SEQUENTIAL RANK TESTS— ONE SAMPLE CASE

RUPERT G. MILLER, JR.
STANFORD UNIVERSITY

1. Introduction

Let $X_1, X_2, \cdots$ be a sequence of independent randon variables, identically distributed according to the continuous c.d.f. F. The null hypothesis is H_0: $F(-x) = 1 - F(x)$, $0 \leqq x < +\infty$; that is, the random variables are symmetrically distributed about zero. Sequential tests of this hypothesis which are based on the signs and ranks of the X_i are studied in this paper.

Sequential rank tests should be particularly useful in medical clinical trials. The one sample case arises naturally when patients are paired for similarity of influential physical traits, and are randomly assigned to one of two possible treatments so that each treatment is given to one member of each pair. The variable X_i is the difference between the treatment effects measured on the ith patient pair. Sequential binomial trials have been valuable in this context and will continue to be so. Rank tests, however, can take advantage of quantitative (nondichotomous) information in each treatment comparison while at the same time making only minimal assumptions about the form of the distribution.

In 1969 Weed, Bradley, and Govindarajulu [4] proposed a sequential likelihood ratio test for this problem. Let $G(x) = P\{|X| < x | X < 0\}$ and $H(x) = P\{X < x | X > 0\}$. They considered the family of distributions whose left and right tails are related by $1 - H(x) = (1 - G(x))^A, A > 0$, and $F(0) = A/(1 + A)$. For this family the null hypothesis becomes $H_0: A = 1$, and an alternative hypothesis is $H_1: A = B$ where B is a specified constant. The likelihood for the signs and rank order of the absolute values of $X_1, \cdots, X_n$ is

$$\binom{n}{n^-}\left(\frac{A}{1+A}\right)^{n^-}\left(\frac{1}{1+A}\right)^{n^+} n^-!\, n^+! \int_{0<x_1<\cdots<x_n}\cdots\int \prod_{i=1}^{n} \{dG(x_i)\}^{\delta_i}\{dH(x_i)\}^{1-\delta_i}, \tag{1}$$

where $\delta_i = 1$ if the X_i with the ith smallest absolute value is negative, $\delta_i = 0$ if it is positive. For $H_0: A = 1$ and $H_1: A = B$, the likelihood ratio simplifies to

$$LR_n = \frac{(B/1 + B)^n 2^n n!}{\prod_{i=1}^{n} [n_i^- + Bn_i^+]}, \tag{2}$$

where n_i^- is the number of X_j such that $X_j < 0$ and $|X_j| \geqq |X_i|$ and n_i^+ is the number of X_j such that $X_j > 0$ and $X_j \geqq |X_i|$. The Wald type sequential test is to continue sampling as long as $a < LR_n < b$, where $0 < a < 1 < b$. If LR_n exceeds the upper bound for some n, terminate sampling and decide in favor of H_1; if LR_n crosses the lower bound, terminate the test and decide H_0. Weed *et al* proved that this test terminates with probability one so the bounds can be approximated by $a = \beta/(1 - \alpha)$ and $b = (1 - \beta)/\alpha$, where α and β are the specified probabilities of errors of the first and second kind, respectively.

Weed *et al* also considered another model with $H(x) = G^A(x)$ and $F(0)$ arbitrary, but in this paper attention will be restricted to the model cited above.

Also in 1969, Miller [3] proposed an *ad hoc* sequential test based on the Wilcoxon signed rank statistic. Let $R_1 < \cdots < R_{n^-}$ be the ranks of the negative X_i, and $S_1 < \cdots < S_{n^+}$ the ranks of the positive X_i in the ordered sequence of the absolute values of the X_i. The Wilcoxon signed rank statistic is $SR_n = \Sigma_{i=1}^{n^+} S_i - \Sigma_{i=1}^{n^-} R_i$. Under H_0 the first two moments of SR_n are $E(SR_n) = 0$ and $\text{Var}(SR_n) = n(n + 1)(2n + 1)/6$. The fixed sample size test suggests sampling as long as

$$|SR_n| \leqq |z|_N^\alpha[n(n + 1)(2n + 1)/6]^{1/2} \tag{3a}$$

and

$$n < N. \tag{3b}$$

If for some n prior or equal to N, $|SR_n|$ exceeds the bound in (3a), reject H_0. If n reaches N without (3a) being violated, accept H_0. The investigator selects the truncation point N and the probability of a type I error α, which determines the critical constant $|z|_N^\alpha$. A table of $|z|_N^\alpha$ for $\alpha = 0.10, 0.05, 0.01$, and $N = 10(5)30(10)60$ is given in [3]. The percentile points $|z|_N^\alpha$ were estimated by Monte Carlo simulation.

This test is computationally easy to perform since the SR_n can be computed sequentially.

$$\begin{aligned} SR_n &= \sum_{i=1}^{n} \sum_{j=1}^{i} \text{sgn}(X_i + X_j) \\ &= SR_{n-1} + \sum_{i=1}^{n} \text{sgn}(X_i + X_n), \end{aligned} \tag{4}$$

where

$$\text{sgn}(X_i + X_j) = \begin{cases} +1 & \text{if } X_i + X_j > 0, \\ -1 & \text{if } X_i + X_j < 0. \end{cases} \tag{5}$$

As the next observation X_n is obtained, it is easy to compare it with the preceding observations in order to compute the term $\Sigma_{i=1}^n \text{sgn}(X_i + X_n)$. The addition of this sum and the previous SR_{n-1} yields SR_n.

Since $\Sigma_{i=n+1}^N i = (N - n)(N + n + 1)/2$, it is possible for the test to terminate sampling prior to N with the acceptance of H_0. If for any n

$$|SR_n| \leqq |z|_N^\alpha[N(N + 1)(2N + 1)/6]^{1/2} - (N - n)(N + n + 1)/2, \tag{6}$$

then it will be impossible for $|SR_n|$ to reach the rejection boundary by time N, and the test can stop with acceptance of H_0. Expression (6) creates an inner acceptance boundary inside the outer rejection boundary (3a).

An analogous one sided test would be to continue sampling as long as $SR_n \leqq z_N^\alpha[n(n+1)(2n+1)/6]^{1/2}$ and $n < N$. However, the critical constants z_N^α have not been computed. For the one sided test the acceptance boundary $SR_n \leqq z_N^\alpha[N(N+1)(2N+1)/6]^{1/2} - (N-n)(N+n+1)/2$ is more apt to create substantial savings in the number of observations than for the two sided test.

In this paper a third test is presented. It is similar to the preceding test, but it employs a linear barrier instead of a square root barrier. Namely, continue sampling as long as

$$|SR_n| \leqq |w|_N^\alpha n, \tag{7a}$$

$$n < N \tag{7b}$$

for the two sided test, or

$$SR_n \leqq w_N^\alpha n \tag{7a$'$}$$

for the one sided test. The investigator specifies N and α, which determine $|w|_N^\alpha$ (Table Ia, b) or w_N^α (Table IIa, b). As in the previous test acceptance of H_0 can occur prior to N if

$$|SR_n| \leqq |w|_N^\alpha N - (N-n)(N+n+1)/2 \tag{8}$$

in the two sided case, or if

$$SR_n \leqq w_N^\alpha N - (N-n)(N+n+1)/2 \tag{8$'$}$$

in the one sided case.

The linear barrier in (7a) or (7a′) can be motivated in two ways. Suppose that the X_i are independently, identically distributed according to $F(x-\Delta)$, which has density $f(x-\Delta)$ symmetric about Δ. For $H_0: \Delta = 0$ versus $H_1: \Delta = \Delta_1$ the likelihood ratio for the signs and rank order of the absolute values of $X_1, \cdots, X_n$ can be written

$$LR_n = 1 + \Delta_1\left[\sum_{i=1}^{n^+} E\left(\frac{-f'(U_{(s_i)})}{f(U_{(s_i)})}\right) - \sum_{i=1}^{n^-} E\left(\frac{-f'(U_{(r_i)})}{f(U_{(r_i)})}\right)\right] + O(\Delta_1^2) \tag{9}$$

as $\Delta_1 \to 0$, where $U_{(1)} < \cdots < U_{(n)}$ are the order statistics from a sample generated by the density $2f(u)$ for $u > 0$. If F is chosen to be the logistic distribution, then

$$E\left(\frac{-f'(U_{(s_i)})}{f'(U_{(s_i)})}\right) = \frac{s_i}{n+1}, \tag{10}$$

and similarly for r_i. Thus,

$$LR_n = 1 + \Delta_1 SR_n/(n+1) + O(\Delta_1^2), \tag{11}$$

and a linear barrier seems appropriate for local shift alternatives. This approach is analogous to the one employed for the two sample problem by W. J. Hall in unpublished work.

A second justification for a linear barrier arises from the approximate normality of SR_n. For moderate or large values of n, SR_n is approximately normally distributed with mean $\mu_n = n(n-1)\theta_2 + n\theta_1$ and variance $\sigma_n^2 = n(n-1)(n-2)\eta_3 + n(n-1)\eta_2 + n\eta_1$. The constants $\theta_1, \theta_2, \eta_1, \eta_2, \eta_3$ depend on F, but not n. In particular, $\theta_1 = P\{X_1 > 0\} - P\{X_1 < 0\}$, and $\theta_2 = [P\{X_1 + X_2 > 0\} - P\{X_1 + X_2 < 0\}]/2$. For translation alternatives $F(x - \Delta)$ the mean μ_n under $H_0: \Delta = -\Delta_1$ is the negative of the mean under $H_1: \Delta = +\Delta_1$; the variance σ_n^2 is the same under both H_0 and H_1. Thus, if it is assumed that SR_n is approximately normally distributed, the likelihood ratio for SR_n under H_0: $\Delta = -\Delta_1$ and $H_1: \Delta = +\Delta_1$ is

$$L_n = \exp\left\{\frac{1}{2\sigma_n^2}(SR_n + \mu_n)^2 - \frac{1}{2\sigma_n^2}(SR_n - \mu_n)^2\right\}, \tag{12}$$

and

$$\log L_n = \frac{2\mu_n SR_n}{\sigma_n^2} \sim \left(\frac{2\theta_2}{\eta_3}\right)\frac{SR_n}{n}, \tag{13}$$

which again suggests a linear barrier.

2. Percentile points for linear barriers

Define $Y_n = SR_n/n$, and

$$W_N = \max\{Y_1, \cdots, Y_N\}, \qquad |W|_N = \max\{|Y_1|, \cdots, |Y_N|\}. \tag{14}$$

In order for the test defined by (7a), (7b) to have size α, the constant $|w|_N^\alpha$ must be the upper α-percentile point of the distribution of $|W|_N$; that is, $P\{|W|_N > |w|_N^\alpha\} = \alpha$. Similarly, w_N^α is defined by $P\{W_N > w_N^\alpha\} = \alpha$.

Both W_N and $|W|_N$ have discrete distributions which have not been treated analytically. However, their distributions can be estimated easily through Monte Carlo simulation. Two thousand W_N and $|W|_N$ were randomly generated for $N = 10(5)30(10)50$. The sample α-percentile points for $\alpha = 0.10, 0.50, 0.01$ are displayed in Table Ia and Table IIa. A pseudorandom number generator [2] was used to generate sequences of uniform variables, which were transformed into W_N or $|W|_N$. Different sequences were used for different values of N.

The stochastic behavior of the signed ranks SR_n, $n = 1, \cdots, N$, can be approximated by a Wiener process. It is well known that under H_0

$$E(Y_n) = 0, \qquad \operatorname{Var}(Y_n) = \frac{n(n+1)(2n+1)}{6n^2} \cong \frac{n}{3}. \tag{15}$$

For $m < n$ the covariance between Y_m and Y_n is

$$\text{(16)} \qquad \operatorname{Cov}(Y_m, Y_n) = \frac{m(n+1)(2m+1)}{6mn} \cong \frac{m}{3},$$

which is easily proved using the representation

$$\text{(17)} \quad \sum_{j=1}^{n} \sum_{i=1}^{j} \operatorname{sgn}(X_i + X_j) = \sum_{j=1}^{m} \sum_{i=1}^{j} \operatorname{sgn}(X_i + X_j) + \sum_{j=m+1}^{n} \sum_{i=1}^{j} \operatorname{sgn}(X_i + X_j)$$

and the covariances

$$\text{(18)} \qquad \begin{aligned} &\operatorname{Cov}[\operatorname{sgn}(X_i + X_j), \operatorname{sgn}(X_k + X_j)] = \tfrac{1}{3}, \\ &\operatorname{Cov}[\operatorname{sgn}(X_j + X_j), \operatorname{sgn}(X_i + X_j)] = \tfrac{1}{2}, \end{aligned}$$

for i, j, k, unequal. The approximate normality of Y_n and the variance-covariance in (15) and (16) suggest that the Y_n are behaving like $W(n)/\sqrt{3}$, where $W(t)$ is a standard Wiener process with zero mean and variance t.

This approximation gives

$$\text{(19)} \qquad \begin{aligned} P\{W_N > c\} &\cong P\{\max_{0 \leqq t \leqq N} W(t)/\sqrt{3} > c\} \\ &= 2P\{W(N) > c\sqrt{3}\} \\ &= 2[1 - \Phi(c\sqrt{3}/\sqrt{N})], \end{aligned}$$

where the second line follows from the reflection principle and Φ is the unit normal c.d.f. The approximation (19) implies $w_N^\alpha \cong \sqrt{N}\, g^{\alpha/2}/\sqrt{3}$, where $g^{\alpha/2}$ is the upper $\alpha/2$-percentile point of the unit normal c.d.f. Values of $\sqrt{N}\, g^{\alpha/2}/\sqrt{3}$ appear in Table IIb.

If the probability of crossing both the positive and negative boundaries by time N is negligible, then $|w|_N^\alpha \cong \sqrt{N}\, g^{\alpha/4}/\sqrt{3}$. The latter values are presented in Table Ib. The probability of a double crossing appears to be negligible for N in the range of the tables, but it will increase with increasing N.

Comparison of Table Ia with Ib and Table IIa with IIb reveals close agreement between the Monte Carlo percentile points and the Wiener approximations. This seems remarkable since a discrete process in discrete time is being approximated by a continuous process in continuous time.

TABLE Ia

VALUES OF $|w|_N^\alpha$ BY MONTE CARLO APPROXIMATION

α \ N	10	15	20	25	30	40	50
.01	5.00	6.23	6.90	7.59	8.50	10.05	11.08
.05	4.10	5.07	5.70	6.50	6.93	7.97	8.82
.10	3.50	4.36	5.06	5.64	6.15	6.90	7.75

TABLE Ib

VALUES OF $|w|_N^2$ BY WIENER APPROXIMATION

α \ N	10	15	20	25	30	40	50
.01	5.12	6.28	7.25	8.10	8.88	10.25	11.46
.05	4.09	5.01	5.79	6.47	7.09	8.18	9.15
.10	3.58	4.38	5.06	5.66	6.20	7.16	8.00

TABLE IIa

VALUES OF w_N^2 BY MONTE CARLO APPROXIMATION

α \ N	10	15	20	25	30	40	50
.01	4.50	5.73	6.30	7.32	7.83	9.38	10.73
.05	3.70	4.30	5.00	5.50	5.96	7.00	7.82
.10	3.10	3.70	4.25	4.61	5.06	5.79	6.61

TABLE IIb

VALUES OF w_N^2 BY WIENER APPROXIMATION

α \ N	10	15	20	25	30	40	50
.01	4.70	5.76	6.65	7.44	8.15	9.41	10.52
.05	3.58	4.38	5.06	5.66	6.20	7.16	8.00
.10	3.00	3.68	4.25	4.75	5.20	6.01	6.72

3. Power and expected sample size

The power and expected stopping time of the test using SR_n with linear barriers was investigated for shift alternatives in the double exponential distribution. Let

$$f(x) = \tfrac{1}{2}e^{-|x-\Delta|}, \qquad -\infty < x < +\infty. \tag{20}$$

This distribution has mean Δ, variance 2 (standard deviation 1.41). The tails of (20), which are heavier than the normal, behave like the logistic distribution for which the Wilcoxon statistic is known to have good local properties. In [3] this distribution was used to examine the power and stopping time for SR_n with square root barriers.

For $N = 20$ the performance of the test was studied at $\Delta = 0, 0.5, 1, 1.5$; for $N = 50$, at $\Delta = 0, 0.25, 0.5, 0.75, 1$. The error probability α was taken to be either 0.05 or 0.01. For each combination of N, α, Δ, five hundred sequences of

TABLE IIIa

MONTE CARLO POWER AND EXPECTED SAMPLE SIZE FOR TWO SIDED TEST, $N = 20$

Power	Δ			
	0	.5	1	1.5
$\alpha = .05$	.046	.39	.88	.99
$\alpha = .01$	.014	.19	.71	.96
Expected n				
$\alpha = .05$	19.8	18.5	14.8	13.0
$\alpha = .01$	20.0	19.5	17.4	15.5

TABLE IVa

MONTE CARLO POWER AND EXPECTED SAMPLE SIZE FOR TWO SIDED TEST, $N = 50$

Power	Δ				
	0	.25	.5	.75	1
$\alpha = .05$	.062	.30	.82	.98	1.0
$\alpha = .01$	.010	.13	.59	.93	1.0
Expected n					
$\alpha = .05$	49.4	46.6	37.2	29.0	24.2
$\alpha = .01$	49.9	48.9	43.5	35.8	29.9

TABLE Va

MONTE CARLO POWER AND EXPECTED SAMPLE SIZE FOR ONE SIDED TEST, $N = 20$

Power	Δ			
	0	.5	1	1.5
$\alpha = .05$	.052	.51	.94	.99
$\alpha = .01$	.020	.32	.78	.98
Expected n				
$\alpha = .05$	14.7	16.1	13.3	11.5
$\alpha = .01$	13.5	16.3	15.7	14.0

double exponential random variables were substituted into the test. For the two sided test the results are displayed in Tables IIIa, IVa; the one sided test results appear in Tables Va, VIa.

TABLE VIa

MONTE CARLO POWER AND EXPECTED SAMPLE SIZE FOR ONE SIDED TEST, $N = 50$

Power	Δ				
	0	.25	.5	.75	1
$\alpha = .05$	.052	.41	.87	.99	1.0
$\alpha = .01$	.006	.14	.59	.92	1.0
Expected n					
$\alpha = .05$	41.9	41.6	32.9	25.7	21.5
$\alpha = .01$	38.8	43.3	41.6	34.3	29.5

For the two sided test the power is the probability of exceeding the rejection boundary in the correct direction. Under these alternative hypotheses just one Monte Carlo sequence ever crossed the rejection boundary in the wrong direction. This occurred for $N = 50$, $\alpha = 0.05$, $\Delta = 0.25$.

For the two sided test the inner acceptance boundary was not used so that the expected sample sizes could be compared with those in [3] for SR_n with square root boundaries. The power of the test is unaffected by the inclusion, or exclusion, of the inner boundary, but the expected sample sizes would be slightly smaller with the inner boundary included. The early acceptance boundary was included for the one sided test.

Comparison of the entries in Tables IIIa, IVa with the corresponding values in [3] for SR_n with square root boundaries leads to the following conclusion. The test with linear barriers has slightly better power than the square root barrier test, but the reverse holds for expected sample sizes. The square root barrier gives expected sample sizes which are slightly smaller than those for the linear barrier. This means that if a sequence $X_1, X_2, \cdots$ is going to reject H_0, it will stop sooner with the square root barrier than the linear one.

As in the null case, the behavior of SR_n, $n = 1, 2, \cdots, N$, can be approximated by a Wiener process. Let $W(t)$ be a Wiener process with drift μt and variance $\sigma^2 t$. Dinges [1] proved that

$$(21) \qquad P\{\max_{0 \leqq t \leqq T} W(t) > c\} = \left(e^{c\mu/\sigma^2}\right)^2 \Phi\left(\frac{-\mu\sqrt{T}}{\sigma} - \frac{\sigma}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{\mu\sqrt{T}}{\sigma} - \frac{c}{\sigma\sqrt{T}}\right).$$

For the distribution (20)

$$(22) \qquad \begin{aligned} E(Y_n) &= nE[\operatorname{sgn}(X_i + X_j)] + O(1), \\ &= \tfrac{1}{2}[1 - (1 + \Delta)e^{-2\Delta}]n + O(1), \end{aligned}$$

TABLE IIIb

WIENER POWER FOR TWO SIDED TEST, $N = 20$

Power	Δ			
	0	.5	1	1.5
$\alpha = .05$	.055	.37	.86	1.00
$\alpha = .01$	.015	.19	.64	.95

TABLE IVb

WIENER POWER FOR TWO SIDED TEST, $N = 50$

Power	Δ				
	0	.25	.5	.75	1
$\alpha = .05$	.061	.32	.80	.98	1.0
$\alpha = .01$	.013	.14	.58	.92	1.0

TABLE Vb

WIENER POWER FOR ONE SIDED TEST, $N = 20$

Power	Δ			
	0	.5	1	1.5
$\alpha = .05$	.053	.50	.93	1.00
$\alpha = .01$	.015	.27	.76	.98

TABLE VIb

WIENER POWER FOR ONE SIDED TEST, $N = 50$

Power	Δ				
	0	.25	.5	.75	1
$\alpha = .05$	.055	.42	.87	.99	1.0
$\alpha = .01$	.009	.16	.62	.92	1.0

$$\text{(23)} \qquad \begin{aligned} \text{Var}\,(Y_n) &= n \text{ Cov}\,[\text{sgn}\,(X_i + X_j), \text{sgn}\,(X_k + X_j)] + O(1), \\ &= \tfrac{1}{3}[5e^{-2\Delta} - (4 + 6\Delta + 3\Delta^2)e^{-4\Delta}]n + O(1). \end{aligned}$$

If μ and σ^2 are taken to be the coefficients of n in (22) and (23), expression (21) gives the entries in Tables IIIb, IVb, Vb, VIb.

Comparison of the *a* and *b* parts of Tables III, IV, V, VI reveals a remarkable agreement between the Monte Carlo approximation to the power and the Wiener approximation.

4. Comparison of tests

An interesting question is how to compare the three tests mentioned in Section 1. One approach is to adjust the structures of the tests so that they have the same power at a fixed alternative. The expected sample sizes can then be directly compared. This cannot be done analytically because workable expressions for the power are not available for the SR_n tests (except for Wiener approximation in the linear case) and for the expected sample sizes nothing is available for any of the tests. However, the comparison can be carried out by Monte Carlo simulation.

Let

$$f(x) = \begin{cases} \frac{A}{1+A} e^{x}, & x < 0, \\ \frac{A}{1+A} e^{-Ax}, & x > 0. \end{cases} \tag{24}$$

This family of unsymmetric double exponential distributions constitutes Lehmann alternatives to which the likelihood ratio test (2) applies. For $N = 20$, $\alpha = 0.05$ five hundred sequences of random variables from the distribution (24) with $A = 1, 0.75, 0.5, 0.25$ were substituted into the one sided SR_n test with linear barrier. The resulting Monte Carlo power and expected sample sizes appear in Table VIIa in the M (Miller) linear line. For $B = 0.5$ the error probability β is 0.432. These constants were then used to define the sequential likelihood ratio test based on (2). An identical number of Monte Carlo sequences were substituted into this test. and the results appear in the W–B–G (Weed–Bradley–Govindarajulu) lines of the table.

TABLE VIIa

COMPARISON OF THE MILLER TESTS ($N = 20$, $\alpha = 0.05$) WITH THE WEED–BRADLEY–GOVINDARAJULU TEST ($\alpha = 0.048$, $\beta = 0.432$, $B = 0.5$) FOR LEHMANN ALTERNATIVES

Power	A: 1	.75	.5	.25
M square root	.048	.18	.51	.95
M linear	.048	.18	.57	.97
W–B–G	.048	.21	.61	.94
Expected n				
M square root	13.4	14.3	13.9	10.1
M linear	14.7	15.9	15.9	12.9
W–B–G	8.4	12.9	14.5	11.3

To prevent unlimited sampling the W–B–G test was truncated at fifty observations. Just a few sequences needed to be truncated. For $A = 1$ there were three out of five hundred, $A = 0.75$ nine, and $A = 0.5$ six. For shift alternatives with $\Delta = 0$ there were two, and with $\Delta = 0.5$ eight.

The critical constant z_N^α for the one sided SR_n test with square root barrier when $N = 20$, $\alpha = 0.05$ was simulated in a special run so that this type of test could be compared with the previous two. (Complete tables of z_N^α have not been computed.) The estimated percentile point was 2.17. The Monte Carlo results for the square root barrier test appear in the M square root line. Both tests based on SR_n used the early acceptance boundary for H_0.

Since the likelihood ratio test is designed for Lehmann alternatives, it seemed fair to evaluate how it performs for shift alternatives where the Wilcoxon statistic is better adapted. Monte Carlo sequences from the distribution (20) were substituted into the three tests, and the outcome is displayed in Table VIIb.

Table VIIb

Comparison of the Miller Tests ($N = 20$, $\alpha = 0.05$) with the Weed–Bradley–Govindarajulu Test ($\alpha = 0.048$, $\beta = 0.432$, $B = 0.5$) for Shift Alternatives

Power	Δ			
	0	.5	1	1.5
M square root	.048	.48	.90	.98
M linear	.052	.51	.94	.99
W–B–G	.034	.51	.87	.96
Expected n				
M square root	13.5	13.9	10.2	8.3
M linear	14.7	16.1	13.3	11.5
W–B–G	8.0	14.3	12.8	10.5

Examination of the values in the tables leads to the following conclusions. The M linear test has slightly better power than the M square root, but the M square root has smaller expected sample sizes. This agrees with the conclusion reached earlier in the two sided case. For both power and expected sample size the W–B–G test performs better than the M square root test for near alternatives whereas the M square root is better for far alternatives. The expected sample size under H_0 is very much smaller for the W–B–G test. The choice of alternative distribution, Lehmann or shift, appears to have little effect.

These conclusions are also reflected in the smallest value of n for which acceptance of H_0 or H_1 can occur. For M square root the earliest time at which acceptance of H_0 can happen is 10 and for H_1 it is 6. With M linear the values are 10 for both H_0 and H_1, and the W–B–G test needs only 2 for H_0 but 9 for H_1. Thus, the square root boundary allows earlier rejection of H_0 than the other two, and the likelihood ratio test permits earlier acceptance of H_0.

Which test should be used? The SR_n tests are much simpler computationally, and they are very simple to explain to the investigator. In medical applications selection of an alternative hypothesis and its associated power is often difficult or extremely arbitrary. A bound on the amount of sampling is usually easier to determine due to limitations of money, time, and so forth. The likelihood ratio test stops very early when H_0 is true, but in a medical setting the far alternatives seem more important. If there is little or no difference between treatments, then continuation of the trial is relatively unimportant from the ethical point of view, but if one treatment is much better than the other, you want to stop the trial as soon as possible.

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