# NOTE ON TECHNIQUES OF EVALUATION OF SINGLE RAIN STIMULATION EXPERIMENTS

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## 1. Introduction

The formulas used in our paper [1] its appendix and in [2], [3], are all based on the theory given in [4] and particularly in [5]. The deduction of these formulas is straightforward, but the formulas themselves are not familiar. Their description in the text of the three papers [1], [2], and [3] would have tended to disrupt the continuity of discussion of the substantive matters treated therein. Therefore, it was decided to compile the present note assembling all the formulas employed and also some extensions that may be useful.

All the techniques employed in our treatment of rain stimulation experiments are asymptotic techniques. In particular, the normal distributions of the test criteria were obtained under a passage to the limit as the number $N$ of observations is indefinitely increased. As far as the distributions under the hypothesis tested are concerned, no special comments are needed. This is not so for the asymptotic distributions of the test criteria that lead to the approximate evaluation of the power of the tests. Here the passage to the limit, invented in 1936 [6], is somewhat peculiar: in parallel with increasing the number $N$ of observations, the parameter $\xi$, characterizing the effectiveness of the treatment, is supposed to tend to zero so that the product $\xi N^{1/2}$ remains constant or, at least, tends to a fixed limit different from zero. Thus, in any particular case in which $N$ is large and $\xi$ small, the asymptotic formula for the power is obtained simply by equating the product $\xi N^{1/2}$ to its presumed limit.

As indicated in [5], this double passage to the limit, which is the basis of what we like to call the method of alternatives infinitely close to the hypothesis tested, while being useful in deducing optimal $C(\alpha)$ tests, provides simplifications of formulas for the asymptotic power which, in some cases, are too sweeping. In what follows, formulas obtained under this double passage to the limit will be described as the first approximation to the power of the tests. The method of obtaining the second approximation to the same power is also described in [5]. The passage to the limit used to obtain the second approximation is a more conventional one. It is based on the assumptions that $\xi$ is fixed and that $N \to \infty$.

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Each optimal $C(\alpha)$ criterion is a sum of $N$ identically distributed random variables. Ordinarily, estimates of nuisance parameters are involved in each summand and, as a result, these summands are not mutually independent. However, because of the fact that the estimates used have stochastic limits as $N \rightarrow \infty$, the difference between the criterion actually computable and the one that could be computed if the values of the nuisance parameters were known tends to zero in probability. In consequence the asymptotic distribution of the criterion, properly normed, appears normal with mean zero and unit variance. The problem of evaluating the second approximation to the power reduces, then, to the evaluation, under a fixed alternative hypothesis, of two quantities: the asymptotic mean of the criterion considered and its asymptotic variance. In principle, this is very simple. However, the process involves certain questions that thus far have not been fully explored.

As found in [4], the first approximation to the power of the test is not dependent upon the identity of the estimators of nuisance parameters, provided they satisfy the condition of being "locally root $N$ consistent." However, this is not so with the second approximation to the power and, in frequent cases where more than one locally root $N$ consistent estimate is available, the question arises as to which of them is preferable. This general question splits into two more particular questions: which of the available estimates insures the larger power of the test and which of these estimates provides the better approximaιion to the power attained by the test.

Several particular cases that have been investigated indicate that the answers to these questions are somewhat unexpected. For example, contrary to our expectation, it appears that in some cases at least a particular estimator which is only locally root $N$ consistent is preferable to another estimator which is consistent in the large, and so forth.

For the above reasons the information regarding the second approximation to power assembled below is not complete and there is no certainty that the formulas given are optimal.

As indicated in [5], we consider a sequence $\{U_N\}$ of experimental units for which precipitation amounts in the target and, perhaps, also some predictor variables are observed. These experimental units may be storms, as in SCUD, or fixed periods of time as in Grossversuch III. As determined by a system of randomization, each of the units $U_N$ may be subject to seeding or not. The randomization may be either in pairs or unrestricted, with a preassigned probability $\pi$ for seeding. As found in [5], under the assumption that, given the predictors, the precipitation amounts corresponding to two members of a randomized pair are conditionally independent, the optimal $C(\alpha)$ criterion corresponding to randomized pairs has the same form as for unrestricted randomization with $\pi = 1/2$. Therefore, only formulas for unrestrictedly randomized experiments need be listed.

Our basic assumption is that, whether seeded or not, to each experimental unit there corresponds a possibly positive probability that the target precipitation will be zero and that this probability may be affected by seeding: the seeding may either "trigger" the rainfall which otherwise would not have fallen, or may prevent the rainfall. Probably it is realistic to assume that the probability of zero rain depends on the values of the predictor variables. However, thus far, this situation has not been treated and the formulas given below depend upon the assumption that the probability of rain in the target does not depend upon the predictors. With these formulas, then, the effect of predictor variables can be studied only through a partitioning of the experimental units into several groups, each characterized by values of predictor variables in some conveniently selected intervals, perhaps "low," "medium," and "high," and so forth.

Our further general assumption is that, given that the target precipitation is not zero, it has a conditional probability density, joint with the predictors if such are available. The specialization of this density determines the several different cases considered below.

Our final general assumption is that, if seeding has an effect on the distribution of nonzero target precipitation, then this effect is multiplicative. This means that, whatever the predictors, the conditional expectation of seeded target rainfall is equal to that not seeded, multiplied by a factor independent of the predictors. It is assumed that seeding has no other effect on the distribution of the nonzero target rainfall.

Tests of three distinct hypotheses are considered, as follows.

$H_1$ is the hypothesis that seeding does not affect the probability, say $\vartheta$, of nonzero rain in the target.

$H_2$ denotes the hypothesis that seeding has no effect on the distribution of nonzero precipitation in the target. In other words, $H_2$ assumes that seeding does not affect the target precipitation averaged per "rainy" experimental unit, which may or may not be accompanied by a change in the frequency of such units.

$H_3$ means the hypothesis that seeding does not affect the target precipitation averaged per experimental unit.

It will be noticed that, in a sense, $H_1$ and $H_2$ are independent: either may be true or false and this does not imply anything on the other. On the other hand, $H_3$ depends on $H_1$ and $H_2$. If both $H_1$ and $H_2$ are true then $H_3$ is true also. However, $H_3$ may be true while both $H_1$ and $H_2$ are false. For example, seeding may trigger precipitation which would not fall otherwise but, at the same time, may decrease the precipitation per rainy day, with the net effect on rainfall per experimental unit being zero. On the other hand, cases may exist where seeding has a positive effect both on the frequency of some rain in the target and on the average rainfall per rainy unit. On occasion these two effects may be slight and difficult to detect, while their combination may be noticeable.

For testing the hypotheses $H_1$ and $H_2$ we give the criteria $Z_1$ and $Z_2$ which are optimal $C(\alpha)$ criteria. The optimal $C(\alpha)$ criterion for testing $H_3$ is rather complicated and is not given here. Instead we give a criterion $Z_3$ which is an easy combination of $Z_1$ and $Z_2$ so adjusted that both the asymptotic significance probability and the asymptotic power can be obtained by following the same rules as those for $Z_1$ and $Z_2$.

Thus, the three criteria $Z_1$, $Z_2$, $Z_3$ are all asymptotically normal and, if the observations yield $Z_i = z_i$, for $i = 1, 2, 3$, the corresponding significance probability has the asymptotic expression

$$P(z_i) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-|z_i|}^{+|z_i|} e^{-x^2/2}\, dx. \tag{1.1}$$

With the number of observations $N$ of order of 50 or higher, this formula is reasonably reliable. Its value can be obtained from any of the many published tables of the normal integral. The asymptotic power, say $\beta(\xi, \alpha)$, of any of the three tests also depends upon the normal integral and we have

$$\beta(\xi, \alpha) = 1 - \frac{1}{\sqrt{2\pi}} \int_{\tau - a\nu(\alpha)}^{\tau + a\nu(\alpha)} e^{-x^2/2}\, dx. \tag{1.2}$$

Here the symbol $\nu(\alpha)$ is the "two tail normal deviate" corresponding to the intended level of significance $\alpha$. In other words, if $z_i$ in formula (1.1) is replaced by $\nu(\alpha)$, the result will be

$$P\{\nu(\alpha)\} = \alpha. \tag{1.3}$$

The symbol $\tau$ is the so called noncentrality parameter

$$\tau = \xi\Delta[N\pi(1 - \pi)]^{1/2}, \tag{1.4}$$

where $\xi$ is a conventional measure of the effectiveness of seeding and $\Delta$ depends upon the hypothesis to be tested and on the design of the experiment. Finally, $a$ is a coefficient generally depending upon $N$, which requires specification in any particular case.

It follows that, for each particular test, the following formulas are needed: the formula for the calculation of $Z$ from the results of the experiment, the specification of $\xi$, the conventional measure of the effect of seeding, and the formulas for $\Delta$ and $a$. As mentioned above, it is intended to provide two approximations for the power of each test. It so happens that the first approximation to $a$ is always unity. Therefore, for each test considered there is need for two formulas for $\Delta$ and for just one for $a$. The first approximation $\Delta$ will be denoted by just this letter, occasionally with an identifying subscript. The second approximations will be denoted by $\Delta^*$ and $a^*$, respectively.

## 2. Optimal $C(\alpha)$ test of hypothesis $H_1$ that seeding does not affect the frequency of rain in the target

In this case, the optimal $C(\alpha)$ criterion is a modification of the classical $\chi$, namely,

$$Z_1 = \frac{n_1 n_4 - n_2 n_3}{[N\pi(1 - \pi)(n_1 + n_2)(n_3 + n_4)]^{1/2}}, \tag{2.1}$$

where $n_1$, $n_2$, $n_3$, and $n_4$ are the numbers of experimental units in a $2 \times 2$ classification as shown below.

| | Seeded | Not | Totals |
|---|---|---|---|
| With rain | $n_1$ | $n_2$ | $n_1 + n_2$ |
| Without rain | $n_3$ | $n_4$ | $n_3 + n_4$ |
| Total | | | $N = n_1 + n_2 + n_3 + n_4$ |

The modification is due to the fact that, in the present case, the probability $\pi$ is a known number.

REMARK. In the numerical computations shown in [1], Yates' correction for continuity was applied to equation (2.1). Thus, the numerator becomes $n_1n_4 - n_2n_3 \pm n/2$, with the plus sign used when $n_1n_4 - n_2n_3$ is positive, the minus sign when it is negative.

The most convenient measure of the effect of seeding on the frequency of days with rain is the difference between the probability of rain with seeding, say $\vartheta_1$, and the probability of rain without seeding, say $\vartheta_0$. Thus, the conventional measure of the effect of seeding may be set $\xi = \vartheta_1 - \vartheta_0$. With this particular convention, the first approximation $\Delta$ is

$$\Delta = [\vartheta_0(1 - \vartheta_0)]^{-1/2}. \tag{2.2}$$

However, in defining the test of the hypothesis $H_3$ it will be convenient to adopt a different convention. Namely, it will be convenient to consider a factor, say $\rho_1 = 1 + \xi_1$, by which the seeding, so to speak, multiplies the no seeding probability of rain $\vartheta_0$, so that $\vartheta_1 = \vartheta_0(1 + \xi_1)$. The new conventional measure of effectiveness of seeding is then

$$\xi_1 = (\vartheta_1 - \vartheta_0)/\vartheta_0. \tag{2.3}$$

If $\xi_1$ is adopted as the conventional measure of the effect of seeding, then the corresponding formula for $\Delta$ will be, say

$$\Delta_1 = [\vartheta_0/(1 - \vartheta_0)]^{1/2}. \tag{2.4}$$

This formula will be used in the sequel.

The second approximation formulas are

$$\Delta^* = [(1 - \pi)\vartheta_1(1 - \vartheta_1) + \pi\vartheta_0(1 - \vartheta_0) + (1 - 2\pi)^2(\vartheta_1 - \vartheta_0)^2]^{-1/2}, \tag{2.5}$$

$$a^* = [\bar{\vartheta}(1 - \bar{\vartheta})]^{1/2}\,\Delta^*, \tag{2.6}$$

with $\bar{\vartheta} = \pi\vartheta_1 + (1 - \pi)\vartheta_0$.

Table I was constructed to illustrate the difference in precision provided by the first and the second approximations to power, the adequacy of the second approximation and the difficulty of detecting the effect of seeding on the frequency of rain. The particular problem considered is typical for the use of the power function: to determine the number, say $N$, of observations insuring a preassigned probability $\beta$ that an indicated effect $\xi$ of seeding will be found significant at a preassigned $100\alpha$ per cent significance level. The preassigned $\xi$, $\alpha$, and $\beta$ characterize the desired precision of the experiment. We choose $\xi = 0.1$, $\alpha = 0.1$, and $\beta = 0.9$, and set $\pi = 1/2$. Formula (1.4), combined with (2.2), yields then

$$N = \frac{(20\tau)^2}{\vartheta_0(1 - \vartheta_0)}, \tag{2.7}$$

which, with $\tau = 2.927$ obtained from tables of the normal distribution, yields the numbers $N_1$ given in the second column of table I. Column four gives the values of $N$, labeled $N_2$, obtained through the use of the second approximation to the power. It is seen that the numbers $N_2$ are always larger than $N_1$. The third column of table I gives the second approximation to power computed

TABLE I

NUMBER OF OBSERVATIONS SUPPOSED TO INSURE $\beta(\vartheta_1 = \vartheta_0 + 0.1, \alpha = 0.1) = 0.9$; SECOND APPROXIMATION TO POWER AND TO $N$; AND EMPIRICAL VALIDATION BY MEANS OF ACTUAL FREQUENCIES OF REJECTION OF $H_1$ IN 1000 MONTE CARLO TRIALS EACH WITH $N_2$ OBSERVATIONS

| | | | | Empirical Validation | | | |
|---|---|---|---|---|---|---|---|
| | | Power (2nd approx.) | | $H_1$ true $\vartheta = \vartheta_0$ | | $H_1$ false $\vartheta_1 = \vartheta_0 + 0.1$ | |
| $\vartheta_0$ (1) | $N_1$ (1st approx.) (2) | with $N_1$ obs. (3) | $N_2$ (2nd approx.) (4) | $\alpha = 0.05$ (5) | $\alpha = 0.10$ (6) | $\alpha = 0.05$ (7) | $\alpha = 0.10$ (8) |
| 0.1 or 0.8 | 308 | 0.794 | 433 | 0.053 | 0.101 | 0.855 | 0.906 |
| 0.2 or 0.7 | 548 | 0.857 | 639 | 0.052 | 0.108 | 0.827 | 0.906 |
| 0.3 or 0.6 | 720 | 0.880 | 776 | 0.044 | 0.088 | 0.821 | 0.909 |
| 0.4 or 0.5 | 822 | 0.893 | 849 | 0.054 | 0.089 | 0.833 | 0.900 |

assuming the number of observations equal to $N_1$. It is seen that for the extreme values of $\vartheta_0$, either 0.1 or 0.8, the result of this calculation is noticeably less than the intended power, namely it is 0.8 against the desired 0.9. With more central values of $\vartheta_0$ this difference becomes negligible. Columns five and six refer to the situation where $H_1$ is true and indicate the precision with which the actual distribution of the criterion $Z_1$ is approximated by the normal. The numbers given in these columns represent actual frequencies with which, in 1000 Monte Carlo experiments, the criterion $|Z_1|$, calculated using $N_2$ observations, exceeded either $\nu(\alpha = 0.05) = 1.96$ or $\nu(\alpha = 0.10) = 1.645$, respectively. It is seen that the observed frequencies agree with those expected. Columns seven and eight give the empirical power of the test corresponding to the case where $\vartheta_1 = \vartheta_0 + 0.1$. Here again the empirical frequencies resulted from 1000 Monte Carlo experiments, each with $N_2$ observations. It is seen that the frequencies in the last column, corresponding to the intended level of significance $\alpha = 0.10$, agree quite well with the intended power of 0.9. This, then, validates the calculations based on the second approximation to the power function. The general conclusion is that the second approximation formula for power is quite reliable over a broad range of values of $\vartheta_0$, likely to cover all cases to be encountered in practical experimentation. The range of approximate validity of the first approximation is substantially narrower.

REMARK. In the present problem there is just one nuisance parameter $\vartheta_0$, the probability of some rain in the target without seeding. The criterion $Z_1$ of formula (2.1) was obtained through the use of a particular estimate of $\vartheta_0$, namely $\hat{\vartheta}_0 = (n_1 + n_2)/N$. If $H_1$ is true, then this estimator is consistent. However, if $H_1$ is not true and $\vartheta_1 \neq \vartheta_0$, then the stochastic limit of $\hat{\vartheta}_0$ is $\pi\vartheta_1 + (1 - \pi)\vartheta_0 = \vartheta_0 + \pi(\vartheta_1 - \vartheta_0)$ and it is seen that $\hat{\vartheta}_0$ is only locally root $N$ consistent. Easy analysis shows that the estimator $n_2/(n_2 + n_3)$ is consistent in the large, and we expected it to be preferable to $\hat{\vartheta}_0$. However, it is not uniformly better than $\hat{\vartheta}_0$. Still another locally root $N$ consistent estimator, namely $n_1(1 - \pi)/(n_1 + n_3) + n_2\pi/(n_2 + n_4)$ may be optimal, but its apparent advantage over $\hat{\vartheta}_0$ appears numerically insignificant.

The final point that table I is meant to illustrate is that differences in the probability of rain between seeded and not seeded experimental units are rather difficult to detect. In several experiments known to us the probability $\vartheta_0$ of rain without seeding is of the order of 0.6. Table I indicates that if seeding increases this probability by one unit in the first decimal, say from 0.6 to 0.7, then, in order to insure the chance of nine in ten of finding this increase significant at the conservative ten per cent, it is necessary to have close to 800 observations, which appears prohibitive. Even if seeding changes the frequency of rain by two units in the first decimal, the requisite number of observations would be about 200. These calculations indicate little hope that an experiment of moderate size will detect the effect of seeding on the frequency of rain per experimental unit. On the other hand, a combination of this effect with that on the average rain per rainy unit may be quite substantial and, hopefully, more easy to detect. This is the motivation for the efforts to test the hypothesis $H_3$.

## 3. Optimal $C(\alpha)$ tests of the hypothesis $H_2$ that seeding does not affect the conditional distribution of rainfall, given that this rainfall is not zero

In this section, the alternative to $H_2$ against which the indicated $C(\alpha)$ tests are optimal is that the effect of seeding is multiplicative. The convenient conventional measure of the effectiveness of seeding is then $\xi = \rho - 1$, where $\rho$ denotes the factor by which the seeding "multiplies" the precipitation that, in any given set of conditions, would have been expected without seeding.

3.1. *Case* (*i*). *No predictor variables are available.* Following our own experience and that of some other authors, that, typically, the distribution of nonzero precipitation can be satisfactorily approximated by the Gamma density

$$\frac{\delta^\gamma}{\Gamma(\gamma)} x^{\gamma-1}e^{-\delta x}, \tag{3.1}$$

we adopt this formula as the basis for our deductions. The optimal $C(\alpha)$ criterion is

$$Z_2 = \frac{\hat{\gamma}^{1/2} n_s n_c (\bar{x}_s - \bar{x}_c)}{(n_s \bar{x}_s + n_c \bar{x}_c)[N\pi(1 - \pi)]^{1/2}}, \tag{3.2}$$

and the first approximation $\Delta$ is

$$\Delta = \gamma^{1/2}. \tag{3.3}$$

Here $\hat{\gamma}$ denotes the maximum likelihood estimate of the shape parameter $\gamma$ in (3.1) so obtained as to be consistent whether the hypothesis tested $H_2$ is true or not. The relevant equation is

$$\log \hat{\gamma} - \frac{\Gamma'(\hat{\gamma})}{\Gamma(\hat{\gamma})} = [n_s(\log \bar{x} - \overline{\log} \, x)_s + n_c(\log \bar{x} - \overline{\log} \, x)_c]/N, \tag{3.4}$$

where all logarithms are natural logarithms, while bars indicate averaging and the subscripts $s$ and $c$ refer to seeded and control experimental units, respectively. Thus, for example, $n_s$ stands for the number of experimental units with some rain which were actually seeded, and $\bar{x}_s$ the average amount of precipitation per such unit. Also $(\log \bar{x} - \overline{\log} \, x)_s$ means the logarithm of $\bar{x}_s$ less the mean $\log x$ computed for seeded experimental units, and so forth.

Equation (3.4) is solved conveniently using the tables due to Chapman [7].

REMARK. The evaluation of Grossversuch III data discussed in [1] are based on an estimate of $\gamma$ which is different from that resulting from equation (3.4). As far as the significance level is concerned, both estimates are asymptotically equivalent. Also, both lead to the same first approximation to the power. However, formulas for the second approximation to power, those given below, are much simpler for the estimate of $\gamma$ obtained through the solution of (3.4).

Formulas for computing the second approximation to the power of the test are:

$$\Delta^* = \frac{\gamma^{1/2}}{[(1 - \pi)(1 + \xi)^2 + \pi + \gamma(1 - 2\pi)^2 \xi^2]^{1/2}} \tag{3.5}$$

and

$$a^* = (1 + \pi\xi)\Delta^*/\gamma^{1/2}. \tag{3.6}$$

The maximum likelihood estimate of the quotient $\rho$ of mean seeded to mean nonseeded precipitation per rainy experimental unit is simply the quotient $\bar{x}_s/\bar{x}_c$. The estimate of the percentage change in precipitation due to seeding is then $(\bar{x}_s/\bar{x}_c - 1)100$.

3.2. *Case (ii). In addition to the rainfall in the target, the observations include some predictor variables.* In this section we consider the cases where the test of the hypothesis of no effect of seeding on the distribution of nonzero target precipitation is performed using some predictor variables. About these predictors it is specifically assumed that their distribution is not affected by seeding. The theory developed in [5] refers to the case where either the target rainfall itself, perhaps measured in inches, or some transformation thereof, has a certain property which it will be convenient to label CNL, connoting conditional normal distribution with linear regression. The exact definition of CNL is: (a) linearity of regression on predictors, (b) given the predictors, conditional normality of the distribution with constant variance.

The formulas given below refer to two alternative situations. Case (iia) is characterized by the assumption that the property CNL is possessed by the nonzero target precipitation itself, measured in inches or millimeters, and so forth. In case (iib) it is assumed that the $r$th root of the target precipitation (for example the square root or the cube root) has the property CNL. In order to simplify the notation, the symbol $y_j$ will be used to denote the particular rainfall variable that has the property CNL. The subscript $j$ will refer to the $j$th experimental unit considered, say to the $j$th storm, and so forth. In case (iia) $y_j$ will mean the target precipitation from the $j$th storm (given that it is not zero) measured in the original units. In case (iib) the same letter $y_j$ will mean the $r$th root of the target precipitation.

The $i$th predictor variable referring to the $j$th experimental unit will be denoted by $x_{ij}$, with $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, N$. Also, it will be convenient to use a single bold face letter $\mathbf{x}_j$, with subscript $j$, to denote the totality of the predictors referring to the $j$th experimental unit, $\mathbf{x}_j = (x_{1j}, x_{2j}, \cdots, x_{mj})$. According to the basic assumptions, given $\mathbf{x}_j$, the expectation of the unseeded precipitation variable is a linear combination of the predictors, say

$$\eta_c(\mathbf{x}_j) = \sum_{i=0}^{m} \alpha_i x_{ij}, \qquad \text{with} \quad x_{0j} \equiv 1, \tag{3.7}$$

where the $\alpha_i$ are unknown nuisance parameters. With seeding, the same expectation is, say,

$$\eta_s(\mathbf{x}_j) = \rho\eta_c(\mathbf{x}_j), \tag{3.8}$$

where $\rho = 1 + \xi$ represents the effect of seeding.

3.3. *Case* (*iia*). *The target precipitation itself possesses the property* CNL. The optimal $C(\alpha)$ criterion for testing $H_3$ has the form

$$Z_2 = \frac{(1 - \pi) \sum_s[y_j - y(\mathbf{x}_j)]y(\mathbf{x}_j) - \pi\sum_c[y_j - y(\mathbf{x}_j)]y(\mathbf{x}_j)}{\bar{\sigma}[\pi(1 - \pi) \sum y^2(\mathbf{x}_j)]^{1/2}} \tag{3.9}$$

where

$$y(\mathbf{x}_j) = \sum_{i=0}^{m} a_i x_{ij}, \tag{3.10}$$

represents an estimate of $\eta(\mathbf{x}_j)$, with $a_i$ standing for a locally root $N$ consistent estimate of $\alpha_i$ and $\bar{\sigma}^2$ a locally root $N$ consistent estimate of $\sigma^2$, the conditional variance of $y$, given the predictors.

In order to obtain the maximum likelihood estimate of the factor, say $\rho$, by which the seeding is supposed to multiply the expected unseeded target precipitation, one has to minimize the sum

$$\sum_c(y_j - b_0 - \sum_{i=1}^{m} b_i x_{ij})^2 + \sum_s[y_j - \rho(b_0 + \sum_{i=1}^{m} b_i x_{ij})]^2. \tag{3.11}$$

The minimization is required with respect to the unrestricted variation of the coefficients $b_i$ and also of $\rho$. Using the digital computer, one begins with a sequence of trial values of $\rho$, say $1 \pm 0.1$, $\pm 0.2$, and so forth. For each such value $\rho$ the expression (3.12) is minimized with respect to the variation of the coefficients $b$, which requires only the solution of a system of $m + 1$ linear equations. Let $\Phi(\rho)$ stand for the minimum of (3.11) so obtained for a given $\rho$. Next, the values of $\Phi(\rho)$ are plotted against $\rho$ and the minimizing $\hat{\rho}$ is obtained either visually or by interpolation. The $\hat{\rho}$ is the maximum likelihood estimate of the factor $\rho$. (See [3], p. 368.)

The asymptotic properties of the test as determined by the "double" passage to the limit, with $\xi N^{1/2}$ tending to a constant, do not depend upon the identity of the estimators of the nuisance parameters, provided all of them are at least locally root $N$ consistent. With any such estimates the first approximation to power is obtained with

$$\Delta^2 = E[\eta^2(\mathbf{x})]/\sigma^2. \tag{3.12}$$

On the other hand, both the actual power of the test and its second approximation do depend upon the estimates of the nuisance parameters. One possibility is to assume that the hypothesis $H_2$ is true and to use all the observations, both with seeding and without, in order to obtain the ordinary least squares estimates of the regression coefficients $\alpha_i$ and of the residual variance $\sigma^2$. Actually, the evaluation of SCUD data reported in appendix B was performed using this method. In this case, the second approximation to the power is obtained using

$$\Delta^* = \frac{E\eta^2}{\{\sigma^2 E\eta^2 + \xi^2\pi(1 - \pi)[E\eta^4 - E^2\eta^2)] + \xi^2(1 - 2\pi)^2 E\eta^4)\}^{1/2}}, \tag{3.13}$$

where, for brevity, $\eta = \eta(\mathbf{X})$ and the expectations $E$ are taken with respect to the variation of the predictors $\mathbf{X}$. Also we have

$$a^* = \bar{\sigma}^*\Delta^*/(E\eta^2)^{1/2}, \tag{3.14}$$

where $\bar{\sigma}^*$ denotes the stochastic limit of $\bar{\sigma}$.

However, if the evaluation includes the maximum likelihood estimate of $\rho$, then one can use in (3.9) the expressions (3.10) with coefficients $a_i$ replaced by $b_i$, the maximum likelihood estimates of $\alpha_i$. Also $\bar{\sigma}^2$ may now be the maximum likelihood estimate of the residual variance. In both cases the estimates will be root $N$ consistent in the large, rather than just locally and one might expect a beneficial effect on the power. However, calculations show that through these changes the only modification in the expression of $\Delta^*$ in (3.13) is that the last term in the denominator is replaced by

$$\xi^2(1 - \pi)^2 E\eta^4(\mathbf{X}). \tag{3.15}$$

Finally, it appears that, by a proper choice of estimates of regression coefficients, the last terms in the denominator in (3.20) can be replaced by zero. For this purpose it is sufficient to set $a_k = b_k[\pi + (1 - \pi)\hat{\rho}]$. With this choice of the locally root $N$ consistent estimates of coefficients $\alpha_k$, as $N$ increases, the increase in the second approximation to the power is at least equal to that which would result from the use of the maximum likelihood estimates of the same coefficients. When $\pi = 1/2$, there is no difference. However, if $\pi \neq 1/2$, the gain in power may be substantial.

3.4. *Case (iib). Property* CNL *is possessed by the rth root of target precipitation.* In this case, the multiplicativity of the effect of seeding, as defined at the outset, implies that the seeding modifies not only the regression of $y$ on $\mathbf{x}$, but also the conditional variance of $y$ given $\mathbf{x}$, and, with obvious notation, we have

$$\eta_s(\mathbf{x}) = \rho^{1/r}\, \eta_c(\mathbf{x}) = q\eta_c(\mathbf{x}), \tag{3.16}$$

say, and

$$\sigma_s = q\sigma_c, \tag{3.17}$$

where, as before, $\rho = 1 + \xi$.

This latter equation causes a considerable modification in the formula for the criterion $Z$, namely,

$$Z_2 = \frac{(1 - \pi) \sum_s f(\mathbf{x}_j, y_j) - \pi \sum_c f(\mathbf{x}_j, y_j) - \bar{\sigma}^2(n_s - N\pi)}{\bar{\sigma}\{\pi(1 - \pi)[2N\bar{\sigma}^2 + \sum y^2(\mathbf{x}_j)]\}^{1/2}}, \tag{3.18}$$

where

$$f(\mathbf{x}_j, y_j) = y_j[y_j - y(\mathbf{x}_j)], \tag{3.19}$$

and all other symbols have the same meaning as in case (iia), except that now $y_j$ means the $r$th root of the target precipitation. In order to simplify the formulas that follow, it will be convenient to use the symbol $\eta_k$ to denote $E\eta^k(\mathbf{X})$.

The first approximation to the power of the test is obtained using the expression

$$\Delta = \{2\sigma^2 + \eta_2\}^{1/2}/\sigma r. \tag{3.20}$$

The maximum likelihood estimate of $\rho = 1 + \xi$ is obtained by a process exactly similar to that indicated for the case (iia) except that in the present situation the function to be minimized is

$$\rho^{2n_s/rN}\{\textstyle\sum_c (y_j - \sum b_i x_{ij})^2 + \sum_s (y_j/\rho^{1/r} - \sum b_i x_{ij})^2\}. \tag{3.21}$$

By substituting the minimizing values $\hat{b}_i$ of $b_i$ in the expression in curly brackets in (3.21) and by dividing the result by $N$ the maximum likelihood estimate $\hat{\sigma}^2$ of $\sigma^2$ is obtained, which is consistent in the large. On the assumption that all the estimates of nuisance parameters used in (3.18) are consistent in the large, for example $\hat{\sigma}^2$, $\hat{\rho}$, and $\hat{b}_i$ obtainable as just described, the formulas for the second approximation to the power are

$$\Delta^* = \frac{(q + 1)\sigma^2 + q\eta_2}{r[A + (q - 1)^2(1 - \pi)B]^{1/2}}, \tag{3.22}$$

with

$$A = 2\sigma^4[(1 - \pi)q^4 + \pi] + \sigma^2\eta_2[(1 - \pi)q^2(2q - 1)^2 + \pi], \tag{3.23}$$

(3.24) $$B = (\eta_4 - \eta_2^2)q^2 + (1 - \pi)[(q + 1)\sigma^2 + q\eta_2]^2,$$

and

(3.25) $$a^* = r\Delta^*\sigma[2\sigma^2 + \eta_2]^{1/2}/[(q + 1)\sigma^2 + q\eta_2].$$

## 4. Combined test of the hypothesis $H_3$ that cloud seeding does not affect the target precipitation averaged per experimental unit

The criterion $Z_3$ advanced for testing $H_3$ is a linear combination of criteria $Z_1$ and $Z_2$ so adjusted as to be sensitive to departures from $H_3$ but not to departures from $H_1$ and $H_2$, if these latter departures are jointly consistent with $H_3$.

Let $A$ and $B$ denote two numbers such that $A^2 + B^2 = 1$. Then the general form of $Z_3$ is $AZ_1 + BZ_2$. Under the "double" passage to the limit discussed earlier, the criteria $Z_1$ and $Z_2$ are independent, normal and have variances equal to unity. As a result, under the same passage to the limit, the asymptotic distribution of $Z_3$ is also normal with unit variance. Consider the case where cloud seeding has a double effect: it multiplies the probability $\vartheta_0$ of some rain in the target by a factor $\rho_1 = 1 + \xi$, and also it multiplies the average of nonzero target precipitation by another factor $\rho_2 = 1 + \xi_2$. As a result, the target precipitation averaged per experimental unit will be multiplied by the product, say

(4.1) $$\rho_3 = \rho_1\rho_2 = 1 + \xi_1 + \xi_2 + \xi_1\xi_2.$$

Under the "double" passage to the limit which we now adopt, both $\xi_1$ and $\xi_2$ are of the order of $N^{-1/2}$ and, therefore, we may write

(4.2) $$\rho_3 = 1 + \eta,$$

with $\eta = \xi_1 + \xi_2$. Our problem is to determine the coefficients $A$ and $B$ so that the expectation of $Z_3$ be asymptotically proportional to $\eta$ and independent of either $\xi_1$ or $\xi_2$.

Under the double passage to the limit the expectations of $Z_1$ and $Z_2$ are proportional to $\xi_1\Delta_1$ and $\xi_2\Delta_2$ where $\Delta_1$ is given by (2.3) and $\Delta_2$ by either (3.3) or (3.12) or (3.20), depending on the availability of predictor variables and, if they are available, on the conditional distribution of the target precipitation. It follows that, whatever $A$ and $B$ might be, the asymptotic mean of $Z_3$ is proportional to

(4.3) $$A\Delta_1\xi_1 + B\Delta_2\xi_2 = (A\Delta_1 - B\Delta_2)\xi_1 + B\Delta_2\eta.$$

In order that this expectation be independent of $\xi_1$ taken by itself, it is sufficient to set

(4.4) $$A = \Delta_2/(\Delta_1^2 + \Delta_2^2)^{1/2},$$
$$B = \Delta_1/(\Delta_1^2 + \Delta_2^2)^{1/2},$$

so that

(4.5) $$Z_3 = (\Delta_2 Z_1 + \Delta_1 Z_2)/(\Delta_1^2 + \Delta_2^2)^{1/2}.$$

The first order approximation to the power of $Z_3$ is obtained from (1.2), with

$$\tau = \eta\Delta_3[\pi(1 - \pi)N]^{1/2}, \tag{4.6}$$

where

$$\Delta_3 = B\Delta_2 = \Delta_1\Delta_2/(\Delta_1^2 + \Delta_2^2)^{1/2}. \tag{4.7}$$

The question that immediately arises in connection with the criterion $Z_3$ is whether, and under what conditions, its power exceeds that of either $Z_1$ or $Z_2$. *A priori* it is obvious that this will not be true in all cases. For example, if the seeding decreases the frequency of rain but increases the average amount of precipitation per rainy observational unit, the net effect of seeding $\eta = \xi_1 + \xi_2$ may be zero or very small in absolute value, while $|\xi_1|$ and $\xi_2$ are considerable. Thus, the situation is interesting when $\xi_1$ and $\xi_2$ are of the same sign, say positive. It is also clear *a priori*, that the falsehood of $H_3$ can be more easily detectable than that of either $H_1$ or $H_2$, when the degrees to which these two hypotheses are false are, so to speak, of comparable magnitude. In other words, if $\xi_1$ is very small compared to $\xi_2$ (or vice versa) then it is intuitively clear that the falsehood of $H_2$ (or that of $H_1$) will be more easily detectable than that of $H_3$. The exact characterization of the situation is obtained by solving two inequalities

$$\begin{aligned} \Delta_3(\xi_1 + \xi_2) &> \Delta_1\xi_1 \\ \Delta_3(\xi_1 + \xi_2) &> \Delta_2\xi_2. \end{aligned} \tag{4.8}$$

The result is

$$\frac{(\Delta_1^2 + \Delta_2^2) - \Delta_1}{\Delta_1} < \frac{\xi_1}{\xi_2} < \frac{\Delta_2}{(\Delta_1^2 + \Delta_2^2)^{1/2} - \Delta_2}. \tag{4.9}$$

If the effects of seeding on the frequency of rain, as measured by $\xi_1$, and on the target precipitation averaged per rainy unit, as measured by $\xi_2$, satisfy the double inequality (4.9), then the criterion $Z_3$ is more powerful than either $Z_1$ or $Z_2$, but not otherwise.

## 5. Concluding remarks

The present note summarizes the techniques developed and used in the Statistical Laboratory. The whole problem of statistical methodology of evaluating rain stimulation experiments is not considered completely solved and the techniques indicated constitute, more or less, a progress report. In addition to various problems mentioned in this note and also in our other contributions to the present Proceedings, we would like to mention the following.

Practically all our techniques are based on the assumption that the possible effect of seeding on rainfall is multiplicative. This assumption was adopted because of occasional pronouncements of knowledgeable meteorologists. However, it must be obvious that the assumption of multiplicativity of the effect of seeding requires verification.

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