ON THE NORMAL APPROXIMATION FOR A CERTAIN CLASS OF STATISTICS

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1. Summary and introduction

We consider the class $C(\alpha)$ of asymptotic tests proposed by Neyman [6]. The term of order $n^{-1/2}$ in the normal approximation for the distributions of the test statistics is obtained. Moreover several theorems on conditional distributions are proved. They are used in deriving the main result but they also seem to be of independent interest.

Neyman [6] proposed a class of asymptotic tests called $C(\alpha)$ for the following statistical problem. Let a random variable (r.v.) X have a distribution depending on parameters $\theta = (\theta_1, \cdots, \theta_s)$ and ξ which take their values in open sets $\Theta \subset R^s$ and $\Xi \subset R^1$ respectively. (We denote by R^s, $s = 1, 2, \cdots$, the space of real row vectors $x = (x_1, \cdots, x_s)$ with the Euclidean norm $\|x\| = (xx')^{1/2}$, a prime denoting the transposition.) The hypothesis $H: \xi = \xi_0$, where $\xi_0 \in \Xi$ is a specified value, is to be tested on the basis of n independent observations $X_1, \cdots, X_n$ of the r.v. (In the sequel, we put $\xi_0 = 0$.) The distribution of X is assumed to have a density $f(x; \theta, \xi)$ (with respect to an appropriate measure) which satisfies certain regularity conditions. The $C(\alpha)$ tests are constructed as follows. Let a function $g(x, \theta)$ be such that

$$E_{\theta,0}\, g(X, \theta) \equiv 0, \qquad E_{\theta,0}\, g^2(X, \theta) = \sigma^2 < \infty, \qquad \theta \in \Theta. \tag{1.1}$$

(The first assumption can always be satisfied by considering $g(x, \theta) - E_{\theta,0}\, g(X, \theta)$ instead of $g(x, \theta)$.) Form the function

$$Z_n(\theta) = \frac{1}{\sigma(\theta)\sqrt{n}} \sum_{i=1}^{n} g(X_i, \theta) \tag{1.2}$$

and let $\hat{\theta}_n$ be a locally root n consistent estimator of θ (which means that $\sqrt{n}(\hat{\theta}_n - \theta_0)$ is bounded in probability where θ_0 is the true value of θ; for a precise definition see [6]). It was shown in [6] that $Z_n(\hat{\theta}_n)$ is asymptotically (0, 1) normally distributed whatever be the true value of θ if and only if $g(x, \theta)$ is orthogonal to the logarithmic derivatives

$$h_j(x, \theta) = \frac{\partial}{\partial \theta_j} \log f(x; \theta, 0), \qquad j = 1, \cdots, s, \tag{1.3}$$

in the sense that

$$E_{\theta,0}\, g(X, \theta) h_j(X, \theta) \equiv 0, \qquad \theta \in \Theta, \quad j = 1, \cdots, s. \tag{1.4}$$

This implies that the test with the critical region $\{|Z_n(\hat{\theta}_n)| > z_{\alpha/2}\}$, where $z_{\alpha/2}$ is defined by $\mathcal{N}(z_{\alpha/2}) = 1 - \alpha/2$, $\mathcal{N}(z)$ being the (0, 1) normal distribution function (d.f.), has the limiting significance level α whatever be the true value of θ. The class of tests of this form was called $C(\alpha)$.

A further result of [6] gives a rule for constructing an asymptotically optimal test of class $C(\alpha)$. Namely, let

$$h_0(x, \theta) = \frac{\partial}{\partial \xi} \log f(x; \theta, \xi)|_{\xi=0} \tag{1.5}$$

and let $g(x, \theta)$ be obtained from $h_0(x, \theta)$ by the orthogonalization process,

$$g(x, \theta) = h_0(x, \theta) - \sum_1^s a_j(\theta) h_j(x, \theta) \tag{1.6}$$

to satisfy (1.4). Then the test of class $C(\alpha)$ with this $g(x, \theta)$ is an asymptotically optimal one.

Several examples of the use of $C(\alpha)$ tests in applied problems are given in [7].

When applying an asymptotic test one always encounters the question of the accuracy of the normal approximation. The standard methods related to the sums of independent random variables are inapplicable for the $C(\alpha)$ test statistics since the use of an estimate instead of θ in (1.2) makes the terms of the sum dependent. In the present paper the correction term of order $n^{-1/2}$ to the normal approximation under the hypothesis H is obtained when $g(X, \theta)$ and some related random variables have densities with respect to the Lebesgue measure. This result is contained in Theorem 2.1 stated in Section 2.

In Section 3, we give two theorems on conditional distributions. Namely, the normalized sum of independent random vectors is considered. Each of the vectors consists of two subvectors and hence so does their sum. The theorems concern the conditional distribution of the first subvector of the sum given the second one. The only paper which we know to deal with the conditional distributions of this kind is that of Steck [8]. In this paper some theorems on the convergence of conditional distributions to the normal have been proved.

In the theorems of Section 3 we restrict ourselves to the case of identically distributed summands with a one dimensional conditioning subvector which is sufficient for the proof of Theorem 2.1. Theorem 3.1 establishes a Lipschitz property for the dependence in variation of the conditional distribution on the value of the conditioning variable.

Theorem 3.2 gives an asymptotic estimate for tail probabilities of conditional distributions.

The proofs of theorems of Sections 2 and 3 are given in Sections 4 to 6. Section 7 contains some concluding remarks.

2. The main theorem

Let the hypothesis H be true and $\theta_0 \in \Theta$ be the true value of θ. Thus we have n independent random variables $X_1, \cdots, X_n$ with a common d.f., $F(x) = F(x; \theta_0, 0)$. (Since the values $\theta = \theta_0$ and $\xi = \xi_0$ will be fixed we shall omit them in our notations.) Denote

$$(2.1)\qquad \begin{aligned} g(x) &= g(x, \theta_0), & g_j(x) &= \left.\frac{\partial g(x, \theta)}{\partial \theta_j}\right|_{\theta=\theta_0}, \\ g_{i,j,\ell}(x, \theta) &= \frac{\partial^2 g(x, \theta)}{\partial \theta_i\, \partial \theta_j}, & g_{i,j}(x) &= g_{i,j}(x, \theta_0), \\ g_{i,j,\ell}(x, \theta) &= \frac{\partial^3 g(x, \theta)}{\partial \theta_i\, \partial \theta_j\, \partial \theta_\ell}; & & i, j, \ell = 1, 2, \cdots, s. \end{aligned}$$

We shall state now the assumptions to be used in the theorem.

ASSUMPTION 1. $Eg(X) = 0, \quad Eg^2(X) = 1, \quad E|g(X)|^3 < \infty.$

ASSUMPTION 2. $Eg_j(X) = 0$ *for* $j = 1, 2, \cdots, s.$

REMARK. The variance of $g(X)$ may be taken equal to 1 by considering $g(x, \theta)/\sigma(\theta)$ instead of $g(x, \theta)$. Under certain regularity conditions Assumption 2 is equivalent to (1.4) in view of the equation

$$(2.2)\qquad 0 = \frac{\partial}{\partial \theta_j} \int g(x, \theta) f(x, \theta)\, dx = \int \frac{\partial g}{\partial \theta_j}\, dF + \int g \frac{\partial \log f}{\partial \theta_j}\, dF.$$

We prefer to use Assumption 2 directly because it does not refer to the dependence of the distribution on θ.

ASSUMPTION 3. $E|g_j(X)|^3 < \infty, \quad j = 1, \cdots, s.$

ASSUMPTION 4. $E|g_{i,j}(X)|^{3/2+\delta} < \infty$ *for some* $\delta > 0; i, j = 1, \cdots, s.$

ASSUMPTION 5. *There exist a neighborhood* $U \subset \Theta$ *of* θ_0 *and a function* $K(x)$ *such that* $|g_{i,j,\ell}(x, \theta)| \leqq K(x)$ *for all* $\theta \in U, i, j, \ell = 1, \cdots, s; E(K(x))^{1+\delta} < \infty$ *for some* $\delta > 0.$

For the convenience of notation, we introduce the following symbol.

DEFINITION 2.1. *Let* $\zeta_1, \zeta_2, \cdots$ *be a sequence of random variables. We shall write* $\zeta_n = \omega(a)$ *if for any* $c > 0$, $P\{|\zeta_n| > cn^a\} = o(n^{-1/2})$ *as* $n \to \infty.$

Now we assume that the estimator $\hat{\theta}_n$ is expressible in the form

$$(2.3)\qquad \sqrt{n}\,(\hat{\theta}_n - \theta_0) = \frac{1}{\sqrt{n}} \sum_1^n h(X_i) + \eta_n$$

where $h(x) = (h_1(x), \cdots, h_s(x)), \eta_n = (\eta_{n,1}, \cdots, \eta_{n,s}).$

ASSUMPTION 6. $Eh_j(X) = 0, \quad E|h_j(X)|^3 < \infty, \quad j = 1, \cdots, s.$

ASSUMPTION 7. $\eta_{n,j} = \omega(-\delta)$ *for some* $\delta > 0; j = 1, \cdots, s.$

Note that δ in Assumptions 4, 5 and 7 need not be the same.

The remaining assumptions concern the joint distribution of the $(2s + 1)$ dimensional random vector $(Y_0, Y) = (Y_0, Y_1, \cdots, Y_{2s})$ where

(2.4) $$Y_0 = g(X), \qquad Y_j = g_j(X), \qquad Y_{s+j} = h_j(X), \qquad j = 1, \cdots, s.$$

Let $\varphi(\tau, t)$ be the characteristic function (c.f.) of (Y_0, Y),

(2.5) $$\varphi(\tau, t) = E \exp\{\tau Y_0 + tY'\}, \qquad \tau \in R^1, t \in R^{2s}.$$

ASSUMPTION 8. *For some* $\gamma > 0$, $\varphi(\tau, t) = O(\|\tau, t\|^{-\gamma})$ *as* $\|\tau, t\| \to \infty$ (*we write* $\|\tau, t\| = (|\tau|^2 + \|t\|^2)^{1/2}$).

ASSUMPTION 9. $E(\Pi_{j=1}^{2s} |Y_j|^{\varepsilon_j}) < \infty$ *for any combination of* $\varepsilon_j = 0$ *or* $1, j = 1, \cdots, 2s$.

ASSUMPTION 10. *Put* $\chi_j(\tau) = E[|Y_j|^3 \exp\{i\tau Y_0\}]$. *There exists an* n_1 *such that the* $\chi_j^{n_1}(\tau)$ *are absolutely integrable on* $R^1, j = 1, \cdots, 2s$.

Denote by B the matrix with elements $b_{i,j} = Eg_{i,j}(X), i, j = 1, \cdots, s$, and by Σ the covariance matrix of (Y_0, Y) with elements

(2.6) $$\sigma_{i,j} = EY_iY_j, \qquad i, j = 0, 1, \cdots, 2s.$$

Note that (Y_0, Y) has zero mean.

Let $(V_0, V_1, \cdots, V_{2s})$ be a normally distributed random vector with zero mean and covariance matrix Σ. Put

(2.7) $$\begin{aligned} S &= (V_1, \cdots, V_s), \qquad & T &= (V_{s+1}, \cdots, V_{2s}), \\ W &= ST' + \tfrac{1}{2}TBT', \qquad & \mu(x) &= E[W \mid V_0 = x]. \end{aligned}$$

Since $\sigma_{0,0} = 1$ (see Assumption 1) we have by the well-known formulae

(2.8) $$E(V_j | V_0 = x) = \sigma_{0,j}x,$$

(2.9) $$\operatorname{Cov}(V_i, V_j | V_0 = x) = \sigma_{i,j} - \sigma_{0,i}\sigma_{0,j}, \qquad i, j = 1, \cdots, 2s.$$

Therefore one can easily derive that

(2.10) $$\begin{aligned} \mu(x) = \sum_{j=1}^{s} \sigma_{j,s+j} + \tfrac{1}{2} \sum_{j,\ell=1}^{s} b_{j\ell}\sigma_{s+j,s+\ell} \\ - (1 - x^2)\left[\sum_{j=1}^{\ell} \sigma_{0,j}\sigma_{0,s+j} + \tfrac{1}{2} \sum_{j,\ell=1}^{s} b_{j,\ell}\sigma_{0,s+j}\sigma_{0,s+\ell}\right]. \end{aligned}$$

Denote by $\hat{\phi}_n(x)$ the d.f. of $Z_n(\hat{\theta}_n)$ where $Z_n(\theta)$ is defined by (1.2). We shall write $\mathcal{N}(x)$ and $n(x)$ for the (0, 1) normal d.f. and its density respectively.

THEOREM 2.1. *Let the Assumptions* 1 *through* 10 *be satisfied. Then*

(2.11) $$\hat{\phi}_n(x) = \phi_n(x) + \varepsilon_n(x)$$

where

(2.12) $$\phi_n(x) = \mathcal{N}(x) + n^{-1/2} n(x)[(\mu_3/6)(1 - x^2) - \mu(x)],$$

with $\mu_3 = EY_0^3$ *and* $n^{1/2}\varepsilon_n(x) \to 0$ *as* $n \to \infty$ *uniformly in* $x \in A$ *for any bounded* $A \in R^1$.

3. Some theorems on conditional distributions

Let $Y_{0,i}, Y_i), Y_i = (Y_{i,1}, \cdots, Y_{i,k}), i = 1, \cdots, n$, be n independently identically distributed random vectors in R^{k+1} and

$$\varphi(\tau, t) = E \exp \{\tau Y_{0,1} + tY_1'\}, \qquad t \in R^k, \tag{3.1}$$

their common c.f. Let

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_{0,i}, \qquad S_n = (S_{n,1}, \cdots, S_{n,k}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i. \tag{3.2}$$

In the following theorem the distribution of $(Y_{0,i}, Y_i)$ will be assumed to satisfy Assumptions 8 and 9 (which should be read in this case with $2s$ replaced by k). The c.f. of (Z_n, S_n) is

$$\varphi^n(n^{-1/2}\tau, n^{-1/2}t) = \varphi_n(\tau, t), \tag{3.3}$$

say. Under Assumption 8, for n sufficiently large, this c.f. is absolutely integrable and (Z_n, S_n) has a joint $(k + 1)$ dimensional density. Denote this density by $p_n(z, x)$ and the marginal density of Z_n by $p_n(z)$. Then

$$p_n(x|z) = \frac{p_n(z, x)}{p_n(z)} \tag{3.4}$$

is the conditional density of S_n given Z_n.

THEOREM 3.1. *Let the Assumptions* 8 *and* 9 (*with k instead of* $2s$) *be satisfied. Then for any bounded $A \subset R^1$ there exist a $K > 0$ and a finite N such that*

$$\int_{R^k} \left| p_n(x|z_1) - p_n(x|z_2) \right| dx \leqq K|z_1 - z_2| \tag{3.5}$$

for all $z_1, z_2 \in A$ and $n \geqq N$.

For the next theorem, consider n independent identically distributed two dimensional random vectors $(X_1, Y_1), \cdots, (X_n, Y_n)$. Denote their joint d.f. by $P(x, y)$ and their marginal distribution function by $P(x)$ and $Q(y)$, respectively. Let

$$Z_n = \frac{1}{\sqrt{n}} \sum_{1}^{n} X_i, \qquad S_n = \frac{1}{\sqrt{n}} \sum_{1}^{n} Y_i. \tag{3.6}$$

Denote

$$M_r = E|Y_1|^r, \qquad M_r(x) = E(|Y_1|^r | X_1 = x). \tag{3.7}$$

For two absolutely integrable functions $f_1(x)$ and $f_2(x)$, denote by $f_1 * f_2(x)$ their convolution,

$$f_1 * f_2(x) = \int f_1(x - v) f_2(v)\, dv, \tag{3.8}$$

and by $f_1^{n*}(x)$ an n fold convolution of $f_1(x)$ with itself.

THEOREM 3.2. *Assume that* (i) $EX_1 = EY_1 = 0$, (ii) $P(x)$ *has a density* $p(x)$, (iii) $M_r < \infty$ *for some* $r > 2$, *and* (iv) *the functions* $p^{v*}(x)$ *and* $[M_r(x)p(x)]^{v*}$ *are bounded for some finite* v. *Then*

$$P\{|S_n| > x | Z_n = z\} = o(x^{-r}n^{1-r/2}) \tag{3.9}$$

as $n \to \infty$, $x/\log n \to \infty$ *uniformly in* $z \in A$ *for any bounded* $A \subset R^1$.

For an unconditional counterpart of this theorem see Lemma 4.2 (iv) below.

4. Proof of Theorem 2.1

We establish first several lemmas. Denote

$$Z_{n,0} = \frac{1}{\sqrt{n}} \sum_1^n g(X_i), \qquad Z_{n,j} = \frac{1}{\sqrt{n}} \sum_1^n g_j(X_i), \qquad Z_{n,s+j} = \frac{1}{\sqrt{n}} \sum_1^n h_j(X_i) \tag{4.1}$$
$$j = 1, \cdots, s,$$

$$S_n = (Z_{n,1}, \cdots, Z_{n,s}), \qquad T_n = (Z_{n,s+1}, \cdots, Z_{n,2s}). \tag{4.2}$$

Let $p_n(z)$ and $p_n(z, x)$, $x \in R^{2s}$, denote the marginal density of $Z_{n,0}$ and the joint density of $(Z_{n,0}, Z_{n,1}, \cdots, Z_{n,2s})$ respectively. Under Assumption 8 they exist for n sufficiently large. Denote by $\varphi_n(\tau, t) = \varphi^n(n^{-1/2}\tau, n^{-1/2}t)$ the c.f. of $(Z_{n,0}, Z_{n,1}, \cdots, Z_{n,2s})$ and by $\varphi_n(\tau) = \varphi_n(\tau, 0)$ the c.f. of $Z_{n,0}$.

LEMMA 4.1. *Under Assumption* 8

(i) $p_n(z) \to \varkappa(z)$ *as* $n \to \infty$ *uniformly in* $z \in R^1$, *and*

(ii) $p_n(z)$ *has a derivative* $p_n'(z)$ *for* n *sufficiently large and* $\limsup_{n\to\infty} \sup_{z\in R^1} |p_n'(z)| < \infty$.

PROOF. For part (i) see, for example, Feller [2], Theorem 2 in Chapter XV.5. Since

$$\sup_z |p_n'(z)| \leqq \frac{1}{2\pi} \int |\tau\varphi_n(\tau)|\, d\tau, \tag{4.3}$$

one can get the proof of (ii) from (5.12), (5.17), (5.18) and (5.19) below. (Actually, $p_n'(z)$ converges to $\varkappa'(z)$ but we state in the lemma only what we need in the proof of the theorem.)

LEMMA 4.2. *Let* $Y_1, \cdots, Y_n$ *be independent identically distributed random variables and* $S_n = n^{-1/2} \sum_1^n Y_i$. *Assume that* $E|Y_i|^r < \infty$ *for some* $r > 0$. *Then*

$$P\{|S_n| > x\} = o(x^{-r}n^{1-r/2}) \qquad as\ n \to \infty, \tag{4.4}$$

provided one of the following conditions is satisfied: (i) $0 < r < 1$, (ii) $1 \leqq r < 2$, $EY_1 = 0$, (iii) $r = 2$, $EY_1 = 0$, $x \to \infty$, (iv) $r > 2$, $EY_1 = 0$, $x/\log n \to \infty$.

PROOF. For the parts (i) and (ii) see Binmore and Stratton [1] (note that $E|Y_1|^r < \infty$ implies $P\{|Y_1| > x\} = o(x^{-r})$ as $x \to \infty$). Let $F_n(x)$ be the d.f. of S_n. Part (iii) follows from the inequality

$$x^2 P\{|S_n| > x\} \leqq \int_{|y| \geqq x} y^2\, dF_n(y) \tag{4.5}$$

and the uniform integrability of y^2 in $F_n(y)$ (see Loève [4], Theorem 11.4.A (iii)). Part (iv) follows from Theorem 1 of Nagaev [5]. This theorem actually provides an inequality which implies (4.4) only with O instead of o. However we shall indicate below (see 6.25) and the subsequent paragraph) a modification of Nagaev's proof which gives o in (4.4).

In terms of the symbol ω (Definition 2.1) we have,

COROLLARY 4.1. *Under the conditions of Lemma* 4.2

$$\text{(i) } S_n = \omega\left(\frac{3-r}{2r}\right) \qquad \textit{if } r < 3,$$

$$\text{(ii) } S_n = \omega(\varepsilon) \qquad \textit{for any } \varepsilon > 0 \textit{ if } r \geqq 3.$$

With the notation (4.1), (4.2), let

$$\begin{aligned} W_n &= S_n T_n' + \tfrac{1}{2} T_n B T_n', \qquad & \mu_n(x) &= E[W_n | Z_{n,0} = x], \\ P\{W_n < x\} &= G_n(x), & P\{W_n < x | Z_{n,0} = z\} &= G_n(x|z). \end{aligned} \tag{4.6}$$

Denote $G(x|z) = P\{W < x | V_0 = z\}$ (see (2.7)).

LEMMA 4.3. *Let Assumptions* 8 *and* 9 *be satisfied. Then, for any bounded* $A \subset R^1$

(i) *there exist a* $K > 0$ *and a finite* N *such that*

$$\sup_x |G_n(x|z_1) - G_n(x|z_2)| \leqq K|z_1 - z_2| \tag{4.7}$$

for all $z_1, z_2 \in A$ *and* $n \geqq N$;

(ii) $\sup_{x \in R^1} |G_n(x|z) - G(x|z)| \to 0$ *uniformly in* $z \in A$;

(iii) $\mu_n(z) \to \mu(z)$ *uniformly in* $z \in A$.

PROOF. Since the inequality $\{W_n < x\}$ determines a Borel set in the sample space of (S_n, T_n), assertion (i) follows from Theorem 3.1. The convergence in assertion (ii) for any fixed z follows directly from Theorem 2.4 of Steck [8]; together with (i), this implies the asserted uniform convergence. Otherwise one could find an $\varepsilon > 0$, a subsequence $\{m\} \subset \{n\}$ and a sequence $\{z_m\}$ approaching a finite limit z_0, say, such that

$$\sup_x |G_m(x|z_m) - G(x|z_m)| > \varepsilon \qquad \text{for all } m. \tag{4.8}$$

Then $\sup_x |G_m(x|z_m) - G_m(x|z_0)|$ would not tend to zero which would contradict (i).

Denote by $a(z)$ and $\Sigma(z)$ the conditional mean and covariance matrix of $(V_1, \cdots, V_{2s})$ given $V_0 = z$ (see (2.7) above) and by $a_n(z)$ and $\Sigma_n(z)$ those of $(Z_{n,1}, \cdots, Z_{n,2s})$ given $Z_{n,0} = z$. Then (iii) follows from the convergence

$$a_n(z) \to a(z), \qquad \Sigma_n(z) \to \Sigma(z) \qquad \text{as } n \to \infty \tag{4.9}$$

uniformly in $z \in A$. This latter convergence can be proved by taking the first and second derivatives of

$$\omega_n(t, z) = \frac{\int e^{-i\tau z}\varphi_n(\tau, t)\, d\tau}{\int e^{-i\tau z}\varphi_n(\tau, 0)\, d\tau}, \tag{4.10}$$

the conditional c.f. of (S_n, T_n), given $Z_{n,0} = z$, at $t = 0$. The technique is quite similar to that used in [8]. It is easily verified that the derivatives of $\varphi_n(\tau, t) = \varphi^n(n^{-1/2}\tau, n^{-1/2}t)$ at $t = 0$ and any fixed τ converge to the corresponding derivatives of the limiting normal c.f. Then the passage to the limit under the integral sign which is justified by the bounded convergence theorem in the same way as in [8] leads to (4.9).

REMARK. The assertions (ii) and (iii) are valid actually under the conditions of Theorem 2.4 in [8] which are weaker than those used here. The proof of (iii) sketched above remains valid in this case; the proof of (ii) requires some standard but rather cumbersome technique.

The following lemma states some properties of the symbol ω (Definition 2.1). The proof is obvious and will be omitted.

LEMMA 4.4.

(i) *If* $\zeta_n = \omega(a)$ *then* $\zeta_n = \omega(a')$ *for any* $a' > a$.

(ii) *If* $\zeta_n = \omega(a)$ *and* $\eta_n = \omega(b)$ *then* $\zeta_n + \eta_n = \omega(\max(a, b))$.

(iii) *If* $\zeta_n = \omega(a)$ *and* $\eta_n = \omega(b)$ *then* $\zeta_n\eta_n = \omega(a + b)$.

Now we proceed to the proof of the theorem. For notational convenience, let $\theta_0 = 0$. Expanding $g(X_i, \hat{\theta}_n)$ by the Taylor formula, we have

$$\begin{aligned} Z_n(\hat{\theta}_n) = {} & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g(X_i) + \frac{1}{\sqrt{n}} \sum_{j=1}^{s} \hat{\theta}_{n,j} \sum_{i=1}^{n} g_j(X_i) \\ & + \frac{1}{2\sqrt{n}} \sum_{j,\ell} \hat{\theta}_{n,j}\hat{\theta}_{n,\ell} \sum_{i=1}^{n} g_{j,\ell}(X_i) + \frac{1}{6\sqrt{n}} \sum_{j,k,\ell} \hat{\theta}_{n,j}\hat{\theta}_{n,k}\hat{\theta}_{n,\ell} \sum_{i=1}^{n} g_{j,k,\ell}(X_i, t_{n,i}\hat{\theta}_n), \end{aligned} \tag{4.11}$$

where $0 \leqq t_{n,i} \leqq 1$. Denote by B_n the matrix with elements

$$b_{n,j,\ell} = \frac{1}{n} \sum_{i=1}^{n} g_{j,\ell}(X_i), \qquad j, \ell = 1, \cdots, s. \tag{4.12}$$

Then using also (4.6) we can write

$$Z_n(\hat{\theta}_n) = Z_{n,0} + n^{-1/2} W_n + n^{-1/2} R_n \tag{4.13}$$

where

$$\begin{aligned} R_n = {} & \eta_n S_n' + \tfrac{1}{2}(T_n + \eta_n)(B_n - B)(T_n + \eta_n)' + (T_n B \eta_n' + \tfrac{1}{2}\eta_n B \eta_n') \\ & + \tfrac{1}{6} \sum_{j,k,\ell} \hat{\theta}_{n,j}\hat{\theta}_{n,k}\hat{\theta}_{n,\ell} \sum_{i=1}^{n} g_{j,k,\ell}(X_i, t_{n,i}\hat{\theta}_n) = R_{n,1} + \tfrac{1}{2}R_{n,2} + R_{n,3} + \tfrac{1}{6}R_{n,4}, \end{aligned} \tag{4.14}$$

say. We shall show now that $R_n = \omega(0)$. By Lemma 5.2 (ii), it is sufficient to show that $R_{n,i} = \omega(0)$, $i = 1, \cdots, 4$.

By Assumptions 4, 5 and 7 we can find an α, $0 < \alpha < \frac{1}{3}$, such that

$$E|g_{i,j}(X)|^{3/2(1-\alpha)} < \infty, \qquad E\big(K(X)\big)^{1/(1-\alpha)} < \infty, \qquad \eta_{n,j} = \omega\left(-\frac{\alpha}{2}\right). \tag{4.15}$$

We have $R_{n,1} = \Sigma_{j=1}^{s} \eta_{n,j} Z_{n,j}$. By Assumption 3 and Corollary 4.1 (ii), we have $Z_{n,j} = \omega(\alpha/2)$. Therefore, $\eta_{n,j} Z_{n,j} = \omega(0)$ by Lemma 4.4, and $R_{n,1} = \omega(0)$.

Consider a term $R_{n,2}^{j,\ell} = (b_{n,j,\ell} - b_{j,\ell})(T_{n,j} + \eta_{n,j})(T_{n,\ell} + \eta_{n,\ell})$ of $R_{n,2}$. By Assumption 4 and Corollary 4.1 (i)

$$b_{n,j,\ell} - b_{j,\ell} = n^{-1/2} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (g_{j,\ell}(X_i) - b_{j,\ell}) = \omega\left(-\frac{1}{2} + \frac{3-r}{2r}\right) \tag{4.16}$$

with $r = 3/2(1-\alpha)$ (see (4.15)). Thus $b_{n,j,\ell} - b_{j,\ell} = \omega(-\alpha)$. By Assumption 6 $T_{n,j} = \omega(\alpha/2)$ and by Lemma 4.4 $T_{n,j} + \eta_{n,j} = \omega(\alpha/2)$. $R_{n,2}^{j,\ell} = \omega(0)$, and $R_{n,2} = \omega(0)$.

In a similar way we obtain $R_{n,3} = \omega(0)$.

Now consider a term

$$R_{n,4}^{j,k,\ell} = \hat{\theta}_{n,j}\hat{\theta}_{n,k}\hat{\theta}_{n,\ell} \sum_{i=1}^{n} g_{j,k,\ell}(X_i, t_{n,i}\hat{\theta}_n) \tag{4.17}$$

of $R_{n,4}$. We have $\hat{\theta}_{n,j} = n^{-1/2}(T_{n,j} + \eta_{n,j}) = \omega\big((\alpha - 1)/2\big)$ and, by Lemma 4.4, $|\hat{\theta}_n| = \omega\big((\alpha-1)/2\big)$. Take a $\delta > 0$ such that $\{\theta : |\theta| \leqq \delta\} \subset U$ (see Assumption 5). Then

$$\begin{aligned} P\{|R_{n,4}^{j,k,\ell}| > c\} &\leqq P\{|R_{n,4}^{j,k,\ell}| > c, \hat{\theta}_n \in U\} + P\{|\hat{\theta}_n| > \delta\} \\ &\leqq P\{\sqrt{n}|\hat{\theta}_{n,j}\hat{\theta}_{n,k}\hat{\theta}_{n,\ell}| n^{-1/2} \sum_{i=1}^{n} K(X_i) > c\} + o(n^{-1/2}). \end{aligned} \tag{4.18}$$

We have

$$\sqrt{n}|\hat{\theta}_{n,j}\hat{\theta}_{n,k}\hat{\theta}_{n,\ell}| = \omega\left(\tfrac{1}{2} + 3\,\frac{\alpha - 1}{2}\right) = \omega\left(\frac{3\alpha}{2} - 1\right). \tag{4.19}$$

Hence it remains to show that

$$n^{-1/2} \sum_{i=1}^{n} K(X_i) = \omega\left(1 - \frac{3\alpha}{2}\right). \tag{4.20}$$

Denote $\kappa = EK(X)$. By Assumption 5 (see (4.15)) and Corollary 4.1 (i),

$$n^{-1/2} \sum_i K(X_i) - \kappa\sqrt{n} = n^{-1/2} \sum_i \big(K(X_i) - \kappa\big) = \omega\left(1 - \frac{3\alpha}{2}\right). \tag{4.21}$$

Since α was chosen to be less than 1/3, $\kappa\sqrt{n} = \omega(1 - 3\alpha/2)$. Therefore (4.21) implies (4.20).

We shall show now that the term $n^{-1/2}R_n$ in (4.13) may be neglected. Denote

$$Z_n^* = Z_{n,0} + n^{-1/2} W_n, \qquad \phi_n^*(x) = P\{Z_n^* < x\}. \tag{4.22}$$

Let (2.11) hold for $\phi_n^*(x)$. Then for an arbitrary $\delta > 0$,

$$(4.23)\qquad \begin{aligned}\hat{\phi}_n(x) &= P\{Z_n^* + n^{-1/2}R_n < x\}\\ &\leqq P\{Z_n^* < x + n^{-1/2}\delta\} + P\{|R_n| > \delta\}\\ &= \phi_n(x + n^{-1/2}\delta) + \varepsilon_n(x + n^{-1/2}\delta) + o(n^{-1/2}).\end{aligned}$$

A similar estimate from below gives

$$(4.24)\qquad \hat{\phi}_n(x) \geqq \phi_n(x - n^{-1/2}\delta) + \varepsilon_n(x - n^{-1/2}\delta) + o(n^{-1/2}).$$

The function $\phi_n(x)$ has a derivative bounded uniformly in n and x, $\phi_n'(x) < C$, say. Therefore

$$(4.25)\qquad |\hat{\phi}_n(x) - \phi_n(x)| \leqq n^{-1/2}C\delta + \bar{\varepsilon}_n(x) + o(n^{-1/2})$$

where $\bar{\varepsilon}_n(x) = \sup\,[|\varepsilon_n(x + u)|;\, |u| \leqq \delta]$. Since $\delta > 0$ is arbitrary, (4.25) implies the assertion of Theorem 2.1 for $\hat{\phi}_n(x)$.

Writing $P_n(x)$ for the d.f. of $Z_{n,0}$ we have from (4.6) and (4.22)

$$(4.26)\qquad \phi_n^*(x) = \int G_n\big((x - z)\sqrt{n}\,|z\big)\, dP_n(z).$$

The next step of the proof will be to show that $\phi_n^*(x)$ may be replaced by

$$(4.27)\qquad \phi_n^{**}(x) = \int G_n\big((x - z)\sqrt{n}\,|x\big)\, dP_n(z),$$

that is, for any $a > 0$

$$(4.28)\qquad \sup_{x\in[-a,a]} |\phi_n^*(x) - \phi_n^{**}(x)| = o(n^{-1/2}) \qquad \text{as } n \to \infty.$$

Put $\delta = n^{-3/8}$ and write the difference of (4.26) and (4.27) as

$$(4.29)\qquad \begin{aligned}&\phi_n^*(x) - \phi_n^{**}(x)\\ &= \left(\int_{x-\delta}^{x+\delta} + \int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty}\right)\big[G_n\big((x - z)\sqrt{n}\,|z\big) - G_n\big((x - z)\sqrt{n}\,|x\big)\big]\, dP_n(z)\\ &= I_1(x) + I_2(x) + I_3(x),\end{aligned}$$

say. Applying Lemma 4.3 (i) with $A = [-a - 1, a + 1]$ and using the fact that $p_n(z)$ is bounded (Lemma 4.1 (i)), we obtain

$$(4.30)\qquad \max_{x\in[-a,a]} |I_1(x)| \leqq K \max p_n(x) \int_{x-\delta}^{x+\delta} |x - z|\, dz = O(\delta^2) = o(n^{-1/2}).$$

Further, $|I_2(x)| \leqq I_{2,1}(x) + I_{2,2}(x)$ where

$$(4.31)\qquad I_{2,1}(x) = \int_{-\infty}^{x-\delta} \big[1 - G_n\big((x - z)\sqrt{n}\,|x\big)\big]\, p_n(z)\, dz,$$

$$(4.32)\qquad I_{2,2}(x) = \int_{-\infty}^{x-\delta} \big[1 - G_n\big((x - z)\sqrt{n}\,|z\big)\big]\, p_n(z)\, dz.$$

Using the inequality

$$(4.33)\qquad 1 - G_n\big((x - z)\sqrt{n}|y\big) \leqq P\{W_n > n^{1/8}|Z_{n,0} = y\} \qquad \text{for } x - z > \delta$$

with $y = x$, we obtain

$$(4.34)\qquad I_{2,1}(x) \leqq P\{W_n > n^{1/8}|Z_{n,0} = x\}.$$

Assumptions 3, 6, 8, and 10 assure the fulfillment of the conditions of Theorem 3.2 for the vectors $(g_j(X_i), g(X_i))$ and $(h_j(X_i), g(X_i)), j = 1, \cdots, s$, with $r = 3$. (One should only note that $\chi_j(\tau)$ in Assumption 10 is the Fourier transform of $M_3(x)p(x)$, and the integrability of $\chi_j^{n_1}(\tau)$ implies the boundedness of $[M_3(y)p(y)]^{n_1*}$.) Therefore

$$(4.35)\quad P\{|Z_{n,j}| > n^{1/16}|Z_{n,0} = x\} = o(n^{-1/2+3/16)}) = o(n^{-1/2}), \quad j = 1, \cdots, 2s.$$

(In relations of this kind we mean that $o(n^{-1/2})$ is uniform in $x \in [-a, a]$ without stating it explicitly.) Now we obtain from the definition of W_n (see (4.6)) and Lemma 4.4 (applied to conditional probabilities) that

$$(4.36)\qquad P\{|W_n| > n^{1/8}|Z_{n,0} = x\} = o(n^{-1/2}).$$

In view of (4.34) this implies

$$(4.37)\qquad I_{2,1}(x) = o(n^{-1/2}).$$

Using (4.33) with $y = z$, we have

$$(4.38)\qquad I_{2,2}(x) \leqq \int_{-\infty}^{x-\delta} P\{W_n > n^{1/8}|Z_{n,0} = z\}\, dP_n(z)$$

$$\leqq \int_{-\infty}^{\infty} P\{W_n > n^{1/8}|Z_{n,0} = z\}\, dP_n(z) = P\{W_n > n^{1/8}\}.$$

This probability is estimated in the same way as (4.34) but Lemma 4.2 is used, rather than Theorem 3.2, which gives

$$(4.39)\qquad I_{2,2}(x) = o(n^{-1/2}).$$

The relations (4.29), (4.30), (4.37) and (4.39) together with a similar estimate for $I_3(x)$ prove (4.28).

Thus we are to prove the theorem for $\phi_n^{**}(x)$ defined by (4.27). Rewrite it in the form

$$(4.40)\qquad \phi_n^{**}(x) = \int P_n(x - z)\, dG_n(z\sqrt{n}|x).$$

Using (4.36) we have

$$(4.41)\quad \int_{|z| > n^{-3/8}} P_n(x - z)\, dG_n(z\sqrt{n}|x) \leqq P\{W_n > n^{1/8}|Z_{n,0} = x\} = o(n^{-1/2}).$$

Therefore (writing again $\delta = n^{-3/8}$)

$$(4.42)\qquad \phi_n^{**}(x) = \int_{-\delta}^{\delta} P_n(x - z)\, dG_n(z\sqrt{n}|x) + o(n^{-1/2}).$$

By Taylor's formula we obtain

$$(4.43)\quad \begin{aligned}\phi_n^{**}(x) = {} & \int_{-\delta}^{\delta} P_n(x - n^{-1/2}\mu_n(x))\, dG_n(z\sqrt{n}\,|\,x) \\ & + \int_{-\delta}^{\delta} (n^{-1/2}\mu_n(x) - z)\, p_n(x - n^{-1/2}\mu_n(x))\, dG_n(z\sqrt{n}\,|\,x) \\ & + \int_{-\delta}^{\delta} (n^{-1/2}\mu_n(x) - z)[p_n(x - z^*) - p_n(x - n^{-1/2}\mu_n(x))]\, dG_n(z\sqrt{n}\,|\,x) \\ & + o(n^{-1/2}) \\ = {} & J_1(x) + J_2(x) + J_3(x) + o(n^{-1/2}),\end{aligned}$$

say, where z^* lies between z and $n^{-1/2}\mu_n(x)$. By virtue of (4.36)

$$(4.44)\quad \begin{aligned}J_1(x) &= P_n(x - n^{-1/2}\mu_n(x))[1 - P\{|W_n| > n^{1/8}\,|\,Z_{n,0} = x\}] \\ &= P_n(x - n^{-1/2}\mu_n(x)) + o(n^{-1/2}).\end{aligned}$$

Now we shall show that $J_2(x)$ and $J_3(x)$ are $o(n^{-1/2})$. First, by Theorem 11.4.A (iii) of Loève [4], the assertions (ii) and (iii) of Lemma 4.3 imply

$$(4.45)\quad \int_{|z|>\delta} z\, dG_n(z\sqrt{n}\,|\,x) = n^{-1/2}\int_{|y|>n^{1/8}} y\, dG_n(y\,|\,x) = o(n^{-1/2}).$$

Up to the factor $p_n(x - n^{-1/2}\mu_n(x))$, which is bounded by Lemma 4.1 (i), $J_2(x)$ is equal to

$$(4.46)\quad \begin{aligned}& n^{-1/2}\mu_n(x)[1 - \int_{|z|>\delta} dG_n(z\sqrt{n}\,|\,x)] \\ & \qquad - [n^{-1/2}\mu_n(x) - \int_{|z|>\delta} z\, dG_n(z\sqrt{n}\,|\,x)],\end{aligned}$$

and by (4.36) and (4.45) we get $J_2(x) = o(n^{-1/2})$.

Finally,

$$(4.47)\quad \begin{aligned}n^{-1/2}\mu_n(x) - z &= O(n^{-3/8}), \\ n^{-1/2}\mu_n(x) - z^* &= O(n^{-3/8}) \text{ for } z \in [-\delta, \delta],\end{aligned}$$

and making use of Lemma 4.1 (ii) we obtain $J_3(x) = O(n^{-3/4}) = o(n^{-1/2})$.

Thus

$$(4.48)\quad \phi_n^{**}(x) = P_n(x - n^{-1/2}\mu_n(x)) + o(n^{-1/2}),$$

or since $\mu_n(x) \to \mu(x)$ and $p_n(x)$ is bounded,

$$(4.49)\quad \phi_n^{**}(x) = P_n(x - n^{-1/2}\mu(x)) + o(n^{-1/2}).$$

By virtue of the well known expansion

$$(4.50)\quad P_n(x) = \mathcal{N}(x) + n^{-1/2}\tfrac{1}{6}\mu_3(1 - x^2)n(x) + o(n^{-1/2})$$

(see, for example, [2], Ch. XVI, § 4), the assertion of the theorem follows from (4.49).

5. Proof of Theorem 3.1

We have

$$\int_{R^k} \left| p_n(x|z_1) - p_n(x|z_2) \right| dx = \int_{R^k} \left| \frac{p_n(z_1, x)}{p_n(z_1)} - \frac{p_n(z_2, x)}{p_n(z_2)} \right| dx \leqq I_1 + I_2, \tag{5.1}$$

where

$$I_1 = \int_{R^k} \frac{p_n(z_1, x)|p_n(z_2) - p_n(z_1)|}{p_n(z_1)\, p_n(z_2)} dx = \frac{|p_n(z_2) - p_n(z_1)|}{p_n(z_2)},$$

$$\begin{aligned} I_2 &= \int_{R^k} \frac{|p_n(z_1, x) - p_n(z_2, x)| p_n(z_1)}{p_n(z_1)\, p_n(z_2)} dx \\ &= \frac{1}{p_n(z_2)} \int_{R^k} |p_n(z_1, x) - p_n(z_2, x)|\, dx = \frac{I}{p_n(z_2)}, \end{aligned} \tag{5.2}$$

say. By Lemma 4.1, $|p_n(z_2) - p_n(z_1)| \leqq C|z_2 - z_1|$ and $p_n(z_2)$ is bounded away from zero for $z_2 \in A$ and n sufficiently large. Thus we need only to obtain an inequality for I similar to (3.5). This will be based on the following lemma, which is an immediate multidimensional extension of Lemma 1.5.1 from Ibragimov and Linnik [3].

LEMMA 5.1. *Let a function $f(x)$, $x \in R^k$, be absolutely integrable in R^k, with Fourier transform*

$$\psi(t) = \int_{R^k} e^{itx'} f(x)\, dx, \qquad t \in R^k, \tag{5.3}$$

which has derivatives

$$\mathscr{D}_{\varepsilon_1, \cdots, \varepsilon_k} \psi(t) = \frac{\partial^{\varepsilon_1 + \cdots + \varepsilon_k}}{\partial t_1^{\varepsilon_1} \cdots \partial t_k^{\varepsilon_k}} \psi(t), \qquad \varepsilon_j = 0 \text{ or } 1, j = 1, \cdots, k, \tag{5.4}$$

with these derivatives (including $\mathscr{D}_{0, \cdots, 0}\, \psi(t) = \psi(t)$) being square integrable in R^k. Then

$$\int_{R^k} |f(x)|\, dx \leqq \frac{1}{2^{k/2}} \left(\sum \int_{R^k} |\mathscr{D}_{\varepsilon_1, \cdots, \varepsilon_k}\, \psi(t)|^2\, dt \right)^{1/2}, \tag{5.5}$$

where the summation is over all possible combinations of $\varepsilon_1, \cdots, \varepsilon_k = 0$ or 1.

Now let $\psi_n(z, \cdot)$ denote the Fourier transform of $p_n(z, \cdot)$. We have

$$\int e^{i\tau z}\, \psi_n(z, t)\, dz = \int e^{i\tau z} \left(\int_{R^k} e^{itx'}\, p_n(z, x)\, dx \right) dz = \varphi_n(\tau, t). \tag{5.6}$$

Therefore, for sufficiently large n,

$$\psi_n(z, t) = \frac{1}{2\pi} \int e^{-i\tau z}\, \varphi_n(\tau, t)\, d\tau, \tag{5.7}$$

the application of the inversion formula being justified by Assumption 8. Let $\mathscr{D}$ stand for one of the operators $\mathscr{D}_{\varepsilon_1, \cdots, \varepsilon_k}$ from (5.4). Then

$$\mathscr{D}\psi_n(z, t) = \frac{1}{2\pi} \int e^{-i\tau z} \mathscr{D}\varphi_n(\tau, t)\, d\tau. \tag{5.8}$$

(The differentiation under the integral sign is justified by Assumption 9 and relations (5.23), (5.24) below.) Furthermore, (5.8) implies

$$|\mathscr{D}(\psi_n(z_1, t) - \psi_n(z_2, t))| \leqq \frac{|z_1 - z_2|}{2\pi} \int |\tau \mathscr{D}\varphi_n(\tau, t)|\, d\tau, \tag{5.9}$$

and in order to obtain the required inequality for I we need by Lemma 5.1 to show that

$$\int_{R^k} a^2_{\varepsilon_1, \cdots, \varepsilon_k}(t)\, dt \leqq K \qquad \text{for all } \varepsilon_1, \cdots, \varepsilon_k = 0 \text{ or } 1, \tag{5.10}$$

where

$$a_{\varepsilon_1, \cdots, \varepsilon_k}(t) = \int |\tau \mathscr{D}_{\varepsilon_1, \cdots, \varepsilon_k} \varphi_n(\tau, t)|\, d\tau \tag{5.11}$$

and K is a constant which does not depend on n, τ and t.

Consider first $a(t) = a_{0, \cdots, 0}(t)$,

$$a(t) = \int |\tau \varphi_n(\tau, t)|\, d\tau. \tag{5.12}$$

In view of (3.3) we need an estimate for $\varphi(\tau, t)$. By Assumption 8 we can find $C > 0$ and $B > 0$ such that

$$|\varphi(\tau, t)| \leqq C\|\tau, t\|^{-\gamma} \qquad \text{for } \|\tau, t\| \geqq B. \tag{5.13}$$

We take B large enough to satisfy the inequality

$$CB^{-\gamma} < 1. \tag{5.14}$$

Furthermore, it follows from Assumption 8 that Σ, the covariance matrix of $(Y_{0,1}, Y_1)$, is nondegenerate (otherwise there would exist $(\tau_0, t_0) \neq (0, 0)$ such that $E(\tau_0 Y_{0,1} + t_0 Y_1')^2 = 0$ and $\varphi(u\tau_0, ut_0) \equiv 1$, $-\infty < u < \infty$). Since $E(Y_{0,1}, Y_1) = 0$, one can find $\lambda > 0$ and $\delta > 0$ such that

$$|\varphi(\tau, t)| \leqq e^{-\lambda\|\tau, t\|^2} \qquad \text{for } \|\tau, t\| < \delta. \tag{5.15}$$

Moreover, $\sup_{\|\tau, t\| \geqq \delta} |\varphi(\tau, t)| < 1$. Therefore, reducing λ if necessary, we get

$$|\varphi(\tau, t)| \leqq e^{-\lambda\|\tau, t\|^2} \qquad \text{for } \|\tau, t\| \leqq B. \tag{5.16}$$

Setting $\tau(t) = \max[(B^2 n - \|t\|^2)^{1/2}, 0]$, write $a(t)$ as

$$a(t) = \left(\int_{|\tau| \leqq \tau(t)} + \int_{|\tau| > \tau(t)}\right) |\tau\varphi_n(\tau, t)|\, d\tau = a^{(1)}(t) + a^{(2)}(t), \tag{5.17}$$

say. Then we have from (3.3), (5.13) and (5.16)

$$a^{(1)}(t) \leqq 2 \int_0^{\tau(t)} \tau e^{-\lambda\|\tau, t\|^2}\, d\tau \leqq 2 \int_0^{\infty} \tau e^{-\lambda\|\tau, t\|^2}\, d\tau, \tag{5.18}$$

$$\begin{aligned} a^{(2)}(t) &\leqq 2 \int_{\tau(t)}^{\infty} \tau \frac{C^n n^{n\gamma/2}}{(\tau^2 + \|t\|^2)^{n\gamma/2}}\, d\tau = \frac{2C^n n^{n\gamma/2}}{(n\gamma - 2)[\tau^2(t) + \|t\|^2]^{(n\gamma/2)-1}} \\ &= \frac{2C^n n}{(n\gamma - 2)B^{n\gamma-2}} \quad \text{for } \|t\|^2 \leqq B^2 n, \\ &= \frac{2C^n n^{n\gamma/2}}{(n\gamma - 2)\|t\|^{n\gamma-2}} \quad \text{for } \|t\|^2 \geqq B^2 n. \end{aligned} \tag{5.19}$$

We can estimate the integrals $\int [a^{(i)}(t)]^2\, dt$, $i = 1, 2$. From (5.18),

$$\begin{aligned} &\int [a^{(1)}(t)]^2\, dt \\ &\leqq 4 \int_{R^k} \int_0^{\infty} \int_0^{\infty} \tau_1 \tau_2 \exp\{-\lambda(\|\tau_1, t\|^2 + \|\tau_2, t\|^2)\}\, d\tau_1\, d\tau_2\, dt < \infty. \end{aligned} \tag{5.20}$$

Let $V_n(B)$ denote the volume of the k dimensional sphere $\|t\|^2 \leqq B^2 n$; then for B fixed, $V_n(B) = O(n^{k/2})$. Hence we obtain from (5.19)

$$\begin{aligned} \int [a^{(2)}(t)]^2\, dt &\leqq \frac{4C^{2n} n^2}{(n\gamma - 2)^2 B^{2n\gamma-4}} V_n(B) + \frac{4C^{2n} n^{n\gamma}}{(n\gamma - 2)^2} \int_{\|t\| \geqq B n^{1/2}} \frac{dt}{\|t\|^{2n\gamma-4}} \\ &= O\big(n^{k/2} (CB^{-\gamma})^{2n}\big) \to 0 \end{aligned} \tag{5.21}$$

in view of (5.14). Now (5.17), (5.20) and (5.21) imply (5.10) for $a(t) = a_{0,\cdots,0}(t)$.

Consider now the general case of (5.10). Suppose without loss of generality that $\varepsilon_1 = \cdots = \varepsilon_\ell = 1$, $\varepsilon_{\ell+1} = \cdots = \varepsilon_k = 0$, $0 < \ell \leqq k$, that is, that $\mathscr{D}_{\varepsilon_1, \cdots, \varepsilon_k} = \partial^\ell / \partial t_1 \cdots \partial t_\ell$ in (5.11). Let $T = \{j_1, \cdots, j_r\}$ be a subset of $\{1, \cdots, \ell\}$. Denote

$$\varphi_T(\tau, t) = \frac{\partial^r}{\partial t_{j_1} \cdots \partial t_{j_r}} \varphi(\tau, t). \tag{5.22}$$

As is easily seen, $(\partial^\ell / \partial t_1 \cdots \partial t_\ell) \varphi_n(\tau, t)$ is the sum of the following terms

$$\begin{aligned} n(n-1) \cdots (n - m + 1)\, \varphi^{n-m}\left(\frac{\tau}{\sqrt{n}}, \frac{t}{\sqrt{n}}\right) \varphi_{T_1}\left(\frac{\tau}{\sqrt{n}}, \frac{t}{\sqrt{n}}\right) \\ \cdots \varphi_{T_m}\left(\frac{\tau}{\sqrt{n}}, \frac{t}{\sqrt{n}}\right) n^{-k/2} \end{aligned} \tag{5.23}$$

where $T_1, \cdots, T_m$ is a partition of the set $\{1, \cdots, \ell\}$ into nonempty disjoint subsets, and the summation is over all possible partitions. Every term (5.23) is estimated as in the case of $a(t)$ above, and we shall only indicate the distinctions which arise.

Split the integral of τ times (5.23) into two parts as in (5.17) and call them again $a^{(1)}(t)$ and $a^{(2)}(t)$. We have for $T = \{j_1, \cdots, j_r\}$

$$|\varphi_T(\tau, t)| \leqq E|Y_{1,j_1} \cdots Y_{1,j_r}| = M_T, \tag{5.24}$$

say, which is finite by Assumption 9. Hence the modulus of (5.23) is bounded from above by

$$n^{m-k/2} M_{T_1} \cdots M_{T_m} \left|\varphi\left(\frac{\tau}{\sqrt{n}}, \frac{t}{\sqrt{n}}\right)\right|^{n-m}. \tag{5.25}$$

This enables us to estimate $a^{(2)}(t)$ just as above. The only difference from (5.21) occurs in the factor of order of $n^{m-k/2}$. This does not matter in the presence of a geometric series term.

Concerning $a^{(1)}(t)$, (5.25) is sufficient when $m < k/2$. Consider the case $m > k/2$. We have

$$\frac{\partial}{\partial t_j} \varphi(0, t)\bigg|_{t=0} = iEY_{1,j} = 0, \qquad j = 1, \cdots, k, \tag{5.26}$$

and, writing for a moment t_0 instead of τ,

$$\left|\frac{\partial^2}{\partial t_i\, \partial t_j} \varphi(t_0, t)\right| \leqq E|Y_{1,i} Y_{1,j}|, \qquad i, j = 0, 1, \cdots, k. \tag{5.27}$$

Hence we can find a constant L such that, for all τ, t

$$\left|\frac{\partial}{\partial t_j} \varphi(\tau, t)\right| \leqq L\|\tau, t\|, \qquad j = 1, \cdots, k. \tag{5.28}$$

Note now that there are at least $2m - k$ sets among $T_1, \cdots, T_m$ containing just one element. In fact, if r is the number of such sets, then the remaining $m - r$ sets contain not less than $2(m - r)$ elements, that is, $k - r \geqq 2(m - r)$ whence $r \geqq 2m - k$. Suppose, to be definite, that $T_1, \cdots, T_{2m-k}$ contain one element each. Then by (5.28)

$$\left|\varphi_{T_j}\left(\frac{\tau}{\sqrt{n}}, \frac{t}{\sqrt{n}}\right)\right| \leqq L\|\tau, t\|n^{-1/2}, \quad j = 1, \cdots, 2m - k. \tag{5.29}$$

Applying (5.24) to the remaining φ_T in (5.23), we obtain for (5.23), up to a constant factor, an upper bound $\|\tau, t\|^{2m-k} \varphi^{n-m}(\tau n^{-1/2}, tn^{-1/2})$. Therefore, proceeding as in (5.16), (5.18) and (5.20), we arrive at an inequality whose right side differs from that of (5.20) by some power of $\|\tau_1, t\|\ \|\tau_2, t\|$ under the integral sign, which does not affect the convergence of the integral. The proof is thus completed.

6. Proof of Theorem 3.2

Without loss of generality assume $\operatorname{Var} X_1 = \operatorname{Var} Y_1 = 1$. Put $u = xn^{1/2}$. All limits will be taken as $n \to \infty$, $x/\log n \to \infty$ (or, equivalently, as $n \to \infty$, $u/\sqrt{n} \log n \to \infty$) unless otherwise stated. We shall prove (3.8) with S_n rather than $|S_n|$. Since the same will be true for $-S_n$, this will imply (3.8). We take and fix an arbitrary bounded $A \subset R^1$ for reference when dealing with the uniformity of convergence.

Let $A_{n,u}$ denote the event

$$(6.1) \qquad \{Y_i < u \quad \text{for all } i = 1, \cdots, n\}.$$

Writing $\bar{A}_{n,u}$ for its complement, we have

$$(6.2) \quad P\{S_n > x \mid Z_n = z\} \leqq P\{S_n > x, A_{n,u} \mid Z_n = z\} + P\{\bar{A}_{n,u} \mid Z_n = z\}.$$

Estimate first the last term in (6.2). We have obviously

$$(6.3) \qquad P\{\bar{A}_{n,u} \mid Z_n = z\} \leqq nP\{Y_n > u \mid Z_n = z\}.$$

We show now that

$$(6.4) \qquad \sup_z P\{Y_n > u \mid Z_n = z\}\, p_n(z) \leqq a_n P\{Y_n > u\}$$

where $a_n \to (2\pi)^{-1/2}$ and hence is bounded. Let $\Delta \subset R^1$ be an arbitrary bounded Borel set. Then

$$(6.5) \quad \int_\Delta P\{Y_n > u \mid Z_n = z\}\, p_n(z)\, dz = P\{Y_n > u, Z_n \in \Delta\}$$

$$= \int_{-\infty}^{\infty} \int_u^{\infty} P\{Z_n \in \Delta \mid X_n = x, Y_n = y\}\, dP(x, y)$$

$$= \int_{-\infty}^{\infty} \int_u^{\infty} P\left\{Z_{n-1} \in \left[\left(\frac{n}{n-1}\right)^{1/2} \Delta - \frac{x}{(n-1)^{1/2}}\right] \Bigg| X_n = x, Y_n = y\right\} dP(x, y)$$

$$\leqq \sup_t P\left\{Z_{n-1} \in \left(\frac{n}{n-1}\right)^{1/2} \Delta - t\right\} P\{Y_n > u\}$$

$$\leqq |\Delta| \left(\frac{n}{n-1}\right)^{1/2} \max_z p_n(z)\, P\{Y_n > u\},$$

where $|\Delta|$ is the Lebesgue measure of Δ. With $a_n = [n/(n-1)]^{1/2} \max_z p_n(z)$ this implies (6.4). The assumption that $p^{\nu *}(z)$ is bounded assures the convergence

$$(6.6) \qquad p_n(z) \to n(z) \qquad \text{uniformly in } z \in R^1.$$

This implies that $a_n \to (2\pi)^{-1/2}$ and moreover that $p_n(z)$ is bounded away from zero on A for n large enough. Since $P\{Y_n > u\} = o(u^{-r})$ we obtain from (6.3) and (6.4) that

(6.7) $$P\{\bar{A}_{n,u} | Z_n = z\} = o(nu^{-r}) \qquad \text{uniformly in } z \in A,$$

that is, this term has the required order.

Consider now the first term in the right side of (6.2). We shall write $\tilde{S}_n$, $\tilde{Z}_n$ for $S_n\sqrt{n}$, $Z_n\sqrt{n}$ (nonnormalized sums). For an event B, let I_B denote the indicator function; instead of $I_{A_{n,u}}$, we shall write $I_{n,u}$. Then for any $h > 0$,

(6.8) $$\begin{aligned} P\{S_n > x, A_{n,u} | Z_n = z\} &= P\{\tilde{S}_n > u, A_{n,u} | Z_n = z\} \\ &\leqq e^{-hu} E[\exp\{h\tilde{S}_n\} I_{n,u} | Z_n = z]. \end{aligned}$$

Put

(6.9) $$d_r(v) = \int_{1/v}^{\infty} y^r \, dQ(y).$$

(6.10) $$c_n = \max\left(n^{-1/2}, d_r(\sqrt{n})\right).$$

(6.11) $$h_{n,u} = -u^{-1} \log(c_n n u^{-r}).$$

Writing $u = \lambda\sqrt{n} \log n$ where $\lambda \to \infty$, we see that

(6.12) $$h_{n,u} = \frac{-\log c_n n + r \log \lambda + r \log(n^{1/2} \log n)}{\lambda\sqrt{n} \log n} = o(n^{-1/2}).$$

Since $c_n \to 0$, we have from (6.11)

(6.13) $$\exp\{-h_{n,u}u\} = o(nu^{-r}).$$

We shall show that

(6.14) $$E[\exp\{h_{n,u}\tilde{S}_n\} I_{n,u} | Z_n = z] \, p_n(z)$$

is bounded uniformly in z and sufficiently large n. Then the theorem will follow from (6.1), (6.7), (6.8), (6.13) and (6.6).

Denote the density of $\tilde{Z}_n$ by $\tilde{p}_n(z)$; $\tilde{p}_n(z\sqrt{n})\sqrt{n} = p_n(z)$. We can rewrite (6.14) as

(6.15) $$E[\exp\{h_{n,u}\tilde{S}_n\} I_{n,u} | \tilde{Z}_n = z\sqrt{n}] \, \tilde{p}_n(z\sqrt{n})\sqrt{n}.$$

This expression will be estimated with the help of the following lemma.

LEMMA 6.1. *Let (U_1, V_1), (U_2, V_2) be independent random vectors, V_i having a density $p_i(v)$, $i = 1, 2$, and let $p(v)$ be the density of $V_1 + V_2$. Put*

(6.16) $$\begin{aligned} f_i(v) &= E[U_i | V_i = v] \, p_i(v), \qquad i = 1, 2, \\ f(v) &= E[U_1 U_2 | V_1 + V_2 = v] \, p(v). \end{aligned}$$

Then

(6.17) $$f(v) = f_1 * f_2(v).$$

PROOF. For any Borel set $B \subset R^1$, put

(6.18) $$\begin{aligned} \pi_i(B) &= E[U_i I_B(V_i)], \qquad i = 1, 2, \\ \pi(B) &= E[U_1 U_2 I_B(V_1 + V_2)]. \end{aligned}$$

Let $P_i(u, v)$ be the d.f. of (U_i, V_i), $i = 1, 2$. Then

$$\pi(B) = \int_{R^2 \times R^2} u_1 u_2 I_B(v_1 + v_2)\, dP_1(u_1, v_1)\, dP_2(u_2, v_2) \tag{6.19}$$
$$= \int_{R^2} I_B(v_1 + v_2)\, d\pi_1\, d\pi_2 = \pi_1 * \pi_2(B).$$

On the other hand,

$$\begin{aligned}\int_B f_i(v)\, dv &= \int_B E[U_i | V_i = v]\, p_i(v)\, dv \\ &= E[U_i I_B(V_i)] = \pi_i(B), \qquad i = 1, 2,\end{aligned} \tag{6.20}$$

That is, $f_i(v)$ is the density of π_i with respect to the Lebesgue measure. Similarly, $f(v)$ is the density of π. Thus (6.19) implies (6.17).

This lemma can be extended in an obvious way to any finite number of vectors (U_i, V_i). Denote

$$f(z; h, u) = E[\exp\{hY_1\}\, I_{\{Y_1 < u\}} | X_1 = z]. \tag{6.21}$$

Putting $U_i = \exp\{hY_i\}\, I_{\{Y_i < u\}}$, $V_i = X_i$ for $i = 1, \cdots, n$, we obtain

$$E[\exp\{h\tilde{S}_n\}\, I_{n,u} | \tilde{Z}_n = z]\, \tilde{p}_n(z) = [f(z; h, u)\, p(z)]^{n*}. \tag{6.22}$$

Comparing this with (6.15), we see that all we need to show is

$$\sup_z\, [f(z; h_{n,u}, u)\, p(z)]^{n*} = O(n^{-1/2}). \tag{6.23}$$

Note that

$$\int f(z; h, u)\, p(z)\, dz = E[\exp\{hY_1\}\, I_{\{Y_1 < u\}}] = R(h, u), \tag{6.24}$$

say. The proof of Theorem 1 in [5] contains the following estimate

$$R(h, u) = 1 + hm_1 + 2\theta_1 h^2 m_2 + \theta_2 K_r d_r(h)\, e^{hu} u^{-r}, \tag{6.25}$$

where $|\theta_1| < 1$, $0 < \theta_2 < 1$, $m_k = EY_1^k$, $k = 1, 2$ and K_r is a function of r only. Actually in that proof $M_r = E|Y_1|^r$ rather than $d_r(h)$ is used, but it appears there in the inequality $1 - Q(x) \leqq M_r x^{-r}$ (in our notation) which is used only for $x > 1/h$ and therefore holds true when M_r is replaced by $d_r(h)$. This is the modification of the proof we referred to in the proof of Lemma 4.2. It follows from (6.10) and (6.12) that $d_r(h_{n,u}) \leqq c_n$ for n large enough. In view of (6.11), (6.12) and $m_1 = EY_1 = 0$, we obtain from (6.25)

$$R(h_{n,u}, u) = 1 + O(n^{-1}). \tag{6.26}$$

Put

$$r_{n,u}(z) = \frac{f(z; h_{n,u}, u)\, p(z)}{R(h_{n,u}, u)}. \tag{6.27}$$

Since $r_{n,u}$ is nonnegative and integrates to unity by (6.23), it is a probability density. By virtue of (6.26), (6.23) is equivalent to

$$\sup_z [r_{n,u}(z)]^{n*} = O(n^{-1/2}). \tag{6.28}$$

First we shall show that

$$\limsup_{n,u} \sup_z [r_{n,u}(z)]^{2\nu *} < \infty. \tag{6.29}$$

On applying (6.25) to the conditional distribution of Y_1 given $X_1 = z$ (see (6.21) and (6.24)), we obtain

$$f(z; h, u) = 1 + hm_1(z) + 2\theta_1 h^2 m_2(z) + \theta_2 K_r M_r(z)\, e^{hu} u^{-r} \tag{6.30}$$

where $m_i(z) = E[Y_1^i | X_1 = z]$, $i = 1, 2$. (Actually, (6.30) corresponds to the version of (6.25) with M_r instead of $d_r(h)$.) It follows from (6.11) and (6.10) that

$$\exp\{h_{n,u}u\}\, u^{-r} = 1/c_n n \leqq n^{-1/2}. \tag{6.31}$$

Moreover,

$$m_i(z) \leqq (M_r(z))^{i/r} \leqq 1 + M_r(z), \qquad i = 1, 2. \tag{6.32}$$

Therefore

$$f(z; h, u) \leqq 1 + h + 2h^2 + (h + 2h^2 + K_r n^{-1/2})\, M_r(z) = \bar{f}(z; h, u), \tag{6.33}$$

say.

Define $\bar{r}_{n,u}(z)$ by (6.27) with f replaced by $\bar{f}$. Then we have from (6.12), (6.26) and (6.33)

$$r_{n,u}(z) \leqq \bar{r}_{n,u}(z) = \alpha_{n,u}p(z) + \beta_{n,u}M_r(z)p(z), \tag{6.34}$$

where $\alpha_{n,u} \to 1$, $\beta_{n,u} \to 0$. Denote by $\rho_{n,u}(t)$, $\bar{\rho}_{n,u}(t)$, $\varphi(t)$ and $\varphi_r(t)$ the c.f. of $r_{n,u}(z)$, $\bar{r}_{n,u}(z)$, $p(z)$ and $M_r(z)p(z)/M_r$, respectively. Put

$$L = \sup_z p^{\nu *}(z), \qquad L_r = \sup_z [M_r(z)p(z)/M_r]^{\nu *}. \tag{6.35}$$

(This is finite by assumption (iv) of Theorem 3.2.) By the Plancherel identity (see, for example, Feller [2], Chapter XV, equation (3.8)),

$$\frac{1}{2\pi}\int |\varphi(t)|^{2\nu}\, dt = \int [p^{\nu *}(z)]^2\, dz \leqq L \int p^{\nu *}(z)\, dz = L. \tag{6.36}$$

Similarly

$$\frac{1}{2\pi}\int |\varphi_r(t)|^{2\nu}\, dt \leqq L_r. \tag{6.37}$$

Using Minkowski's inequality, we have from (6.34), (6.36) and (6.37)

$$(6.38)\qquad \frac{1}{2\pi}\int |\bar{\rho}_{n,u}(t)|^{2\nu}\,dt \leqq [\alpha_{n,u}L^{1/2\nu} + \beta_{n,u}M_r L_r^{1/2\nu}]^{2\nu} = L_{n,u},$$

say, and $\alpha_{n,u} \to 1$, $\beta_{n,u} \to 0$ imply $L_{n,u} \to L$. Furthermore

$$(6.39)\qquad \sup_z\, [\bar{r}_{n,u}(z)]^{2\nu *} \leqq \frac{1}{2\pi}\int |\bar{\rho}_{n,u}(t)|^{2\nu}\,dt \leqq L_{n,u}.$$

Obviously, $r^{k*}_{n,u}(z) \leqq \bar{r}^{k*}_{n,u}(z)$ for any $k = 1, 2, \cdots$, whence $\sup_z [r_{n,u}(z)]^{2\nu *} \leqq L_{n,u}$ which proves (6.29). Applying again the Plancherel identity we obtain

$$(6.40)\qquad \frac{1}{2\pi}\int |\rho_{n,u}(t)|^{4\nu}\,dt \leqq L_{n,u}.$$

By virtue of the inequality

$$(6.41)\qquad \sup_z\, [r_{n,u}(z)]^{n*} \leqq \frac{1}{2\pi}\int |\rho_{n,u}(t)|^{n}\,dt = \frac{1}{2\pi}J_{n,u},$$

say, in order to prove (6.28) we need to show that $J_{n,u} = O(n^{-1/2})$. The relations (6.26) and (6.30) give

$$(6.42)\qquad \int |r_{n,u}(z) - p(z)|\,dz \to 0.$$

For any density $r(z)$, we shall write $r^{(2)}(z) = r * r^{-}(z)$ where $r^{-}(z) = r(-z)$. Then (6.42) implies

$$(6.43)\qquad \int |r^{(2)}_{n,u}(z) - p^{(2)}(z)|\,dz \to 0.$$

Take an arbitrary $\delta > 0$ and put

$$(6.44)\qquad b_{n,u} = \int_{|z| \leqq 1/\delta} z^2 r^{(2)}_{n,u}(z)\,dz.$$

Then (6.43) implies

$$(6.45)\qquad b_{n,u} \to \int_{|z| \leqq 1/\delta} z^2 p^{(2)}(z)\,dz.$$

Taking into account that the c.f. of $r^{(2)}_{n,u}(z)$ is $|\rho_{n,u}(t)|^2$ and using the first of the truncation inequalities [4], 12.4.B′, we obtain for $|t| \leqq \delta$

$$(6.46)\qquad |\rho_{n,u}(t)|^2 \leqq 1 - \tfrac{1}{3}t^2 b_{n,u} \leqq \exp\{-\tfrac{1}{3}t^2 b_{n,u}\}$$

or

$$(6.47)\qquad |\rho_{n,u}(t)|^n \leqq \exp\{-\tfrac{1}{6}nt^2 b_{n,u}\}.$$

Further, (6.42) implies that $\rho_{n,u}(t) \to \varphi(t)$ uniformly in $t \in R^1$, whence

$$(6.48)\qquad \sup_{|t| \geqq \delta} |\rho_{n,u}(t)| \to \sup_{|t| \geqq \delta} |\varphi(t)| < 1.$$

Split the integral $J_{n,u}$ in (6.41) into the integrals over $\{|t| \leqq \delta\}$ and $\{|t| > \delta\}$ and use (6.47) in the first and the inequality

$$|\rho_{n,u}(t)|^n \leqq [\sup_{|t| \geqq \delta} |\rho_{n,u}(t)|]^{n-4\delta} |\rho_{n,z}(t)|^{4v} \tag{6.49}$$

in the second of them. Then by virtue of (6.40), (6.45) and (6.48) we obtain $J_{n,u} = O(n^{-1/2})$ which was to be proved.

7. Concluding remarks

After strengthening certain assumptions in Theorem 2.1, the same proof, somewhat refined, could give for $\varepsilon_n(x)$ an estimate $O(n^{-\beta})$ with $\frac{1}{2} < \beta < 1$. However it is impossible to obtain the naturally expected order n^{-1} by the present method. For this reason we restrict ourselves to the assertion that $\varepsilon_n(x) = o(n^{-1/2})$.

Though it is not explicitly stated in (2.12), the function $\phi_n(x)$ depends on θ_0 because μ_3 and $\mu(x)$ do, thus $\phi_n(x) = \phi_n(x, \theta_0)$, say. There is no such dependence (and the d.f. of $Z_n(\hat{\theta}_n)$ does not depend on θ_0 at all) when θ is the location scale parameter and has an appropriate invariance property. In the general case we cannot determine the critical value $z_{n,\alpha}$ from the equation $\phi_n(z, \theta_0) = 1 - \alpha$ since θ_0 is unknown. It may be shown, however, that under certain smoothness of the dependence of μ_3 and $\mu(x)$ on θ the critical value $\hat{z}_{n,\alpha}$, determined from the equation $\phi_n(z, \hat{\theta}_n) = 1 - \alpha$, has the property that

$$P\{Z_n(\hat{\theta}_n) > \hat{z}_{n,\alpha}\} = \alpha + o(n^{-1/2}), \tag{7.1}$$

that is, it provides the same order of approximation as with known θ_0.

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