SOME TECHNIQUES OF SUMMARY EVALUATIONS OF SEVERAL INDEPENDENT EXPERIMENTS

ROBERT B. DAVIES and PREM S. PURI
UNIVERSITY OF CALIFORNIA, BERKELEY

1. Introduction

Because of the notorious frequency with which rain stimulation experiments fail to yield statistically significant results it is important to develop efficient methods for summary evaluations. In the present note two related but distinct problems are treated. In both cases a number s of independent experiments $E_1, E_2, \cdots, E_s$ are considered, each conducted to investigate the presence of certain effects. Specifically, the ith experiment is concerned with a parameter ξ_i which is a measure of a certain effect. The first problem is the summary test of the hypothesis, H_0 say, that all the parameters ξ_i are zero; that is,

$$\xi_1 = \xi_2 = \cdots = \xi_s = 0. \tag{1.1}$$

The second is to test the hypothesis, H_1 say, that all the parameters ξ_i have the same value ξ which however the hypothesis H_1 does not specify; that is,

$$\xi_1 = \xi_2 = \cdots = \xi_s = \xi. \tag{1.2}$$

The first test of the hypothesis H_0 has been proposed by R. A. Fisher [1] and later studied by E. S. Pearson [2] and used by J. Neyman and E. L. Scott [3] in the present *Proceedings*. Obviously the efficiency of the test of H_0 must depend upon the information regarding the experiments $E_1, E_2, \cdots, E_s$ that the test utilizes. Fisher's test is very broad and is based only on the exact significance probabilities resulting from the individual experiments. It is therefore plausible that an alternative summary test using more information about the experiments covered might be more powerful.

The problem of testing H_1 is familiar in the general domain of the analysis of variance, also due to Fisher. Here the separate experiments $E_1, E_2, \cdots, E_s$ are equivalent to "blocks" and the hypothesis H_1 is that of no block-treatment interaction. As is well known, the analysis of variance tests are applicable when the observable variables are normal with fixed variance. The test given below is deduced for a particular situation where these assumptions do not hold.

All formulas given in the present note are specializations of the results of

This paper was supported by the National Institute of Health, Public Health Service (Grant GM-10525).

B. R. Bhat and B. N. Nagnur [4] and of W. J. Bühler and P. S. Puri [5] concerning $C(\alpha)$ tests.

2. Summary test of the hypothesis H_0

It is assumed that for experiment E_i a criterion Z_i is computed to test the hypothesis that $\xi_i = 0$. It is further assumed that, whether $\xi_i = 0$ or not, the criterion Z_i is (at least asymptotically) normal with asymptotic variance equal to one and asymptotic mean equal to τ_i, and that $\tau_i = 0$ if and only if $\xi_i = 0$. In this case, a criterion suggested by the theory of $C(\alpha)$ tests for testing H_0 against the alternative $\sum \tau_i^2 > 0$ has critical region

$$\sum_{i=1}^{s} Z_i^2 = \chi^2 > \nu_s(\alpha), \tag{2.1}$$

where $\nu_s(\alpha)$ is chosen to give the desired significance level α. The distribution of χ^2 is asymptotically a chi square distribution with s degrees of freedom, central under the hypothesis H_0 and with noncentrality parameter $\sum \tau_i^2$ otherwise. While the indicated test criterion (2.1) is plausible its general optimality is subject to question and is being investigated.

3. Test of the identity of treatment effects

The method of deducing the proposed test is as follows. Select a fixed number ξ and deduce an optimal $C(\alpha)$ test, with statistic Z_i to test the hypothesis that in the ith experiment the effect of the treatment studied is represented by the number ξ. Whatever be the situation, the test statistic Z_i has an asymptotically normal distribution with variance unity. Next use all the s experiments to obtain a locally root n consistent estimate $\hat{\xi}$ of the effect of treatment under the express assumption that this effect is the same in all the s experiments. Finally, substitute $\hat{\xi}$ instead of the hypothetical ξ in the Z_i computed as indicated above. Then the proposed test for H_1, that in all s experiments the effects of the treatments studied are the same, has the critical region

$$Z_0 > \nu_{s-1}(\alpha), \tag{3.1}$$

where

$$Z_0 = \sum_{i-1}^{s} Z_i^2 - \left(\sum_{i=1}^{s} a_i Z_i\right)^2 \Big/ \sum_{i=1}^{s} a_i^2 \tag{3.2}$$

and where the a_i are such that the means of the Z_i in the basic $C(\alpha)$ tests are given (asymptotically) by

$$\tau_i = a_i(\xi_i - \xi). \tag{3.3}$$

When maximum likelihood estimators are used for obtaining $\hat{\xi}$ and for estimating the parameters in the basic tests, the test statistic reduces to

$$Z_0 = \sum_{i=1}^{s} Z_i^2. \tag{3.4}$$

If the hypothesis H_1 is true then Z_0 has asymptotically a central chi square distribution with $s - 1$ degrees of freedom. Otherwise the asymptotic form is a noncentral chi square distribution with noncentrality parameter inf $\sum_{i=1}^{s} \tau_i^2$ where the infimum is taken over values of ξ. Here again the problem of optimality is open and is being investigated. However, the test is optimal in the sense of the paper of Bühler and Puri [5].

4. Test of the identity of the effect of seeding on the nonzero precipitation amounts in *s* independent trials

Consider s independent rain stimulation experiments and suppose that in the ith of these the distribution of the normal nonzero precipitation amounts has the Gamma density

$$\frac{\delta_i^{\gamma_i}}{\Gamma(\gamma_i)} x^{\gamma_i - 1} e^{-\delta_i x}. \tag{4.1}$$

Assume further that the result of seeding, if any, is strictly multiplicative and is limited to change in the parameter δ_i while the shape parameter γ_i remains unchanged. The hypothesis H_1 to be tested is that irrespective of the values of γ_i and δ_i which may vary from one experiment to the next, the seeding alters the mean nonzero rain by the same factor in all the s experiments.

Pursuing the ideas outlined above, the description of the test requires the following formulas: the formula for computing the Z_i appearing in (3.2) or (3.4), the formula for estimating the nuisance parameters γ_i and δ_i for $i = 1, 2, \cdots, s$ and the formula for estimating ρ the effect of seeding assumed multiplicative and the same in all s experiments.

A convenient formula for Z_i is

$$Z_i^2 = \hat{\gamma}_i \eta_i^2 / [n_i \pi_i (1 - \pi_i)], \tag{4.2}$$

where

$$\eta_i = n_{is} n_{ic} (\bar{x}_{is} - \hat{\rho} \bar{x}_{ic}) / (n_{is} \bar{x}_{is} + \hat{\rho} n_{ic} \bar{x}_{ic}), \tag{4.3}$$

and where the subscript i refers to the ith experiment, and the subscripts c and s to control units and those seeded, respectively. The $\bar{x}$ stands for arithmetic means of nonzero precipitation amounts, while n indicates the number of relevant units of observation (with nonzero precipitation) with $n_i = n_{ic} + n_{is}$ and π_i represents the probability that an experimental unit will be seeded.

The nuisance parameters requiring estimation are the shape parameters γ_i and the multiplicative effect parameter ρ. Note that an estimate $\bar{x}_i$ for γ_i/δ_i has already been substituted in equation (4.3). The relevant equations are

$$\log_e \hat{\gamma}_i - \Gamma'(\hat{\gamma}_i)/\Gamma(\hat{\gamma}_i) = \log_e [(n_{ic} x_{ic} \hat{\rho} + n_{is} x_{is})/n_i] - \overline{\log_e x_i} - n_{is}(\log_e \hat{\rho})/n_i, \tag{4.4}$$

for $i = 1, 2, \cdots, s$ where the logarithms are natural logarithms. The last equation is

$$\sum_{i=1}^{s} \hat{\gamma}_i \eta_i = 0. \tag{4.5}$$

An explicit solution of this system is not available, but it can easily be solved by a digital computer. With these equations maximum likelihood estimates of the parameters are obtained (under H_1: $\xi_i = \xi$) and hence the test statistic may be found using equation (3.4);

$$Z_0 = \sum_{i=1}^{s} Z_i^2. \tag{4.6}$$

As mentioned in [3], this test has been used with reference to Grossversuch III.

REFERENCES

[1] R. A. Fisher, *Statistical Methods for Research Workers*, Edinburgh and London, Oliver and Boyd, 1932 and later editions.

[2] E. S. Pearson, "The probability integral transformation for testing goodness of fit and combining independent tests of significance," *Biometrika*, Vol. 30 (1938), pp. 134–148.

[3] J. Neyman and E. L. Scott, "Some outstanding problems relating to rain modification," *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California, 1967, Vol. 5, pp. 293–325.

[4] B. R. Bhat and B. N. Nagnur, "Locally asymptotically most stringent tests and Langrangian multiplier tests with linear hypotheses," *Biometrika*, Vol. 52 (1965), pp. 459–68.

[5] W. J. Bühler and P. S. Puri, "On optimal asymptotic tests of composite hypotheses with several constraints," *Z. Wahrschein Verw. Gebiete.*, Vol. 5 (1966), pp. 71–88.