 1]$. In particular, $H_G^{-1}(t) = t$ so that H_G is the uniform distribution on $[0, 1]$. When it is clear from the context which distribution we are transforming, we will simply write H^{-1} for H_F^{-1}.

By (1.1) $F \underset{c}{<} G$ implies $f(x)/g[G^{-1}F(x)]$ is nondecreasing in x for $0 < F(x) < 1$ or $g[G^{-1}(t)]/f[F^{-1}(t)]$ is nonincreasing in t, $0 \leqq t \leqq 1$. Since

$$\frac{d}{dt} H^{-1}(t) = \frac{g[G^{-1}(t)]}{f[F^{-1}(t)]}, \tag{2.2}$$

it follows that H^{-1} is concave on $[0, 1]$ or H is convex on the interval where $0 < H < 1$ if and only if $F \underset{c}{<} G$. Hence, using transformation (2.1) we can reduce our problem (1.2) and (1.3) to that of testing

$$H_0: H(x) \quad \text{linear for} \quad 0 < H(x) < 1 \tag{2.3}$$

versus

$$H_1: H(x) \quad \text{convex and not linear for} \quad 0 < H(x) < 1. \tag{2.4}$$

The following result from Barlow and van Zwet [2] will be needed.

LEMMA 2.1. *If F, $G \in \mathscr{F}$, if gG^{-1} is uniformly continuous on* $[0, 1)$, *if* $\int_0^\infty x\, dF(x) < \infty$, *and if* $F^{-1}(1) < \infty$, *or* $gG^{-1}(y)/(1 - y)$ *is bounded on* $(0, 1)$, *or* $F \underset{c}{<} G$, *then* $H^{-1}(1) < \infty$.

If $G(x) = 1 - \exp\{-x\}$ for $x \geqq 0$, then

$$H^{-1}(t) = \int_0^{F^{-1}(t)} [1 - F(u)]\, du. \tag{2.5}$$

In this case $H^{-1}(1) = \int_0^\infty x\, dF(x)$. If $F \underset{c}{<} G$, then $\int_0^\infty x\, dF(x) < \infty$ is automatically satisfied.

In testing $H_0: F \underset{c}{=} G$ versus $H_1: F \underset{c}{\lneqq} G$ we will be interested in tests ϕ that have isotonic power with respect to c-ordering: that is $F_1 \underset{c}{<} F_2$ implies $\beta_\phi(F_1) \geqq \beta_\phi(F_2)$ where $\beta_\phi(F)$ is the power of the test ϕ when F is the true distribution. One advantage of the transformation H_F^{-1} is that it transforms c-ordering into stochastic ordering

THEOREM 2.1. *If $F_1 \underset{c}{<} F_2$, F_1, F_2, $G \in \mathscr{F}$ and if Lemma* 2.1 *holds then*

$$\frac{H_{F_1}^{-1}(t)}{H_{F_1}^{-1}(1)} \geqq \frac{H_{F_2}^{-1}(t)}{H_{F_2}^{-1}(1)} \qquad 0 \leqq t \leqq 1. \tag{2.6}$$

If, in addition $F_2 \underset{c}{<} G$, then

$$\frac{H_{F_2}^{-1}(t)}{H_{F_2}^{-1}(1)} \geqq t, \qquad 0 \leqq t \leqq 1. \tag{2.7}$$

PROOF. Note that $F_1 \underset{c}{<} F_2$ implies

$$\frac{f_1(x)}{f_2[F_2^{-1}F_1(x)]} \quad \text{increasing in } x, \tag{2.8}$$

or

$$\frac{f_1[F_1^{-1}(u)]}{f_2[F_2^{-1}(u)]} \quad \text{increasing in } u. \tag{2.9}$$

Hence,

$$(2.10)\qquad \frac{H_{F_1}^{-1}(t)}{H_{F_1}^{-1}(1)} - \frac{H_{F_2}^{-1}(t)}{H_{F_2}^{-1}(1)} = \int_0^t \left[\frac{1}{H_{F_1}^{-1}(1)} \frac{gG^{-1}(u)}{f_1[F_1^{-1}(u)]} - \frac{1}{H_{F_2}^{-1}(1)} \frac{gG^{-1}(u)}{f_2[F_2^{-1}(u)]}\right] du$$

$$= \int_0^t \left[\frac{1}{H_{F_1}^{-1}(1)} \frac{f_2F_2^{-1}(u)}{f_1F_1^{-1}(u)} - \frac{1}{H_{F_2}^{-1}(1)}\right] \frac{gG^{-1}(u)}{f_2F_2^{-1}(u)} du$$

$$\stackrel{\text{def}}{=} \int_0^t h(u) \frac{gG^{-1}(u)}{f_2F_2^{-1}(u)} du.$$

Since $\int_0^1 h(u) \left(gG^{-1}(u)\right)/\left(f_2F_2^{-1}(u)\right) du = 0$ and $h(u)$ changes sign at most once and from positive to negative values if at all, it follows that

$$(2.11)\qquad \int_0^t h(u) \frac{gG^{-1}(u)}{f_2F_2^{-1}(u)} du \geqq 0.$$

The second inequality follows from

$$(2.12)\qquad H_G^{-1}(t) = t.$$

Q.E.D.

Since G is assumed known we can estimate H_F^{-1} by substituting the empirical distribution F_n for F; that is,

$$(2.13)\qquad H_n^{-1}(t) = H_{F_n}^{-1}(t) \stackrel{\text{def}}{=} \int_0^{F_n^{-1}(t)} gG^{-1}F_n(u)\, du$$

and

$$(2.14)\qquad H_n^{-1}\left(\frac{i}{n}\right) = \int_0^{X_{i:n}} gG^{-1}F_n(u)\, du = \sum_{j=1}^{i} gG^{-1}\left(\frac{j-1}{n}\right)(X_{j:n} - X_{j-1:n}),$$

where $X_{i:n}$ is the ith order statistic in a sample of size n from F and $X_{0:n} \equiv 0$. If $G(x) = 1 - \exp\{-x\}$ for $x \geqq 0$, then

$$(2.15)\qquad H_n^{-1}\left(\frac{i}{n}\right) = n^{-1} \sum_{j=1}^{i} (n - j + 1)(X_{j:n} - X_{j-1:n}),$$

that is, n^{-1} times the "total time on test" until the ith ordered observation from F.

The following result was proved in Barlow and van Zwet [2].

THEOREM 2.2. *If F, $G \in \mathscr{F}$ and*

(i) $\int_0^\infty x\, dF(x) < \infty$,

(ii) gG^{-1} *is uniformly continuous on* $[0, 1)$,

(iii) *either* $F^{-1}(1) < \infty$, $gG^{-1}(y)/(1 - y)$ *is bounded on* $(0, 1)$, *or* $F \underset{c}{<} G$ *and there exists* η, $0 < \eta < 1$, *such that for* $\eta \leqq y < 1$, $gG^{-1}(y)$ *is nonincreasing and* $gG^{-1}(y)/(1 - y)$ *is nondecreasing in* y, *then for* $n \to \infty$

$$\sup_{x \geq 0} \left| \int_0^x g[G^{-1}F_n(u)]\,du - \int_0^x g[G^{-1}F(u)]\,du \right| \to 0 \quad \textit{almost surely.} \tag{2.16}$$

2.1. *Order statistics from H.* The "total time on test" statistics, $H_n^{-1}(1/n) \leqq H_n^{-1}(2/n) \leqq \cdots \leqq H_n^{-1}((n-1)/n)$, "behave" asymptotically like order statistics from H. To see this let $U_{i:n}$ be the ith order statistic from the uniform distribution on $[0, 1]$. Then

$$Z_{i:n} \underset{\text{st}}{=} H^{-1}(U_{i:n}) \overset{\text{def}}{=} \int_0^{F^{-1}(U_{i:n})} gG^{-1}F(u)\,du \underset{\text{st}}{=} \int_0^{X_{i:n}} gG^{-1}F(u)\,du \tag{2.17}$$

will be distributed as the ith order statistic in a random sample of size n from the distribution H. Since we do not know F, $Z_{i:n}$, $i = 1, 2, \cdots, n$, are unobservable except in the case $G(x) = x$ for $0 \leqq x \leqq 1$. From Theorem 2.2 we see that

$$\left| H_n^{-1}\left(\frac{i}{n}\right) - Z_{i:n} \right| \to 0 \tag{2.18}$$

almost surely and uniformly in i/n, $1 \leqq i \leqq n$. This observation suggests various tests for our transformed problem

$$H_0: H(x) \quad \text{linear for} \quad 0 < H(x) < 1 \tag{2.19}$$

versus

$$H_1: H(x) \quad \text{convex and not linear for} \quad 0 < H(x) < 1 \tag{2.20}$$

based on the "total time on test" statistics. Since our problem is clearly scale invariant, we consider tests based on the studentized statistics

$$W_{F_n}\left(\frac{i}{n}\right) \overset{\text{def}}{=} W_{i:n} \overset{\text{def}}{=} H_n^{-1}\left(\frac{i}{n}\right) / H_n^{-1}(1). \tag{2.21}$$

Distributions which are c-ordered have studentized statistics which are stochastically ordered. This result is the basis for the isotonicity of tests to be considered in this paper.

THEOREM 2.3. *If $F, K, G \in \mathscr{F}$ and $F \underset{c}{<} K \underset{c}{<} G$, then*

$$W_{F_n}\left(\frac{i}{n}\right) \underset{\text{st}}{\geqq} W_{K_n}\left(\frac{i}{n}\right) \underset{\text{st}}{\geqq} W_{G_n}\left(\frac{i}{n}\right) \tag{2.22}$$

where $\underset{\text{st}}{\geqq}$ denotes stochastic ordering and F_n, K_n, G_n are empirical distributions corresponding to independent random samples of size n from F, K, and G, respectively.

PROOF. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be an ordered sample from F. Let $V_{i:n} = K^{-1}F(X_{i:n})$ and note that

$$\frac{V_{i:n} - V_{i-1:n}}{X_{i:n} - X_{i-1:n}} \tag{2.23}$$

is nondecreasing in i since $K^{-1}F$ is convex. Hence,

$$gG^{-1}\left(\frac{i-1}{n}\right)(V_{i:n} - V_{i-1:n})/gG^{-1}\left(\frac{i-1}{n}\right)(X_{i:n} - X_{i-1:n}) \overset{\text{def}}{=} \frac{\beta_i}{\alpha_i}, \tag{2.24}$$

is also nondecreasing in i, where

$$\begin{aligned} \beta_i &= gG^{-1}\left(\frac{i-1}{n}\right)(V_{i:n} - V_{i-1:n}), \\ \alpha_i &= gG^{-1}\left(\frac{i-1}{n}\right)(X_{i:n} - X_{i-1:n}). \end{aligned} \tag{2.25}$$

Define $\psi(0) = 0$, $\psi(\alpha_1 + \cdots + \alpha_i) = \beta_1 + \cdots + \beta_i$, $1 \leqq i \leqq n$. Define $\psi(x)$ elsewhere on $[0, \alpha_1 + \cdots + \alpha_n]$ by linear interpolation between successive points defined above. Note that

$$\frac{\psi(\alpha_1 + \cdots + \alpha_i) - \psi(\alpha_1 + \cdots + \alpha_{i-1})}{(\alpha_1 + \cdots + \alpha_i) - (\alpha_1 + \cdots + \alpha_{i-1})} = \frac{\beta_i}{\alpha_i} \tag{2.26}$$

is increasing in i, so that ψ is a convex function on $[0, \alpha_1 + \cdots + \alpha_n]$. Since $\psi(0) = 0$, ψ is also starshaped, that is $\psi(x)/x$ is nondecreasing in x. Hence

$$\frac{\psi[\sum_1^r \alpha_i]}{\sum_1^r \alpha_i} = \frac{\sum_1^r \beta_i}{\sum_1^r \alpha_i} \tag{2.27}$$

is nondecreasing in r.

Inequalities (2.22) follow by noting that

$$H_{F_n}^{-1}\left(\frac{i}{n}\right) = \sum_{j=1}^{i} gG^{-1}\left(\frac{j-1}{n}\right)(X_{j:n} - X_{j-1:n}) = \sum_{j=1}^{i} \alpha_j \tag{2.28}$$

and

$$H_{K_n}^{-1}\left(\frac{i}{n}\right) = \sum_{j=1}^{i} gG^{-1}\left(\frac{j-1}{n}\right)(V_{j:n} - V_{j-1:n}) = \sum_{j=1}^{i} \beta_j. \tag{2.29}$$

Hence,

$$\frac{H_{K_n}^{-1}\left(\frac{i}{n}\right)}{H_{F_n}^{-1}\left(\frac{i}{n}\right)} \leqq \frac{H_{K_n}^{-1}(1)}{H_{F_n}^{-1}(1)} \tag{2.30}$$

implies

$$\frac{H_{F_n}^{-1}\left(\frac{i}{n}\right)}{H_{F_n}^{-1}(1)} \geqq \frac{H_{K_n}^{-1}\left(\frac{i}{n}\right)}{H_{K_n}^{-1}(1)}. \tag{2.31}$$

Stochastic ordering follows by noting that $(V_{1:n}, \cdots, V_{n:n})$ is stochastically equal to an *independent* ordered sample from K. This establishes the first stochastic inequality in (2.22). The second inequality follows similarly. *Q.E.D.*

The above proof is similar to that for Lemma 3.7 (i) Barlow and Proschan [5].

DEFINITION. *A test ϕ based on $X_1, X_2, \cdots, X_n$ is monotonic if*

$$\phi(X_1, \cdots, X_n) = \begin{cases} 1 & \textit{if} \quad T(X_1, \cdots, X_n) > c_{n,\alpha} \\ 0 & \textit{otherwise}, \end{cases} \tag{2.32}$$

where T is nondecreasing coordinatewise.

DEFINITION. *A test ϕ is* isotonic *with respect to* c*-ordering if $F_1 \underset{c}{<} F_2$ and $X = (X_1, \cdots, X_n)(Y = (Y_1, \cdots, Y_n))$ is a random sample from F_1 (F_2) implies $\phi(X) \underset{st}{\geqq} \phi(Y)$.*

THEOREM 2.4. *Monotonic tests based on $W_{1:n}, W_{2:n}, \cdots, W_{n:n}$ are* isotonic *tests with respect to* c*-ordering. Isotonic tests of* c*-ordering have isotonic power with respect to* c*-ordering; that is, $F_1 \underset{c}{<} F_2 \underset{c}{<} G$ implies*

$$\beta_\phi(F_1) \geqq \beta_\phi(F_2) \geqq \beta_\phi(G), \tag{2.33}$$

where $\beta_\phi(F)$ is the power of ϕ when the true distribution is F.

PROOF. This is an immediate consequence of Theorem 2.3. *Q.E.D.*

3. Tests for convex orderings

Note that $F \in \mathscr{F}$ implies $F^{-1}(0) = 0$ which in turn implies $H^{-1}(0) = 0$. Under the conditions of Theorem 2.2, $H_n^{-1}(1) \to H^{-1}(1)$ almost surely as $n \to \infty$. For the purpose of asymptotic comparison of competing tests we may suppose that $H^{-1}(1) = 1$. This simplifies the discussion somewhat. The problem of testing for c-ordering becomes

$$H_0: H(t) = t \quad \text{on} \quad [0, 1] \tag{3.1}$$

versus

$$H_1: H \quad \text{convex on} \quad [0, 1]. \tag{3.2}$$

We are in effect testing that H is the uniform distribution on $[0, 1]$ versus the alternative that H has an increasing density (when $H^{-1}(1)$ is known).

3.1. *General scores statistics.* If we consider the problem in which the alternative to the uniform distribution is specified, then one can maximize the power by using the Neyman-Pearson lemma. Let $h(t) = d\,H(t)/dt$. If $Z_{1:n} < Z_{2:n} < \cdots < Z_{n:n}$ are the order statistics from H, then the Most Powerful (MP) level α test would reject when

$$\sum_{i=1}^{n} \log h(Z_{i:n}) > k_{n,\alpha}. \tag{3.3}$$

Since $W_{i:n}$, $1 \leqq i \leqq n$, "behave" asymptotically like order statistics from H we are led to consider statistics of the form

$$T_n(J) = n^{-1} \sum_{i=1}^{n} J[W_{i:n}], \tag{3.4}$$

where J is an increasing function on $[0, 1]$. (Note that since H is convex, h is increasing and so is $J(x) = \log h(x)$.) The corresponding test would reject H_0

for large values of the statistic. Tests based on such statistics are isotonic and hence have isotonic power by Theorem 2.4.

DEFINITION. *The test* ψ *corresponding to* $J(x) = x$, *for which*

$$\psi[W_{1:n}, \cdots, W_{n:n}] = \begin{cases} 1 & \text{if } n^{-1} \sum_{i=1}^{n} W_{i:n} > k_{n,\alpha} \\ 0 & \text{otherwise}, \end{cases} \tag{3.5}$$

is called the uniform scores *test and* $n^{-1} \Sigma_{i=1}^{n} W_{i:n}$ *(or* $n^{-1} \Sigma_{i=1}^{n-1} W_{i:n}$ *since* $W_{n:n} \equiv 1$*) is called the* cumulative total time on test statistic.

REMARK 3.1. Suppose $G(x) = 1 - \exp\{-x\}$ for $x \geqq 0$. Then

$$W_{i:n} = \frac{\sum_{j=1}^{i} (n - j + 1)(X_{j:n} - X_{j-1:n})}{\sum_{j=1}^{n} (n - j + 1)(X_{j:n} - X_{j-1:n})} \tag{3.6}$$

and $n^{-1} \Sigma_{i=1}^{n-1} W_{i:n}$ is the *cumulative total time on test statistic* studied by Nadler and Eilbott [17], Bickel and Doksum [7] and Barlow and Proschan [4].

Other general scores tests are Fisher's test for the problem of combining tests with

$$J(x) = \log x, \tag{3.7}$$

the Pearson or exponential scores test with

$$J(x) = -\log(1 - x), \tag{3.8}$$

and the normal scores test with

$$J(x) = \Phi^{-1}(x). \tag{3.9}$$

where Φ is the $N(0, 1)$ distribution.

We will show that the uniform scores test (when G is uniform or exponential) is asymptotically minimax over a certain natural class of alternatives determined by the Kolmogorov distance and with respect to a class of tests including all of the above examples.

Tests based on general scores statistics where J is increasing on $[0, 1]$ are clearly unbiased since they have isotonic power as noted previously.

CONDITION 3.1. *The following regularity conditions are assumed to hold for* J*:* J *has the continuous derivative* J' *on* $(0, 1)$ *and* $\int_0^1 J^2(x)\, dx < \infty$.

To show that such tests are consistent we need the following result.

THEOREM 3.1. *If* $F, G \in \mathscr{F}$, *if the conditions of Theorem 2.2 hold, if* J *is uniformly continuous on* $[0, 1]$, *and if* $\int_0^1 J[(H^{-1}(u))/[(H^{-1}(1))]\, du < \infty$, *then*

$$n^{-1} \sum_{i=1}^{n} J[W_{i:n}] \to \int_0^1 J\left[\frac{H^{-1}(u)}{H^{-1}(1)}\right] du \tag{3.10}$$

almost surely as $n \to \infty$.

PROOF. Without loss of generality we may assume $H^{-1}(1) = 1$. Let $Z_{1:n} \leqq \cdots \leqq Z_{n:n}$ be order statistics from H. By the strong law of large numbers,

$$n^{-1} \sum_{i=1}^{n} J[Z_{i:n}] \to \int_0^1 J\left[\frac{H^{-1}(u)}{H^{-1}(1)}\right] du \tag{3.11}$$

almost surely as $n \to \infty$. Since J is uniformly continuous and $|W_{i:n} - Z_{i:n}| \to 0$ uniformly in i/n and almost surely as $n \to \infty$, by Theorem 2.2 we have that

$$n^{-1} \sum_{i=1}^{n} \{J[W_{i:n}] - J[Z_{i:n}]\} \to 0 \tag{3.12}$$

almost surely as $n \to \infty$. *Q.E.D.*

Consistency of general scores tests follows from Theorem 3.1 and the observation that $F \underset{c}{<} G$ and $F \underset{c}{\neq} G$ implies

$$\int_0^1 J\left[\frac{H^{-1}(u)}{H^{-1}(1)}\right] du > \int_0^1 J(u)\, du \tag{3.13}$$

by Theorem 2.1 (if J is strictly increasing).

Note that $\mu(H) = \int_0^1 (H^{-1}(u))/(H^{-1}(1))\, du = 1/2$ when $F \underset{c}{=} G$.

3.2. *The integral criterion.* Another class of tests that are natural for our problem are those based on one sided distance functions; that is, functions which measure the "distance" between $H_n^{-1}(x)/H_n^{-1}(1)$ and x. The integral criterion is one such statistic; that is,

$$\int_0^1 \left[\frac{H_n^{-1}(u)}{H_n^{-1}(1)} - u\right] dM_n(u) \overset{\text{def}}{=} n^{-1} \sum_{i=1}^{n} \left[W_{i:n} - \frac{i}{n}\right] L\left(\frac{i}{n}\right) \tag{3.14}$$

where $L(u) \geqq 0$. The corresponding test would reject H_0 for large values of the statistic. An equivalent statistic is

$$n^{-1} \sum_{i=1}^{n} L\left(\frac{i}{n}\right) W_{i:n}. \tag{3.15}$$

Such statistics are called systematic statistics. When $L(i/n) \equiv 1$ we have the cumulative total time on test statistic. Bickel and Doksum [7] studied selected types of such statistics for $G(x) = 1 - \exp\{-x\}$ for $x \geqq 0$. In Section 7 we prove the asymptotic equivalence of these statistics to certain general scores statistics when $G(x) = 1 - \exp\{-x\}$. It is clear that such statistics lead to isotonic and hence unbiased tests.

If L satisfies Condition 3.1, $H^{-1}(1) = 1$ and $\int_0^1 xL[H(x)]\, dH(x) < \infty$, then

$$n^{-1} \sum_{i=1}^{n} L\left(\frac{i}{n}\right) W_{i:n} \to \int_0^1 xL[H(x)]\, dH(x) \tag{3.16}$$

almost surely as $n \to \infty$. This can be proved using Theorem 2.2 and the method of proof used by Moore [16] in proving his Theorem 1.1 (that is, Theorem 4.1 in this paper). Consistency follows from the observation that $F \underset{c}{<} G$ and $F \underset{c}{\neq} G$ imply

$$\int_0^1 xL[H(x)]\,dH(x) > \int_0^1 xL(x)\,dx \tag{3.17}$$

by Theorem 2.1 since $L(x) \geqq 0$.

3.3. *The D_n^+ test.* The one sided Kolmogorov statistic suggests the one sided distance function

$$D_n^+ = \sup_{1 \leqq i \leqq n} \left[W_{i:n} - \frac{i}{n} \right] \tag{3.18}$$

for use in the convex ordering problem. The corresponding test ϕ, would reject H_0 if $D_n^+ > c_{n,\alpha}$ where $c_{n,\alpha}$ is determined by H_0. By Theorem 2.3 this test will have isotonic power, since $F_1 \underset{c}{<} F_2$ implies

$$\beta_\phi(F_1) = P_{F_1}[D_n^+ > c_{n,\alpha}] \geqq P_{F_2}[D_n^+ > c_{n,\alpha}] = \beta_\phi(F_2). \tag{3.19}$$

Intuitively, any test based on a one sided distance function will have isotonic power by Theorem 2.3.

When G is the exponential distribution, the distribution of D_n^+ under H_0 is the same as that of the one sided Kolmogorov statistic since

$$W_{i:n} \underset{\text{st}}{=} U_{i:n-1} \tag{3.20}$$

where $U_{i:n-1}$ is the ith order statistic in a sample of size $n - 1$ from a uniform distribution on [0, 1]. Birnbaum and Tingey [9] computed the exact distribution for D_n^+ under H_0. For large n we can use the well-known result

$$\lim_{n \to \infty} P_G \left\{ n^{1/2} \sup_{1 \leqq i \leqq n} \left[W_{i:n} - \frac{i}{n} \right] \leqq t \right\} = 1 - e^{-2t^2} \tag{3.21}$$

for $t \geqq 0$.

Seshadri, Csörgö, and Stephens [19] consider the D_n^+ test among other omnibus tests for exponentiality.

4. Asymptotic distribution of the cumulative total time on test statistic: general G

The *cumulative total time on test statistic* is

$$n^{-1} \sum_{i=1}^{n-1} \frac{H_n\left(\frac{i}{n}\right)}{H_n^{-1}(1)} \overset{\text{def}}{=} n^{-1} \sum_{i=1}^{n-1} W_{i:n}. \tag{4.1}$$

We reject the null hypothesis (that is, $F \underset{c}{=} G$) for large values of the statistic.

We seek the asymptotic distribution of

$$T_n \overset{\text{def}}{=} n^{1/2}\left[n^{-1}\sum_{i=1}^{n-1} W_{i:n} - \mu(H)\right] \tag{4.2}$$

under the general alternative distribution F, where

$$\mu(H) \overset{\text{def}}{=} \int_0^1 \frac{H^{-1}(u)}{H^{-1}(1)}\,du. \tag{4.3}$$

To obtain the asymptotic distribution of T_n we use the following result (see D. S. Moore [16]).

Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the order statistics from F and

$$S_n = n^{-1}\sum_{i=1}^{n} L\left(\frac{i}{n}\right) X_{i:n}, \tag{4.4}$$

$$\sigma^2 = \sigma^2(F) = 2\iint_{s<t} L[F(s)]L[F(t)]F(s)[1 - F(t)]\,ds dt. \tag{4.5}$$

THEOREM 4.1. (D. S. Moore [16]). *If* $\sigma^2 < \infty$ *and*

(i) $E|X| = \int_0^1 |F^{-1}(u)|\,du < \infty$,

(ii) *L is continuous on $[0, 1]$ except for jump discontinuities at $a_1, \cdots, a_M$, and L' is continuous and of bounded variation on $[0, 1] - \{a_1, \cdots, a_M\}$, then*

$$\mathscr{L}\left\{n^{1/2}\left[S_n - \int_{-\infty}^{\infty} xL[F(x)]\,dF(x)\right]\right\} \to N(0, \sigma^2). \tag{4.6}$$

Stigler [20], Corollary 4.1 gives weaker conditions for asymptotic normality of sums of the form $S_n = \Sigma_{i=1}^n c_{i,n}X_{i:n}$.)

To use this result note that

$$\begin{aligned} H_n^{-1}\left(\frac{i}{n}\right) &\overset{\text{def}}{=} \int_0^{F_n^{-1}(i/n)} gG^{-1}F_n(u)\,du = \int_0^{X_{i:n}} gG^{-1}F_n(u)\,du \\ &= \sum_{j=1}^{i} gG^{-1}\left(\frac{j-1}{n}\right)(X_{j:n} - X_{j-1:n}), \end{aligned} \tag{4.7}$$

where $X_0 \equiv 0$. Using (4.7) we see that

$$\begin{aligned} &\sum_{i=1}^{n-1} H_n^{-1}\left(\frac{i}{n}\right) \\ &= \sum_{i=1}^{n-2}\left\{gG^{-1}\left(\frac{i-1}{n}\right) - (n-i-1)\left[gG^{-1}\left(\frac{i}{n}\right) - gG^{-1}\left(\frac{i-1}{n}\right)\right]\right\}X_{i:n} \\ &\quad + gG^{-1}\left(\frac{n-2}{n}\right)X_{n-1:n}. \end{aligned} \tag{4.8}$$

Let

$$S_n = n^{-1} \sum_{i=1}^{n-1} H_n^{-1}\left(\frac{i}{n}\right) - \mu(H) H_n^{-1}(1). \tag{4.9}$$

Then, assuming $gG^{-1}(1) = 0$ (so that $gG^{-1}((n-1)/n) \to 0$),

$$S_n = n^{-1} \sum_{i=1}^{n} \left\{ gG^{-1}\left(\frac{i-1}{n}\right) - n\left(1 - \frac{i+1}{n} - \mu(H)\right)\left[gG^{-1}\left(\frac{i}{n}\right) - gG^{-1}\left(\frac{i-1}{n}\right)\right]\right\} X_{i:n}. \tag{4.10}$$

Assuming that $\psi(u) = gG^{-1}(u)$ has a continuous derivative on $[0, 1]$, we may approximate S_n by $n^{-1} \sum_{i=1}^{n} L(i/n)X_{i:n}$ where $L(u) = \psi(u) - (1 - u - \mu(H))\psi'(u)$.

To apply Theorem 4.1 we wish to show that for this weight function

$$\int_0^\infty xL[F(x)]\, dF(x) = 0. \tag{4.11}$$

LEMMA 4.1. *If $F, G \in \mathscr{F}$, if $\int_0^\infty x\, dF(x) < \infty$, $F \underset{c}{<} G$, $g(0) < \infty$, and if ψ' is continuous on $(0, 1)$, then*

$$H^{-1}(1) = -\int_0^\infty x\psi'[F(x)]\, dF(x), \tag{4.12}$$

where $\psi(u) = gG^{-1}(u)$.

PROOF. Recall that $H^{-1}(1) = \int_0^\infty gG^{-1}F(u)\, du$. Integrating the right expression by parts we find

$$\int_0^\infty gG^{-1}F(x)\, dx = xgG^{-1}F(x)\Big|_0^\infty - \int_0^\infty x\psi'[F(x)]\, dF(x). \tag{4.13}$$

Now

$$\lim_{x\to\infty} xgG^{-1}F(x) = \lim_{x\to\infty} x\frac{f(x)}{r(x)} = 0, \tag{4.14}$$

since $F \underset{c}{<} G$ implies $r(x) = f(x)/gG^{-1}F(x)$ is nondecreasing and $\int_0^\infty xf(x)\, dx < \infty$ by assumption. *Q.E.D.*

LEMMA 4.2. *Under the conditions of Lemma 4.1, $gG^{-1}(1) < \infty$, and $F^{-1}(0) = 0$,*

$$\int_0^\infty xL[F(x)]\, dF(x) \overset{\text{def}}{=} \int_0^\infty x\{\psi[F(x)] - (1 - F(x) - \mu(H))\psi'[F(x)]\}\, dF(x) = 0. \tag{4.15}$$

PROOF. By Lemma 4.1 and Equation (4.3) the definition of $\mu(H)$,

$$(4.16)\qquad \int_0^\infty xL(x)\,dF(x)$$
$$= \int_0^\infty x\{\psi[F(x)] - [1 - F(x)]\psi'[F(x)]\}\,dF(x) - \int_0^1 H^{-1}(u)\,du.$$

Integrating by parts, we find that

$$(4.17)\qquad \int_0^\infty x[1 - F(x)]\psi'[F(x)]\,dF(x) = \int_0^\infty F^{-1}(u)[1 - u]\psi'(u)\,du$$
$$= -\int_0^\infty [1 - F(x)]\psi[F(x)]\,dx$$
$$+ \int_0^\infty x\psi[F(x)]\,dF(x).$$

Hence,

$$(4.18)\qquad \int_0^\infty xL[F(x)]\,dF(x) = \int_0^\infty [1 - F(x)]gG^{-1}F(x)\,dx - \int_0^1 H^{-1}(u)\,du.$$

Now

$$(4.19)\qquad \int_0^1 H^{-1}(u)\,du \stackrel{\text{def}}{=} \int_0^1 \left[\int_0^{F^{-1}(u)} gG^{-1}F(s)\,ds\right] du$$
$$= \int_0^\infty [1 - F(x)]gG^{-1}F(x)\,dx$$

by another integration by parts. It follows that

$$(4.20)\qquad \int_0^\infty xL[F(x)]\,dF(x) = 0$$

as claimed. *Q.E.D.*

It follows from Theorem 4.1 and Lemma 4.2 that

$$(4.21)\qquad \mathscr{L}\{n^{1/2}S_n\} \to N(0, \sigma^2(F)),$$

where

$$(4.22)\qquad \sigma^2(F) = 2\int_0^1 \left[\int_0^t \frac{[\psi(s) - (1 - s - \mu(H))\psi'(s)]}{fF^{-1}(s)}\, s\,ds\right]$$
$$\cdot \frac{[\psi(t) - (1 - t - \mu(H))\psi'(t)]}{fF^{-1}(t)}(1 - t)\,dt.$$

Since $T_n = S_n/H_n^{-1}(1)$, an application of Slutsky's theorem (Cramér [11]) gives us the following result.

THEOREM 4.2. *Assume the conditions of Theorem 2.2. In addition, assume* $\psi(u) = gG^{-1}(u)$ *has a continuous derivative on* $[0, 1]$, $\psi(1) = 0, F^{-1}(0) = 0,$ *and*

$\sigma^2(F) < \infty$, *then*

$$\mathscr{L}\left\{n^{1/2}\left[n^{-1}\sum_{i=1}^{n-1} W_{i:n} - \mu(H)\right]\right\} \to N\left(0, \frac{\sigma^2(F)}{[H^{-1}(1)]^2}\right), \tag{4.23}$$

where $\mu(H) = \int_0^1 H^{-1}(u)\,du/H^{-1}(1)$ *and* $\sigma^2(F)$ *is given by* (4.22).

EXAMPLE. Let $G(x) = 1 - e^{-x}$ for $x \geqq 0$. Then

$$L(u) = 2(1 - u) - \zeta,\ \psi(u) = 1 - u, \tag{4.24}$$

$$n^{-1}\sum_{i=1}^{n-1} W_{i:n} - \mu(H) = n^{-1}\sum_{i=1}^{n}\frac{\left[2\left(1 - \frac{i}{n}\right) - \mu(H)\right]X_{i:n}}{H_n^{-1}(1)}, \tag{4.25}$$

We reject the exponential null hypothesis for large values of the statistic. Under the null hypothesis

$$\sum_{i=1}^{n-1} W_{i:n} \underset{st}{=} \sum_{i=1}^{n-1} U_i, \tag{4.26}$$

where U_i, $i = 1, \cdots, n - 1$, are independent uniform random variables on $[0, 1]$. It follows that

$$\mathscr{L}\left[(12n)^{1/2}\left\{n^{-1}\sum_{i=1}^{n-1} W_{i:n} - 1/2\right\}\right] \to N(0, 1) \tag{4.27}$$

under the null hypothesis.

In general, if $F \underset{c}{<} G$ and $G(x) = 1 - e^{-x}$ for $x \geqq 0$, then

$$\mathscr{L}\left\{n^{1/2}\left[n^{-1}\sum_{i=1}^{n-1} W_{i:n} - \mu(H)\right]\right\} \to N(0, \sigma^2(F)), \tag{4.28}$$

where

$$\sigma^2(F) = 2\int_0^1\left[\int_0^v \frac{\{2(1 - u) - \mu(H)\}}{fF^{-1}(u)}\,u\,du\right] \cdot \frac{[2(1 - v) - \mu(H)]}{fF^{-1}(v)}(1 - v)\,dv. \tag{4.29}$$

It can be verified that $\sigma^2(G) = 1/12$.

In the case $G(x) = x$ for $0 \leqq x \leqq 1$,

$$\sigma^2(F) = 2\int_0^1\left[\int_0^t s\,dF^{-1}(s)\right](1 - t)\,dF^{-1}(t) \tag{4.30}$$

and again $\sigma^2(G) = 1/12$.

In both cases $n^{-1}\sum_{i=1}^{n-1} W_{i:n}$ is asymptotically equivalent in distribution to $n^{-1}\sum_{i=1}^{n-1} U_i$ when $F \underset{c}{=} G$. This is *not* true for arbitrary G.

The result for the exponential case, (4.28), was first obtained by Nadler and Eilbott [17] by a different and more tedious argument.

5. Alternative classes of distributions based on the Kolmogorov distance

For this discussion we assume that $H^{-1}(1) = 1$. Consider the problem $H_0: H(t) = t$ versus $H_1: H(t)$ convex for $t \in [0, 1]$. (Note that $H(t) \leqq t$ under H_1.) If C is a class of level α tests for this problem and Ω is a class of alternatives H with H convex, the $\psi \in C$ is said to be *minimax* over Ω and C if and only if it maximizes the minimum power, that is, if and only if

$$\inf_{H \in \Omega} \beta_\psi(H) = \sup_{\phi \in C} \left[\inf_{H \in \Omega} \beta_\phi(H)\right]. \tag{5.1}$$

It is clear that Ω cannot be taken to be all H with H convex since for this class, the infima in (5.1) would be α and all tests in C would be minimax. Thus, the alternatives in Ω must be "separated." Birnbaum [8], Chapman [10], Doksum [12], and others have considered alternatives separated by the Kolmogorov distance, that is, alternatives H, with H convex in this case, and $\sup_{t \in [0,1]} [t - H(t)] \geqq \Delta$. Here, $\Omega(\Delta)$ will denote the class of H with H convex and $\sup_{t \in [0,1]} [t - H(t)] \geqq \Delta$.

5.1. *Extremal classes.* The following distributions have Kolmogorov distance Δ (that is, $\sup_{t \in [0,1]} [t - H(t)] = \Delta$) and are convex on $[0, 1]$ with $H^{-1}(1) = 1$:

$$H_{u,\Delta}(t) = \begin{cases} a_1 t, & 0 \leqq t \leqq u, \quad \Delta \leqq u \leqq 1, \\ 1 - a_2(1 - t), & u \leqq t \leqq 1, \quad \Delta \leqq u \leqq 1, \end{cases} \tag{5.2}$$

where $a_1 = (u - \Delta)/u$, $a_2 = 1 + \Delta/(1 - u)$. See Figure 1.

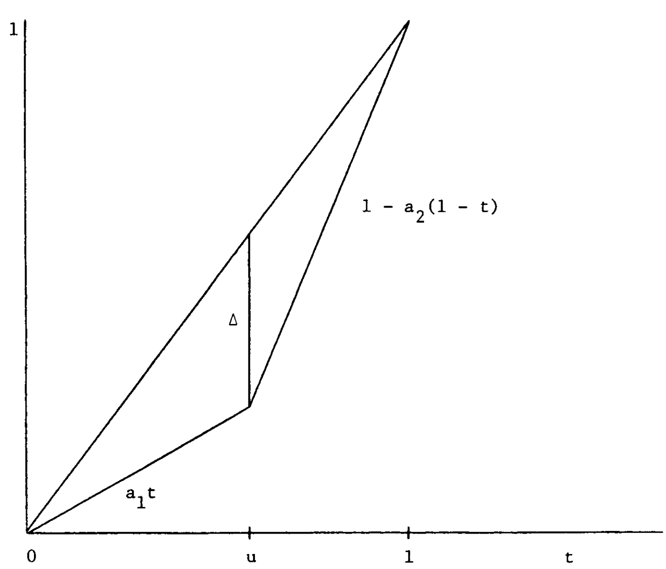

FIGURE 1
Graph of $H_{u,\Delta}$.

We will also need to use

$$(5.3)\qquad H_{u,\Delta}^{-1}(y) = \begin{cases} y/a_1, & 0 \leqq y \leqq u - \Delta, \quad \Delta \leqq u \leqq 1, \\ 1 - (1 - y)/a_2, & u - \Delta \leqq y \leqq 1, \quad \Delta \leqq u \leqq 1. \end{cases}$$

Let

$$(5.4)\qquad h_{u,\Delta}(t) = \begin{cases} a_1, & 0 \leqq t \leqq u \\ a_2, & u \leqq t \leqq 1 \end{cases}$$

denote the density of $H_{u,\Delta}(t)$. The distribution $F(F \underset{c}{<} G)$ corresponding to $H_{u,\Delta}$ has the form

$$(5.5)\qquad G_{u,\Delta}(x) = \begin{cases} G(a_1 x), & 0 \leqq x \leqq x_0 \\ G[a_1 x_0 + a_2(x - x_0)], & x_0 \leqq x < \infty \end{cases}$$

where $x_0 = G^{-1}(u - \Delta)/a_1$. To verify this, compute

$$(5.6)\qquad H_{G_{u,\Delta}}^{-1}(y) = \int_0^{G_{u,\Delta}^{-1}(y)} gG^{-1}G_{u,\Delta}(x)\,dx$$

$$= \int_0^{G_{u,\Delta}^{-1}(y)} g(a_1 x)\,dx = \frac{G_{u,\Delta}G_{u,\Delta}^{-1}(y)}{a_1} = \frac{y}{a_1}$$

for $0 \leqq y \leqq u - \Delta$. A similar calculation verifies the assertion for $u - \Delta \leqq y \leqq 1$.

The following lemma is a consequence of the fact that

$$(5.7)\qquad \inf_{H \in \Omega(\Delta)} H(t) = \inf_{\Delta \leqq u \leqq 1} H_{u,\Delta}(t).$$

LEMMA 5.1. *The distributions* $\{H_{u,\Delta}\}$, $0 \leqq \Delta \leqq u \leqq 1$, *are least favorable in* $\Omega(\Delta)$ *for the class of monotone tests in the sense that if* ϕ *is a monotone test and* $W_{i:n}$ *is replaced by* $Z_{i:n}$, *then*

$$(5.8)\qquad \inf_{H \in \Omega(\Delta)} \beta_\phi(H) = \inf_{\Delta \leqq u \leqq 1} \beta_\phi(H_{u,\Delta}).$$

PROOF. Suppose $H \in \Omega(\Delta)$ and $\Delta = u - H(u)$. Then $H_{u,\Delta}(x) \geqq H(x)$ for $0 \leqq x \leqq 1$ which in turn implies $\beta_\phi(H_{u,\Delta}) \leqq \beta_\phi(H)$ if ϕ is a monotone test. *Q.E.D.*

Let $r(x) = f(x)/gG^{-1}F(x)$ be the generalized failure rate function corresponding to F, so that $dH^{-1}(t)/dt = 1/r[F^{-1}(t)]$. We claim that the Kolmogorov distance applied to transforms of distributions provides a reasonable way of separating distributions having different failure rate variation. Suppose that F has transform $H_F \in \Omega(\Delta)$ and $\Delta = u - H(u)$. Then since $H^{-1}(t)$ is concave

$$(5.9)\qquad \sup_{0 \leqq t \leqq 1} r[F^{-1}(t)] - \inf_{0 \leqq t \leqq 1} r[F^{-1}(t)]$$

$$= \sup_{0 \leqq t \leqq 1} \left[\frac{d}{dt} H^{-1}(t)\right]^{-1} - \inf_{0 \leqq t \leqq 1} \left[\frac{d}{dt} H^{-1}(t)\right]^{-1}$$

$$\geq \sup_{0 \leq t \leq 1} \left[\frac{d}{dt} H_{u,\Delta}^{-1}(t)\right]^{-1} - \inf_{0 \leq t \leq 1} \left[\frac{d}{dt} H_{u,\Delta}^{-1}(t)\right]^{-1}$$

$$= a_2 - a_1 = \frac{\Delta}{u(1-u)} \geq 4\Delta.$$

Hence, large values of Δ correspond to large failure rate variation.

5.2. *Contiguity.* The concept of contiguous alternatives plays a crucial role in Sections 7 and 8. (See LeCam [14], Hájek [13].) Let $\{H_\nu, K_\nu\}\nu \geq 1$ be a sequence of similar testing problems. In this sequence the νth testing problem concerns n_ν observations $X_1, \cdots, X_{n_\nu}$ with $n_\nu \to \infty$. In our setup H_ν depends on ν through n_ν only, whereas K_ν depends on the parameters Δ_{n_ν}, in addition. Our problem is to determine

$$\lim_{\nu \to \infty} \beta(\alpha, H_\nu, K_\nu) = \beta(\alpha), \qquad 0 \leq \alpha \leq 1, \tag{5.10}$$

where the sequence $\{\Delta_n\}$ will be chosen so that $\alpha < \beta(\alpha) < 1$. The concept of contiguous alternatives will be useful in computing (5.10) in Sections 7 and 8.

DEFINITION. *A sequence* $\{g_{u,\Delta_n}\}$ *is said to be* contiguous *to* $g_{u,0}$ *(in the sense of LeCam-Hájek) if for any sequence of random variables* $R_n(X_1, \cdots, X_n)$, $R_n \to 0$ *in* P_0 *probability implies* $R_n \to 0$ *in* P_{Δ_n} *probability where* P_θ *denotes the probability distribution of* $X_1, \cdots, X_n$ *if* $g_{u,\theta}$ *is true.*

The following conditions implying contiguity for sequences when $\lim_{n\to\infty} n^{1/2}\Delta_n = c$ for some $0 \leq c < \infty$ can be found in Bickel and Doksum [7]:

$$\text{(5.11)} \quad \begin{aligned} &\text{(a)} \quad \partial g_{u,\Delta}(x)/\partial\Delta \neq 0 \quad \text{whenever} \quad g_{u,\Delta}(x) > 0, \\ &\text{(b)} \quad \int_0^\infty \sup \{[\partial g_{u,\Delta}(x)/\partial\Delta]^2 [g_{u,\Delta}(x)]^{-1} : 0 \leq \Delta \leq \delta\}\, dx < \infty \end{aligned}$$

for some $\delta > 0$.

It is easy to verify that (5.11) (b) holds for $g_{u,\Delta}(x) = \partial G_{u,\Delta}(x)/\partial x$ (where $G_{u,\Delta}$ is defined by (5.5) and $G(x) = 1 - e^{-x}$) for some $\delta > 0$ such that $0 < \delta < u < 1$. Hence, by condition (5.11) $\{g_{u,\Delta_n}\}$ are contiguous alternatives to $g_{u,0}$ if $0 < u < 1$. This fact will be used in Sections 7 and 8.

6. Asymptotic minimax property of the cumulative total time on test statistic: uniform case

In this section we assume that $G(x) = x$ for $0 \leq x \leq 1$ so that $H^{-1}(t) = F^{-1}(t)$. We also assume that $H^{-1}(1) = F^{-1}(1) = 1$. Our problem then is

$$H_0 : H(t) = t, \qquad t \in [0, 1] \tag{6.1}$$

versus

$$H_1 : H(t) \quad \text{convex}, \qquad t \in [0, 1]. \tag{6.2}$$

Let $Z_1, Z_2, \cdots, Z_n$ be independent observations from H. We study statistics of the form

$$n^{-1} \sum_{i=1}^{n} J[Z_i]. \tag{6.3}$$

We show that the test ψ, corresponding to $J(x) = x$ is asymptotically minimax. The function $J(x) = x$ corresponds to the cumulative total time on test statistic.

6.1. *Asymptotic properties of the cumulative total time on test statistic.* Consider the test

$$\psi(Z) = \begin{cases} 1 & \text{if} \quad (12n)^{1/2}\left[n^{-1} \sum_{i=1}^{n} Z_i - 1/2\right] > k_{n,\alpha}, \\ 0 & \text{otherwise,} \end{cases} \tag{6.4}$$

where $k_{n,\alpha}$ is defined by $\beta_\psi(U) \stackrel{\text{def}}{=} P_U[\psi(Z) > k_{n,\alpha}] = \alpha$ and $U(t) = t, t \in [0, 1]$.

Computing $\mu = E(Z)$ under $H_{u,\Delta}$ we find

$$\mu = \mu(\Delta, u) = \int_0^u x \frac{u - \Delta}{u} dx + \int_u^1 x \left[\frac{1 - u + \Delta}{1 - u}\right] dx = \frac{1 + \Delta}{2} \tag{6.5}$$

and $\sigma^2 = \sigma^2(\Delta, u) = EZ^2 - \mu^2$ or

$$\sigma^2 = \frac{1}{12} + \frac{\Delta}{3}(1 + u) - \frac{(2\Delta - \Delta^2)}{4} \geqq 0.01 \tag{6.6}$$

for $0 \leqq \Delta \leqq 0.01$.

In order to compute asymptotic quantities such as

$$\lim_{n \to \infty} \inf_{\Delta \leqq u \leqq 1} \beta_\psi(H_{u,\Delta}) \tag{6.7}$$

the Berry-Esseen theorem (Loève, p. 288) will be needed. Doksum [12] made a similar application to a related problem. Applied to the random variables $Z_1, \cdots, Z_n$, it states that if $\mu = E(Z_i)$, $E(Z_i - \mu)^2 = \sigma^2$, $E|Z_i - \mu|^3 = \beta$, and H_n^* is the distribution of $\Sigma_{i=1}^n (Z_i - \mu)/n^{1/2}\sigma$, then there exists a constant $K < \infty$ such that for all x

$$|H_n^*(x) - \Phi(x)| \leqq \frac{K\beta}{n^{1/2}\sigma^{3/2}}. \tag{6.8}$$

If H_n^* is the distribution of $\Sigma_{i=1}^n (Z_i - \mu)/n^{1/2}\sigma$ under $H_{u,\Delta}$, then $|Z_i - \mu| \leqq 1$ implies

$$\beta = E|Z_i - \mu|^3 \leqq 1. \tag{6.9}$$

Then (6.6), (6.8) and (6.9) imply

$$|H_n^*(x) - \Phi(x)| \leqq \frac{1000K}{n^{1/2}} \qquad \text{for all} \quad \Delta \in [0, 0.01], \tag{6.10}$$

for all $u \in [\Delta, 1]$, and for all x, where Φ is the $N(0, 1)$ distribution.

For the alternatives $H_{u,\Delta}$ and for the test corresponding to the statistic given by (6.3) we have

$$\beta_\psi(H_{u,\Delta}) = P\left\{(12n)^{1/2}\left[n^{-1}\sum_{i=1}^{n} Z_i - 1/2\right] \geqq k_{n,\alpha} \middle| H_{u,\Delta}\right\} \tag{6.11}$$

$$\beta_\psi(H_{u,\Delta}) = P\left\{n^{1/2}\left[n^{-1}\frac{\sum_{i=1}^{n}(Z_i-\mu)}{\sigma}\right] \geqq \frac{k_{n,\alpha}-(3n)^{1/2}\Delta}{(12)^{1/2}\sigma} \middle| H_{u,\Delta}\right\}. \tag{6.12}$$

This and (6.10) imply

$$\left|\beta_\psi(H_{u,\Delta}) - \Phi\left(\frac{-k_{n,\alpha}+(3n)^{1/2}\Delta}{(12)^{1/2}\sigma}\right)\right| \leqq \frac{1000K}{n^{1/2}} \tag{6.13}$$

for all $\Delta \in [0, 0.01]$ and $u \in [\Delta, 1]$.

LEMMA 6.1. *The cumulative total time on test, ψ, satisfies*

(i) $\inf_{H\in\Omega(\Delta_n)} \beta_\psi(H)$ *tends to a limit between α and one as $n \to \infty$ if and only if* $\lim_{n\to\infty} n^{1/2}\Delta_n = c > 0$.

(ii) *For each sequence $\{\Delta_n\}$ such that $\lim_{n\to\infty} n^{1/2}\Delta_n = c > 0$ one has*

$$\lim_{n\to\infty}\left[\inf_{H\in\Omega(\Delta_n)} \beta_\psi(H)\right] = \Phi(-k_\alpha + c\,3^{1/2}), \tag{6.14}$$

where k_α is defined by $\Phi(k_\alpha) = 1 - \alpha$.

PROOF. By Lemma 4.1

$$\inf_{H\in\Omega(\Delta)} \beta_\psi(H) = \inf_{\Delta\leqq u\leqq 1} \beta_\psi(H_{u,\Delta}). \tag{6.15}$$

Let $u_0 = u_0(\Delta, n)$ be such that

$$\beta_\psi(H_{u_0,\Delta}) = \inf_{\Delta\leqq u\leqq 1} \beta_\psi(H_{u,\Delta}) \overset{\text{def}}{=} \beta_\psi(\Delta). \tag{6.16}$$

Now (6.13) implies that

$$\left|\beta_\psi(\Delta) - \Phi\left(\frac{-k_{n,\alpha}+(3n)^{1/2}\Delta}{12^{1/2}\sigma(\Delta, u_0)}\right)\right| \leqq \frac{1000K}{n^{1/2}} \tag{6.17}$$

where $\Delta \in [0, 0.01]$. From (6.6), one has that $\sigma^2(\Delta, u_0) \to 1/12$ as $\Delta \to 0$. Moreover, $-k_{n,\alpha} \to -k_\alpha$ with k_α satisfying $\Phi(-k_\alpha) = \alpha$. Thus, (6.17) implies that $\beta_\psi(\Delta_n)$ tends to a limit between α and one if and only if $(3n)^{1/2}\Delta_n \to c$ for some $c > 0$. This implies (*i*). Furthermore, when $n^{1/2}\Delta_n \to c$, then $\beta_\psi(\Delta_n) \to \Phi(-k_\alpha + 3^{1/2}c)$ which is (ii). *Q.E.D.*

6.2. *Asymptotic properties of general scores statistics.* The general scores statistics look like

$$T_n(J) = n^{-1}\sum_{i=1}^{n} J(Z_i). \tag{6.18}$$

Consider the test

$$\phi = \begin{cases} 1 & \text{if } n^{1/2}[T_n(J) - \mu_J]/\sigma_J > k_{n,\alpha}, \\ 0 & \text{otherwise}, \end{cases} \tag{6.19}$$

where $\mu_J = \int_0^1 J(x)\,dx$, $\sigma_J^2 = \int_0^1 J^2(x)\,dx - \mu_J^2$, and $k_{n,\alpha}$ is defined by $\beta_\phi(U) = \alpha$ where U is the uniform distribution on $[0, 1]$.

LEMMA 6.2. *If* $\{\Delta_n\}$ *satisfies* $\lim_{n\to\infty} n^{1/2}\Delta_n = c > 0$, *and* J *satisfies Condition* 3.1, *then*

$$\lim_{n\to\infty} \beta_\phi(H_{u,\Delta_n}) = \Phi\left(-k_\alpha + \frac{c\left[-\frac{1}{u}\int_0^u J(x)\,dx + \frac{1}{1-u}\int_u^1 J(x)\,dx\right]}{\sigma_J}\right). \tag{6.20}$$

PROOF. Let $E_J(\Delta, u)$ and $V_J(\Delta, u)$ denote $E[T_n(J)|H_{u,\Delta}]$ and Var $[T_n(J)|H_{u,\Delta}]$. Then by definition

$$E_J(\Delta, u) = \left(\frac{u - \Delta}{u}\right)\int_0^u J(x)\,dx + \left(\frac{1 - u + \Delta}{1 - u}\right)\int_\mu^1 J(x)\,dx \tag{6.21}$$

by (5.2) and

$$V_J(\Delta, u) = n^{-1}\left\{\left(\frac{u - \Delta}{u}\right)\int_0^u J^2(x)\,dx + \left(\frac{1 - u + \Delta}{1 - u}\right)\int_u^1 J^2(x)\,dx - E_J^2(\Delta, u)\right\}. \tag{6.22}$$

Since in Condition 3.1 we assumed $\int_0^1 J^2(x)\,dx < \infty$, we see that $E_J(\Delta, u) < \infty$ and $V_J(\Delta, u) < \infty$ so long as $0 < u < 1$. Note that $V_J(\Delta, u) \to V_J(0, u) = n^{-1}\sigma_J^2 = n^{-1}[\int_0^1 J^2(x)\,dx - (\int_0^1 J(x)\,dx)^2]$ as $\Delta \to 0^+$. Thus, the central limit theorem implies that for each sequence $\{\Delta_n\}$ with $\Delta_n \to 0$ and $0 < u < 1$,

$$\lim_{n\to\infty} P\left[n^{1/2}\left(\frac{T_n(J) - E_J(\Delta_n, u)}{\sigma_J}\right) \leqq t\right] = \Phi(t). \tag{6.23}$$

From (6.19) we have

$$\beta_\phi(H_{\Delta,u}) = P\left\{-n^{1/2}\left[\frac{T_n(J) - E_J(\Delta, u)}{\sigma_J}\right] \leqq -k_{n,\alpha} + \frac{n^{1/2}}{\sigma_J}[E_J(\Delta, u) - \mu_J]\right\}. \tag{6.24}$$

Using the definition of μ_J and $E_J(\Delta, u)$ we see that

$$E_J(\Delta, u) - \mu_J = -\frac{\Delta}{u}\int_0^u J(x)\,dx + \frac{\Delta}{1 - u}\int_u^1 J(x)\,dx. \tag{6.25}$$

Since $\lim_{n\to\infty} n^{1/2}\Delta_n = c > 0$ we see that

$$\lim_{n\to\infty} n^{1/2}[E_J(\Delta_n, u) - \mu_J] = c\left[-\frac{1}{u}\int_0^u J(x)\,dx + \frac{1}{1-u}\int_u^1 J(x)\,dx\right] \tag{6.26}$$

which completes the proof of (6.20). *Q.E.D.*

We would like to show that

$$\inf_{0<u<1}\left[\frac{-\frac{1}{u}\int_0^u J(x)\,dx + \frac{1}{1-u}\int_u^1 J(x)\,dx}{\sigma_J}\right] \tag{6.27}$$

is maximized for $J(x) = x$ since this would imply that the cumulative total time on test statistic maximizes the minimum power. (Note that (6.27) is unchanged if we replace J by $aJ + b$ when $a > 0$.)

The following lemma was communicated to the authors by W. R. van Zwet.

LEMMA 6.3. (W. R. van Zwet)

$$A_J = \inf_{0<u<1}\left[\frac{-u^{-1}\int_0^u J(x)\,dx + (1-u)^{-1}\int_u^1 J(x)\,dx}{\sigma_J}\right] \tag{6.28}$$

is maximized among all square integrable J on (0, 1) *by* $J(x) = x$ *where*

$$\sigma_J^2 = \int_0^1 J^2(x)\,dx - \left[\int_0^1 J(x)\,dx\right]^2. \tag{6.29}$$

PROOF. Since the value of A_J remains unchanged if J is replaced by $aJ + b$, $a > 0$, we may assume

$$\int_0^1 J(x)\,dx = \int_0^1 x\,dx = 1/2 \tag{6.30}$$

and

$$\int_0^1 J^2(x)\,dx = \int_0^1 x^2\,dx = 1/3. \tag{6.31}$$

Let $J(x) = x + K(x)$. Then

$$\int_0^1 K(x)\,dx = 0, \tag{6.32}$$

$$\int_0^1 [K^2(x) + 2xK(x)]\,dx = 0, \tag{6.33}$$

and

$$A_J = A_I + \inf_{0<u<1}\frac{-u^{-1}\int_0^u K(x)\,dx + (1-u)^{-1}\int_u^1 K(x)\,dx}{(1/12)^{1/2}}, \tag{6.34}$$

where $I(x) = x$ and $A_I = 3^{1/2}$ (for $J = I$ the infimum is assumed at every u!). Suppose the proposition were false, then we would have for some K satisfying (6.32) and (6.33)

$$\inf_{0<u<1} \left[-u^{-1} \int_0^u K(x)\,dx + (1-u)^{-1} \int_u^1 K(x)\,dx \right] = \inf_{0<u<1} \frac{1}{u(1-u)} \int_u^1 K(x)\,dx > 0, \tag{6.35}$$

and hence

$$\int_u^1 K(x)\,dx > 0 \qquad \text{for all} \quad 0 < u < 1. \tag{6.36}$$

However,

$$\int_0^1 xK(x)\,dx = \int_0^1 dx \int_x^1 K(y)\,dy \tag{6.37}$$

and hence (6.36) would imply that $\int_0^1 xK(x)\,dx > 0$ which contradicts (6.33). *Q.E.D.*

We have proved a minimax result for testing that F is uniform versus F convex.

THEOREM 6.1. *Let $G(x) = x$, $0 \leqq x \leqq 1$, in the problem* (1.2) *and* (1.3). *If J is square integrable, satisfies Condition* 3.1 *and ϕ is the level α general scores test associated with J, then*

$$\lim_{n\to\infty} [\inf_{H\in\Omega(\Delta_n)} \beta_\psi(H)] \geqq [\limsup_{n\to\infty} \inf_{H\in\Omega(\Delta_n)} \beta_\phi(H)]; \tag{6.38}$$

that is, the level α cumulative total time on test statistic corresponding to $J(x) = x$ is minimax in the class of tests whose weight functions satisfy the conditions above.

Doksum [12] showed that $J(x) = x$ provides a minimax test over the class of those J satisfying Condition 3.1 and over the class of stochastically ordered alternatives determined by the Kolomogorov distance.

It follows from Theorem 6.1 that, in the minimax sense, the uniform scores test is better than tests based on Fisher's weights ($J(x) = \log x$), better than the Pearson or exponential scores test and better than the normal scores test.

7. Asymptotic normality and efficiency of statistics based on total time on test statistics: exponential case

Bickel and Doksum [7] and Bickel [6] considered four classes of statistics for testing $H_0: F(x) = G_\lambda(x) = 1 - \exp\{-\lambda x\}$ against IFR alternatives. These four types of statistics were shown to be asymptotically equivalent and it was shown that each of the classes contains asymptotically most powerful statistics for parametric alternatives. We now show that the statistics

$$T_n(J) = n^{-1} \sum_{i=1}^n J(W_{i:n}) \tag{7.1}$$

based on the total time on test statistics are asymptotically equivalent to the four classes of statistics in [6] and [7]. Consequently, for a given parametric family $\{F_\theta\}$ of IFR distributions, it is possible to find a $J = J_{F_\theta}$ such that the test that rejects H_0 for large values of $T_n(J)$ is asymptotically most powerful.

Note that under H_0, $W_{1:n}, \cdots, W_{n-1:n}$ are distributed as the order statistics of a sample of size $n - 1$ from the uniform distribution on [0, 1]. Using well-known results (for example, [16]) on linear combinations of order statistics in reverse, we have that $T_n(J)$ is asymptotically equivalent to

$$S_n(J) = n^{-1} \sum_{i=1}^{n} J'\left(\frac{i}{n+1}\right) W_{i:n}. \tag{7.2}$$

More precisely, if we define

$$\mu_J = \int_0^1 J(x)\,dx, \quad \text{and} \quad \sigma_J^2 = \int_0^1 J^2(x)\,dx - \mu_J^2, \tag{7.3}$$

then we have the following Lemma.

LEMMA 7.1. *Suppose that* $0 < \sigma^2 < \infty$, *and that* J' *satisfies condition* (ii) *of Theorem* 4.1, *then*

$$n^{1/2}\{[S_n(J) - (J(1) - \mu_J)] - [T_n(J) - \mu_J]\} \tag{7.4}$$

converges to zero in probability under H_0.

Next note that since $n\bar{X}\ W_{i:n} = \Sigma_{j=1}^{i} (n - j + 1)(X_{j:n} - X_{j-1:n})$ if we set $D_j = (n - j + 1)(X_{j:n} - X_{j-1:n})$ and

$$V_n(J) = -(n\bar{X})^{-1} \sum_{i=1}^{n} J\left(\frac{i}{n+1}\right) D_j \tag{7.5}$$

then

$$n^{1/2}[S_n(J) - [V_n(J) + J(1)]] \tag{7.6}$$

tends to zero in probability under H_0. For a given parametric family $\{F_\theta\}$ of distributions, it is shown in [6] that there exists a function $a(u) = a_{F_\theta}(u)$ (see [6], Equation (2.9)) such that the test that rejects H_0 for large values of $V_n(a)$ is asymptotically most powerful for $\{F_\theta\}$. Using this, Lemma 7.1, and the definition of contiguity, we have

THEOREM 7.1. *If* $J = a$ *satisfies the conditions of Lemma* 7.1 *and* $\{F_\theta\}$ *satisfies the conditions of Corollary* 2.1 *of Bickel* [6], *then the test that rejects* H_0 *for large values of* $T_n(a)$ *is asymptotically most powerful among all similar tests.*

Bickel and Doksum ([7], Section 7) show that $n^{1/2}\ \bar{X} V_n(J)$ can be approximated by a sum $\Sigma_{i=1}^{n} h(X_i)$, of independent, identically distributed random variables. We now proceed to give a similar approximation using a different derivation. Note that we can write

$$-\bar{X} V_n(J) \tag{7.7}$$

$$= n^{-1} \sum_{i=1}^{n} \left\{\left(1 - \frac{i-1}{n}\right)\left[n\left(J\left(\frac{i}{n+1}\right) - J\left(\frac{i+1}{n+1}\right)\right)\right] + J\left(\frac{i+1}{n+1}\right)\right\} X_{i:n}.$$

It follows that if we define

$$L(u) = L_J(u) = (1 - u)J'(u) - J(u) \tag{7.8}$$

and

$$W_n(J) = n^{-1} \sum_{i=1}^{n} L\left(\frac{i}{n+1}\right) X_{i:n}, \tag{7.9}$$

then

$$n^{1/2}\left(W_n(J) - \bar{X} V_n(J)\right) \tag{7.10}$$

and

$$n^{1/2}\left(\frac{W_n(J)}{\bar{X}} - V_n(J)\right) \tag{7.11}$$

tend to zero in probability under H_0.

We will need

$$\xi_W = \int_0^\infty x L[F(x)]\, dF(x) \tag{7.12}$$

and

$$\tau_W^2 = 2 \iint_{0<s<t<\infty} L[F(s)]L[F(t)]F(s)[1 - F(t)]\, ds\, dt. \tag{7.13}$$

If we apply Moore's approximation [16] to $W_n(J)$ we get

LEMMA 7.2. *If* $\tau_W^2 < \infty$ *if* $\mu_{F^{-1}} < \infty$, *and if* L *satisfies condition* (ii) *of Theorem* 4.1, *then* $n^{1/2}\{[W_n(J) - \xi_W] - Q_n(J)\}$ *tends to zero in probability, where*

$$Q_n(J) = n^{-1} \sum_{i=1}^{n} B_F(X_i) \tag{7.14}$$

and

$$B_F(x) = \int_0^x F(t)L[F(t)]\, dt - \int_x^\infty [1 - F(t)]L[F(t)]\, dt. \tag{7.15}$$

This result establishes the asymptotic equivalence under contiguous alternatives of all the statistics in this section with sums of independent, identically distributed random variables. For the computations of asymptotic power we need some lemmas.

LEMMA 7.3. *If the conditions of Lemma* 7.2 *hold, then* $E[B_F(X)|F] = 0$.

PROOF. If we define $I_x(t) = 1(0)$ if $x \leqq t (x > t)$, then

$$B_F(x) = \int_0^\infty [F(t) - I_x(t)]L[F(t)]\, dt. \tag{7.16}$$

The result follows since $E[I_X(t)|F] = F(t)$. *Q.E.D.*

LEMMA 7.4. *If* $F(x) = 1 - \exp\{-x\}$, *then*

$$B_F(x) = J[F(x)] - \int_0^x J[F(t)]\,dt. \tag{7.17}$$

PROOF. Note that $d\{[1 - F(x)]J[F(x)\}/dx = L[F(x)]f(x)$ and that $f(x) = 1 - F(x)$. Thus integrating by parts

$$\begin{aligned} &\int_0^x F(t)L[F(t)]\,dt \\ &= \int_0^x F(t)[1 - F(t)]^{-1}\,d[(1 - F(t))J[F(t)]] \\ &= F(x)J[F(x)] - \int_0^x [1 - F(t)]J[F(t)]\,d[F(t)[1 - F(t)]^{-1}] \\ &= F(x)J[F(x)] - \int_0^x J[F(t)]\,dt, \end{aligned} \tag{7.18}$$

where the last equality follows from $F(t)[1 - F(t)]^{-1} = e^t - 1$.

Similarly,

$$\begin{aligned} &\int_x^\infty [1 - F(t)]L[f(t)]\,dt \\ &= \int_x^\infty d[(1 - F(t)]J[F(t)]] = -(1 - F(x))J[F(x)]. \end{aligned} \tag{7.19}$$

Q.E.D.

LEMMA 7.5. (i) *If* $F(x) = G_\lambda(x) = 1 - \exp\{-\lambda x)$, *then* $B_{G_\lambda}(x) = B_{G_1}(\lambda x)/\lambda$.

(ii) *If* $J(u) = u$, *then* $B_{G_1}(x) = 2G_1(x) - x$.

PROOF. Part (i) follows by setting $x = \lambda t$ in the definition of $B_{G_\lambda}(x)$. Part (ii) is immediate. *Q.E.D.*

LEMMA 7.6. *If* $t[1 - F(t)]J[F(t)] \to 0$ *as* $t \to \infty$, *then*

(i) $\xi_W = -\int_0^\infty [1 - F(t)]J[F(t)]\,dt$;

(ii) *if* $F(x) = G_1(x)$, *then* $\xi_W = -\mu_J$.

PROOF.

$$\xi_W = \int_0^\infty t[1 - F(t)]J'[F(t)]\,dF(t) - \int_0^\infty tJ[F(t)]\,dF(t). \tag{7.20}$$

But

$$\begin{aligned} \int_0^\infty t[1 - F(t)]J'[F(t)]\,dF(t) &= \int_0^\infty t[1 - F(t)]\,dJ[F(t)] \\ &= -\int_0^\infty [1 - F(t)]J[F(t)]\,dt \\ &\quad + \int_0^\infty tJ[F(t)]\,dF(t) \end{aligned} \tag{7.21}$$

by integration by parts. Part (i) follows.

To show (ii), note that if we set $u = F(t) = G_1(t)$, then

$$\frac{du}{dt} = \exp\{-t\} = 1 - u; \tag{7.22}$$

thus

$$\int_0^\infty [1 - F(t)]J[F(t)]\,dt = \int_0^1 J(u)\,du. \tag{7.23}$$

Q.E.D.

If we now put together the results of this section, we have that

$$n^{1/2}[T_n(J) - \mu_J] - n^{1/2}[(\bar{X})^{-1}[Q_n(J) - \mu_J] + \mu_J] \tag{7.24}$$

converges to zero under G_1 and under contiguous alternatives, where

$$Q_n(J) = n^{-1}\sum_{i=1}^{n} B_{G_1}(X_i). \tag{7.25}$$

Let

$$\begin{aligned}\mu(F, J) &= \lim_{n\to\infty}\{\bar{X}^{-1}[Q_n(J) - \mu_J] + \mu_J\} \\ &= [\mu(F^{-1})]^{-1}\Bigg[\int_0^\infty J[G_1(t)]\,dF(t) \\ &\qquad - \int_0^\infty [1 - F(t)]J[G_1(t)]\,dt - \mu_J\Bigg] + \mu_J.\end{aligned} \tag{7.26}$$

We have shown the following theorem.

THEOREM 7.2. *If the conditions of Lemma* 7.1 *and* 7.2 *are satisfied, and if* $\{F_n\}$ *is a sequence of alternative distributions contiguous to* G_1, *then*

$$\frac{n^{1/2}}{\sigma_J}[T_n(J) - \mu_J - \mu(F_n, J)] \tag{7.27}$$

converges in law to a standard normal variable.

REMARK 7.1. Theorem 7.2 can be used to obtain the results of Theorem 7.1. Let

$$\phi = \begin{cases} 1 & \text{if } \dfrac{n^{1/2}}{\sigma_J}[T_n(J) - \mu_J] > k_{n,\alpha}, \\ 0 & \text{otherwise.}\end{cases} \tag{7.28}$$

and

$$\begin{aligned}B_\phi(G_{u,\Delta}) = P_{G_{u,\Delta}}\Bigg\{&\frac{n^{1/2}}{\sigma_J}[T_n(J) - \mu_J - \mu(G_{u,\Delta}, J)]_\Delta \\ &> k_{n,\alpha} - \frac{1}{\sigma_J}\mu(G_{u,\Delta}, J)\Bigg\}\end{aligned} \tag{7.29}$$

where $G_{u,\Delta}$ is defined by (5.5) and $G(x) = 1 - e^{-x}$. If $\lim_{n\to\infty} n^{1/2}\Delta_n = c > 0$, then using Theorem 7.2 it is easy to verify that

$$\lim_{n\to\infty} \beta_\phi(G_{u,\Delta_n}) = \Phi\left\{-k_\alpha + \frac{c}{\sigma_J}\left[-\frac{1}{u}\int_0^u J(x)\,dx + \frac{1}{1-u}\int_u^1 J(x)\,dx\right]\right\}. \tag{7.30}$$

8. Asymptotic minimax property of the cumulative total time on test statistic: exponential case

In this section we again consider the null hypothesis $H_0: F(x) = G_\lambda(x) = 1 - \exp\{-\lambda x\}$. Let

$$T_n(J) = n^{-1}\sum_{i=1}^{n-1} J(W_{i:n}) \tag{8.1}$$

where J is increasing, $\mu_J = \int_0^1 J(x)\,dx < \infty$, $\sigma_J^2 = \int_0^1 J^2(x)\,dx - \mu_J^2 < \infty$, and

$$W_{i:n} = \frac{\sum_{j=1}^{i}(n-j+1)(X_{j:n} - X_{j-1:n})}{\sum_{i=1}^{n}(n-j+1)(X_{j:n} - X_{j-1:n})} \tag{8.2}$$

in this case. Consider the test

$$\phi = \begin{cases} 1 & \text{if } \dfrac{n^{1/2}}{\sigma_J}[T_n(J) - \mu_J] > k_{n,\alpha}, \\ 0 & \text{otherwise,} \end{cases} \tag{8.3}$$

where $k_{n,\alpha}$ is defined by $\beta_\phi(U) = \alpha$ and U is the uniform distribution on $[0, 1]$.

Let F have transform H (see (2.1)) and

$$H_1^{-1}(t) = \frac{H^{-1}(t)}{H^{-1}(1)}. \tag{8.4}$$

Let $\Omega_1(\Delta)$ be the class of distributions $F \in \mathscr{F}$ for which

$$\sup_{\Delta < u < 1} [u - H_1(u)] \geqq \Delta \tag{8.5}$$

and H is convex on $[0, 1]$. Note that if $F \in \Omega_1(\Delta)$ there does not necessarily exist $u \in [\Delta, 1]$ such that $F \underset{c}{<} G_{u,\Delta}$. If we restrict ourselves to totally ordered classes of distributions in $\Omega_1(\Delta)$ containing some $G_{u,\Delta}$, then we have that $F \in \Omega_1(\Delta)$ implies $F \underset{c}{<} G_{u,\Delta}$. The union of these classes is the class $\Gamma(\Delta)$ of F in $\Omega_1(\Delta)$ for which there exists as $u \in [\Delta, 1]$ such that $F \underset{c}{<} G_{u,\Delta}$. Suppose that $F \in \Gamma(\Delta)$. Since ϕ is an isotonic test we have by Theorem 2.4, that

$$\begin{aligned} \beta_\phi(F) &= P\left[\frac{n^{1/2}}{\sigma_J}[T_n(J) - \mu_J] > k_{n,\alpha} \middle| F\right] \\ &\geqq P\left[\frac{n^{1/2}}{\sigma_J}[T_n(J) - \mu_J] > k_{n,\alpha} \middle| G_{u,\Delta}\right] = \beta_\phi(G_{u,\Delta}) \end{aligned} \tag{8.6}$$

for some $u \in [\Delta, 1]$. Hence,

$$\inf [\beta_\phi(F): F \in \Gamma(\Delta)] = \inf [\beta_\phi(G_{u,\Delta}): u \in [\Delta, 1]]. \tag{8.7}$$

Let $T_n^{(1)} = n^{-1} \Sigma_{i=1}^{n-1} W_{i:n}$ and

$$\psi = \begin{cases} 1 & \text{if } (12n)^{1/2}[T_n^{(1)} - 1/2] \geqq k_{n,\alpha}, \\ 0 & \text{otherwise,} \end{cases} \tag{8.8}$$

that is, the test based on the cumulative total time on test statistic. Let $\mu(H_1) = [H^{-1}(1)]^{-1} \int_0^1 H^{-1}(u)\, du$ and $\sigma_1^2(H) = \sigma^2(F)[H^{-1}(1)]^{-2}$ where $\sigma^2(F)$ is given by (4.29).

LEMMA 8.1. *If* $\lim_{n\to\infty} n^{1/2}\Delta_n = c$, *then*

$$\lim_{n\to\infty} [\inf \beta_\psi(F): F \in \Gamma(\Delta_n)] = \Phi(-k_\alpha + 3^{1/2}c). \tag{8.9}$$

PROOF. Assume $H^{-1}(1) = 1$. By definition

$$\begin{aligned} \beta_\psi(G_{u,\Delta}) &= P_{u,\Delta}[(12n)^{1/2}(T_n^{(1)} - 1/2) \geqq k_{n,\alpha}] \\ &= P_{u,\Delta}[(12n)^{1/2}[(T_n^{(1)} - 1/2) - (\mu(H_{u,\Delta}) - 1/2)] \\ &\quad \geqq k_{n,\alpha} - (12n)^{1/2}[\mu(H_{u,\Delta}) - 1/2]]. \end{aligned} \tag{8.10}$$

For the distribution G_{u,Δ_n}, $\Delta_n = c/n^{1/2}$, and the statistic

$$S_n^{(1)} = n^{-1} \sum_{i=1}^{n} J_0\left(\frac{i}{n}\right) X_{i:n}, \quad J_0(t) = 2(1 - t) \tag{8.11}$$

we find that the error term I_{2n} of Moore [16] is zero, while the second one, I_{3n}, tends to zero in probability uniformly in $u \in [\Delta_n, 1]$, that is,

$$\sup_u P_{u,\Delta_n}(|I_{3n}| > \varepsilon) \to 0 \quad \text{as} \quad n \to \infty \tag{8.12}$$

for each $\varepsilon > 0$. Thus, $n^{-1}[S_n^{(1)} - \mu(H_{u,\Delta})]$ can be expressed as a sum of independent, identically distributed random variables with third moments plus a term that tends to zero uniformly in u. Using the representation

$$n^{1/2}[T_n^{(1)} - \mu(H_{u,\Delta})] = \frac{n^{1/2}}{\bar{X}}[S_n^{(1)} - \bar{X}\mu(H_{u,\Delta})], \tag{8.13}$$

we find that the same thing is true for

$$n^{1/2}[T_n^{(1)} - \mu(H_{u,\Delta})]. \tag{8.14}$$

If we now apply a uniform version of Slutsky's theorem and the Berry-Esseen theorem, we have that

$$\sup_{u\in[\Delta_n, 1]} |\beta_\psi(G_{u,\Delta_n}) - \{1 - \Phi[k_{n,\alpha} - (12n)^{1/2}(\mu(H_{u,\Delta_n}) - 1/2)]\}| \to 0 \quad \text{as} \quad n \to \infty. \tag{8.15}$$

The result now follows by the computations leading to Lemma 6.1. *Q.E.D.*

From Lemma 8.1, Equation (7.29), and Lemma 6.3, we obtain:

THEOREM 8.1. *If J and the conditions of Lemmas 7.1 and 7.2 are satisfied, then*

$$\lim_{n\to\infty} [\inf \beta_{\psi}(F): F \in \Gamma(\Delta_n)] \geqq \limsup_{n\to\infty} [\inf \beta_{\phi}(F): F \in \Gamma(\Delta_n)] \tag{8.16}$$

for each sequence $\{\Delta_n\}$ satisfying $\lim_{n\to\infty} n^{1/2}\Delta_n = c$, for some $c \in [0, \infty)$.

Thus, we have shown that the cumulative total time on test statistic is asymptotically minimax over $\Gamma(\Delta)$ in each of the classes of statistics of Section 7.

REMARK 8.1. Bickel and Doksum [7] considered the problem of Section 7. exponentiality against four totally ordered families of distributions. They determined the studentized asymptotically most powerful linear spacings test for each family. The cumulative total time on test statistic is the asymptotically most powerful linear spacings test for the family of densities

$$f_{\theta}(x) = [1 + \theta(1 - e^{-x})] \exp\{-[x + \theta(x + e^{-x} - 1)]\} \tag{8.17}$$

where $f_0(x) = e^{-x}$ and $f_{\theta}(x)$ has increasing failure rate for $\theta > 0$. Bickel [6] showed that this test is in fact asymptotically equivalent to the level α test which is most powerful against the family $\{f_{\theta}(x)\}$ among all tests which are similar and level α.

Bickel and Doksum [7] computed the asymptotic efficiency of each linear spacings test considered relative to each family of distributions for comparison purposes ([7], Table 6.1). Let $e(W_i, j)$ denote the asymptotic efficiency of Bickel and Doksum's test W_i relative to the jth family, $j = 1, \cdots, 4$, of distributions (W_1 corresponds to the cumulative total time on test statistic). It is easy to verify from ([7], Table 6.1) that

$$\min_j e(W_1, j) = \max_i \min_j e(W_i, j); \tag{8.18}$$

that is, the cumulative total time on test statistic maximizes the minimum asymptotic efficiency. This observation suggested the minimax property proved in Theorem 8.1.

REMARK 8.2. It is possible to prove a minimax result for the cumulative total time on test statistic and the class of alternatives $\Omega_1(\Delta)$ rather than $\Gamma(\Delta)$ if we worked with $\inf_F \lim_{n\to\infty} \beta_{\psi}(F)$ rather than $\lim_{n\to\infty} \inf_F \beta_{\psi}(F)$. This is clear since the asymptotic power of

$$T_n^{(1)} = n^{-1} \sum_{i=1}^{n-1} W_{i:n} \tag{8.19}$$

depends only on $\mu(H_1)$ and

$$\inf_{F\in\Omega(\Delta)} \mu(H_1) = \inf_{u\in[\Delta, 1]} \mu(H_{u,\Delta}). \tag{8.20}$$

However, we would then have to assume that the distributions F in $\Omega(\Delta)$ satisfied conditions such that $T_n^{(1)}$ is asymptotically normal. Such conditions are given in [6] and [7] for parametric families of distributions.

9. Asymptotic minimax property of the cumulative total time on test statistic: general G

If we consider the problem of testing that $F = G$ where G is completely specified then a natural class of statistics are those of the form

$$\phi^* = \begin{cases} 1 & \text{if } \dfrac{n^{1/2}}{\sigma_J}\left[n^{-1}\sum_{i=1}^{n} J[G(X_i)] - \mu_J\right] > k_{n,\alpha}, \\ 0 & \text{otherwise.} \end{cases} \tag{9.1}$$

For $J(x) = x$, $\phi^* = \psi^*$ is the uniform scores test.

Suppose $G^{-1}(1) = F^{-1}(1) = 1$ and $F \underset{c}{<} G$. Then $F(x) \leqq G(x)$ for $0 \leqq x \leqq 1$ and if J is increasing, ϕ^* will provide an isotonic test for our problem.

If we confine attention to the extremal class of alternative distributions defined in (5.5) then we can prove the following asymptotic minimax theorem.

THEOREM 9.1. *If J is square integrable and*

$$\frac{1}{gG^{-1}(y)} g\left[\left(1 - \frac{\Delta}{u}\right) G^{-1}(y)\right] \to 1 \tag{9.2}$$

uniformly in $y \in [0, 1]$ as $\Delta \to 0$, then

$$\lim_{n\to\infty} [\inf_{0<u<1} \beta_{\psi^*}(G_{u,\Delta_n})] \geqq \limsup_{n\to\infty} [\inf_{0<u<1} \beta_{\phi^*}(G_{u,\Delta_n})], \tag{9.3}$$

where $\lim_{n\to\infty} n^{1/2}\Delta_n = c > 0$.

The proof is similar to that of Theorem 6.1. In this case, $E_J(\Delta, u) = E_{G_{u,\Delta}}[J[G(X)]]$ becomes

$$\begin{aligned} E_J(\Delta, u) = {} & \frac{(u - \Delta)}{u}\int_0^u J(y)\left\{\frac{1}{gG^{-1}(y)} g\left[\left(\frac{u - \Delta}{u}\right) G^{-1}(y)\right]\right\} dy \\ & + \left[1 + \frac{\Delta}{1 - u}\right]\int_u^1 J(y)\left\{\frac{1}{gG^{-1}(y)} g[a_1x_0 + a_2(G^{-1}(y) - x_0)]\right\} dy \end{aligned} \tag{9.4}$$

where $a_1 = (u - \Delta)/u$, $a_2 = 1 + \Delta/(1 - u)$ and $x_0 = G^{-1}(u - \Delta)/a_1$. As before

$$\lim_{n\to\infty} n^{1/2}[E_J(\Delta_n, u) - \mu_J] = c\left[-\frac{1}{u}\int_0^u J(y)\, dy + \frac{1}{1 - u}\int_u^1 J(y)\, dy\right]. \tag{9.5}$$

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