APPLICATIONS OF CONTIGUITY TO MULTIPARAMETER HYPOTHESES TESTING

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1. Summary and introduction

Consider a Markov process whose probability law depends on a k dimensional ($k \geqq 2$) parameter θ. The parameter space Θ is assumed to be an open subset of R^k. For each positive integer n, we consider the surface E_n defined by $(z - \theta_0)'\Gamma(z - \theta_0) = d_n$ for some sequence $\{d_n\}$ with $0 < d_n = O(n^{-1})$; Γ is a certain positive definite matrix.

For testing the hypothesis $H: \theta = \theta_0$ against the alternative $A: \theta \neq \theta_0$, a sequence of tests is constructed which, asymptotically, possesses the following optimal properties within a certain class of tests. It has best average power over E_n with respect to a certain weight function; it has constant power on E_n and is most powerful within the class of tests whose power is (asymptotically) constant on E_n. Finally, it enjoys the property of being asymptotically most stringent.

In this paper, we are dealing with the problem of testing the hypothesis $H: \theta = \theta_0$ when the underlying process is Markovian. The parameter θ varies over a k dimensional open subset of R^k denoted by Θ. Since the alternatives consist of all $\theta \in \Theta$ which are different from θ_0, one would not possibly expect to construct a test whose power would be "best" for each particular alternative. Therefore interest is centered on tests whose power is optimal over suitably chosen subsets of Θ. The class of subsets of Θ considered here consists of the surfaces of ellipsoids centered at θ_0. The question then arises as to which restricted class of tests one could search and still obtain an optimal test. The discussion detailed in Section 5 produces a class of tests, denoted by $\bar{\mathscr{F}}$, which consists of those tests each of which is the indicator function of the complement of a certain closed, convex set. The precise definition of $\bar{\mathscr{F}}$ is given in (4.4) and the arguments leading to it are due to Birnbaum [1] and Matthes and Truax [14]. The main steps of these arguments are summarized in an appendix for easy reference.

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The main results derived in this paper are of a local character. In order to give a brief description of them, we introduce the following notation. For each positive integer n, consider the surface E_n defined by

$$E_n = \{z \in R^k; (z - \theta_0)'\Gamma(z - \theta_0) = d_n\}, \tag{1.1}$$

where $0 < d_n = O(n^{-1})$ and Γ is a certain positive definite matrix. Also let ζ be a positive valued function defined on $R^k - \{\theta_0\}$ whose (surface) integral over each E_n is equal to 1. This function is given in (3.10) and (3.9). Then the test ϕ defined below by (4.5) has the following optimal properties. The power function of ϕ, weighted by the function ζ and integrated over E_n, is asymptotically largest when the competitor tests lie in $\mathscr{F}$. This is made precise in Theorems 4.1 and 6.1, the latter being a certain uniform version of the former. Next, the power of ϕ is asymptotically constant on E_n and the test ϕ is asymptotically most powerful on E_n among those tests in $\mathscr{F}$ which have asymptotically constant power on E_n. This is the content of Theorem 4.2. Finally, the test ϕ is asymptotically most stringent according to Theorems 8.1 and 8.2. Again, the latter of these theorems is a certain uniform version of the former.

In Section 9, the extra Assumption 5 is added under which the test ϕ is globally optimal in the sense that its power tends to 1, as $n \to \infty$, under nonlocal alternatives.

The hypothesis testing problem considered here has been considered by Wald [24] who also provided a solution to it (see Theorems I, II and III in Wald's paper). However, the discussion and solution to be presented here differ from those of Wald in the following respects. The assumptions made here are substantially weaker than those used by Wald. In particular, while Wald's results are formulated in terms of the maximum likelihood estimate, the present paper makes no reference to its existence. The present method of attacking the problem is that of utilizing available results obtained in Roussas [19], [20] and Johnson and Roussas [8], [9], which in turn were derived by exploiting the concept of contiguity introduced by LeCam [10]. (See also LeCam [11] and Hájek and Šidák [7], Chapter VI, and Roussas [21], [22].) As a consequence, the approach employed here is different and much less cumbersome than that of Wald. Finally, the present results also include the Markov case, whereas Wald's results were established for the independent, identically distributed case only. However, the classical methods have been used by Wald [24] and also Neyman [15] for testing composite hypotheses.

A test statistic similar to the one used here was also proposed by Rao [16] for the independent, identically distributed case and under the standard assumptions (of pointwise differentiability and so forth). However, no asymptotically optimal properties of the test were discussed except for its asymptotic distribution under the hypothesis being tested. Finally, some general results of a similar nature have been obtained by Chibisov [2]. The same author (Chibisov [3]) also obtained some average power type asymptotically optimal results in connection with the problem of testing a distribution function. An earlier version of the present paper appeared as a Technical Report, Roussas [23].

The relevant notation and assumptions are presented in Section 2. Auxiliary results necessary for the formulation of the main results are obtained in Section 3 and subsequent sections.

2. Notation and assumptions

Let Θ be a k dimensional open subset of R^k and for each $\theta \in \Theta$, consider the probability space $(\mathscr{X}, \mathscr{A}, P_\theta)$; $(\mathscr{X}, \mathscr{A}) = \Pi_{j=0}^{\infty}(R_j, \mathscr{B}_j)$, where $(R_j, \mathscr{B}_j) = (R, \mathscr{B})$ denotes the Borel real line and P_θ is the probability measure induced on $\mathscr{A}$ by a probability measure $p_\theta(\cdot)$ on $\mathscr{B}$ and a transition probability measure $p_\theta(\cdot\,;\cdot)$ defined on $R \times \mathscr{B}$. For each $\theta \in \Theta$, the coordinate process $\{X_n\}$, $n \geqq 0$, n an integer, is a Markov process with initial measure $p_\theta(\cdot)$ and transition measure $p_\theta(\cdot\,;\cdot)$.

Let $\mathscr{A}_n$ denote the σ-field induced by the random variables $X_0, X_1, \cdots, X_n$ and let $P_{n,\theta}$ denote the restriction of P_θ to $\mathscr{A}_n$. By the assumptions to be made below, the following quantities exist and are well defined up to null sets. For $\theta, \theta^* \in \Theta$, let

$$\frac{dP_{0,\theta^*}}{dP_{0,\theta}} = q(X_0; \theta, \theta^*), \tag{2.1}$$
$$\frac{dP_{1,\theta^*}}{dP_{1,\theta}} = q(X_0, X_1; \theta, \theta^*).$$

Also set

$$q(X_j \mid X_{j-1}; \theta, \theta^*) = q(X_{j-1}, X_j; \theta, \theta^*)/q(X_{j-1}; \theta, \theta^*) \tag{2.2}$$

and

$$\phi_j(\theta, \theta^*) = [q(X_j \mid X_{j-1}; \theta, \theta^*)]^{1/2}. \tag{2.3}$$

It follows that

$$\frac{dP_{n,\theta^*}}{dP_{n,\theta}} = q(X_0; \theta, \theta^*) \prod_{j=1}^{n} \phi_j^2(\theta, \theta^*). \tag{2.4}$$

The Assumptions stated below are extracted from those used by one of the present authors in another paper (see Roussas [19]) and are stated here for the sake of completeness.

ASSUMPTION 1. *For each $\theta \in \Theta$, the Markov process $\{X_n\}$, $n \geqq 0$ is (strictly) stationary and metrically transitive (ergodic).* (See, for example, Doob [4], pp. 191, 460).

ASSUMPTION 2. *The probability measures $\{P_{n,\theta}; \theta \in \Theta\}$ are mutually absolutely continuous for all $n \geqq 0$.*

ASSUMPTION 3. (i) *For each $\theta \in \Theta$, the random function $\phi_1(\theta, \theta^*)$ is differentiable in quadratic mean (q.m.) with respect to θ^* at the point (θ, θ) when P_θ is employed.* (See, for example, Loève [13] or LeCam [10].)

Let $\dot{\phi}_1(\theta)$ be the derivative in q.m. of $\phi_1(\theta, \theta^*)$ with respect to θ^* at (θ, θ). Then (ii) $\dot{\phi}_1(\theta)$ is *$\mathscr{A}_1 \times \mathscr{C}$ measurable, where $\mathscr{C}$ is the σ-field of Borel subsets of Θ.*

Let $\Gamma(\theta)$ be the covariance function defined by

$$\Gamma(\theta) = 4\mathscr{E}_\theta[\dot{\phi}_1(\theta)\dot{\phi}_1'(\theta)]. \tag{2.5}$$

Then, (iii) *$\Gamma(\theta)$ is positive definite for every $\theta \in \Theta$.*

ASSUMPTION 4. *For each $\theta \in \Theta$, $q(X_0, X_1; \theta, \theta^*) \to 1$ in $P_{1,\theta}$ probability as $\theta^* \to \theta$.*

REMARK 2.1. In the independent, identically distributed case, Assumption 1 is automatically satisfied (see, for example, Doob [4], p. 460). The random function $\phi_1(\theta, \theta^*)$ is equal to $[q(X_1; \theta, \theta^*)]^{1/2}$ and Assumption 4 is redundant (following from Assumption 3 (i)).

For later reference, we now introduce the k dimensional random vector $\Delta_n(\theta)$ which plays a fundamental role in this paper. Actually, $\Delta_n(\theta_0)$ replaces the maximum likelihood estimate, as will become apparent in the sequel.

$$\Delta_n(\theta) = \frac{2}{\sqrt{n}} \sum_{j=1}^{n} \dot{\phi}_j(\theta), \tag{2.6}$$

where $\dot{\phi}_j(\theta)$ is given in Assumption 3.

In closing this section, we should like to mention that all results in this paper (except for Theorem 9.1) will be derived under the basic Assumptions 1 to 4 and this will not be mentioned again explicitly. Also the notation $P_{n,\theta}$ and P_θ will be used interchangeably and all limits will be taken as $\{n\}$, or subsequences thereof, converges to infinity unless otherwise specified.

3. Further notation and preliminary results

We recall that the problem of interest is that of testing $H: \theta = \theta_0$. In the sequel, dependence of various quantities on θ_0 will not be explicitly indicated. For instance, we shall write Γ, Δ_n rather than $\Gamma(\theta_0)$, $\Delta_n(\theta_0)$ and so forth.

Since the matrix Γ is positive definite there exists a nonsingular matrix M such that

$$M'M = \Gamma. \tag{3.1}$$

From (3.1), it immediately follows that

$$(M^{-1})'\Gamma M^{-1} = M\Gamma^{-1}M' = I, \tag{3.2}$$

where I is the $k \times k$ unit matrix.

Consider the following transformation of R^k onto itself

$$M(z - \theta_0) = v - \theta_0, \tag{3.3}$$

where $v = u + \theta_0 - M\theta_0$ and $u = Mz$. For $c > 0$, let $E(c)$ be the surface (of an ellipsoid) defined by

$$E(c) = \{z \in R^k;\ (z - \theta_0)'\Gamma(z - \theta_0) = c\}. \tag{3.4}$$

Then the transformation (3.3) sends the surface $E(c)$ onto the surface (of a sphere) $S(c)$, where

$$S(c) = \{z \in R^k;\ (z - \theta_0)'(z - \theta_0) = c\}. \tag{3.5}$$

For $z \in R^k$, with $z \neq \theta_0$, set $c(z) = (z - \theta_0)'\Gamma(z - \theta_0)$. Then this quantity is positive, since Γ is positive definite by assumption. By means of $E(c(z))$ and $S(c(z))$, define the function ξ as in Wald [24], p. 445. Namely, for any $\rho > 0$, define $\omega(z, \rho)$ by

$$\omega(z, \rho) = \{u \in E(c(z));\ \|u - z\| \leqq \rho\} \tag{3.6}$$

and let $\omega'(z, \rho)$ be the image of the set $\omega(z, \rho)$ under the transformation (3.3). Also denote by $A(\omega(z, \rho))$ and $A(\omega'(z, \rho))$ the areas of the sets $\omega(z, \rho)$ and $\omega'(z, \rho)$, respectively. Then the function ξ is defined as follows

$$\xi(z) = \lim \frac{A(\omega'(z, \rho))}{A(\omega(z, \rho))} \quad \text{as} \quad \rho \to 0. \tag{3.7}$$

Thus one has the positive valued function ξ defined on $R^k - \{\theta_0\}$ and it can be seen that its explicit form is

$$\xi(z) = \frac{[|\Gamma|(z - \theta_0)'\Gamma(z - \theta_0)]^{1/2}}{\|\Gamma(z - \theta_0)\|}, \qquad z \neq \theta_0. \tag{3.8}$$

REMARK 3.1. The significance of the function ξ defined by (4.6) may be seen from the relation

$$\int_{E(c)} \xi(z)\, dA = \text{area of } S(c) \text{ to be denoted by } A(c), \tag{3.9}$$

where $E(c)$ and $S(c)$ are defined by (3.4) and (3.5), respectively, and the integral in (3.9) is a surface integral.

By setting

$$\zeta(c; z) = \frac{\xi(z)}{A(c)}, \qquad z \in E(c), \tag{3.10}$$

one obtains a weight function $\zeta(c; \cdot)$ (integrating to 1) over each one of the surfaces $E(c)$.

In the sequel, we will be interested in parameter points θ_n of the form

$$\theta_n = \theta_0 + \frac{h}{\sqrt{n}}, \qquad h \in R^k. \tag{3.11}$$

Also the h eventually, will be required to satisfy the condition $h'\Gamma h = c_n$, $c_n > 0$, so that $(\theta_n - \theta_0)'\Gamma(\theta_n - \theta_0) = c_n/n$ which we denote by d_n. Since $\theta_0 \in \Theta$ and

Θ is open, there exists a $d_0 > 0$ such that the surface $(\theta - \theta_0)'\Gamma(\theta - \theta_0) = d_0$ lies in Θ. We have

(3.12) $$E_n = E(d_n) = \{z \in R^k; (z - \theta_0)'\Gamma(z - \theta_0) = d_n\}$$

with $d_n = c_n/n$, and let

(3.13) $$E_n^* = E^*(c_n) = \{z \in R^k; z'\Gamma z = c_n\}.$$

Choose c_n satisfying the requirement

(3.14) $$0 < c_n \leqq d_0 \text{ for all } n, \qquad 0 < d_n \leqq d_0 \text{ for all } n.$$

Then, with θ_n being the form (3.11), it follows that

(3.15) $$\theta_n \in E_n \quad \text{if and only if } h \in E_n^*, E_n \subset \Theta \quad \text{for all } n.$$

REMARK 3.2. Let $z_n = \theta_0 + z/\sqrt{n}$. Then from (3.8) it follows that $\xi(z_n) = \xi(z)$. In particular, if θ_n is given by (3.11), then $\xi(\theta_n) = \xi(h)$, where $\theta_n \in E_n \subset \Theta$, so that $h \in E_n^*$. (See relations (3.12) to (3.15).)

4. Formulation of some of the main results

In the present paper, the problem we are interested in is that of testing the hypothesis $H: \theta = \theta_0$, for some fixed parameter point θ_0, against the alternative $A: \theta \neq \theta_0$. To this end, let

(4.1) $$\mathscr{L}_h = N(\Gamma h, \Gamma), \qquad h \in R^k$$

and define the set D as follows

(4.2) $$D = \{z \in R^k; z'\Gamma^{-1}z \leqq d\}, \qquad \mathscr{L}_0(D) = 1 - \alpha, \qquad 0 < \alpha < 1.$$

Also define the class $\mathscr{C}$ by

(4.3) $$\mathscr{C} = \{C \in \mathscr{B}^k; C \text{ is closed and convex}\},$$

and set

(4.4) $$\mathscr{F} = \{\psi; \psi = I[C^c], C \in \mathscr{C}\},$$

where I is the indicator of the set in the brackets.

In particular, set

(4.5) $$\phi = I[D^c].$$

Then, clearly, $D \in \mathscr{C}$, so that $\phi \in \mathscr{F}$.

All tests herein will depend on the random vector Δ_n defined by (2.6) for $\theta = \theta_0$ and for reasons to be explained in Section 5, we may confine ourselves to tests in $\mathscr{F}$.

For $\psi \in \mathscr{F}$, that is $\psi = I[C^c]$ for some $C \in \mathscr{C}$, set

(4.6) $$\beta_n(\theta; C^c) = \beta_n(\theta; \psi) = \mathscr{E}_\theta\psi(\Delta_n).$$

REMARK 4.1. As will be seen in the theorems to be stated below, the sequence of tests $\{\phi(\Delta_n)\}$ possesses certain optimal asymptotic properties, where ϕ is defined by (4.5). Also with θ_n and $\mathscr{L}_h$ defined by (3.11) and (4.1), respectively, it will be shown later (see Lemma 6.1 (i)) that

$$\mathscr{E}_{\theta_0}\phi(\Delta_n) = P_{\theta_0}(\Delta_n \in D^c) \to \mathscr{L}_0(D^c). \tag{4.7}$$

Thus, if we decide to restrict attention to tests in the class $\bar{\mathscr{F}}$ of asymptotic level of significance α—which we shall do—we must have $\mathscr{L}_0(D^c) = \alpha$. This fact provides the justification for the equation $\mathscr{L}_0(D) = 1 - \alpha$ employed in (4.2).

Unless otherwise explicitly specified in all that follows and for each n, we shall consider only parameter points θ_n of the form $\theta_n = \theta_0 + h/\sqrt{n}$ with $h \in E_n^*$, so that $\theta_n \in E_n$ (see (3.12) and (3.13) for the definition of E_n and E_n^*). Although θ_n and h_n would be a more appropriate notation, we shall simply write θ and h, when no confusion is possible, with the understanding that $\theta = \theta_0 + h/\sqrt{n}$ and $h \in E_n^*$, so that $\theta \in E_n$. This will somewhat simplify an already cumbersome notation.

The first main result in this section is presented in the following theorem.

THEOREM 4.1. *Let E_n be defined by* (3.12) *with c_n satisfying* (3.14) *and let $\zeta_n = \zeta(d_n; \cdot)$ be defined by* (3.10). *Also let ϕ be given by* (4.5) *and let $\{\psi_n\}$ be any sequence of tests in $\bar{\mathscr{F}}$ of asymptotic level of significance α. Then one has*

$$\liminf\left[\int_{E_n} \beta_n(\theta; \phi)\zeta_n(\theta)\,dA - \int_{E_n} \beta_n(\theta; \psi_n)\zeta_n(\theta)\,dA\right] \geqq 0, \tag{4.8}$$

where $\beta_n(\theta; \phi)$ and $\beta_n(\theta; \psi_n)$ are defined by (4.6).

It is clear that one can take the sup over c_n belonging to a compact set before taking the lim inf in (4.8). This follows immediately since one can obtain (6.11) in the proof of Theorem 4.1 by passing to a subsequence of ellipsoids. (For a uniform version of the result just presented and also its interpretation, the reader is referred to Theorem 6.1.)

The second main result herein is the following one.

THEOREM 4.2. *Let E_n, ϕ, $\{\psi_n\}$, $\beta_n(\theta; \phi)$ and $\beta_n(\theta; \psi_n)$ be as in Theorem* 4.1. *Then one has* (i) $\lim \{\sup [\beta_n(\theta; \phi); \theta \in E_n] - \inf [\beta_n(\theta; \phi); \theta \in E_n]\} = 0$

and (ii) $\liminf \{\inf [\beta_n(\theta; \phi) - \beta_n(\theta; \psi_n); \theta \in E_n]\} \geqq 0$
for any tests ψ_n as described above and for which $\beta_n(\theta; \psi_n)$ satisfies (i).

As in the previous theorem, lim inf may be replaced by lim inf sup in (ii), since by passing to a subsequence of ellipsoids we can obtain (7.14).

The interpretation of the theorem is clear. Part (i) states that the power of the test ϕ on the surfaces E_n is asymptotically constant. The second part asserts that, within the class of tests whose power on E_n is asymptotically constant, the test ϕ is asymptotically most powerful.

The formulation (and proof) of the third main result in the present paper is deferred to Section 8, since it requires substantial additional notation.

5. Restriction to the class of tests $\bar{\bar{\mathscr{F}}}$

Suppose that we are interested in testing the hypothesis $H: \theta = \theta_0$ against the alternative $A: \theta \neq \theta_0$ at asymptotic level of significance α. All tests are to be based on the random vector Δ_n and power is to be calculated under P_{θ_n}, where $\theta_n = \theta_0 + h/\sqrt{n}$. This can be done without loss of generality by Theorem 6.1 in Johnson and Roussas [9]. By Theorem 6.3 in [9], one has that for any tests ψ_n (not necessarily in $\bar{\bar{\mathscr{F}}}$) and any bounded subset B of R^k,

$$\sup\left[\left|\mathscr{E}_{\theta_n}\psi_n(\Delta_n) - \mathscr{E}_{\theta_n}\psi_n(\Delta_n^*)\right|; h \in B\right] \to 0, \tag{5.1}$$

where Δ_n^* is an appropriate truncated version of Δ_n defined by (4.6) in the last reference above. Therefore from an asymptotic point of view, it suffices to base any tests on the random vector Δ_n^* rather than Δ_n. This is so regardless of whether we are interested in pointwise power or average power (see Theorem 4.1). Power is still to be calculated under P_{θ_n}. Next, by virtue of (5.2) in Johnson and Roussas [9], one has

$$dR_{n,h}/dP_{\theta_0} = \exp\{-B_n(h) + h'\Delta_n^*\}, \tag{5.2}$$

where $\exp\{B_n(h)\} = \mathscr{E}_{\theta_0}(\exp\{h'\Delta_n^*\})$, $h \in R^k$. In the family of probability measures $R_{n,h'}$, the parameter is h and its range is all of R^k. Since for any $\theta \in \Theta$, $\theta = \theta_0 + h/\sqrt{n}$ for some $h \in R^k$, namely, $h = \sqrt{n}(\theta - \theta_0)$, it follows that $\theta = \theta_0$ if and only if $h = 0$. For $h = 0$, it follows from (5.2) that $R_{n,0} = P_{\theta_0}$.

By Theorem 5.1 in Johnson and Roussas [9],

$$\sup\left(\|P_{\theta_n} - R_{n,h}\|; h \in B\right) \to 0, \tag{5.3}$$

where B is any bounded subset of R^k. Thus for any tests $\psi_{n'}$, one has

$$\begin{aligned}\sup\{|\mathscr{E}_{\theta_n}\psi_n(\Delta_n^*) - \mathscr{E}[\psi_n(\Delta_n^*)|R_{n,h}]|; h \in B\}& \\ \leqq \sup\left(\|P_{\theta_n} - R_{n,h}\|; h \in B\right) &\to 0.\end{aligned} \tag{5.4}$$

Therefore it follows that, from an asymptotic power viewpoint, powers may be calculated under $R_{n,h}$ rather than P_{θ_n}. Furthermore this is true regardless of whether our interest lies in pointwise power or average power in the sense of Theorem 4.1.

In order to summarize: the original hypothesis testing problem $H: \theta = \theta_0$ against $A: \theta \neq \theta_0$ at asymptotic level of significance α, where tests are to be based on the random vector Δ_n and power is to be calculated under P_{θ_n}, may be replaced by the equivalent hypothesis testing problem $H^*: h = 0$ against $A^*: h \neq 0$ at asymptotic level of significance α, in connection with the family of probability measures defined by (5.2), where tests are to be based on the random vector Δ_n^* and power is to be calculated under $R_{n,h}$.

Now we introduce the random vector Z_n^*, where

$$Z_n^* = \Gamma^{-1}\Delta_n^*. \tag{5.5}$$

Also the following notation is needed.

$$(5.6)\qquad \mathscr{L}^*_{n,\theta} = \mathscr{L}(\Delta^*_n | P_\theta), \quad \theta \in \Theta, \qquad \mathscr{L}^*_{n,h} = \mathscr{L}(\Delta^*_n | R_{n,h}), \quad h \in R^k$$

and

$$(5.7)\qquad L^*_{n,h} = \mathscr{L}(Z^*_n | R_{n,h}), \qquad h \in R^k.$$

The following result holds true.

LEMMA 5.1. (i) *Let* $\mathscr{L}^*_{n,\theta_0}$ and $\mathscr{L}^*_{n,h}$ *be defined by* (5.6). *Then for every* $h \in R^k$, one has $\mathscr{L}^*_{n,h} \ll \mathscr{L}^*_{n,\theta_0}$ *and*

$$(5.8)\qquad \frac{d\mathscr{L}^*_{n,h}}{d\mathscr{L}^*_{n,\theta_0}} = \exp\{-B_n(h) + h'z\}, \qquad z \in R^k.$$

(ii) *Let* $L^*_{n,h}$ *be defined by* (5.7). *Then for every* $h \in R^k$, *one has* $L^*_{n,h} \ll L^*_{n,0}$ *and*

$$(5.9)\qquad \frac{dL^*_{n,h}}{dL^*_{n,0}} = \exp\{-B_n(h) + h'\Gamma z\}, \qquad z \in R^k.$$

PROOF. (i) For $A \in \mathscr{B}^k$ and by virtue of (5.6) and (5.2), one has

$$\begin{aligned}(5.10)\qquad \mathscr{L}^*_{n,h}(A) = R_{n,h}(\Delta^*_n \in A) &= \int_{(\Delta^*_n \in A)} \exp\{-B_n(h) + h'\Delta^*_n\}\, dP_{\theta_0} \\ &= \int_A \exp\{-B_n(h) + h'z\}\, d\mathscr{L}(\Delta^*_n | P_{\theta_0}) \\ &= \int_A \exp\{-B_n(h) + h'z\}\, d\mathscr{L}^*_{n,\theta_0},\end{aligned}$$

as was asserted.

(ii) With A as above and by virtue of (5.2), (5.5) and (5.6), one has

$$\begin{aligned}(5.11)\qquad L^*_{n,h}(A) = R_{n,h}(Z^*_n \in A) &= \int_{(Z^*_n \in A)} \exp\{-B_n(h) + h'\Delta^*_n\}\, dP_{\theta_0} \\ &= \int_{(Z^*_n \in A)} \exp\{-B_n(h) + h'\Gamma Z^*_n\}\, dP_{\theta_0} \\ &= \int_A \exp\{-B_n(h) + h'\Gamma z\}\, d\mathscr{L}(Z^*_n | P_{\theta_0}) \\ &= \int_A \exp\{-B_n(h) + h'\Gamma z\}\, d\mathscr{L}(Z^*_n | R_{n,0})\end{aligned}$$

because $R_{n,0} = P_{\theta_0}$ by means of (5.2). Now since $\mathscr{L}(Z^*_n | R_{n,0}) = L^*_{n,0}$, we have

$$(5.12)\qquad L^*_{n,h}(A) = \int_A \exp\{-B_n(h) + h'\Gamma z\}\, dL^*_{n,0},$$

as was asserted.

From (5.5), it follows that, in testing the hypothesis last described, our tests may be based on the random vector Z^*_n rather than Δ^*_n.

Now for each n, the family of probability densities

$$\frac{dL^*_{n,h}}{dL^*_{n,0}} = \exp\{-B_n(h) + h'\Gamma z\}, \qquad z \in R^k \tag{5.13}$$

is of the form (A.9) in the Appendix. Therefore, Corollary A.1 in the Appendix applies and we conclude that an arbitrary test ψ'_n based on Z^*_n may be replaced by a test ψ_n based on Z^*_n of the form (A.4). Thus for each n, we consider tests ψ_n based on Z^*_n such that

$$\psi_n(z) = \begin{cases} 1 & \text{if } z \in C^c_n \\ 0 & \text{if } z \in C^0_n \text{ for some } C_n \in \mathscr{C}, \end{cases} \tag{5.14}$$

where $\mathscr{C}$ is given by (4.3); the test may be arbitrary (measurable) on C^b_n, and $\mathscr{E}[\psi_n(Z^*_n)|R_{n,0}] \to \alpha$.

Now it would be convenient to avoid arbitrariness of tests ψ_n on C^b_n; for instance, it would be convenient to set $\psi_n(z) = 0$ for $z \in C^b_n$, so that $\psi_n(z) = I[C^c_n]$. In order for this modification to be valid, we would have to show that by changing the test ψ_n on C^b_n in any arbitrary (measurable) way, both its asymptotic power and size remain intact. That this is, in fact, the case is the content of Lemma 5.3. In order to be able to prove the lemma, some additional notation and some preliminary results are needed. To this end, let

$$\mathscr{C}^* = \{C \in \mathscr{B}^k;\ C \text{ is convex}\} \tag{5.15}$$

and also set

$$\mathscr{L}_{n,\theta} = \mathscr{L}(\Delta_n|P_\theta), \qquad \theta \in \Theta. \tag{5.16}$$

Then by Theorem 6.2 in Johnson and Roussas [9], for $\theta^*_n = \theta_0 + h_n/\sqrt{n}$ with $h_n \to h \in R^k$, $\mathscr{L}_{n,\theta^*_n} \Rightarrow \mathscr{L}_{h'}$, where $\mathscr{L}_h$ is defined by (4.1) and $\Rightarrow$ denotes weak convergence of probability measures. Also $P_{\theta^*_n}(\Delta^*_n \neq \Delta_n) \to 0$ by Proposition 4.1 in Johnson and Roussas [9]. Therefore $\mathscr{L}^*_{n,\theta^*_n} \Rightarrow \mathscr{L}_{h'}$, where $\mathscr{L}'_{n,\theta^*_n}$ is given by (5.6). On the other hand, $\|P_{\theta^*_n} - R_{n,h_n}\| \to 0$, as was mentioned before, so that $\mathscr{L}^*_{n,h_n} \Rightarrow \mathscr{L}_h$. That is, we have

$$\mathscr{L}^*_{n,\theta^*_n} \Rightarrow \mathscr{L}_h, \qquad \mathscr{L}^*_{n,h_n} \Rightarrow \mathscr{L}_h. \tag{5.17}$$

The lemma below shows that these convergences are uniform over the class $\mathscr{C}^*$. More precisely, we have the following result.

LEMMA 5.2. *Let θ_n, $\mathscr{L}_h$, $\mathscr{L}^*_{n,\theta_n}$, $\mathscr{L}^*_{n,h}$ and $\mathscr{C}^*$ be defined by* (3.11), (4.1), (5.6) *and* (5.15), *respectively. Then for any bounded subset B of R^k, one has*

$$\text{(i)} \qquad \sup\{\sup[|\mathscr{L}^*_{n,\theta_n}(C) - \mathscr{L}_h(C)|;\ C \in \mathscr{C}^*];\ h \in B\} \to 0$$

and

$$\text{(ii)} \qquad \sup\{\sup[|\mathscr{L}^*_{n,h}(C) - \mathscr{L}_h(C)|;\ C \in \mathscr{C}^*];\ h \in B\} \to 0.$$

PROOF. (i) The proof is by contradiction. Set

$$\delta_n(h) = \sup [|\mathscr{L}^*_{n,\theta_n}(C) = \mathscr{L}_h(C)|; C \in \mathscr{C}^*] \tag{5.18}$$

and suppose that $\sup [\delta_n(h); h \in B] \not\to 0$. Then there is a subsequence $\{m\} \subseteqq \{n\}$ and $h_m \in B$ such that $\delta_m(h_m) \to \delta$, for some $\delta > 0$. Equivalently,

$$\sup [|\mathscr{L}^*_{m,\theta^*_m}(C) - \mathscr{L}_{h_m}(C)|; C \in \mathscr{C}^*] \to \delta, \tag{5.19}$$

where $\theta^*_m = \theta_0 + h_m/\sqrt{m}$. Let $\{h_r\} \subseteqq \{h_m\}$ be such that $h_r \to t \in R^k$. Then one has, by virtue of (5.17), $\mathscr{L}^*_{r,\theta^*_r} \Rightarrow \mathscr{L}_t$. Thus Theorem 4.2 in Rao [18] applies and gives

$$\sup [|\mathscr{L}^*_{r,\theta^*_r}(C) - \mathscr{L}_t(C)|; C \in \mathscr{C}^*] \to 0. \tag{5.20}$$

On the other hand, we clearly have

$$\sup [|\mathscr{L}_{h_r}(C) - \mathscr{L}_t(C)|; C \in \mathscr{C}^*] \to 0. \tag{5.21}$$

Relations (5.20) and (5.21) then imply that

$$\sup [|\mathscr{L}^*_{r,\theta^*_r}(C) - \mathscr{L}_{h_r}(C)|; C \in \mathscr{C}^*] \to 0. \tag{5.22}$$

However, this contradicts (5.19) with m replaced by r.

(ii) We have

$$\begin{aligned} \|\mathscr{L}^*_{n,\theta_n} - \mathscr{L}^*_{n,h}\| &= 2 \sup [|\mathscr{L}^*_{n,\theta_n}(A) - \mathscr{L}^*_{n,h}(A)|; A \in \mathscr{B}^k] \\ &= 2 \sup [|P_{\theta_n}(\Delta^*_n \in A) - R_{n,h}(\Delta^*_n \in A)|; A \in \mathscr{B}^k] \\ &\leqq 2 \sup [|P_{\theta_n}(E) - R_{n,h}(E)|; E \in \mathscr{A}_n] = \|P_{\theta_n} - R_{n,h}\|. \end{aligned} \tag{5.23}$$

But $\sup (\|P_{\theta_n} - R_{n,h}\|; h \in B) \to 0$. Therefore,

$$\sup (\|\mathscr{L}^*_{n,\theta_n} - \mathscr{L}^*_{n,h}\|; h \in B) \to 0. \tag{5.24}$$

Clearly, (5.24) implies that

$$\sup \{\sup [|\mathscr{L}^*_{n,\theta_n}(C) - \mathscr{L}^*_{n,h}(C)|; C \in \mathscr{C}^*]; h \in B\} \to 0. \tag{5.25}$$

This last convergence together with the first part of the lemma yields the desired conclusion.

The result just obtained is a strengthening of Lemma 2.1 in Chibisov [2] in that taking the sup over h, we allow h to vary over bounded rather than compact sets.

LEMMA 5.3. *Let Z^*_n, $L^*_{n,h}$ and $\mathscr{C}^*$ be defined by* (5.5), (5.7) *and* (5.15), *respectively. Then for any bounded subset B of R^k and any sets $C_n \in \mathscr{C}^*$, one has*

$$\sup [R_{n,h}(Z^*_n \in C^b_n); h \in B] = \sup [L^*_{n,h}(C^b_n); h \in B] \to 0. \tag{5.26}$$

PROOF. For any $A \in \mathscr{B}^k$ one has $Z^*_n \in A$ if and only if $\Delta^*_n \in \hat{A}$, where

$$\hat{A} = \{u \in R^k; u = \Gamma z, z \in A\}. \tag{5.27}$$

Let A^0 and $\bar{A}$ denote the interior and the closure, respectively, of the set A.

We then have, by means of (2.3) and (1.1)

$$(5.28)\qquad \begin{aligned} L^*_{n,h}(C^b_n) &= \mathscr{L}^*_{n,h}(\hat{C}^b_n) = \mathscr{L}^*_{n,h}(\hat{C}^-_n) - \mathscr{L}^*_{n,h}(\hat{C}^0_n) \\ &= [\mathscr{L}^*_{n,h}(\hat{C}^-_n) - \mathscr{L}_h(\hat{C}^-_n)] - [\mathscr{L}^*_{n,h}(\hat{C}^0_n) - \mathscr{L}_h(\hat{C}^0_n)] \end{aligned}$$

for any $C_n \in \mathscr{C}^*$. The equality $\mathscr{L}_h(\hat{C}^-_n) = \mathscr{L}_h(\hat{C}^0_n)$ holds because $C_n \in \mathscr{C}^*$ if and only if $\hat{C}_n \in \mathscr{C}^*$ and the boundary of any convex set in R^k has k dimensional Lebesgue measure zero and hence $\mathscr{L}_h$ measure zero. It is also well known that both the closure and the interior of a convex set are also convex.

Taking the sup of both sides of (5.28) as h varies in B, one obtains the desired result from Lemma 5.2 (ii).

Returning now to the discussion following the definition of the test ψ_n by (5.14), we conclude that we may restrict ourselves to tests ψ_n based on Z^*_n and having the following form

$$(5.29)\qquad \psi_n = I[C^c_n] \qquad \text{for some } C_n \in \overline{\mathscr{C}}$$

and

$$(5.30)\qquad \mathscr{E}[\psi_n(Z^*_n)|R_{n,0}] \to \alpha.$$

6. Proof of the first main result

For the proof of the first theorem, we shall need some additional notation and also some preliminary results. Set

$$(6.1)\qquad Z_n = \Gamma^{-1}\Delta_n$$

and

$$(6.2)\qquad L_{n,\theta} = \mathscr{L}(Z_n|P_\theta), \qquad \theta \in \Theta.$$

Then $Z_n \in A$ if and only if $\Delta_n \in \hat{A}$, where $\hat{A}$ is given by (5.27). Also set

$$(6.3)\qquad L_h = N(h, \Gamma^{-1}), \qquad h \in R^k.$$

One then has the following result.

Lemma 6.1. (i) *Let* $\mathscr{L}_h$, $\mathscr{C}^*$ and $\mathscr{L}_{n,\theta}$ *be defined by* (4.1), (5.15) *and* (5.16), *respectively. Then*

$$(6.4)\qquad \sup\{\sup[|\mathscr{L}_{n,\theta_n}(C) - \mathscr{L}_h(C)|; C \in \mathscr{C}^*]; h \in B\} \to 0,$$

where θ_n *is given by* (3.11) *and B is any bounded subset of* R^k.

(ii) *Let* θ_n, $\mathscr{C}^*$ *and B be as above and let* $L_{n,\theta}$ *and* L_h *be defined by* (6.2) *and* (6.3), *respectively. Then*

$$(6.5)\qquad \sup\{\sup[|L_{n,\theta_n}(C) - L_h(C)|; C \in \mathscr{C}^*]; h \in B\} \to 0.$$

Proof. (i) The proof is similar to that of Lemma 5.2 (i) and the details are left to the reader.

(ii) For $A \in \mathscr{B}^k$ we have $L_{n,\theta_n}(A) = \mathscr{L}_{n,\theta_n}(\hat{A})$, where $\hat{A}$ is given by (5.27) and

$A \in \mathscr{C}^*$ ($A \in \mathscr{C}$) if and only if $\hat{A} \in \mathscr{C}^*$ ($\hat{A} \in \mathscr{C}$). It is also readily seen that

$$L_h(A) = \mathscr{L}_h(\hat{A}). \tag{6.6}$$

Therefore

$$\begin{aligned} \sup [|L_{n,\theta_n}(C) - L_h(C)|; C \in \mathscr{C}^*] &= \sup [|\mathscr{L}_{n,\theta_n}(\hat{C}) - \mathscr{L}_h(\hat{C})|; \hat{C} \in \mathscr{C}^*] \\ &= \sup [|\mathscr{L}_{n,\theta_n}(C) - \mathscr{L}_h(C)|; C \in \mathscr{C}^*]. \end{aligned} \tag{6.7}$$

Then taking the sup of both sides of this last relation as h varies in B and utilizing the first part of the lemma we obtain the desired result.

From (6.1) it follows that $\Delta_n \in A$ if and only if $Z_n \in \tilde{A}$, where

$$\tilde{A} = \{u \in R^k; u = \Gamma^{-1}z, z \in A\}. \tag{6.8}$$

Therefore by setting

$$\beta_n(\theta; A) = P_\theta(\Delta_n \in A), \qquad \tilde{\beta}_n(\theta; \tilde{A}) = P_\theta(Z_n \in \tilde{A}), A \in \mathscr{B}^k, \tag{6.9}$$

we have

$$\beta_n(\theta; A) = \tilde{\beta}_n(\theta; \tilde{A}), \qquad \theta \in \Theta, A \in \mathscr{B}^k. \tag{6.10}$$

We may now proceed with the proof of the first main result.

PROOF OF THEOREM 4.1. The proof is by contradiction. Suppose that (4.8) is not true. Then there is a subsequence $\{m\} \subseteqq \{n\}$ for which

$$\int_{E_m} \beta_m(\theta; \phi)\zeta_m(\theta)\, dA - \int_{E_m} \beta_m(\theta; \psi_m)\zeta_m(\theta)\, dA \to \delta, \quad \text{for some } \delta < 0. \tag{6.11}$$

By employing the notation in (4.6), this is rewritten as follows

$$\int_{E_m} \beta_m(\theta; D^c)\zeta_m(\theta)\, dA - \int_{E_m} \beta_m(\theta; C_m^c)\zeta_m(\theta)\, dA \to \delta, \tag{6.12}$$

or

$$\int_{E_m} \beta_m(\theta; C_m)\zeta_m(\theta)\, dA - \int_{E_m} \beta_m(\theta; D)\zeta_m(\theta)\, dA \to \delta. \tag{6.13}$$

By virtue of (6.10), this becomes

$$\int_{E_m} \tilde{\beta}_m(\theta; \tilde{C}_m)\zeta_m(\theta)\, dA - \int_{E_m} \tilde{\beta}_m(\theta; \tilde{D})\zeta_m(\theta)\, dA \to \delta. \tag{6.14}$$

Now set

$$\beta(h; A) = L_h(A), \qquad h \in R^k, A \in \mathscr{B}^k, \tag{6.15}$$

where L_h is given by (6.3).

Then on account of (6.9) and (6.15), Lemma 6.1 (ii) implies that for arbitrary sets $D_m \in \mathscr{C}^*$

$$\sup [|\tilde{\beta}_m(\theta_m; D_m) - \beta(h; D_m)|; h \in B] \to 0 \tag{6.16}$$

for any bounded subset B of R^k. In particular,

$$\sup\,[|\tilde{\beta}_m(\theta_m; D_m) - \hat{\beta}(h; D_m)|; h \in E_m^*] \to 0, \tag{6.17}$$

where E_m^* is given by (3.13).

At this point we set

$$\hat{\beta}(h; A) = \hat{\beta}(\sqrt{m}(\theta_m - \theta_0); A) = \beta^*(\theta_m; A), \qquad A \in \mathscr{B}^k \tag{6.18}$$

and we recall that, by (3.15), $h \in E_m^*$ if and only if $\theta_m \in E_m$. The convergence in (6.17) then becomes

$$\sup\,[|\tilde{\beta}_m(\theta_m; D_m) - \beta^*(\theta_m; D_m)|; \theta_m \in E_m] \to 0, \tag{6.19}$$

or

$$\sup\,[|\tilde{\beta}_m(\theta; D_m) - \beta^*(\theta; D_m)|; \theta \in E_m] \to 0. \tag{6.20}$$

Utilizing (6.20) with D_m replaced by C_m and D successively, we obtain

$$\int_{E_m} \tilde{\beta}_m(\theta; \tilde{C}_m)\zeta_m(\theta)\, dA - \int_{E_m} \beta^*(\theta; \tilde{C}_m)\zeta_m(\theta)\, dA \to 0 \tag{6.21}$$

and

$$\int_{E_m} \tilde{\beta}_m(\theta; \tilde{D})\zeta_m(\theta)\, dA - \int_{E_m} \beta^*(\theta; \tilde{D})\zeta_m(\theta)\, dA \to 0. \tag{6.22}$$

From (6.14), (6.21) and (6.22), we obtain

$$\int_{E_m} \beta^*(\theta; \tilde{C}_m)\zeta_m(\theta)\, dA - \int_{E_m} \beta^*(\theta; \tilde{D})\zeta_m(\theta)\, dA \to \delta, \tag{6.23}$$

or equivalently,

$$\int_{E_m} \beta^*(\theta; \tilde{D}^c)\zeta_m(\theta)\, dA - \int_{E_m} \beta^*(\theta; \tilde{C}_m^c)\zeta_m(\theta)\, dA \to \delta. \tag{6.24}$$

Thus for all sufficiently large m, $m \geqq m_1$, say, we have

$$\int_{E_m} \beta^*(\theta; \tilde{D}^c\zeta_m(\theta)\, dA < \int_{E_m} \beta^*(\theta; \tilde{C}_m^c)\zeta_m(\theta)\, dA + \frac{\delta}{2} \tag{6.25}$$

(recall that $\delta < 0$), or by means of (6.18),

$$\begin{aligned}\int_{E_m} &\hat{\beta}(\sqrt{m}(\theta - \theta_0); \tilde{D}^c)\zeta_m(\theta)\, dA \\ &< \int_{E_m} \hat{\beta}(\sqrt{m}(\theta - \theta_0); \tilde{C}_m^c)\zeta_m(\theta)\, dA + \frac{\delta}{2} \qquad \text{for all } m \geqq m_1.\end{aligned} \tag{6.26}$$

Let $A_m = \int_{E_m}\xi(z)\, dA$ (see also (3.9)). Then, on account of (3.10), (6.26) becomes

$$(6.27)\qquad \int_{E_m} \hat{\beta}(\sqrt{m}(\theta - \theta_0); \tilde{D}^c)\xi(\theta)\, dA$$
$$< \int_{E_m} \hat{\beta}(\sqrt{m}(\theta - \theta_0); \tilde{C}^c_m)\xi(\theta)\, dA + \frac{\delta A_m}{2} \qquad \text{for all } m \geqq m_1.$$

Set

$$(6.28)\qquad \sqrt{m}(\theta - \theta_0) = h \qquad \text{so that} \quad \theta = \theta_0 + h/\sqrt{m}, \qquad h \in E^*_m,$$

where E^*_m is given by (3.13). Then by virtue of (6.18) and Remark 3.2, the inequality in (6.27) becomes

$$(6.29)\qquad \int_{E^*_m} \hat{\beta}(h; \tilde{D}^c)\xi(h)\, dA < \int_{E^*_m} \hat{\beta}(h; \tilde{C}^c_m)\xi(h)\, dA + \frac{\delta A_m}{2|J_m|} \qquad \text{for all } m \geqq m_1,$$

where the scaling factor J_m results from the transformation in (6.28). It is not hard to show that $|J_m| = m^{-(k-1)/2}$, whereas A_m, which is the surface area of the sphere with radius $(c_m/m)^{1/2}$ corresponding to E_m in (3.12), is equal to

$$(6.30)\qquad \frac{2\pi^{k/2}}{\Gamma\left(\frac{k}{2}\right)} \frac{c_m^{(k-1)/2}}{m^{(k-1)/2}},$$

as is well known. Therefore

$$(6.31)\qquad \frac{A_m}{2|J_m|} = \frac{\pi^{k/2}}{\Gamma\left(\frac{k}{2}\right)} c_m^{(k-1)/2}.$$

Now, on the basis of (3.14) and by passing to a subsequence if necessary, we may assume that $c_m \to c \geqq 0$. First consider the case that $c > 0$. Then for all sufficiently large m, $m \geqq m_2$, say, we have $c_m \geqq c/2$, so that $A_m/2|J_m| \geqq \delta_1$, where

$$(6.32)\qquad \delta_1 = \frac{\pi^{k/2}}{\Gamma\left(\frac{k}{2}\right)} \left(\frac{c}{2}\right)^{(k-1)/2}$$

Hence for $m \geqq m_3 = \max\{m_1, m_2\}$, (6.29) becomes

$$(6.33)\qquad \int_{E^*_m} \hat{\beta}(h; \tilde{D}^c)\xi(h)\, dA < \int_{E^*_m} \hat{\beta}(h; \tilde{C}^c_m)\xi(h)\, dA + \delta_2,$$

where

$$(6.34)\qquad \delta_2 = \delta\delta_1 < 0.$$

At this point, we recall that $\hat{\beta}(h; \tilde{C}^c_m) = L_h(\tilde{C}^c_m)$ by (6.15). On the other hand, it is clear from (6.8) that $\tilde{A}^c = A^{c\sim}$, whereas $A^{\sim\wedge} = A$, as it follows in an obvious manner from (5.27) and (6.8). Therefore one has $L_h(\tilde{C}^c_m) = L_h(C^{c\sim}_m)$ and, by (6.6), this is equal to $\mathscr{L}_h(C^c_m)^{\sim\wedge} = \mathscr{L}_h(C^c_m)$. Summarizing

$$(6.35)\qquad \hat{\beta}(h; \tilde{C}^c_m) = L_h(\tilde{C}^c_m) = \mathscr{L}_h(C^c_m).$$

By (5.16) $\mathscr{L}_{m,\theta_m}(C_m^c) = P_{\theta_m}(\Delta_m \in C_m^c)$, so that $\mathscr{L}_{m,0}(C_m^c) = P_{\theta_0}(\Delta_m \in C_m^c)$ and this converges to α. Then Lemma 6.1(i), in conjunction with (6.36), gives $L_0(_m^{c\sim}) \to \alpha$. Now let $C \in \mathscr{F}$ be such that $L_0(C^c) = \alpha$. Then $L_0(C_m^{c\sim}) - L_0(C^c) \to 0$ and from this it also follows that

$$\sup\left[|L_h(C_m^{c\sim}) - L_h(C^c)|;\, h \in B\right] \to 0, \tag{6.36}$$

for any bounded subset B of R^k.

Since E_m^* remains bounded as $m \to \infty$, it follows that for all sufficiently large $m, m \geqq m_4$, say, one has $L_h(C_m^{c\sim}) \leqq L_h(C^c) + \varepsilon$. This, together with (6.15), gives

$$\int_{E_m^*} \hat{\beta}(h; C_m^{c\sim})\xi(h)\, dA \leqq \int_{E_m^*} \hat{\beta}(h; C^c)\xi(h)\, dA + \varepsilon A_m, \qquad m \geqq m_4. \tag{6.37}$$

Combining this inequality with (6.33), we obtain that for $m \geqq m_5 = \max\{m_3, m_4\}$,

$$\int_{E_m^*} \hat{\beta}(h; \tilde{D}^c)\xi(h)\, dA < \int_{E_m^*} \hat{\beta}(h; C^c)\xi(h)\, dA + \varepsilon A_m + \delta_2. \tag{6.38}$$

From the expression of A_m given above, it follows that $A_m \to 0$. This result, together with (6.34), implies that for $m \geqq m_6$, some m_6, and some $\delta_3 < 0$,

$$\int_{E_m^*} \hat{\beta}(h; \tilde{D}^c)\xi(h)\, dA < \int_{E_m^*} \hat{\beta}(h; \tilde{C}^c\xi)\xi(h)\, dA + \delta_3. \tag{6.39}$$

By (6.15), $\hat{\beta}(h; A)$ is the power of the test $\psi = I[A]$, based on the random vector Z whose distribution, under h, is $L_h = N(h, \Gamma^{-1})$ (see (6.3)). On account of (6.8), the set $\tilde{D}$ is given by

$$\tilde{D} = \{u \in R^k;\, u = \Gamma^{-1}z,\, z \in D\}. \tag{6.40}$$

By taking into consideration (4.2), one has $\tilde{D} = \{u \in R^k;\, u'\Gamma u \leqq d\}$. Applying (6.6) with $A = \tilde{D}$ and $h = 0$ and also utilizing (4.2), we obtain $L_0(\tilde{D}) = 1 - \alpha$. That is $\tilde{D} = \{u \in R^k;\, u'\Gamma u \leqq d\}$, $L_0(\tilde{D}) = 1 - \alpha$, and $L_0(C) = 1 - \alpha$. Also since $E_m^* = \{h \in R^k;\, h'\Gamma h = c_m\}$, we have that both $\tilde{D}$ and E_m^*(for $m \geqq m_2$) are of the type required by Proposition II in Wald [24] for testing the hypothesis $h = 0$ in connection with the distribution $L_h = N(h, \Gamma^{-1})$. Hence relation (6.39) cannot hold true. The desired result is then established.

In order to complete the proof of the theorem, we have to show that its conclusion is true if $c = 0$; that is, if $c_m \to 0$. In this case, by substituting h_m for h in $h'\Gamma h = c_m$, we find that $h_m \to 0$, or equivalently $\sqrt{m}(\theta_m - \theta_0) \to 0$. Then repeating the arguments employed in the last paragraph of the proof of Theorem 4.1 in Johnson and Roussas [8], we obtain $\|P_{\theta_m} - P_{\theta_0}\| \to 0$. From this and by a simple contradiction argument, we also get $\sup(\|P_{\theta_m} - P_{\theta_0}\|;\, \theta_m \in E_m) \to 0$. Therefore uniformly in ψ_m in $\mathscr{F}$ and $\theta_m \in E_m$, one has $\beta_m(\theta_m; \psi_m) - \beta_m(\theta_0; \psi_m) \to 0$. Hence $\beta_m(\theta_m; \psi_m) \to \alpha$ uniformly in $\psi_m \in \mathscr{F}$ and $\theta_m \in E_m$, since $\beta_m(\theta_0; \psi_m) \to \alpha$. Applying this result for $\psi_m = \phi$, we obtain $\beta_m(\theta_m; \phi) - \beta_m(\theta_m; \psi_m) \to 0$ uniformly in $\psi_n \in \mathscr{F}$ and $\theta_m \in E_m$, so that

$$\int_{E_m} \beta_m(\theta; \phi)\zeta_m(\theta)\, dA - \int_{E_m} \beta_m(\theta; \psi_m)\zeta_m(\theta)\, dA \to 0. \tag{6.41}$$

Thus the left side of (4.8) (with lim inf replaced by lim) is equal to zero. The proof is completed.

From Lemma 6.1(i), it follows that from asymptotic power viewpoint (both in the pointwise and the average power sense), rather than considering asymptotic level α tests, we may restrict ourselves, for each n, to tests lying in the class $\mathscr{F}_0$ defined below by (8.2). In this case, the proof of Theorem 4.1 is considerably simpler in that one may deduce the desired contradiction from (6.29). This is so because $\tilde{\beta}(0; \tilde{C}_m^c) = \alpha$. Also, in this case, one may formulate and prove a uniform version of Theorem 4.1. More precisely, one has the following result.

THEOREM 6.1. *With the same notation as that employed in Theorem* 4.1 *one has*

$$\liminf\left\{\inf\left[\int_{E_n} \beta_n(\theta; \phi)\zeta_n(\theta)\, dA - \int_{E_n} \beta_n(\theta; \psi)\zeta_n(\theta)\, dA;\ \psi \in \mathscr{F}_0\right]\right\} = 0. \tag{6.42}$$

PROOF OF THEOREM 6.1. Suppose that (6.42) is not true and let the left side of it be equal to some $\delta < 0$. (Clearly, δ may not be positive.) Then there is a subsequence $\{m\} \subseteqq \{n\}$ such that

$$\inf\left[\int_{E_m} \beta_m(\theta; \phi)\zeta_m(\theta)\, dA - \int_{E_m} \beta_m(\theta; \psi)\zeta_m(\theta)\, dA;\ \psi \in \mathscr{F}_0\right] = \delta. \tag{6.43}$$

From this it follows that there exists a sequence $\{\psi_m\}$ of tests in $\mathscr{F}_0$ such that

$$\int_{E_m} \beta_m(\theta; \phi)\zeta_m(\theta)\, dA - \int_{E_m} \beta_m(\theta; \psi_m)\zeta_m(\theta)\, dA \to \delta. \tag{6.44}$$

This is the same as relation (6.11) and a repetition of the arguments used in the proof of Theorem 4.1, leads us to (6.29). The desired contradiction then follows as indicated above.

7. Proof of the second main result

The following inequalities will be useful in the proof of the second theorem.

Let $\{\alpha_j, j \in I\}$ and $\{\beta_j, j \in I\}$ be any collections of bounded real numbers and let I be any index set. Then the following inequalities hold.

$$|\sup(\alpha_j; j \in I) - \sup(\beta_j, j \in I)| \leqq \sup(|\alpha_j - \beta_j|; j \in I) \tag{7.1}$$

and

$$|\inf(\alpha_j; j \in I) - \inf(\beta_j; j \in I)| \leqq \sup(|\alpha_j - \beta_j|; j \in I). \tag{7.2}$$

A few more facts will be needed before we proceed with the proof of the theorem.

Let Δ stand for the identity mapping in R^k. Also $L_h = N(h, \Gamma^{-1})$ by (6.3). Therefore $\mathscr{L}(\Delta | L_h) = N(h, \Gamma^{-1})$ and hence (see, for example, Rao [17], p. 152)

$$\mathscr{L}(\Delta'\Gamma\Delta \mid L_h) = \chi^2_{k,\delta(h)}. \tag{7.3}$$

where $\delta(h) = h'\Gamma h$. From the definition (6.8) of $\tilde{A}$, it is immediate that $\tilde{A}^c = A^{c\sim}$. On the other hand, as was mentioned in the proof of Theorem 4.1, $A^{\sim\wedge} = A$, where $\hat{A}$ is given by (5.27). Utilizing these facts with $A = D^{c\sim} = \tilde{D}^c$), relation (6.6) becomes

$$L_h(D^{c\sim}) = \mathscr{L}_h(D^c). \tag{7.4}$$

The quantities $\mathscr{L}_h$ and D are defined by (4.1) and (4.2), respectively. From (7.3) and (7.4), one obtains

$$\mathscr{L}_h(D^c) = 1 - P[\chi^2_{k,\delta(h)} \leqq d] = \text{constant on each } E^*_n, \tag{7.5}$$

where E^*_n is given by (3.13).

Finally, let $h \in E^*_n$ and let $\theta_n = \theta_0 + h/\sqrt{n}$, so that $\theta_n \in E_n$, where E_n is defined by (3.12). Then by virtue of (6.15) and (6.18), we have $L_h(A) = \beta^*(\theta_n; A)$. Taking $A = D^{c\sim}$ and employing (7.4), one obtains $\beta^*(\theta_n; D^{c\sim}) = \mathscr{L}_h(D^c)$. This, together with (7.5) implies then

$$\beta^*(\theta; D^{c\sim}) = \text{constant on each } E_n. \tag{7.6}$$

We may now start with the proof of the result.

PROOF OF THEOREM 4.2. (i) By employing (7.6), we have

$$\begin{aligned} &|\sup[\beta_n(\theta;\phi); \theta \in E_n] - \inf[\beta_n(\theta;\phi); \theta \in E_n]| \\ &\qquad \leqq |\sup[\beta_n(\theta;\phi); \theta \in E_n] - \sup[\beta^*(\theta; D^{c\sim}); \theta \in E_n]| \\ &\qquad\quad + |\inf[\beta_n(\theta;\phi); \theta \in E_n] - \inf[\beta^*(\theta; D^{c\sim}); \theta \in E_n]| \end{aligned} \tag{7.7}$$

and by means of (7.1) and (7.2), the right side above is bounded above by

$$\begin{aligned} &2\sup[|\beta_n(\theta;\phi) - \beta^*(\theta; D^{c\sim})|; \theta \in E_n] \\ &\qquad = 2\sup[|\beta_n(\theta; D^c) - \beta^*(\theta; D^{c\sim})|; \theta \in E_n] \\ &\qquad = 2\sup[|\beta_n(\theta; D) - \beta^*(\theta; \tilde{D})|; \theta \in E_n]. \end{aligned} \tag{7.8}$$

Set $\theta_n = \theta_0 + h/\sqrt{n}$ with $h \in E^*_n$. Then on account of (6.9) and (5.16), one has

$$\beta_n(\theta_n; D) = P_{\theta_n}(\Delta_n \in D) = \mathscr{L}_{n,\theta_n}(D). \tag{7.9}$$

From (5.27) and (6.8), if follows that $D^{\sim\wedge} = D$. Therefore, by virtue of (6.18), (6.15) and (6.6), we have

$$\beta^*(\theta_n; \tilde{D}) = \hat{\beta}(h; \tilde{D}) = L_h(\tilde{D}) = \mathscr{L}_h(D). \tag{7.10}$$

Hence

$$\begin{aligned} &2\sup[|\beta_n(\theta; D) - \beta^*(\theta; \tilde{D})|; \theta \in E_n] \\ &\qquad = 2\sup[|\mathscr{L}_{n,\theta_n}(D) - \mathscr{L}_h(D)|; h \in E^*_n]. \end{aligned} \tag{7.11}$$

Thus

$$(7.12)\qquad \left|\sup [\beta_n(\theta; \phi); \theta \in E_n] - \inf [\beta_n(\theta; \phi); \theta \in E_n]\right| \leqq 2 \sup [|\mathscr{L}_{n,\theta_n}(D) - \mathscr{L}_h(D)|; h \in E_n^*].$$

Since the expression on the right side above converges to zero by Lemma 6.1(i), the proof of part (i) is completed.

(ii) We first show that

$$(7.13)\qquad \liminf \{\sup [\beta_n(\theta; \phi) - \beta_n(\theta; \psi_n); \theta \in E_n]\} \geqq 0.$$

The proof is by contradiction. Suppose that (7.13) is not true and let the left side of it be equal to some $\delta < 0$. Then there is a subsequence $\{m\} \subseteqq \{n\}$ such that for all sufficiently large m, $m \geqq m_1$, say, one has

$$(7.14)\qquad \sup [\beta_m(\theta; \phi) - \beta_m(\theta; \psi_m); \theta \in E_m] < \frac{\delta}{2}.$$

This is equivalent to

$$(7.15)\qquad \beta_m(\theta; \phi) - \beta_m(\theta; \psi_m) < \frac{\delta}{2} \qquad \text{for all } \theta \in E_m \text{ and all } m \geqq m_1,$$

or

$$(7.16)\qquad [\beta_m(\theta; \phi) - \beta_m(\theta; \psi_m)]\zeta_m(\theta) < \frac{\delta}{2}\zeta_m(\theta) \qquad \text{for all } \theta \in E_m \text{ and all } m \geqq m_1.$$

Hence

$$(7.17)\qquad \liminf \left[\int_{E_m} \beta_m(\theta; \phi)\zeta_m(\theta)\, dA - \int_{E_m} \beta_m(\theta; \psi_m)\zeta_m(\theta)\, dA\right] \leqq \frac{\delta}{2},$$

since $\int_{E_m} \zeta_m(\theta)\, dA = 1$, and this contradicts (4.8).

We now continue as follows

$$(7.18)\qquad \begin{aligned} &\inf [\beta_n(\theta; \phi) - \beta_n(\theta; \psi_n); \theta \in E_n] \\ &\quad = \inf \{\beta_n(\theta; \phi) + [-\beta_n(\theta; \psi_n)]; \theta \in E_n\} \\ &\quad \geqq \inf [\beta_n(\theta; \phi); \theta \in E_n] + \inf [-\beta_n(\theta; \psi_n); \theta \in E_n] \\ &\quad = \inf [\beta_n(\theta; \phi); \theta \in E_n] - \sup [\beta_n(\theta; \psi_n); \theta \in E_n]. \end{aligned}$$

Adding and subtracting appropriate quantities, the last expression on the right side above becomes

$$(7.19)\qquad \begin{aligned} &\inf [\beta_n(\theta; \phi); \theta \in E_n] - \sup [\beta_n(\theta; \psi_n); \theta \in E_n] \\ &\quad = -\{\sup [\beta_n(\theta; \phi); \theta \in E_n] - \inf [\beta_n(\theta; \phi); \theta \in E_n]\} \\ &\qquad - \{\sup [\beta_n(\theta; \psi_n); \theta \in E_n] - \inf [\beta_n(\theta; \psi_n); \theta \in E_n]\} \\ &\qquad + \{\sup [\beta_n(\theta; \phi); \theta \in E_n] - \inf [\beta_n(\theta; \psi_n); \theta \in E_n]\}. \end{aligned}$$

The third of these terms is further written as $\sup[\beta_n(\theta; \phi); \theta \in E_m] + \sup[-\beta_n(\theta; \psi_m); \theta \in E_n]$ and this is bounded below by $\sup[\beta_n(\theta; \phi) - \beta_n(\theta; \psi_n); \theta \in E_n]$. Combining these results, we obtain then

$$\begin{aligned}(7.20)\qquad \inf[\beta_n(\theta; \phi) - \beta_n(\theta; \psi_n); \theta \in E_n] &\\ \geqq -\{\sup[\beta_n(\theta; \phi); \theta \in E_n] &- \inf[\beta_n(\theta; \phi); \theta \in E_n]\}\\ -\{\sup[\beta_n(\theta; \psi_n); \theta \in E_n] &- \inf[\beta_n(\theta; \psi_n); \theta \in E_n]\}\\ +\sup[\beta_n(\theta; \phi) &- \beta_n(\theta; \psi_n); \theta \in E_n].\end{aligned}$$

Now letting $n \to \infty$ (7.20) we have that the limit of the first term on the right side is equal to zero by the first part of the theorem, the limit of the second term on the same side is equal to zero, by assumption, and the lim inf of the third term on the same side is $\geqq 0$ by (7.13). This establishes (ii) and hence the theorem itself.

8. Formulation and proof of the third main result

Up to this point we have dealt with tests ψ_n in $\bar{\mathscr{F}}$ defined by (4.4), depending on the random vector Δ_n and having asymptotic level of significance α. In this section, we are going to further restrict the class $\bar{\mathscr{F}}$ of tests by introducing another class contained in $\bar{\mathscr{F}}$ and denoted by $\mathscr{F}_0$. The reasons for this restriction are implicit in the definition of the envelope power functions by (8.13) and (8.17) and also Lemma 8.2 below which is needed in the proof of Theorem 8.1. However, in order for the restriction under question to be legitimate, we must show that, asymptotically, nothing is lost in the process, either in terms of power or in terms of asymptotic level of significance.

Set

$$(8.1)\qquad \mathscr{C}_0 = \{C \in \mathscr{B}^k; C \text{ is closed, convex and } \mathscr{L}_0(C) = 1 - \alpha\},$$

where $\mathscr{L}_0$ is given by (4.1), and define the class $\mathscr{F}_0$ by

$$(8.2)\qquad \mathscr{F}_0 = \{\psi; \psi = I[C^c], C \in \mathscr{C}_0\}.$$

Since $\mathscr{L}(\Delta_n | P_{\theta_0}) = \mathscr{L}_{n,\theta_0} \Rightarrow N(0, \Gamma) = \mathscr{L}_0$ by Theorem 3.2.1 in Roussas [19], we have that $\mathscr{L}_{n,\theta_0}(C_n) - \mathscr{L}_0(C_n) \to 0$ for any sets $C_n \in \mathscr{C}^*$; this is so by Theorem 4.2 in Rao [18]. Thus $\omega_n = I[C_n^c]$ in $\mathscr{F}_0$, implies that $\mathscr{L}_0(C_n) = 1 - \alpha$ and the last convergence above gives $\mathscr{L}_{n,\theta_0}(C_n^c) \to \alpha$. That is, tests in $\mathscr{F}_0$ are of asymptotic level of significance α. Thus it suffices for us to show that every test ψ_n in $\bar{\mathscr{F}}$ which is of asymptotic level of significance α, can be replaced, from asymptotic power point of view, by tests ω_n in $\mathscr{F}_0$. More precisely, it suffices to establish the following result.

LEMMA 8.1. *For any sequence of tests $\{\psi_n\}$ in $\bar{\mathscr{F}}$ for which $\mathscr{E}_{\theta_0}\psi_n(\Delta_n) \to \alpha$, there is a sequence of tests ω_n in $\mathscr{F}_0$ such that*

$$(8.3)\qquad \sup[|\mathscr{E}_{\theta_n}\psi_n(\Delta_n) - \mathscr{E}_{\theta_n}\omega_n(\Delta_n)|; h \in B] \to 0,$$

where $\theta_n = \theta_0 + h/\sqrt{n}$ and B is any bounded subset of R^k.

PROOF. Since $\psi_n \in \bar{\mathscr{F}}$ and $\mathscr{E}_{\theta_0}\psi_n(\Delta_n) \to \alpha$, we have $\psi_n = I[C_n^c]$ for some $C_n \in \bar{\mathscr{F}}$ and $\mathscr{L}_{n,\theta_0}(C_n) \to 1 - \alpha$. Setting $h = 0$ in Lemma 6.1 (i), we obtain $\mathscr{L}_{n,\theta_0}(C_n) - \mathscr{L}_0(C_n) \to 0$, so that

$$\mathscr{L}_0(C_n) \to 1 - \alpha. \tag{8.4}$$

Thus for all sufficiently large n, $n \geqq n_1$, say, $\mathscr{L}_0(C_n) > 0$. This implies that for $n \geqq n_1$, the sets C_n are k dimensional since otherwise their k dimensional Lebesgue measure and hence $\mathscr{L}_0$ measure would be zero.

Then for each $n \geqq n_1$, consider the following modification of the set C_n: if $\mathscr{L}_0(C_n) < 1 - \alpha$, enlarge C_n, so that it remains closed and convex and $\mathscr{L}_0(C_n) = 1 - \alpha$. If $\mathscr{L}_0(C_n) > 1 - \alpha$, shrink the set C_n until $\mathscr{L}_0(C_n) = 1 - \alpha$. If $\mathscr{L}_0(C_n) = 1 - \alpha$, the set C_n is left intact. Denote the resulting set by $C_{n,0}$. Then $C_{n,0}$ is closed and convex and

$$\mathscr{L}_0(C_{n,0}) = 1 - \alpha. \tag{8.5}$$

Thus setting $\omega_n = I[C_{n,0}^c]$, we have $\omega_n \in \mathscr{F}_0$. From (8.4) and (8.5), it follows that

$$\mathscr{L}_0(C_n) - \mathscr{L}_0(C_{n,0}) \to 0. \tag{8.6}$$

From the process of arriving at $C_{n,0}$, it follows that $C_{n,0} \subseteqq C_n$ or $C_{n,0} \supseteqq C_n$. Therefore for each $h \in R^k$, we have

$$\mathscr{L}_h(C_n \Delta C_{n,0}) = \mathscr{L}_h(C_n - C_{n,0}) = \mathscr{L}_h(C_n) - \mathscr{L}_h(C_{n,0}) \quad \text{if } C_{n,0} \subseteqq C_n \tag{8.7}$$

and

$$\mathscr{L}_h(C_n \Delta C_{n,0}) = \mathscr{L}_h(C_{n,0} - C_n) = \mathscr{L}_h(C_{n,0}) - \mathscr{L}_h(C_n) \quad \text{if } C_{n,0} \supseteqq C_n. \tag{8.8}$$

Thus

$$\mathscr{L}_h(C_n \Delta C_{n,0}) = |\mathscr{L}_h(C_n) - \mathscr{L}_h(C_{n,0})|. \tag{8.9}$$

For $h = 0$ $\mathscr{L}_h(C_n \Delta C_{n,0}) \to 0$ by (8.6). On the other hand, $\mathscr{L}_h \ll \mathscr{L}_0$ for every $h \in R^k$. Thus $\mathscr{L}_h(C_n \Delta C_{n,0}) \to 0$. It can be further seen that for any bounded subset B of R^k, one has

$$\sup\,[\mathscr{L}_h(C_n \Delta C_{n,0}); h \in B] \to 0. \tag{8.10}$$

Therefore, (8.9) and (8.10) give

$$\sup\,[|\mathscr{L}_h(C_n) - \mathscr{L}_h(C_{n,0})|; h \in B] \to 0. \tag{8.11}$$

Now, for any B as described above, one has

$$\begin{aligned}
\sup\,[|\mathscr{E}_{\theta_n}\psi_n(\Delta_n) - \mathscr{E}_{\theta_n}\omega_n(\Delta_n)|; h \in B] \\
&= \sup\,[|\mathscr{L}_{n,\theta_n}(C_n) - \mathscr{L}_{n,\theta_n}(C_{n,0})|; h \in B] \\
&\leqq \sup\,[|\mathscr{L}_{n,\theta_n}(C_n) - \mathscr{L}_h(C_n)|; h \in B] \\
&\quad + \sup\,[|\mathscr{L}_{n,\theta_n}(C_{n,0}) - \mathscr{L}_h(C_{n,0})|; h \in B] \\
&\quad + \sup\,[|\mathscr{L}_h(C_n) - \mathscr{L}_h(C_{n,0})|; h \in B],
\end{aligned} \tag{8.12}$$

and each one of the terms on the right side above converges to zero on account of Lemma 6.1(i) and (8.11). The proof of Lemma 8.1 is completed and we have the justification for confining ourselves to the class of tests, $\mathscr{F}_0$.

Before we are able to formulate the third main result, we shall have to introduce a further piece of notation. To this end, let $\beta_n(\theta; \psi) = \mathscr{E}_\theta \psi(\Delta_n)$, as given in (4.6) and suppose that the test ψ lies in $\mathscr{F}_0 = \{\psi; \psi = I[C^c], C \in \mathscr{C}_0\}$, where $\mathscr{C}_0 = \{C \in \mathscr{B}^k; C$ is closed, convex and $\mathscr{L}_0(C) = 1 - \alpha\}$; we also recall that $\mathscr{L}_0 = N(0, \Gamma)$. Next define the modified *envelope power function* $\beta_n(\theta; \alpha)$ by

$$(8.13) \qquad \beta_n(\theta; \alpha) = \sup \{\beta_n(\theta; \psi); \psi \in \mathscr{F}_0\}.$$

It is to be noted that the tests involved in the definition of the modified envelope power function $\beta_n(\theta; \alpha)$ are not those of exact level α for finite n but, as with ϕ, they have level α under the limit distribution $\mathscr{L}_0$.

Then the third main result in this paper is as follows.

THEOREM 8.1. *Let E_n and ϕ be defined by* (3.12) *and* (4.5), *respectively, and let ψ_n be any tests in $\mathscr{F}_0$, where $\mathscr{F}_0$ is given in* (8.2). *Also let $\beta_n(\theta; \phi)$ and $\beta_n(\theta; \psi_n)$ be defined by* (4.6) *and let $\beta_n(\theta; \alpha)$ be given by* (8.13). *Then one has*

$$(8.14) \qquad \limsup \{\sup [\beta_n(\theta; \alpha) - \beta_n(\theta; \phi); \theta \in E_n] - \sup [\beta_n(\theta; \alpha) - \beta_n(\theta; \psi_n); \theta \in E_n]\} \leqq 0.$$

Again we could replace lim sup by lim sup sup since relation (8.28) in the proof is obtained by passing to a subsequence of ellipsoids.

The interpretation of the theorem is that within the class $\mathscr{F}_0$, the test ϕ is *asymptotically most stringent* on E_n. We recall that in the present framework and for each n, the test ϕ_n would be said to be *most stringent* on E_n within the class $\mathscr{F}_0$, if the quantity $\sup [\beta_n(\theta; \alpha) - \beta_n(\theta; \psi_n); \theta \in E_n]$ were minimized for $\psi_n = \phi_n$.

For the proof of Theorem 8.1, a couple of auxiliary results will be needed. For their formulation, let us recall once again that $\mathscr{L}_h = N(\Gamma h, \Gamma)$ and set

$$(8.15) \qquad \bar{\beta}(h; A) = \mathscr{L}_h(A), h \in R^k, A \in \mathscr{B}^k.$$

For each n, let $h \in E_n^*$ and transform h to $\theta \in E_n$ through the transformation $\theta = \theta_0 + h/\sqrt{n}$. We recall that E_n and E_n^* are given by (3.12) and (3.13), respectively. Set

$$(8.16) \qquad \bar{\beta}(h; A) = \bar{\beta}(\sqrt{n}(\theta - \theta_0); A) = \beta'(\theta; A).$$

Next by means of $\beta'(\theta; A)$, define the *envelop power function* $\beta'(\theta; \alpha)$ as follows

$$(8.17) \qquad \beta'(\theta; \alpha) = \sup [\beta'(\theta; C^c); C \in \mathscr{C}_0].$$

The first auxiliary result is given in the following lemma.

LEMMA 8.2. *The function $\beta'(\theta; \alpha)$ defined by* (8.17) *remains constant on each E_n, where E_n is given by* (3.12).

PROOF. Clearly, $\beta'(\theta;\alpha) = \bar{\beta}(h;\alpha)$, where

$$\bar{\beta}(h;\alpha) = \sup\,[\bar{\beta};\,C^c);\,C \in \mathscr{C}_0],\, h = \sqrt{n}(\theta - \theta_0). \tag{8.18}$$

Thus it suffices to show that $\bar{\beta}(h;\alpha)$ stays constant on each E_n^*, where E_n^* is given by (3.13). With M defined by (3.1), consider the transformation $t = Mh$. Then, by (3.2), E_n^* is transformed into $S_n^* = \{t \in R^k;\, t't = \|t\|^2 = c_n\}$. Also the class of sets $\mathscr{C}_0$ is transformed into the class of sets $\mathscr{C}_*$, where

$$\mathscr{C}_* = \{C \in \mathscr{B}^k;\, C \text{ is closed, convex and } N_0(C) = 1 - \alpha\} \tag{8.19}$$

and

$$N_t = N(MM't,\, M\Gamma M'), \qquad t \in R^k. \tag{8.20}$$

Therefore, by setting

$$\begin{aligned} \beta^0(t;A) &= N_t(A), \qquad t \in R^k, \quad A \in \mathscr{B}^k, \\ \beta^0(t;\alpha) &= \sup\,[\beta^0(t;\,C^c);\,C \in \mathscr{C}_*], \end{aligned} \tag{8.21}$$

it suffices to show that $\beta^0(t;\alpha)$ is constant on each S_n^*. To obtain a contradiction, suppose that this is not so. Then there exist t_1, $t_2 \in S_n^*$ for which $\beta^0(t_1;\alpha) \neq \beta^0(t_2;\alpha)$ and let

$$\beta^0(t_1;\alpha) < \beta^0(t_2;\alpha). \tag{8.22}$$

From the definition of $\beta^0(t;\alpha)$, there exists a set $C = \mathscr{C}_*$ such that

$$\beta^0(t_1;\alpha) < \beta^0(t_2;\,C^c). \tag{8.23}$$

Now, clearly, $\beta^0(t_2;\,C^c) = N_{t_2}(C^c)$ is equal to the $N(t_2, I)$ measure of the set $(MM')^{-1}C^c$, and by symmetry, this is equal to the $N(t_1, I)$ measure of D, where D is the symmetric image of $(MM')^{-1}C^c$ with respect to the hyperplane through the origin that is perpendicular to the line segment connecting the points t_1 and t_2. But the $N(t_1, I)$ measure of D is equal to $N_{t_1}((MM')D)$, and, clearly, $(MM')D$ is the complement of a closed convex set, C_0^c, say. Then by symmetry, one clearly has $N_0(C_0^c = 1 - \alpha$, so that $C_0 \in \mathscr{C}_*$, and also $\beta^0(t_2;\,C^c) = \beta^0(t_1;\,C_0^c)$. Then (8.23) gives $\beta^0(t_1;\alpha) < \beta^0(t_1;\,C_0^c)$. However, this contradicts the definition of $\beta^0(t_1;\alpha)$ by (8.21). We reach the same conclusion if the inequality in (8.22) is reversed. Thus the proof of the lemma is completed.

The second auxiliary result referred to above is the following lemma. This lemma, as well as the one just established, are of some interest in their own right.

LEMMA 8.3. *Let* $\beta_n(\theta;\alpha)$ *and* $\beta'(\theta;\alpha)$ *be defined by* (8.13) *and* (8.17), *respectively. Then for each n, one has*

(i) $\sup\,[|\beta_n(\theta;\alpha) - \beta'(\theta;\alpha)|;\,\theta \in E_n] \to 0$

and

(ii) $\sup\,[\beta_n(\theta;\alpha);\,\theta \in E_n] - \inf\,[\beta_n(\theta;\alpha);\,\theta \in E_n] \to 0$,

where E_n *is given by* (3.12).

PROOF. (i) In the first place, relation (7.1) justifies the inequality below

$$(8.24)\quad \begin{aligned}&|\beta_n(\theta;\alpha) - \beta'(\theta;\alpha)|\\ &\quad = |\sup[\beta_n(\theta;\psi);\psi\in\mathscr{F}_0] - \sup[\beta'(\theta;C^c);C\in\mathscr{C}_0]|\\ &\quad = |\sup[\beta_n(\theta;C^c);C\in\mathscr{C}_0] - \sup[\beta'(\theta;C^c);C\in\mathscr{C}_0]|\\ &\quad \leqq \sup[|\beta_n(\theta;C^c) - \beta'(\theta;C^c)|;C\in\mathscr{C}_0]\\ &\quad = \sup[|\beta_n(\theta;C) - \beta'(\theta;C)|;C\in\mathscr{C}_0],\end{aligned}$$

and this last expression is equal to $\sup[|\mathscr{L}_{n,\theta}(C) - \mathscr{L}_h(C)|;C\in\mathscr{C}_0]$, where $h = \sqrt{n}(\theta - \theta_0)$, since $\beta_n(\theta;C) = P_\theta(\Delta_n\in C) = \mathscr{L}_{n,\theta}(C)$ and $\beta'(\theta;C) = \bar{\beta}(h;C) = \mathscr{L}_h(C)$. That is, with $h = \sqrt{n}(\theta - \theta_0)$,

$$(8.25)\quad |\beta_n(\theta;\alpha) - \beta'(\theta;\alpha)| = \sup[|\mathscr{L}_{n,\theta}(C) - \mathscr{L}_h(C)|;C\in\mathscr{C}_0].$$

Hence

$$(8.26)\quad \begin{aligned}&\sup[|\beta_n(\theta;\alpha) - \beta'(\theta;\alpha)|;\theta\in E_n]\\ &\qquad = \sup\{\sup[|\mathscr{L}_{n,\theta}(C) - \mathscr{L}_h(C)|;C\in\mathscr{C}_0];h\in E_n^*\},\end{aligned}$$

and the expression on the right side converges to zero by Lemma 6.1(i).

(ii) Letting $\theta\in E_n$ and utilizing Lemma 8.2 and inequalities (7.1) and (7.2), one has

$$(8.27)\quad \begin{aligned}&|\sup[\beta_n(\theta;\alpha);\theta\in E_n] - \inf[\beta_n(\theta;\alpha);\theta\in E_n]|\\ &\quad \leqq |\sup[\beta_n(\theta;\alpha);\theta\in E_n] - \beta'(\theta;\alpha)| + |\inf[\beta_n(\theta;\alpha);\theta\in E_n] - \beta'(\theta;\alpha)|\\ &\quad = |\sup[\beta_n(\theta;\alpha);\theta\in E_n] - \sup[\beta'(\theta;\alpha);\theta\in E_n]|\\ &\qquad\qquad + |\inf[\beta_n(\theta;\alpha);\theta\in E_n] - \inf[\beta'(\theta;\alpha);\theta\in E_n]\\ &\quad \leqq 2\sup[|\beta_n(\theta;\alpha) - \beta'(\theta;\alpha)|;\theta\in E_n],\end{aligned}$$

and this last expression tends to zero by part (i). This establishes the lemma.

We may now proceed with the proof of the third main result.

PROOF OF THEOREM 8.1. The proof is by contradiction. Suppose that the theorem is not true and let the left side of (8.14) be equal to $4\delta > 0$. Then there exists a subsequence $\{m\}\subseteqq\{n\}$ for which

$$(8.28)\quad \begin{aligned}&\sup[\beta_m(\theta;\alpha) - \beta_m(\theta;\phi);\theta\in E_m]\\ &\qquad > \sup[\beta_m(\theta;\alpha) - \beta_m(\theta;\psi_m);\theta\in E_m] + 3\delta\end{aligned}$$

for all sufficiently large m, $m \geqq m_1$, say.

The left side of the inequality above is bounded from above by

$$(8.29)\quad \sup[\beta_m(\theta;\alpha);\theta\in E_m] - \inf[\beta_m(\theta;\phi);\theta\in E_m]$$

and its right side is bounded from below by

$$(8.30)\quad \inf[\beta_m(\theta;\alpha);\theta\in E_m] - \inf[\beta_m(\theta;\psi_m);\theta\in E_m] + 3\delta$$

by virtue of (7.2).

By means of (8.29) and (8.30), relation (8.28) gives that, for all $m \geqq m_1$

$$(8.31)\qquad \sup[\beta_m(\theta;\alpha);\theta\in E_m] - \inf[\beta_m(\theta;\phi);\theta\in E_m] > \inf[\beta_m(\theta;\alpha);\theta\in E_m] - \inf[\beta_m(\theta;\psi_m);\theta\in E_m] + 3\delta,$$

or

$$(8.32)\qquad \sup[\beta_m(\theta;\alpha);\theta\in E_m] - \inf[\beta_m(\theta;\alpha);\theta\in E_m] > \inf[\beta_m(\theta;\phi);\theta\in E_m] - \inf[\beta_m(\theta;\psi_m);\theta\in E_m] + 3\delta.$$

By Lemma 8.3 (ii), the left side of (8.32) tends to zero and hence it remains less than δ for all sufficiently large m, $m \geqq m_2$, say. On the other hand, Theorem 4.2(i) yields

$$(8.33)\qquad \inf[\beta_n(\theta;\phi);\theta\in E_m] > \sup[\beta_m(\theta;\phi);\theta\in E_m] - \delta$$

for all sufficiently large m, $m \geqq m_3$, say. On account of these facts, inequality (8.32) then becomes

$$(8.34)\qquad \delta > \sup[\beta_m(\theta;\phi);\theta\in E_m] - \inf[\beta_m(\theta;\psi_m);\theta\in E_m] + 2\delta,$$

or

$$(8.35)\qquad \inf[\beta_m(\theta;\psi_m);\theta\in E_m] - \delta = \inf[\beta_m(\theta;\psi_m) - \delta;\theta\in E_m] > \sup[\beta_m(\theta;\phi);\theta\in E_m]$$

for all $m \geqq m_4 \geqq \max\{m_1, m_2, m_3\}$. Hence

$$(8.36)\qquad \beta_m(\theta;\phi_m) - \delta > \beta_m(\theta;\phi)$$

for all $m \geqq m_4$ and every $\theta \in E_m$. Therefore

$$(8.37)\qquad \int_{E_m}\beta_m(\theta;\phi)\zeta_m(\theta)\,dA - \int_{E_m}\beta_m(\theta;\psi_m)\zeta_m(\theta)\,dA < -\delta$$

for all $m \geqq m_4$, and this implies that

$$(8.38)\qquad \liminf\left[\int_{E_m}\beta_m(\theta;\phi)\zeta_m(\theta)\,dA - \int_{E_m}\beta_m(\theta;\psi_m)\zeta_m(\theta)\,dA\right] < -\delta.$$

However, this contradicts Theorem 4.1. The desired result follows.

The following uniform version of Theorem 8.1 is also true.

THEOREM 8.2. *With the same notation as that employed in Theorem* 8.1, *one has*

$$(8.39)\qquad \limsup\left\{\sup[\beta_n(\theta;\alpha) - \beta_n(\theta;\phi);\theta\in E_n] - \inf\{\sup[\beta_n(\theta;\alpha) - \beta_n(\theta;\psi);\theta\in E_n];\psi\in\mathscr{F}_0\}\right\} = 0.$$

PROOF. Suppose that the theorem is not true and let the left side of (8.39) be equal to some $\delta > 0$. (Clearly, δ may not be negative.) Then there exists a subsequence $\{m\} \subseteqq \{n\}$ such that

$$\sup [\beta_m(\theta; \alpha) - \beta_m(\theta; \psi); \theta \in E_m] - \inf \{\sup [\beta_m(\theta; \alpha) - \beta_m(\theta; \psi); \theta \in E_m]; \psi \in \mathscr{F}_0\} \to \delta. \tag{8.40}$$

Thus for $\varepsilon > 0$ and all $m \geqq m_5$, say, one has

$$\begin{aligned} \sup [\beta_m(\theta; \alpha) - \beta_m(\theta; \phi); \theta \in E_m] - \delta - \varepsilon &< \inf \{\sup [\beta_m(\theta; \alpha) - \beta_m(\theta; \psi); \theta \in E_m]; \psi \in \mathscr{F}_0\} \\ &< \sup [\beta_m(\theta; \alpha) - \beta_m(\theta; \phi); \theta \in E_m] - \delta + \varepsilon. \end{aligned} \tag{8.41}$$

Therefore, for each $m \geqq m_5$, there exists a test $\psi_m \in \mathscr{F}_0$ such that

$$\begin{aligned} \sup [\beta_m(\theta; \alpha) - \beta_m(\theta; \phi); \theta \in E_m] - \delta - \varepsilon &< \sup [\beta_m(\theta; \alpha) - \beta_m(\theta; \psi_m); \theta \in E_m] \\ &< \sup [\beta_m(\theta; \alpha) - \beta_m(\theta; \phi); \theta \in E_m] - \delta + \varepsilon, \end{aligned} \tag{8.42}$$

or equivalently

$$\delta - \varepsilon < \sup [\beta_m(\theta; \alpha) - \beta_m(\theta; \phi); \theta \in E_m] - \sup [\beta_m(\theta; \alpha) - \beta_m(\theta; \psi_m); \theta \in E_m] < \delta + \varepsilon, \tag{8.43}$$

provided $m \geqq m_5$.

It follows that

$$\sup [\beta_m(\theta; \alpha) - \beta_m(\theta; \phi); \theta \in E_m] - \sup [\beta_m(\theta; \alpha) - \beta_m(\theta; \psi_m); \theta \in E_m] \to \delta \ (> 0). \tag{8.44}$$

However, this result contradicts (8.14). The proof of the theorem is completed.

9. Behavior of the power under nonlocal alternatives

Recall that $\phi = I[D^c]$, where D is given by (4.2). Also recall that the power of the test ϕ, based on $\Delta_n = \Delta_n(\theta_0)$, has been denoted by $\beta_n(\theta; \phi)$, $\theta \in \Theta$. Then the theorems formulated and proved in the previous sections, provide us with some optimal properties of the test ϕ. However, these properties are local in character, since the alternatives are required to lie close to the hypothesis being tested; actually, they are required to converge to θ_0 and at a specified rate.

The underlying basic Assumptions 1 to 4 employed throughout this paper, do not suffice for establishing optimal properties of the power function at alternatives removed from θ_0 or not converging to it at the specified rate. This can be done, however, under the following additional condition, Assumption 5.

ASSUMPTION 5. *Consider a sequence* $\{\theta_n\}$ *with* $\theta_n \in \Theta$ *for all* n. *Then* $\|\Delta_n(\theta_0)\| \to \infty$ *in* P_{n,θ_n} *probability whenever* $\|\sqrt{n}(\theta_n - \theta_0)\| \to \infty$.

The following result can now be established.

THEOREM 9.1. *Under Assumptions* 1 *to* 5, *for testing the hypothesis* $H: \theta = \theta_0$ *against the alternative* $A: \theta \neq \theta_0$ *at asymptotic level of significance* α, *the test defined by* (4.5) *possesses the optimal properties mentioned in Theorems* 4.1, 4.2, 6.1, 8.1, 8.2 *and also has the property that its power converges to* 1, *that is*, $\beta_n(\theta_n; \phi) \to 1$, *whenever* $\|\sqrt{n}(\theta_n - \theta_0)\| \to \infty$.

PROOF. The proof is immediate. Since Γ is positive definite, so is Γ^{-1}. Thus there exists a positive number p such that $z'\Gamma^{-1}z \geqq p\|z\|^2$ for all $z \in R^k$. Therefore

$$\beta_n(\theta_n; \phi) = \mathscr{E}_{\theta_n}\phi(\Delta_n) = P_{\theta_n}(\Delta_n \in D^c) = P_{\theta_n}(\Delta_n'\Gamma^{-1}\Delta_n > d). \tag{9.1}$$

However, this last quantity is greater than or equal to $P_{\theta_n}(p\|\Delta_n\|^2 > d)$ which converges to 1 by Assumption 5. Thus $\beta_n(\theta_n; \phi \to 1$, as was to be seen.

APPENDIX

At the beginning of Section 5, it was pointed out that for testing the hypothesis $H: \theta = \theta_0$ against the alternative $A: \theta \neq \theta_0$, it suffices to consider the class of tests based only on Δ_n^*. Furthermore, each such test function is essentially the indicator of the complement of a closed, convex set in R^k. The reason for this is that the distribution $\mathscr{L}_{n,h}^*$ of Δ_n^*, under $R_{n,h}$ (defined in (5.6)), is of the standard exponential form, so that results obtained in Birnbaum [1] and Matthes and Truax [14] apply (see also Theorems 1.1 and 1.2 in Chibisov [2]). The purpose of this appendix is to elaborate further on this point.

In order to simplify the notation, in all that follows we shall omit the subscript n, since there is no danger of confusion.

From Lemma 5.1(i), one has

$$\frac{d\mathscr{L}_h^*}{d\mathscr{L}_0^*} = \exp\{-B(h) + h'z\}, \qquad z, h \in R^k, \tag{A.1}$$

where $\mathscr{L}_h^* = \mathscr{L}(\Delta^*|R_h)$. Then the hypothesis testing problem above, described in terms of the family (A.1), becomes

$$H^*: h = 0 \qquad \text{against} \quad A^*: h \neq 0. \tag{A.2}$$

For any test ϕ, the associated risk corresponding to the usual zero-one loss function is

$$R_\phi(0) = \beta_\phi(0), \qquad R_\phi(h) = 1 - \beta_\phi(h) \qquad \text{for} \quad h \neq 0, \tag{A.3}$$

where $\beta_\phi(0)$ is the size of the test ϕ and $\beta_\phi(h)$ is its power at h.

The Bayes, or global risk, with respect to any prior probability distribution W on $\mathscr{B}^k$ that is associated with the test ϕ is given by

$$r(\phi, W) = \int R_\phi(h)\, dW. \tag{A.4}$$

The following theorem is obtained in Birnbaum [1].

THEOREM A.1. *In connection with the family* (A.1). *for testing the hypothesis* $H^*: h = 0$ *against the alternative* $A^*: h \neq 0$ *on the basis of* Δ^*, *any Bayes test* ϕ_W *with respect to the prior distribution* W, *is given by*

$$\phi_W(z) = \begin{cases} 1 & \text{if } z \in C_W^c \\ 0 & \text{if } z \in C_W^0. \end{cases} \tag{A.5}$$

where C_W *is a closed, convex set in* R^k. *The test may be defined in an arbitrary (but measurable) manner on the boundary* C_W^b *of* C_W.

PROOF. If w_0 is the mass assigned to $\{0\}$ by W, one has from (A.4)

$$\begin{aligned} r(\phi, W) &= \int R_\phi(h)\, dW = w_0[2\beta_\phi(0) - 1] + \int [1 - \beta_\phi(h)]\, dW \\ &= (1 - w_0) + 2w_0 \int \phi(z)\, d\mathscr{L}_0^* \\ &\quad - \int \left[\int \phi(z) \exp\{-B(h) + h'z\}\, d\mathscr{L}_0^*\right] dW \\ &= (1 - w_0) + \int \left[2w_0 - \int \exp\{-B(h) + h'z\}\, dW\right] \phi(z)\, d\mathscr{L}_0^*. \end{aligned} \tag{A.6}$$

Clearly, the Bayes risk is minimized by the test

$$\phi_W(z) = \begin{cases} 1 & \text{if } z \in A_1 = \left[z \in R^k; 2w_0 < \int \exp\{-B(h) + h'z\}\, dW\right] \\ 0 & \text{if } z \in A_2 = \left[z \in R^k: 2w_0 > \int \exp\{-B(h) + h'z\}\, dW\right]; \end{cases} \tag{A.7}$$

ϕ_W may be defined arbitrarily (but in a measurable way) on the set

$$A_3 = \left[z \in R^k: 2w_0 = \int \exp\{-B(h) + h'z\}\, dW\right] \tag{A.8}$$

which is the boundary of $A_2 \cup A_3$. From the definition of $\exp\{B(h)\}$ in (5.1), it easily follows that $\exp\{-B(h)\}$ is bounded over bounded sets of h in R^k. Therefore for any probability measure W on $\mathscr{B}^k$, one defines a σ-finite measure μ_W on $\mathscr{B}^k$ as follows

$$\mu_W(B) = \int_B \exp\{-B(h)\}\, dW. \tag{A.9}$$

Then

$$\int \exp\{-B(h) + h'z\}\, dW = \int \exp\{h'z\}\, d\mu_W = \int \exp\{z'h\}\, d\mu_W \tag{A.10}$$

and by Theorem 9 on p. 52 in Lehmann [12], it follows that $\int \exp\{-B(h) + h'z\}\, dW$ is continuous (as a function of z). Next by using the inequality

$$\exp\{\lambda u + (1 - \lambda)v\} \leqq \lambda \exp\{u\} + (1 - \lambda) \exp\{v\} \quad 0 < \lambda < 1 \tag{A.11}$$

with strict inequality unless $u = v$, one has that

$$\text{(A.12)} \qquad \int \exp\{-B(h) + h'[\lambda z_1 + (1-\lambda)z_2]\}\, dW$$
$$\leqq \lambda \int \exp\{-B(h) + h'z_1\}\, dW + (1-\lambda)\int \exp\{-B(h) + h'z_2\}\, dW$$
$$\leqq 2w_0 \qquad \text{whenever} \quad z_1, z_2 \in A_2 \cup A_3 = C_W.$$

Thus C_W is a convex set. It is also closed, since $\int \exp\{-B(h) + h'z\}\, dW$ is continuous, as was shown above.

According to the weak compactness theorem for tests, for any given sequence of tests $\{\phi_n\}$, there is a subsequence $\{\phi_m\}$, which converges weakly to a test ϕ in the sense that

$$\text{(A.13)} \qquad \int \phi_m g\, d\mathscr{L}_0^* \to \int \phi g\, d\mathscr{L}_0^*$$

for every $\mathscr{L}_0^*$ integrable function g defined on R^k into R. A proof of this theorem can be found in Lehmann [12], pp. 354–356.

REMARK A.1. As a consequence of the weak compactness theorem stated above, one has that, if $\{\phi_m\}$ converges weakly to ϕ, then $\beta_{\phi_m}(h) \to \beta_\phi(h)$ for every $h \in R^k$. This follows from (A.13) above by replacing $g(x)$ by the $\mathscr{L}_0^*$ integrable function $\exp\{-B(h) + h'z\}$.

Now consider the class of tests ϕ of the following form

$$\text{(A.14)} \qquad \phi(z) = \begin{cases} 1 & \text{if } z \in C^c \\ 0 & \text{if } z \in C^0, \end{cases}$$

where C is a closed, convex set in R^k. The test may be defined in an arbitrary (but measurable) manner on the boundary C^b of C. The Bayes tests given by (A.5) are also of the form (A.14). However, in the following, we will be interested in tests of the form (A.14) which may not correspond to any prior W on $\mathscr{B}^k$.

We shall show below that the weak limit of a sequence of tests, each one of which is of the form (A.5), is also of the same form. To this end, denote by S_r the closed, solid sphere of radius r centered at the origin, and for any two closed subsets of S_r, A and B, consider their Hausdorff distance, $d(A, B)$, defined as follows

$$\text{(A.15)} \qquad d(A, B) = \inf\{\varepsilon > 0; A \subset N_\varepsilon(B), B \subset N_\varepsilon(A)\}.$$

Here $N_\varepsilon(A) = \{y \in R^k; (y - z)'(y - z) < \varepsilon \text{ for some } z \in A\}$ and similarly for $N_\varepsilon(B)$.

The following standard result on convex sets, which is established in Eggleston [5], p. 64, will also be needed.

THEOREM A.2. (Blaschke selection theorem). *Given any sequence of closed, convex subsets of S_r, $\{C_n\}$, there exists a subsequence $\{C_m\}$ and a nonvoid, convex subset C of S_r such that $d(C_m, C) \to 0$.*

By utilizing Theorem A.2, one can establish the following result, as in Matthes and Truax [14], p. 684.

THEOREM A.3. *If* $\{\phi_m\}$ *is a sequence of tests of the form* (A.14) *which converges weakly to the test* ϕ *in the sense of* (A.13), *then* ϕ *is also of the same form a.s.* $[\mathscr{L}_0^*]$.

The following definition refers to the essential completeness of a class of tests, namely,

DEFINITION A.1. *A class of tests is said to be essentially complete for testing the hypothesis* $H^*: h = 0$ *against the alternative* $A^*: h \neq 0$ *in the family* (A.1), *if for any test* ψ *not in the class there is a test* ϕ *in the class for which* $\beta_\psi(0) \geqq \beta_\phi(0)$ *and* $\beta_\psi(h) \leqq \beta_\phi(h)$ *for* $h \neq 0$.

We now show that tests of the form (A.14) form an essentially complete class. More precisely,

THEOREM A.4. *For testing the hypothesis* $H^*: h = 0$ *against the alternative* $A^*: h \neq 0$ *in the family* (A.1), *the class of tests of the form* (A.14) *is essentially complete when* W *varies over the class of all probability distributions on* $\mathscr{B}^k$.

PROOF. Assume that the test ψ is not of the form (A.14) and define the new risk R_ϕ^* associated with a test ϕ as follows

$$R_\phi^*(h) = R_\phi(h) - R_\psi(h), \qquad h \in R^k, \tag{A.16}$$

where R_ϕ and R_ψ are given by (A.3) (with ϕ replaced by ψ for the latter). Let $\{h_n\}$ be a dense sequence in R^k with $h_1 = 0$ and distinct terms. We claim that for each $j = 1, 2, \cdots$, there exists a test ϕ_j of the form (A.5) with

$$R_{\phi_j}^*(h_i) \leqq 0, \qquad i = 1, 2, \cdots, j. \tag{A.17}$$

Suppose for a moment that (A.17) has been established. Then by considering the sequence $\{\phi_j\}$ there is a subsequence $\{\phi_m\}$ which converges weakly to a test ϕ, by the weak compactness theorem. Furthermore, the test ϕ is also of the form (A.14) a.s. $[\mathscr{L}_0^*]$. This is so by Theorem A.3. Now according to (A.17), $R_{\phi_m}^*(h_i) \leqq 0$ for all i and m, or equivalently, $R_{\phi_m}(h_i) \leqq R_\psi(h_i)$ for all i and m, as follows from (A.16). From Remark A.1, one then has that $\beta_\psi(0) \geqq \beta_\phi(0)$ and $\beta_\psi(h_i) \leqq \beta_\phi(h_i)$ for $i = 2, 3, \cdots$. By the fact that the power function of any test is continuous (by Theorem 9 on p. 52 in Lehmann [12]) and the choice of the sequence $\{h_i\}$, one has that $\beta_\psi(h) \leqq \beta_\phi(h)$ for all $h \in R^k$, $h \neq 0$.

Thus it suffices to establish (A.17). For each $j = 1, 2, \cdots$, consider the risk set, $C_j(R^*)$ consisting of points of the form $(R_\phi^*(h_1), \cdots, R_\phi^*(h_j))'$ for some test ϕ. It is then an easy matter to show that $C_j(R^*)$ is closed and convex. As is well known, the minimax test ϕ_j, say, is obtained by finding the point in $C_j(R^*)$ on the equiangular line with the smallest coordinates. To this end, define d_j by $d_j = \inf\{\delta; \delta(1, \cdots, 1)' \in C_j(R^*)\}$ and set

$$C(d_j) = \{z = (x_1, \cdots, x_j)' \in R^j; x_i < d_j, i = 1, \cdots, j\}. \tag{A.18}$$

Then $C(d_j)$ and $C_j(R^*)$ are disjoint, convex sets. By the separating hyperplane theorem (see, for example, Ferguson [6], pp. 73–74), it follows that there exists hyperplane $w'z = a$ with $w'z \geqq a$ for $z \in C_j(R^*)$, $w'z < a$ for $z \in C(d_j)$ and $w'z = a$ for $z = d_j(1, \cdots, 1)'$. The coordinates w_i, $i = 1, \cdots, j$ of w are $\geqq 0$ because otherwise $x_i \to -\infty$ would lead to a contradiction. Therefore we may as well assume that $(w_i, \cdots, w_j)'$ is a probability distribution over $\{h_1, \cdots, h_j\}$. Since $w'z \geqq a$ for $z \in C_j(R^*)$ and $w'z = a$ for $z = d_j(1, \cdots, 1)'$, it follows that the minimax test ϕ_j, corresponding to the point $d_j(1, \cdots, 1)'$, is also Bayes relative to the distribution $(w_1, \cdots, w_j)'$ and hence ϕ_j is of the form (A.14). Next from (A.16) it follows that $R^*_\psi(h_i) = 0$, $i = 1, \cdots, j$, and since ϕ_j is minimax, one has that $R^*_{\phi_j}(h_i) \leqq 0$ for all $i = 1, \cdots, j$. The claim in (A.17) is established and the proof of the theorem is completed.

To Theorem A.4, there is the following corollary.

COROLLARY A.1. *For testing the hypothesis* $H^*: h = 0$ *against the alternative* $A^*: h \neq 0$ *in the family of probability densities given in Lemma* 5.1(ii), *namely*

$$\frac{dL_h^*}{dL_0^*} = \exp\{-B(h) + h'\Gamma z\}, \qquad z, h \in R^k, \tag{A.19}$$

where $L_h^* = \mathscr{L}(\Gamma^{-1}\Delta^* | R_h)$, *the class of tests of the form* (A.14) *is essentially complete.*

PROOF. The transformation $t = \Gamma h$ brings the family (A.18) into the form (A.1) and then Theorem A.4 applies.

This appendix is closed with the following remark.

REMARK A.2. As follows from Theorem A.4 and also Corollary A.1, when testing the hypothesis $H^*: h = 0$ against $A^*: h \neq 0$, for any test ψ not of the form (A.14), there exists a test of that form with no smaller power and no larger size whatever the size of ψ. Consequently, whatever criterion is proposed in terms of power, it is possible to restrict ourselves to members of the class of tests of the form (A.14).

REFERENCES

[1] A. BIRNBAUM, "Characterization of complete classes of tests of some multiparametric hypotheses, with applications to likelihood ratio tests," *Ann. Math. Statist.*, Vol. 26 (1955), pp. 21–36.

[2] D. M. CHIBISOV, "A theorem on admissible tests and its application to an asymptotic problem of testing hypotheses," *Theor. Probability Appl.*, Vol. 12 (1967), pp. 90–103.

[3] ———, "Transition to the limiting process for deriving asymptotically optimal tests," *Sankhyā Ser. A*, Vol. 31 (1969), pp. 241–258.

[4] J. L. DOOB, *Stochastic Processes*, New York, Wiley, 1953.

[5] H. G. EGGLESTON, *Convexity*, Cambridge, Cambridge University Press, 1966.

[6] T. S. FERGUSON, *Mathematical Statistics*, New York, Academic Press, 1967.

[7] J. HÁJEK and Z. ŠIDÁK, *Theory of Rank Tests*, New York, Academic Press, 1967.

[8] R. A. JOHNSON and G. G. ROUSSAS, "Asymptotically most powerful tests in Markov processes," *Ann. Math. Statist.*, Vol. 40 (1969), pp. 1207–1215.

[9] ———, "Asymptotically optimal tests in Markov processes," *Ann. Math. Statist.*, Vol. 41 (1970), pp. 918–938.

[10] L. LECAM, "Locally asymptotically normal families of distributions," *Univ. Calif. Publ. Statist.*, Vol. 3 (1960), pp. 37–98.

[11] ———, "Likelihood functions for large numbers of independent observations," *Research Papers in Statistics*, edited by F. N. David, New York, Wiley, 1966.

[12] E. L. LEHMANN, *Testing Statistical Hypotheses*, New York, Wiley, 1959.

[13] M. LOÈVE, *Probability Theory*, 3rd ed., Princeton, Van Nostrand, 1963.

[14] T. K. MATTHES and D. R. TRUAX, "Tests of composite hypotheses for the multivariate exponential family," *Ann. Math. Statist.*, Vol. 38 (1967), pp. 681-697.

[15] J. NEYMAN, "Optimal asymptotic tests of composite statistical hypotheses," *Probability and Statistics* (edited by Ulf Grenander), New York, Wiley, 1959.

[16] C. R. RAO, "Large sample tests of statistical hypotheses concerning several parameters with applications to problems of estimation," *Proc. Cambridge Philos. Soc.*, Vol. 44 (1948), pp. 50-57.

[17] ———, *Linear Statistical Inference and Its Applications*, New York, Wiley, 1965.

[18] R. R. RAO, "Relations between weak and uniform convergence of measures with applications," *Ann. Math. Statist.*, Vol. 33 (1962), pp. 659-680.

[19] G. G. ROUSSAS, "Asymptotic inference in Markov processes," *Ann. Math. Statist.*, Vol. 36 (1965), pp. 978-992.

[20] ———, "Some applications of the asymptotic distribution of likelihood functions to the asymptotic efficiency of estimates," *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, Vol. 10 (1968), pp. 252-260.

[21] ———, "On the concept of contiguity and related theorems," Aarhus Universitet Preprint Series 1968/69 No. 33, 28 pp.

[22] ———, "The usage of the concept of contiguity in discussing some statistical problems," Aarhus Universitet Preprint Series 1968/69 No. 34, 42 pp.

[23] ———, "Multiparameter asymptotically optimal tests for Markov processes," Aarhus Universitet Preprint Series 1968/69 No. 42, 26 pp.

[24] A. WALD, "Tests of statistical hypotheses concerning several parameters when the number of observations is large," *Trans. Amer. Math. Soc.*, Vol. 54 (1943), pp. 426-482.