ESTIMATING THE TRAJECTORY OF A POPULATION

NATHAN KEYFITZ
UNIVERSITY OF CHICAGO

1. Introduction

The population trajectory with which this paper is concerned is the changes in numbers of people which would result from certain birth and death rates, specific by age and sex, when these are applied to a given initial age distribution. Considering either sex, let the probability that a person aged x to $x + 4$ years at last birthday will survive for five years be ${}_5L_{x+5}/{}_5L_x$; the age specific fertility rates for age x to $x + 4$ be m_x; the number of individuals alive at the time t be ${}_5K_x^{(t)}$. (In general the superscript on the upper right in parentheses will refer to time, that on the lower right to the initial age of the interval, that on the lower left to the length of the interval.) This paper will estimate the path of ${}_5K_x^{(t)}$ subject to two restrictions: (a) that the age specific rates of birth and death are constant; and (b) that their application is without any random variation, which is to say the argument will be entirely in terms of expected values. The extension of the method to rates varying in time, and for probabilistic as well as for deterministic models, is important, and one hopes that it will attract the attention of the good minds needed to cope with it.

The conditions of birth and death set up in the preceding paragraph enable us to show the relation between the population at time $t + 1$ and that at time t, where t is in units of five years, as a set of linear, first order, homogeneous, difference equations with constant coefficients

$$
\begin{aligned}
(1.1)\qquad \frac{{}_5L_0}{2\ell_0}\{[{}_5K_{15}^{(t)} + {}_5K_{15}^{(t+1)}]m_{15} + [{}_5K_{20}^{(t)} + {}_5K_{20}^{(t+1)}]m_{20} \\
+ \cdots + [{}_5K_{40}^{(t)} + {}_5K_{40}^{(t+1)}]m_{40}\} &= {}_5K_0^{(t+1)} \\
\frac{{}_5L_5}{{}_5L_0}\,{}_5K_0^{(t)} &= {}_5K_5^{(t+1)} \\
\vdots \qquad & \qquad \vdots \\
\frac{{}_5L_{85}}{{}_5L_{80}}\,{}_5K_{80}^{(t)} &= {}_5K_{85}^{(t+1)}.
\end{aligned}
$$

The project of which this is a part received initial financing from the Ford Foundation and the Population Council, and a grant from the National Science Foundation has supported the actual computation.

In the first equation of the set, the age specific fertility rates m_{15}, and so forth, are applied to the average number of persons alive in age groups over the period, $[{}_5K_{15}^{(t)} + {}_5K_{15}^{(t+1)}]/2$; the factor allowing for the children born over five years surviving to the end of the period is ${}_5L_0/\ell_0$. The survivorships in the left side of the equations below the first can be substituted for ${}_5K_{15}^{(t+1)}$, and so forth, on the left side of the first equation. When this is done the entire set can be written as

$$L\bar{K}^{(t)} = \bar{K}^{(t+1)}, \tag{1.2}$$

where $\bar{K}^{(t)}$ is the (vertical) vector of unknowns at time t,

$$\bar{K}^{(t)} = \begin{Bmatrix} {}_5K_0^{(t)} \\ {}_5K_5^{(t)} \\ \cdot \\ \cdot \\ \cdot \\ {}_5K_{85}^{(t)} \end{Bmatrix}, \tag{1.3}$$

and L is the matrix of the coefficients in the difference equations, fully specified in the set (1.1).

There are two approaches to (1.2). The most obvious is to think of it as an equation in scalars, and solve by

$$\begin{aligned} L\bar{K}^{(0)} &= \bar{K}^{(1)} \\ L\bar{K}^{(1)} &= \bar{K}^{(2)}. \end{aligned} \tag{1.4}$$

This is what is done in the conventional population projection, when the initial age specific rates are held constant, and each point of time is obtained from the preceding. It can be somewhat broadened by writing the solution in the form $\bar{K}^{(t)} = L^t\bar{K}^{(0)}$, which enables one to study separately the effects of the pattern of fertility and mortality contained in L and L^t, in separation from the initial age distribution $\bar{K}^{(0)}$.

Another approach to the solution of (1.1) or (1.2) is to analyze the matrix of the coefficients in terms of its latent roots. This is what is done in factor analysis. In our case it will turn out that the first three roots contain most of the meaning; this being so, there may be some simplification, and hence understanding, to be gained by the more analytical form of solution.

It is convenient to express the following argument in terms of a partition of L. In fact L and its powers may be expressed as four square submatrices by a split at a point which corresponds to the highest age of reproduction

$$L = \begin{bmatrix} M & 0 \\ A & B \end{bmatrix}, \qquad L^2 = \begin{bmatrix} M^2 & 0 \\ AM + BA & B^2 \end{bmatrix}, \qquad L^t = \begin{bmatrix} M^t & 0 \\ A_t & B^t \end{bmatrix}. \tag{1.5}$$

The important feature of the partitioning is that the upper right submatrix is zero, and it remains zero at all positive integral powers of L, which may be verified by application of the rules of matrix multiplication. If now the age vector $\{\bar{K}\}$ also be split at the same highest age of reproduction, say as

(1.6)
$$\bar{K} = \begin{Bmatrix} K \\ D \end{Bmatrix},$$

then $\bar{K}^{(t)} = L^t\bar{K}^{(0)}$ becomes

(1.7)
$$\bar{K}^{(t)} = \begin{Bmatrix} K^{(t)} \\ D^{(t)} \end{Bmatrix} = \begin{bmatrix} M^t & 0 \\ A_t & B^t \end{bmatrix} \begin{Bmatrix} K^{(0)} \\ D^{(0)} \end{Bmatrix} = \begin{Bmatrix} M^tK^{(0)} \\ A_tK^{(0)} + B^tD^{(0)} \end{Bmatrix},$$

from which it is evident that the A, B, and D, referring to the ages beyond reproduction, never affect the younger ages. Our subsequent work will be in terms of the matrix M^t and the vector $\{K^{(t)}\}$ covering the fertile ages only, rather than L^t and $\{\bar{K}^{(t)}\}$ which deal with the whole of life. We start with a numerical example in 15 year age groups, so that a 3×3 matrix will describe ages 0 to 44 inclusive.

To derive a usable 3×3 matrix we start with M, compiled in five year age groups, and cube it to obtain M^3 (this is the only part done by computer). Table I shows M^3 for Taiwan females, 1961, in which a partitioning into nine submatrices is sketched out. In parallel to the notation introduced earlier, where $MK^{(0)} = K^{(1)}$, we write the same symbols with a bar below when they refer to the 3×3 matrix, that is to the condensation into 15 year age groups; thus $\underline{M}\underline{K}^{(0)} = \underline{K}^{(1)}$ stands for

(1.8)
$$\begin{bmatrix} \underline{m}_{11} & \underline{m}_{12} & \underline{m}_{13} \\ \underline{m}_{21} & \underline{m}_{22} & \underline{m}_{23} \\ \underline{m}_{31} & \underline{m}_{32} & \underline{m}_{33} \end{bmatrix} \begin{bmatrix} \underline{k}_1 \\ \underline{k}_2 \\ \underline{k}_3 \end{bmatrix} = \begin{bmatrix} \underline{k}_1^{(1)} \\ \underline{k}_2^{(1)} \\ \underline{k}_3^{(1)} \end{bmatrix}.$$

To find the $\underline{m}_{ij}$ from the $m_{ij}^{(3)}$ and the $\underline{k}_i$, we have two conditions to meet.

The first is that each age group in the population as projected by the small matrix be equal to the sum of the corresponding three ages in the population as projected three times by the large one, for example, $\underline{k}_1^{(1)} = k_1^{(3)} + k_2^{(3)} + k_3^{(3)}$, since both sides stand for the population under 15 years of age, 15 years in time after the zero point.

The second condition is that the cohorts already alive at time zero each move into the following 15 year age group over the 15 year period of projection by the small matrix. This last is met by making each element of the small matrix depend only on the corresponding partition of M^3; thus, $\underline{m}_{11}$ will depend only on the upper left 3×3 of M^3 for example. The equation representing the two conditions is

(1.9)
$$m_{11}^{(3)}k_1 + m_{12}^{(3)}k_2 + m_{13}^{(3)}k_3 + m_{21}^{(3)}k_1 + m_{22}^{(3)}k_2 + m_{23}^{(3)}k_3 + m_{31}^{(3)}k_1 + m_{32}^{(3)}k_2 + m_{33}^{(3)}k_3 = \underline{m}_{11}(k_1 + k_2 + k_3),$$

and on solving for $\underline{m}_{11}$ we have

(1.10)
$$\underline{m}_{11} = \{[m_{11}^{(3)} + m_{21}^{(3)} + m_{31}^{(3)}]k_1 + [m_{12}^{(3)} + m_{22}^{(3)} + m_{32}^{(3)}]k_2 + [m_{13}^{(3)} + m_{23}^{(3)} + m_{33}^{(3)}]k_3\}/(k_1 + k_2 + k_3).$$

A similar argument applies to the remainder of the $\underline{m}_{ij}$.

The construction of the 3×3 matrix now requires only a decision on the k's. One possibility is the use of the initial given population in five year age groups,

TABLE I

TAIWAN FEMALES 1961
MATRIX M, M^2, AND M^3

Power 1, Year 5

	0.	0.	0.0524870	0.3402551	0.6842584	0.6814820	0.4660676	0.2739481	0.0933682
	0.9787761	0.	0.	0.	0.	0.	0.	0.	0.
	0.	0.9955489	0.	0.	0.	0.	0.	0.	0.
	0.	0.	0.9958544	0.	0.	0.	0.	0.	0.
	0.	0.	0.	0.9931794	0.	0.	0.	0.	0.
	0.	0.	0.	0.	0.9914279	0.	0.	0.	0.
	0.	0.	0.	0.	0.	0.9898080	0.	0.	0.
	0.	0.	0.	0.	0.	0.	0.9871292	0.	0.
	0.	0.	0.	0.	0.	0.	0.	0.9830797	0.
$K^{(0)}$	941917.	865780.	628352.	451027.	452576.	397055.	350994.	291156.	231310.
$K^{(1)}$	1031655.	921926.	861926.	625747.	447951.	448696.	393008.	346476.	286230.

Power 2, Year 10

	0.	0.0522534	0.3388445	0.6795913	0.6756403	0.4613174	0.2704221	0.0917884	0.
	0.	0.	0.0513731	0.3330335	0.6697358	0.6670183	0.4561758	0.2681338	0.0913866
	0.9744195	0.	0.	0.	0.	0.	0.	0.	0.
	0.	0.9914218	0.	0.	0.	0.	0.	0.	0.
	0.	0.	0.9890620	0.	0.	0.	0.	0.	0.
	0.	0.	0.	0.9846657	0.	0.	0.	0.	0.
	0.	0.	0.	0.	0.9813232	0.	0.	0.	0.
	0.	0.	0.	0.	0.	0.9770683	0.	0.	0.
	0.	0.	0.	0.	0.	0.	0.9704267	0.	0.
$K^{(2)}$	1175256.	1009759.	917822.	858353.	621479.	444111.	444123.	387950.	340614.

Power 3, Year 15

	0.0511444	0.3373363	0.6767740	0.6710320	0.4573630	0.2676660	0.0906070	0.	0.
	0.	0.0511444	0.3316529	0.6651677	0.6613006	0.4515265	0.2646827	0.0898403	0.
	0.	0.	0.0511444	0.3315512	0.6667547	0.6640493	0.4541454	0.2669403	0.0909798
	0.9703799	0.	0.	0.	0.	0.	0.	0.	0.
	0.	0.9846596	0.	0.	0.	0.	0.	0.	0.
	0.	0.	0.9805837	0.	0.	0.	0.	0.	0.
	0.	0.	0.	0.9746300	0.	0.	0.	0.	0.
	0.	0.	0.	0.	0.9686928	0.	0.	0.	0.
	0.	0.	0.	0.	0.	0.9605361	0.	0.	0.
$K^{(3)}$	1413211.	1150312.	1005265.	914017.	852499.	616152.	439584.	438407.	381386.

but this offends the objective of making projection matrices depend only on the mortality and fertility, and not on the initial conditions of age distribution. Hence we have chosen the k's for (1.10) as those age distributions which would be reached by the continued operation of the matrix M. These, known as the stable population (see below), are only available to within a multiplicative constant, and it is seen from (1.10) that they are only needed to within such a constant.

The $\underline{M}$ resulting from equations such as (1.10) is

$$(1.11)\qquad \underline{M} = \begin{bmatrix} 0.44324 & 1.62701 & 0.46043 \\ 0.97791 & 0 & 0 \\ 0 & 0.96872 & 0 \end{bmatrix}$$

for Taiwan females 1961. The interpretation to be put on the elements of the matrix is essentially unchanged: $\underline{m}_{11}$ or 0.44324 is the number of girl children alive at the end of the 15 year period expected to be born to a girl child now 0 to 14 years of age at last birthday; $\underline{m}_{21}$ or 0.97791 is the probability of survival over the next fifteen years of the same girl randomly selected out of the initial population 0 to 14 years of age. The method is perfectly general and could be used conveniently on the computer to proceed from single years of age to five year age groups, that is, to condense a 45 × 45 matrix to one 9 × 9.

Thinking again of the set of linear recurrence equations in the k's, that is, $\underline{K}^{(1)} = \underline{M}K^{(0)}$, the condition of stability is met if the operation of the equations (or multiplication by the equivalent matrix) on a distribution of a population among the several ages produces the same distribution for the next time period except for a multiplicative constant λ. The three equations, $\underline{M}\underline{K}^{(t)} = \underline{K}^{(t+1)}$, are written out for the Taiwan females 1961 as

$$\begin{aligned} 0.44324\underline{k}_1^{(t)} + 1.62701\underline{k}_2^{(t)} + 0.46043\underline{k}_3^{(t)} &= \underline{k}_1^{(t+1)}, \\ 0.97791\underline{k}_1^{(t)} \qquad\qquad\qquad &= \underline{k}_2^{(t+1)}, \\ 0.96872\underline{k}_2^{(t)} \qquad\quad &= \underline{k}_3^{(t+1)}. \end{aligned} \tag{1.12}$$

At stability,

$$\underline{k}_1^{(t+1)} = \lambda\underline{k}_1^{(t)}, \qquad \underline{k}_2^{(t+1)} = \lambda\underline{k}_2^{(t)}, \qquad \underline{k}_3^{(t+1)} = \lambda\underline{k}_3^{(t)}. \tag{1.13}$$

Substituting these values for the three $\underline{k}^{(t+1)}$ on the right side of (1.12), and then transposing the terms in λ to the left side, the condition for consistency of the three simultaneous equations in $\underline{k}_1$, $\underline{k}_2$, and $\underline{k}_3$ is seen to be the determinantal relation

$$\begin{vmatrix} 0.44324 - \lambda & 1.62701 & 0.46043 \\ 0.97791 & -\lambda & 0 \\ 0 & 0.96872 & -\lambda \end{vmatrix} = 0. \tag{1.14}$$

Values of λ satisfying (1.14) will give nontrivial solutions in the k's. Expanding the determinant in (1.14), we have

$$\lambda^3 - 0.44324\lambda^2 - 1.59107\lambda - 0.43617 = 0, \tag{1.15}$$

which is referred to as the characteristic equation of the system of recurrence equations and of the matrix of its coefficients. The roots of (1.15) are

$$\begin{aligned} \lambda_1 &= 1.6045, \\ \lambda_2 &= -0.8357, \\ \lambda_3 &= -0.3256. \end{aligned} \tag{1.16}$$

That there can be at most one positive root of (1.15) follows from Descartes' rule of signs; in fact, the equation which emerges in such analysis always has only one change of sign, after the term in the highest power of λ, whether the matrix be 3 × 3, 9 × 9, 45 × 45, or some other size.

Entering λ_1 in the equations (1.12) and (1.13) and solving for the stable age distribution $\underline{k}_1$, $\underline{k}_2$, $\underline{k}_3$ gives, to within a multiplicative constant,

$$\{\underline{K}_1\} = \begin{Bmatrix} \underline{k}_1 \\ \underline{k}_2 \\ \underline{k}_3 \end{Bmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{Bmatrix} 1{,}130{,}000 \\ 688{,}800 \\ 415{,}800 \end{Bmatrix}, \tag{1.17}$$

and similarly, by entering λ_2 and λ_3 in the same equations (1.12) and (1.13),

$$\{\underline{K}_2\} = \begin{Bmatrix} +0.6984 \\ -0.8172 \\ +0.9473 \end{Bmatrix}, \qquad \{\underline{K}_3\} = \begin{Bmatrix} +0.1117 \\ -0.3358 \\ +1.0000 \end{Bmatrix}. \tag{1.18}$$

It turns out, however, that although the set of λ's is complete, the columns of $\{\underline{K}_1\}$, $\{\underline{K}_2\}$, and $\{\underline{K}_3\}$ derived from them are not the only stable vectors. There is a set of rows, again one corresponding to each λ, which we will call $[\underline{H}_1]$, $[\underline{H}_2]$, and $[\underline{H}_3]$, such that

$$[\underline{H}_i]\underline{M} = \lambda_i[\underline{H}_i], \qquad i = 1, 2, 3. \tag{1.19}$$

For the first of the λ's the equations are $\underline{M}[\underline{H}_1] = \lambda_1[\underline{H}_1]$ or

$$\begin{aligned} 0.44324h_1 + 0.97791h_2 \qquad\qquad &= 1.6045h_1, \\ 1.62701h_1 + \qquad\qquad 0.96872h_3 &= 1.6045h_2, \\ 0.46043h_1 \qquad\qquad\qquad\qquad &= 1.6045h_3. \end{aligned} \tag{1.20}$$

The resulting $[\underline{H}_1]$, a horizontal stable vector, is

$$[\underline{H}_1] = [2.5744 \quad 3.0566 \quad 0.7388], \tag{1.21}$$

and the other two horizontal stable vectors are

$$\begin{aligned} [\underline{H}_2] &= [0.6984 \quad -0.9137 \quad -0.3848], \\ [\underline{H}_3] &= [1.0 \quad -0.7862 \quad -1.4141]. \end{aligned} \tag{1.22}$$

We continue with the arithmetic, promising to show later the logic that lies behind it. Corresponding to each of λ_1, λ_2, and λ_3, a stable 3×3 matrix can be found, called a spectral component Z, which enjoys the same property as the eigenvectors $\underline{M}\underline{Z} = \lambda\underline{Z}$. In fact $\underline{Z}_i$ is nothing more than the column vector $\{\underline{K}_i\}$ multiplied by the row vector $[\underline{H}_i]$ and normalized by dividing by the scalar $[\underline{H}_i]\{\underline{K}_i\}$,

$$\underline{Z}_i = \frac{\{\underline{K}_i\}[\underline{H}_i]}{[\underline{H}_i]\{\underline{K}_i\}}, \qquad i = 1, 2, 3. \tag{1.23}$$

For Taiwan females 1961

$$\underline{Z}_1 = \frac{\{\underline{K}_1\}[\underline{H}_1]}{[\underline{H}_1]\{\underline{K}_1\}} = \begin{bmatrix} 0.5467 & 0.6491 & 0.1569 \\ 0.3332 & 0.3956 & 0.0956 \\ 0.2011 & 0.2388 & 0.0577 \end{bmatrix}. \tag{1.24}$$

Applying the same argument to the negative roots, we secure for each of them a stable row vector and a stable column vector, and hence stable matrices $\underline{Z}_2$ and $\underline{Z}_3$

$$Z_2 = \begin{bmatrix} 0.5607 & -0.7336 & -0.3089 \\ -0.6561 & 0.8583 & 0.3615 \\ 0.7605 & -0.9950 & -0.4190 \end{bmatrix},$$

(1.25)

$$Z_3 = \begin{bmatrix} -0.1074 & 0.0845 & 0.1520 \\ +0.3229 & -0.2539 & -0.4571 \\ -0.9616 & 0.7562 & 1.3613 \end{bmatrix}.$$

The stable matrices $\underline{Z}_1$, $\underline{Z}_2$, and $\underline{Z}_3$ constitute a useful decomposition of $\underline{M}$; not only is

(1.26) $$\underline{M} = \lambda_1\underline{Z}_1 + \lambda_2\underline{Z}_2 + \lambda_3\underline{Z}_3$$

but, much more generally, for any polynomial function f,

(1.27) $$f(\underline{M}) = f(\lambda_1)\underline{Z}_1 + f(\lambda_2)\underline{Z}_2 + f(\lambda_3)\underline{Z}_3.$$

This is Sylvester's theorem. Applying it to $f(\underline{M}) = \underline{M}^t$,

(1.28) $$\underline{M}^t = \lambda_1^t\underline{Z}_1 + \lambda_2^t\underline{Z}_2 + \lambda_3^t\underline{Z}_3.$$

Since $|\lambda_2|/|\lambda_1| = 0.203$, and $|\lambda_3|/|\lambda_1| = 0.521$, we can expect that $\underline{M}^t$ will be approximated more and more closely by $\lambda_1^t\underline{Z}_1$ as t increases.

Let us check this by calculating $\underline{M}^t$ directly. When one wants to find a high power of a scalar it is easiest to square it, then square the square, and so forth, and so with a matrix. The first, second, fourth, eighth, sixteenth, and thirty second powers of $\underline{M}$ for Taiwan females 1961 are shown below (table II). Some

TABLE II

POWERS OF $\underline{M}$ FOR TAIWAN FEMALES 1961

$$\underline{M} = \begin{bmatrix} 0.4432 & 1.6270 & 0.4604 \\ 0.9779 & 0 & 0 \\ 0 & 0.9687 & 0 \end{bmatrix} \qquad \underline{M}^2 = \begin{bmatrix} 1.7875 & 1.1671 & 0.2041 \\ 0.4334 & 1.5910 & 0.4502 \\ 0.9473 & 0 & 0 \end{bmatrix}$$

$$\underline{M}^4 = \begin{bmatrix} 3.8942 & 3.9430 & 0.8902 \\ 1.8908 & 3.0372 & 0.8048 \\ 1.6933 & 1.1056 & 0.1933 \end{bmatrix} \qquad \underline{M}^8 = \begin{bmatrix} 24.127 & 28.314 & 6.812 \\ 14.468 & 17.570 & 4.283 \\ 9.011 & 10.248 & 2.434 \end{bmatrix}$$

$$\underline{M}^{16} = \begin{bmatrix} 1053.2 & 1250.4 & 302.2 \\ 641.9 & 762.2 & 184.2 \\ 387.6 & 460.2 & 111.2 \end{bmatrix} \qquad \frac{\underline{M}^{32}}{1000} = \begin{bmatrix} 2029 & 2409 & 582 \\ 1237 & 1468 & 355 \\ 747 & 887 & 214 \end{bmatrix}$$

suggestion of stabilization appears in $\underline{M}^4$. By $\underline{M}^{32}$ the result is stable to about four significant digits; the ratio of the jth element in the ith row of $\underline{M}^{33}$ to the corresponding element of $\underline{M}^{32}$ is

(1.29) $$\frac{\underline{m}_{ij}^{(33)}}{\underline{m}_{ij}^{(32)}} = \begin{matrix} 1.6044 & 1.6043 & 1.6045 \\ 1.6043 & 1.6044 & 1.6042 \\ 1.6044 & 1.6044 & 1.6043 \end{matrix}, \qquad \begin{matrix} i = 1, 2, 3, \\ j = 1, 2, 3. \end{matrix}$$

Thus any two elements of $\underline{M}^{32}$ would provide the dominant root. If we know $\underline{m}_{11}^{(32)}$ and $\underline{m}_{21}^{(32)}$, then taking account of the zeros in $\underline{M}$,

$$\lambda_1 = \frac{\underline{m}_{21}^{(33)}}{\underline{m}_{21}^{(32)}} = \frac{m_{21}\underline{m}_{11}^{(32)}}{\underline{m}_{21}^{(32)}} = \frac{0.9779\underline{m}_{11}^{(32)}}{\underline{m}_{21}^{(32)}} = 1.6043, \tag{1.30}$$

as appears in the second row first element of the array above. Conversely, knowing $\underline{Z}_1$, we could have multiplied it by $\lambda_1^{32} = (1.6044)^{32} = 3,715,500$ and found

$$\lambda_1^{32}\underline{Z}_1 = 1000\begin{bmatrix} 2031 & 2412 & 583 \\ 1238 & 1470 & 355 \\ 747 & 887 & 214 \end{bmatrix}, \tag{1.31}$$

which is nearly identical to $\underline{M}^{32}$ as shown in table II.

2. General analysis of the matrix

Having worked a simple arithmetical example, we are now ready to retrace our steps and go over the same argument in more systematic fashion. Again we think of a characteristic or stable vector, a fixed point in the space of the age distribution expressed in homogeneous coordinates, defined by the property of being unaltered when premultiplied by the matrix operator M. To find it, we solve the set of linear equations $M\{K\} = \lambda\{K\}$, or set out more fully,

$$\begin{aligned} m_{11}k_1 + m_{12}k_2 + \cdots + m_{1n}k_n &= \lambda k_1, \\ m_{21}k_1 + m_{22}k_2 + \cdots + m_{2n}k_n &= \lambda k_2, \\ &\vdots \\ m_{n1}k_1 + m_{n2}k_2 + \cdots + m_{nn}k_n &= \lambda k_n, \end{aligned} \tag{2.1}$$

where the m's are the elements of M, and $k_1, k_2, \cdots, k_n$ are the number of persons in the first age group, the second age group, and so forth, represented as a vertical vector

$$\{K\} = \begin{Bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{Bmatrix}. \tag{2.2}$$

A set of k's which are not all zero may be found which satisfy the set (2.1) only if the several equations are consistent, the condition for which is

$$\begin{vmatrix} m_{11} - \lambda & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} - \lambda & \cdots & m_{2n} \\ \vdots & & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} - \lambda \end{vmatrix} = 0, \tag{2.3}$$

or more compactly

$$|M - \lambda I| = 0. \tag{2.4}$$

This is a polynomial equation of the nth degree in λ, for which there are n roots, in our application all distinct. We call them $\lambda_1, \lambda_2, \cdots, \lambda_n$. The latent roots for Mexico females 1960 and United States females 1962 are given in table III.

TABLE III

LATENT ROOTS FOR MEXICO FEMALES 1960 AND UNITED STATES FEMALES 1962

	Mexico 1960	United States 1962
λ_1	1.1830	1.0988
λ_2, λ_3	$0.4406 \pm 0.7918i$	$0.3060 \pm 0.7910i$
λ_4, λ_5	$0.0042 \pm 0.7353i$	$0.0264 \pm 0.5330i$
λ_6, λ_7	$0.4478 \pm 0.4925i$	$-0.4095 \pm 0.3910i$
λ_8, λ_9	$0.5885 \pm 0.1747i$	$-0.4723 \pm 0.1619i$

When we replace λ by one of these, say λ_i, in the set (2.1), we could find a solution in the k, say the vector,

$$\{K_i\} = \begin{Bmatrix} k_{1i} \\ \cdot \\ \cdot \\ \cdot \\ k_{ni} \end{Bmatrix}, \qquad i = 1, 2, \cdots, n, \tag{2.5}$$

where one of the elements of the vector is arbitrary. It is also easy to show that the elements of $\{K_i\}$ are proportional to the transpose of the cofactors of $|M - \lambda I|$. But these general methods are not necessary for the particular matrix M with which we are dealing.

The elements of the stable vectors are readily calculable by recurrence from M and the several λ_i. For the vector equation $M\{K_i\} = \lambda_i\{K_i\}$ may be expressed as a series of recurrence relations. Because the relevant nonzero elements of M are in the subdiagonal, it follows from row by column multiplication that $(m_{j+1,j})(k_j) = \lambda_i k_{j+1}$ for the jth row ($j > 1$), or

$$k_{j+1} = \frac{m_{j+1,j}}{\lambda_i} k_j, \tag{2.6}$$

the subdiagonal elements being survivorships

$$m_{j+1,j} = \frac{{}_5L_{5(j)}}{{}_5L_{5(j-1)}}. \tag{2.7}$$

If we arbitrarily take k_1 as ${}_5L_0/\sqrt{\lambda_i}$, and apply the recurrence equation successively, then the stable vector corresponding to λ_i is

$$\{K_i\} = \begin{Bmatrix} {}_5L_0\lambda_i^{-1/2} \\ {}_5L_5\lambda_i^{-3/2} \\ {}_5L_{10}\lambda_i^{-5/2} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}. \tag{2.8}$$

The n vectors $\{K_i\}$ are the stable populations. If they are arranged side by side to constitute a matrix

$$K = [K_1K_2\cdots K_n] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}, \tag{2.9}$$

then, because of the fact that the column components of the matrix K are $\{K_i\}$, and $M\{K_i\} = \{K_i\}\lambda_i$, it follows that

$$MK = K \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{2.10}$$

or $MK = K\Lambda$, where Λ stands for the diagonal matrix of the roots of the characteristic equation. For the right side of (2.10) on the row by column rule will give a matrix whose first column is λ_1K_1, whose second is λ_2K_2, and so forth, and from (2.1) the columns of MK will be equal to these. From (2.10) on multiplying on the right by K^{-1}, we obtain

$$MKK^{-1} = K\Lambda K^{-1} \tag{2.11}$$

or $M = K\Lambda K^{-1}$.

The factoring of M indicated in (2.11) is important in what follows. For $M^2 = (K\Lambda K^{-1})(K\Lambda K^{-1}) = K\Lambda(K^{-1}K)\Lambda K^{-1} = K\Lambda^2K^{-1}$, and repeating the procedure proves that $M^t = K\Lambda^tK^{-1}$, if t is integral > 0. Thus, to raise M to a power, given K and K^{-1}, one need merely raise Λ, whose powers are calculated simply as

$$\Lambda^t = \begin{bmatrix} \lambda_1^t & 0 & \cdots & 0 \\ 0 & \lambda_2^t & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \cdots & \lambda_n^t \end{bmatrix}. \tag{2.12}$$

This applies only when the roots are distinct, a condition found empirically to hold in demographic work.

All of what precedes can be said also for multiplication on the left by a new set of vectors $[H_i]$, horizontal this time. Corresponding to (2.1) we would have

$[H]M = \lambda[H]$, and the condition of consistency would be the same equation in λ, $|M - \lambda I| = 0$, producing the same latent roots. For each latent root λ_i we could find $[H_i]$ determined, except for a constant multiplier, by the set of linear equations (2.13), or more compactly $[H_i]M = \lambda_i[H_i]$. These give a new set of vectors $[H_i]$ not related in any obvious way to the $\{K_i\}$, but in fact a very simple relation is shown below to connect the two sets.

Recurrence equations may be obtained for the successive elements of the stable horizontal vector as for the vertical vector, but the equations are somewhat more complicated. For any λ the equations are (where only m_{1j} and $m_{i+1,i}$ can be nonzero)

$$(2.13)\qquad [h_1\, h_2\, h_3 \cdots h_n]\begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{21} & m_{22} & m_{23} & \cdots & m_{2n} \\ m_{31} & m_{32} & m_{33} & \cdots & m_{3n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ m_{n1} & m_{n2} & m_{n3} & \cdots & m_{nn} \end{bmatrix} = \lambda[h_1\, h_2\, h_3 \cdots h_n],$$

so that the typical equation is

$$(2.14)\qquad h_1 m_{1i} + h_{i+1} m_{i+1,i} = \lambda h_i.$$

One of the $\underline{h}$ is at our choice, and it seems convenient to put $h_1 = 1$. Then

$$(2.15)\qquad h_{i+1} = \frac{\lambda h_i - m_{1i}}{m_{i+1,i}} = \frac{\lambda h_i - m_{1i}}{{}_5L_{5i}/{}_5L_{5(i-1)}}, \qquad = \frac{{}_5L_{5(i-1)}}{{}_5L_{5i}}(\lambda h_i - m_{1i}).$$

Given the life table and the matrix M this is easily worked out for any one of the λ. For complex λ_r, $[H_r]$ will consist of complex elements, except for the arbitrary h_1.

There is a stable row vector corresponding to each of the λ's, and if the $[H_i]$ are arrayed one beneath the other to make

$$(2.16)\qquad H = \begin{Bmatrix} H_1 \\ \cdot \\ \cdot \\ \cdot \\ H_n \end{Bmatrix},$$

then

$$(2.17)\qquad \begin{Bmatrix} H_1 \\ H_2 \\ \cdot \\ \cdot \\ \cdot \\ H_n \end{Bmatrix}\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}\begin{Bmatrix} H_1 \\ H_2 \\ \cdot \\ \cdot \\ \cdot \\ H_n \end{Bmatrix}$$

or $HM = \Lambda H$. Multiplying by H^{-1} on the left gives

$$(2.18)\qquad M = H^{-1}\Lambda H,$$

this being the same equation as (2.11), since it will be shown that $HK = I$.

We only know the stable vectors to within a multiplicative constant. If the column vector for the ith root as originally calculated is $\{\bar{K}_i\}$, and the row vector for the same root is $[\bar{H}_i]$, then it may be convenient to write

$$[H_i] = \frac{[\bar{H}_i]}{([\bar{H}_i]\{\bar{K}_i\})^{1/2}}, \qquad \{K_i\} = \frac{\{\bar{K}_i\}}{([\bar{H}_i]\{\bar{K}_i\})^{1/2}}, \tag{2.19}$$

the expression within the square root being a scalar quantity. The product $[H_i]\{K_i\}$ in (2.19) equals one.

The stable vectors possess an important orthogonality property: if $i \neq j$, then $[H_j]\{K_i\} = 0$. For

$$M\{K_i\} = \lambda_i\{K_i\} \tag{2.20}$$

by the definition of the stable column vector. Multiplying (2.20) on the left by $[H_j]$ gives

$$[H_j]M\{K_i\} = \lambda_i[H_j]. \tag{2.21}$$

But it is also true by the definition of the stable row vector that

$$[H_j]M = \lambda_j[H_j]. \tag{2.22}$$

Multiplying (2.22) on the right by $\{K_i\}$ gives

$$[H_j]M\{K_i\} = \lambda_j[H_j]\{K_i\}. \tag{2.23}$$

Equations (2.21) and (2.23) are the same on the left and differ by having scalars λ_i and λ_j respectively on the right, where $\lambda_i \neq \lambda_j$. Subtracting (2.23) from (2.21) and dividing through by $\lambda_i - \lambda_j$ gives

$$[H_j]\{K_i\} = 0. \tag{2.24}$$

Among other uses, this result enables us to express any age distribution, say the arbitrary column of frequencies $\{K'\}$, as a sum of the stable vectors each multiplied by a constant

$$\{K'\} = c_1\{K_1\} + c_2\{K_2\} + \cdots + c_n\{K_n\}. \tag{2.25}$$

To find c_i premultiply (2.25) by the normalized row vector $[H_i]$; the result is

$$[H_i]\{K'\} = c_i. \tag{2.26}$$

In terms of the unnormalized vectors

$$c_i = \frac{[\bar{H}_i]\{K'\}}{[\bar{H}_i]\{K_i\}}. \tag{2.27}$$

Equation (2.25) makes the analysis of changes easy in age distribution under a given regime of fertility and mortality. Multiplying (2.25) by M on the left,

$$\begin{aligned} M\{K'\} &= c_1M\{K_1\} + c_2M\{K_2\} + \cdots + c_nM\{K_n\} \\ &= \lambda_1c_1\{K_i\} + \lambda_2c_2\{K_2\} + \cdots + \lambda_nc_n\{K_n\}, \end{aligned} \tag{2.28}$$

since $M\{K_1\} = \lambda_1\{K_1\}$ and so forth; by a t-fold repetition of the multiplication,

$$M^t\{K'\} = \lambda_1^tc_1\{K_1\} + \lambda_2^tc_2\{K_2\} + \cdots + \lambda_n^tc_n\{K_n\}, \tag{2.29}$$

where t is integral. Insofar as λ_1 is larger than any other of the λ's in absolute value, the first term on the right side of (2.29) will be of increasing relative magnitude. The age distribution will approach closer and closer to $c_1\lambda_1^t\{K_1\}$, which is called "the" stable population, irrespective of the shape of the original age distribution $\{K'\}$. This is the ergodic theorem—the tendency of a population to forget its past ages under the action of a fixed regime of mortality and fertility. Strong ergodicity—to which the preceding discussion has been confined—concerns the way in which the forgetting takes place under a fixed regime of mortality and fertility. Weak ergodicity [7], due to Coale and Lopez, which we will study elsewhere, concerns a given changing regime of mortality and fertility.

By virtue of the orthogonality of the stable vectors, $\bar{H}\bar{K}$ is a diagonal matrix, and when the vectors are in their normal form, $HK = I_n$. For the ith row of H is $[H_i]$, and the jth column of K is K_j; the product of these is unity when $i = j$ and zero when $i \neq j$. Hence $K^{-1} = H$, and recalling (2.11), $M = K\Lambda K^{-1}$, we have, on substituting H for K^{-1},

$$M = K\Lambda H. \tag{2.30}$$

Now Λ may be looked on as a sum of matrices each containing one element,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}. \tag{2.31}$$

Substituting this sum for Λ in (2.30), gives

$$\begin{aligned} M &= \lambda_1\{K_1\}[H_1] + \lambda_2\{K_2\}[H_2] + \cdots + \lambda_n\{K_n\}[H_n] \\ &= \lambda_1 Z_1 + \lambda_2 Z_2 + \cdots + \lambda_n Z_n, \end{aligned} \tag{2.32}$$

all other components vanishing.

The same argument that we have here used to decompose $M = K\Lambda K^{-1}$ also applies to $M^t = K\Lambda^t K^{-1}$. Substituting $K^{-1} = H$, we have $M^t = K\Lambda^t H$, and decomposing Λ^t into n matrices each with a single nonzero term λ_i^t gives

$$M^t = K\Lambda^t H = \sum_i \lambda_i^t\{K_i\}[H_i]. \tag{2.33}$$

Equation (2.33) may be multiplied by a constant scalar, say c_t. If a number of equations such as (2.33) for values $t, t-1, \cdots, 1$, are each multiplied by an arbitrary constant and then added, we have

$$\begin{aligned} &c_t M^t + c_{t-1}M^{t-1} + \cdots + c_0 I_n \\ &\qquad = \sum_i (c_t\lambda_i^t + c_{t-1}\lambda_i^{t-} + \cdots + c_0)\{K_i\}[H_i], \end{aligned} \tag{2.34}$$

which is to say that $f(M)$, any polynomial function of the matrix M, may be expanded as

$$f(M) = f(\lambda_1)Z_1 + f(\lambda_2)Z_2 + \cdots + f(\lambda_n)Z_n, \tag{2.35}$$

where the $Z_i = \{K_i\}[H_i]$ are known as spectral operators. If $f(\lambda)$ is the characteristic function $|M - \lambda I|$, then

$$f(\lambda_1) = f(\lambda_2) = \cdots = f(\lambda_n) = 0, \tag{2.36}$$

and hence, $f(M) = 0$. This proves, for distinct λ, the remarkable Cayley-Hamilton theorem that a matrix satisfies its own characteristic equation.

Spectral operators are idempotent: they are equal to powers of themselves. The Z_i satisfy this condition, for

$$\begin{aligned} Z_i^2 &= (\{K_i\}[H_i])(\{K_i\}[H_i]) = K_i([H_i]\{K_i\})[H_i] \\ &= \{K_i\}[H_i], \end{aligned} \tag{2.37}$$

since we normalized according to (2.19) to make $[H_i]\{K_i\} = 1$. By repeating the argument, $Z_i^k = Z_i$, where k is any positive integer.

Spectral operators are orthogonal. Z_i and Z_j also satisfy the condition

$$Z_iZ_j = \{K_i\}[H_i]\{K_j\}[H_j] = \{K_i\}([H_i]\{K_j\})[H_j] = 0, \tag{2.38}$$

since the expression $[H_i]\{K_j\}$ in parentheses is zero by the orthogonality property (2.24), with $i \neq j$.

Finally the sum of the spectral operators is the unit matrix. The total of our Z's,

$$\sum_i Z_i = \{K_1\}[H_1] + \{K_2\}[H_2] + \cdots + \{K_n\}[H_n], \tag{2.39}$$

turns out to be equal to KH. For KH may be broken down by separating out the column components of K and the row components of H into separate matrices

$$K = [K_1\, 0 \cdots 0] + [0\, K_2 \cdots 0] + \cdots + [0\, 0 \cdots K_n], \tag{2.40}$$

$$H = \begin{Bmatrix} H_1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ H_2 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} + \cdots + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ H_n \end{Bmatrix} \tag{2.41}$$

and multiplying; $\{K_i\}[H_i] = Z_i$ are the only nonzero terms in the product; this proves that $\sum Z_i = HK$, and we know that $HK = I$.

3. Iterative methods of calculation

The first of the spectral components Z_1 of the 11×11 matrix for United States males 1960 is shown in table IV. Since λ_1 is larger in modulus than the other roots, and since by (2.33)

$$M^t = \lambda_1^t Z_1 + \lambda_2^t Z_2 + \lambda_3^t Z_3 + \cdots, \tag{3.1}$$

TABLE IV

PRINCIPAL SPECTRAL COMPONENT Z_1 FOR UNITED STATES MALES 1960

$M =$

0.	0.	0.	0.051057	0.515253	0.613432	0.410644	0.232723	0.113051	0.043990	0.024790
0.994486	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.997540	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.995489	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.992114	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.991139	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.990865	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.988114	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.981799	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.970469	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.951451	0.

$\frac{M^{128}}{\lambda_1^{128}} = Z_1 =$

0.160506	0.179141	0.199327	0.222245	0.240380	0.185754	0.108710	0.055410	0.024597	0.009434	0.003585
0.143810	0.160506	0.178592	0.199126	0.215375	0.166431	0.097402	0.049646	0.022038	0.008453	0.003212
0.129246	0.144251	0.160506	0.178960	0.193564	0.149576	0.087538	0.044619	0.019806	0.007597	0.002887
0.115918	0.129376	0.143955	0.160506	0.173603	0.134152	0.078511	0.040018	0.017764	0.006813	0.002589
0.103612	0.115642	0.128672	0.143467	0.155174	0.119911	0.070176	0.035769	0.015878	0.006090	0.002314
0.092522	0.103263	0.114900	0.128110	0.138564	0.107075	0.062665	0.031941	0.014178	0.005438	0.002066
0.082595	0.092185	0.102572	0.114366	0.123698	0.095588	0.055942	0.028514	0.012657	0.004855	0.001845
0.073529	0.082066	0.091314	0.101813	0.110121	0.085096	0.049801	0.025384	0.011268	0.004322	0.001642
0.065040	0.072591	0.080771	0.090058	0.097407	0.075271	0.044052	0.022453	0.009967	0.003823	0.001453
0.056867	0.063470	0.070622	0.078741	0.085167	0.065813	0.038516	0.019632	0.008715	0.003343	0.001270
0.048747	0.054406	0.060537	0.067497	0.073005	0.056415	0.033016	0.016829	0.007470	0.002865	0.001089

Z_1 is computed readily by dividing M^t by λ_1^t to obtain

$$\frac{M^t}{\lambda_1^t} = Z_1 + \frac{\lambda_2^t}{\lambda_i} Z_2 + \cdots, \tag{3.2}$$

so that if t is sufficiently large, $Z_1 = M^t/\lambda_1^t$. For the United States male 1960 data λ_1^{128} is 600156.21. It is easy to verify arithmetically on Z_1 the property $Z_1^2 = Z_1$.

From the fact that $Z_i = \{K_i\}[H_i]$, its rows are proportional to one another; it can have no nonvanishing determinant of second or higher order. The same applies to any other matrix which is related to it by multiplication of all terms by a constant. In particular, $M^t = \lambda_1^t Z_1$ must be of rank unity when t is large enough. The computer program for the calculation of Z_1 was controlled by the magnitude of the second order determinant in the upper left of M^t, this determinant being evaluated for each of the successive squarings of M. When t was such that

$$\begin{vmatrix} m_{11}^{(t)} & m_{12}^{(t)} \\ m_{21}^{(t)} & m_{22}^{(t)} \end{vmatrix} \tag{3.3}$$

became less than 0.000001 the program stopped the squaring, calculated λ_1 by

$$\sum_{i,j} m_{ij}^{(t+1)} \Big/ \sum_{i,j} m_{ij}^{(t)}, \tag{3.4}$$

and then divided all terms of M^t by λ_1^t to obtain Z_1.

We are likely to be interested in more than the first latent root and its corresponding vectors, and it happens that similar methods can be applied to find the next pair of roots. To use the analogue of $Z_1 = M^t/\lambda_1^t$, it is first necessary to remove λ_1. We need merely take $M - \lambda_1\{K_1\}[H_1]$, that is to say, subtract the first of the spectral components from the original matrix, to obtain

$$\lambda_2\{K_2\}[H_2] + \lambda_3\{K_3\}[H_3] + \cdots + \lambda_n\{K_n\}[H_n] = N, \tag{3.5}$$

say. There are again a number of options, but suppose we proceed by taking N to a high power. It is not necessary to go as high as for the first latent root, because the drop in absolute value from the third to the fourth roots is greater than from the first to the second. The ratio of the third to the fourth roots for Australian females in 1962 is 1.673, which is more than the square of 1.238, the ratio of the first to the second. Less than half the power we used before will do—say the 32nd.

Though N is a real matrix, most or all of its roots are complex, and we will have to pick them up in pairs. We use Sylvester's theorem again to express a special matrix function in terms of the corresponding function of the latent roots. If

$$N = M - \lambda_1 Z_1 = \lambda_2 Z_2 + \lambda_3 Z_3 + \cdots + \lambda_n Z_n, \tag{3.6}$$

then

$$f(N) = f(\lambda_2)Z_2 + f(\lambda_3)Z_3 + \cdots + f(\lambda_n)Z_n, \tag{3.7}$$

where f is any polynomial function. Hence choosing $f(N) = (N - \lambda_2 I)(N - \lambda_3 I)N^t$, we have

$$(3.8)\qquad (N - \lambda_2 I)(N - \lambda_3 I)N^t \doteq (\lambda_2 - \lambda_2)(\lambda_2 - \lambda_3)\lambda_2^t Z_2 + (\lambda_3 - \lambda_2)(\lambda_3 - \lambda_3)\lambda_3^t Z_3 \doteq 0,$$

where t is large enough that λ_4, and so forth, are negligible compared to λ_2^t and λ_3^t.

Expanding the left side of (3.8), we can write it in the form

$$(3.9)\qquad N^{t+2} - (\lambda_2 + \lambda_3)N^{t+1} + \lambda_2\lambda_3 N^t = 0.$$

Equation (3.8) applies not only to entire matrices, but to each term separately as well. If the jth element of the ith row of N^t is $n_{ij}^{(t)}$, then (3.8) gives us

$$(3.10)\qquad n_{ij}^{(t+2)} - (\lambda_2 + \lambda_3)n_{ij}^{(t+1)} + (\lambda_2\lambda_3)n_{ij}^{(t)} = 0.$$

Any two elements will provide two equations such as (3.10) so that we may solve for the two unknowns λ_2 and λ_3. Suppose that we take the first and second elements of the first row, then

$$(3.11)\qquad \begin{aligned} n_{11}^{(t+2)} - (\lambda_2 + \lambda_3)n_{11}^{(t+1)} + (\lambda_2\lambda_3)n_{11}^{(t)} &= 0,\\ n_{12}^{(t+2)} - (\lambda_2 + \lambda_3)n_{12}^{(t+1)} + (\lambda_2\lambda_3)n_{12}^{(t)} &= 0, \end{aligned}$$

or

$$(3.12)\qquad \lambda_2 + \lambda_3 = \frac{\begin{vmatrix} n_{11}^{(t+2)} & n_{11}^{(t)} \\ n_{12}^{(t+2)} & n_{12}^{(t)} \end{vmatrix}}{\begin{vmatrix} n_{11}^{(t+1)} & n_{11}^{(t)} \\ n_{12}^{(t+1)} & n_{12}^{(t)} \end{vmatrix}}, \qquad \lambda_2\lambda_3 = \frac{\begin{vmatrix} n_{11}^{(t+1)} & -n_{11}^{(t+2)} \\ n_{12}^{(t+1)} & -n_{12}^{(t+2)} \end{vmatrix}}{\begin{vmatrix} n_{11}^{(t+1)} & n_{11}^{(t)} \\ n_{12}^{(t+1)} & n_{12}^{(t)} \end{vmatrix}}.$$

Thus, in order to find λ_2 and λ_3, we seek the roots of the quadratic $x^2 - ax + b$ where a is the ratio of determinants given for $\lambda_2 + \lambda_3$ in (3.12) and b is the ratio of determinants for $\lambda_2\lambda_3$. To put the same matter more compactly, the quadratic whose roots are λ_2 and λ_3 is

$$(3.13)\qquad \begin{vmatrix} x^2 & x & 1 \\ n_{11}^{(t+2)} & n_{11}^{(t+1)} & n_{11}^{(t)} \\ n_{12}^{(t+2)} & n_{12}^{(t+1)} & n_{12}^{(t)} \end{vmatrix} = 0,$$

where $n_{11}^{(t)}$ and $n_{12}^{(t)}$ may be any two elements of the power of N. For the first three roots simultaneously, the equation must be

$$(3.14)\qquad \begin{vmatrix} x^3 & x^2 & x & 1 \\ m_{11}^{(t+3)} & m_{11}^{(t+2)} & m_{11}^{(t+1)} & m_{11}^{(t)} \\ m_{12}^{(t+3)} & m_{12}^{(t+2)} & m_{12}^{(t+1)} & m_{12}^{(t)} \\ m_{13}^{(t+3)} & m_{13}^{(t+2)} & m_{13}^{(t+1)} & m_{13}^{(t)} \end{vmatrix} = 0.$$

Equation (3.13) for Australian females 1962 comes out to

$$(3.15)\qquad \begin{vmatrix} x^2 & x & 1 \\ .02008 & -.04730 & -.06963 \\ .05166 & .02016 & .04749 \end{vmatrix} = 0,$$

or $x^2 - .7241x + .7803 = 0$. Thus, $x = -.3620 \pm .8058i$. The directly calculated λ_2 and λ_3 are $.3616 \pm .8046i$.

The calculation of the second and third spectral components is not difficult from this point. The matrix from which the first spectral component has been subtracted out, $N = M - \lambda_1 Z_1 = M - \lambda_1\{K_1\}[H_1]$, when raised to the tth power is equal to $N^t \doteqdot \lambda_2^t Z_2 + \lambda_3^t Z_3$; to the $(t+1)$th power it is $N^{t+1} \doteqdot \lambda_2^{t+1} Z_2 + \lambda_3^{t+1} Z_3$. Subtracting the first of these multiplied by λ_3 from the second gives

$$N^{t+1} - \lambda_3 N^t = (\lambda_2^{t+1} - \lambda_3 \lambda_2^t) Z_2 \tag{3.16}$$

so that

$$Z_2 = \frac{N^{t+1} - \lambda_3 N^t}{\lambda_2^{t+1} - \lambda_3 \lambda_2^t}. \tag{3.17}$$

One can work out from (3.17) the ith row and jth column of Z_2, and fill in the remainder of Z_2 with

$$z_{kl}^{(2)} = \frac{z_{kj}^{(2)} z_{il}^{(2)}}{z_{ij}^{(2)}}, \tag{3.18}$$

taking advantage of the fact that since Z_2 is of rank one all of its determinants of the second order must vanish, including

$$\begin{vmatrix} z_{ij}^{(2)} & z_{il}^{(2)} \\ z_{kj}^{(2)} & z_{kl}^{(2)} \end{vmatrix} = z_{ij}^{(2)} z_{kl}^{(2)} - z_{il}^{(2)} z_{kj}^{(2)} = 0. \tag{3.19}$$

Equation (3.18) is simply a rearrangement of (3.19).

4. Direct machine computation of the spectral decomposition

Tables V to XIII show an actual computation, the data being the officially published estimate of the age distribution of U. S. females for mid-1963 and the births and deaths of the calendar year 1963. The original matrix which corresponds to the conventional population projection of (1.1) is table V; the number of women exposed to the risk of childbearing was the arithmetic average of the number of women in each age group at the beginning and end of the five year period; the subdiagonal elements are ${}_5L_{x+5}/{}_5L_x$ from an abridged life table.

TABLE V

PROJECTION MATRIX M FOR THE UNITED STATES 1963 FEMALES UP TO AGE 45

0.	0.	0.092137	0.366868	0.495232	0.346164	0.186575	0.078636	0.017885
0.995736	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.998591	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.998065	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.997004	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.996028	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.994866	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.992404	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.988459	0.

Table VI exhibits the odd numbered latent roots and their powers up to the fifteenth. Each even numbered root is the conjugate of the next following odd root, and this is true for powers and other functions of the roots. Inspection of the table shows how rapidly the roots, after the first, and especially after the

TABLE VI

LATENT ROOTS λ_j OF THE PROJECTION MATRIX FOR THE UNITED STATES FEMALES 1963 WITH THEIR POWERS UP TO THE FIFTEENTH

Each even numbered root is the conjugate of the following odd numbered root. All roots beyond the first are complex.

	Powers of Eigenvalues				
	Dominant	Number 3		Number 5	
1	1.089351	0.311546	−0.783686i	0.019564	−0.532708i
2	1.186686	−0.517103	−0.488308i	−0.283395	−0.020844i
3	1.292717	−0.543782	0.253116i	−0.016648	0.150559i
4	1.408223	0.028950	0.505011i	0.079878	0.011814i
5	1.534049	0.404790	0.134646i	0.007856	−0.042321i
6	1.671118	0.231631	−0.275279i	−0.022391	−0.005013i
7	1.820434	−0.143569	−0.267288i	−0.003109	0.011830i
8	1.983091	−0.254198	0.029240i	0.006241	0.001887i
9	2.160282	−0.056279	0.208321i	0.001128	−0.003288i
10	2.353306	0.145725	0.109007i	−0.001729	−0.000665i
11	2.563576	0.130827	−0.080242i	−0.000388	0.000908i
12	2.792634	−0.022126	−0.127527i	0.000476	0.000224i
13	3.042159	−0.106834	−0.022391i	0.000129	−0.000249i
14	3.313978	−0.050831	0.076749i	−0.000130	−0.000074i
15	3.610086	0.044311	0.063746i	−0.000042	0.000068i
		Number 7		Number 9	
1		−0.406258	−0.391959i	−0.469528	−0.157209i
2		0.011414	0.318473i	0.195742	0.147628i
3		0.120191	−0.133856i	−0.068698	−0.100088i
4		−0.101295	0.007270i	0.016521	0.057794i
5		0.044001	0.036750i	0.001329	−0.029733i
6		−0.003471	−0.032177i	−0.005298	0.013752i
7		−0.011202	0.014433i	0.004650	−0.005624i
8		0.010208	−0.001473i	−0.003067	0.001910i
9		−0.004724	−0.003403i	0.001740	−0.000414i
10		0.000586	0.003234i	−0.000882	−0.000079i
11		0.001030	−0.001543i	0.000402	0.000176i
12		−0.001023	0.000223i	−0.000161	−0.000146i
13		0.000503	0.000310i	0.000053	0.000094i
14		−0.000083	−0.000323i	−0.000010	−0.000052i
15		−0.000093	0.000164i	−0.000004	0.000026i

third, diminish when taken to powers. The fifteenth power of the real root is $\lambda_1^{15} = (1.08935)^{15} = 3.610$; the absolute value of the fifteenth power of the second and third roots is $|\lambda_2^{15}| = |\lambda_3^{15}| = 0.07763$; the absolute value of the fifteenth power of the largest of the remaining roots is $|\lambda_6^{15}| = |\lambda_7^{15}| = 0.000188$.

Next in the sequence of calculations is the set of vertical stable vectors $\{K_i\}$,

$i = 1, 2, 3, \cdots, 9$ (table VII). These might have been obtained by the evaluation of the cofactors in $|M - \lambda I|$, but it was easier to program the recurrence equations (2.6) relating each element to the preceding one. The same programming served for the real and complex roots; no special instruction is required

TABLE VII

United States Females 1963

Stable Vertical Vectors $\{K_j\}$, $j = 1, 2, \cdots, 9$

The assembly of the $\{K_j\}$ is the matrix $K = [\{K_1\}\ \{K_2\} \cdots \{K_9\}]$. The vectors of this and the succeeding table have been normalized so that $HK = I$ with maximum error of 0.00002.

$\{K_1\}$	$\{K_2\}$		$\{K_3\}$	
0.953246	1.046381	−0.062243i	1.046381	0.062243i
0.871327	0.388111	−1.175221i	0.388111	1.175221i
0.798732	−1.123362	−0.941121i	−1.123362	0.941121i
0.731800	−1.526125	0.823965i	−1.526125	−0.823965i
0.669763	0.238689	2.036422i	0.238689	−2.036422i
0.612386	2.339125	0.626532i	2.339125	−0.626532i
0.559270	1.706194	−2.291171i	1.706194	2.291171i
0.509498	−1.763717	−2.861749i	−1.763717	2.861749i
0.462310	−3.880584	0.681882i	−3.880584	−0.681882i

$\{K_4\}$		$\{K_5\}$		$\{K_6\}$	
0.393668	−0.349736i	0.393668	0.349736i	0.849793	−0.280286i
−0.625858	−0.758827i	−0.625858	0.758827i	−1.421988	0.684958i
−1.463577	1.119454i	−1.463577	−1.119454i	0.968951	2.618488i
1.993980	2.815341i	1.993980	−2.815341i	1.981539	−4.521115i
5.398902	−3.533609i	5.398902	3.533609i	−8.062657	3.316461i
−6.227818	−10.323288i	−6.227818	10.323288i	14.300550	5.666204i
−19.680001	10.908085i	−19.680001	−10.908085i	−11.203706	−24.685070i
18.949112	37.358612i	18.949112	−37.358612i	−15.956632	44.905494i
70.516446	−32.571008i	70.516446	32.571008i	74.701472	−37.186530i

$\{K_7\}$		$\{K_8\}$		$\{K_9\}$	
0.849793	0.280286i	0.516277	−0.571833i	0.516277	0.571833i
−1.421988	0.684958i	−1.349616	0.760813i	−1.349616	−0.760813i
0.968951	−2.618488i	3.068173	−0.590800i	3.068173	0.590800i
1.981539	4.521115i	−6.242596	−0.834318i	−6.242596	0.834318i
−8.062657	−3.316461i	11.386013	5.583910i	11.386013	−5.583910i
14.300550	−5.666204i	−18.152471	−17.923239i	−18.152471	17.923239i
−11.203706	24.685070i	23.151608	45.728608i	23.151608	−45.728608i
−15.956632	−44.905494i	−14.901487	−101.642290i	−14.901487	101.642290i
74.701472	37.186530i	−36.214495	201.853754i	−36.214495	−201.853754i

for the latter in Fortran IV other than the interpretation of certain symbols as complex numbers. Table VIII shows the horizontal stable vectors $[H_i]$, $i = 1, 2, 3, \cdots, 9$. As these were originally calculated from (2.15), they contained arbitrary elements; the program found $[\overline{H}_i]\{\overline{K}_i\}$ and then divided each element of $[\overline{H}_i]$ and $\{\overline{K}_i\}$ by the square root of the scalar product, $([H_i]\{K_i\})^{1/2}$, to normalize, that is, to find $[H_i]$ and $\{K_i\}$.

TABLE VIII

STABLE HORIZONTAL VECTORS $[H_j]$, $j = 1, 2, \cdots, 9$, DISPLAYED VERTICALLY AS $[H_j]'$

$[H_1]'$		$[H_2]'$		$[H_3]'$	
0.203602		0.169287	0.100739*i*	0.169287	−0.100739*i*
0.222744		−0.026319	0.164755*i*	−0.026319	−0.164755*i*
0.242989		−0.137510	0.030746*i*	−0.137510	−0.030746*i*
0.246418		−0.082694	−0.107676*i*	−0.082694	0.107676*i*
0.194323		−0.003495	−0.135716*i*	−0.003495	0.135716*i*
0.111297		0.021519	−0.095289*i*	0.021519	0.095289*i*
0.051024		0.022897	−0.047941*i*	0.022897	0.047941*i*
0.017730		0.013220	−0.015908*i*	0.013220	0.015908*i*
0.003343		0.003312	−0.002547*i*	0.003312	0.002547*i*
$[H_4]'$		$[H_5]'$		$[H_6]'$	
0.078613	0.003199*i*	0.078613	−0.003199*i*	0.088880	0.105023*i*
−0.000167	0.042120*i*	−0.000167	−0.042120*i*	−0.077604	−0.007863*i*
−0.022472	0.000736*i*	−0.022472	−0.000736*i*	0.034658	−0.027262*i*
−0.008091	−0.012275*i*	−0.008091	0.012275*i*	−0.011606	0.015012*i*
−0.022527	−0.005741*i*	−0.022527	0.005741*i*	−0.033878	−0.049325*i*
−0.036459	−0.013752*i*	−0.036459	0.013752*i*	−0.010963	−0.045431*i*
−0.020707	−0.020906*i*	−0.020707	0.020906*i*	−0.008550	−0.022310*i*
−0.003966	−0.012129*i*	−0.003966	0.012129*i*	−0.004398	−0.013989*i*
0.000204	−0.002632*i*	0.000204	0.002632*i*	0.000284	−0.004350*i*
$[H_7]'$		$[H_8]'$		$[H_9]'$	
0.088880	−0.105023*i*	0.093229	0.060131*i*	0.093229	−0.060131*i*
−0.077604	0.007863*i*	−0.053455	−0.013635*i*	−0.053455	0.013635*i*
0.034658	0.027262*i*	0.027281	−0.002004*i*	0.027281	0.002004*i*
−0.011606	−0.015012*i*	−0.021125	−0.000311*i*	−0.021125	0.000311*i*
−0.033878	0.049325*i*	−0.024308	−0.025311*i*	−0.024308	0.025311*i*
−0.010963	0.045431*i*	−0.030900	−0.021803*i*	−0.030900	0.021803*i*
−0.008550	0.022310*i*	−0.014410	−0.015516*i*	−0.014410	0.015516*i*
−0.004398	0.013989*i*	−0.008252	−0.006247*i*	−0.008252	0.006247*i*
0.000284	0.004350*i*	−0.002504	−0.003129*i*	−0.002504	0.003129*i*

The program then assembled the vertical vectors into K and the horizontal ones into H, and worked out HK and showed it to six places of decimals (not reproduced here); there were only two off diagonal elements which did not show zero to five places, and the diagonal elements were equally close to unity. This verifies the calculation up to this point.

The first spectral component is $Z_1 = \{K_1\}[H_1]$; its upper left term is obtained by multiplying the first elements of $\{K_1\}$ and $[H_1]$, that is, $(0.953246)(0.203602) = 0.194083$, as shown in table IX, and similarly for its other elements. Only the odd spectral components, with the imaginary parts of each element separately displayed, are given below; the second component is the same as the third, except that the signs of the imaginary part are reversed, and similarly for the other even numbered components.

TABLE IX

UNITED STATES FEMALES 1963

SPECTRAL COMPONENTS $Z_j = \{K_j\} [H_j]$

Each even numbered component is the conjugate of the following odd numbered component.

Idempotent Number 1, Z_1								
0.194083	0.212330	0.231628	0.234897	0.185237	0.106094	0.048638	0.016901	0.003186
0.177404	0.194083	0.211723	0.214711	0.169319	0.096976	0.044458	0.015449	0.002913
0.162624	0.177913	0.194083	0.196822	0.155212	0.088897	0.040754	0.014162	0.002670
0.148996	0.163004	0.177819	0.180329	0.142205	0.081447	0.037339	0.012975	0.002446
0.136365	0.149186	0.162745	0.165042	0.130150	0.074543	0.034174	0.011875	0.002239
0.124683	0.136405	0.148803	0.150903	0.119000	0.068157	0.031246	0.010858	0.002047
0.113869	0.124574	0.135897	0.137814	0.108679	0.062245	0.028536	0.009916	0.001869
0.103735	0.113488	0.123802	0.125549	0.099007	0.056706	0.025996	0.009034	0.001103
0.094127	0.102977	0.112336	0.113921	0.089837	0.051454	0.023589	0.008197	0.001545
Real Part of Idempotent Number 3, Z_3								
0.183409	−0.017285	−0.141974	−0.093231	−0.012105	0.016586	0.020975	0.012843	0.003307
0.184093	0.183409	−0.017236	−0.158637	−0.160853	−0.103634	−0.047455	−0.013565	−0.001708
−0.095363	0.184621	0.183409	−0.008441	−0.123799	−0.113852	−0.070840	−0.029822	−0.006117
−0.341359	−0.095586	0.184524	0.214922	0.117160	0.045674	0.004558	−0.007068	−0.002955
−0.164740	−0.341794	−0.095434	0.199535	0.275542	0.199185	0.103093	0.035551	0.005977
0.332868	−0.164788	−0.340916	−0.125969	0.076855	0.110037	0.083596	0.040890	0.009342
0.519647	0.332577	−0.164174	−0.387795	−0.316913	−0.181607	−0.070774	−0.013892	−0.000185
−0.010285	0.517908	0.330517	−0.162293	−0.382221	−0.310646	−0.177579	−0.068841	−0.013129
−0.725626	−0.010210	0.512653	0.394322	0.106107	−0.018531	−0.056164	−0.040453	−0.011114
Imaginary Part of Idempotent Number 3, Z_3								
−0.094874*i*	−0.174035*i*	−0.040731*i*	0.107523*i*	0.141793*i*	0.101048*i*	0.051590*i*	0.017469*i*	0.002871*i*
0.159852*i*	−0.094874*i*	−0.173538*i*	−0.055393*i*	0.048565*i*	0.062272*i*	0.045516*i*	0.021710*i*	0.004880*i*
0.272486*i*	0.160310*i*	−0.094874*i*	−0.198784*i*	−0.155748*i*	−0.086792*i*	−0.032306*i*	−0.005429*i*	0.000255*i*
0.014254*i*	0.273124*i*	0.160226*i*	−0.196190*i*	−0.204240*i*	−0.163154*i*	−0.092030*i*	−0.035170*i*	−0.006616*i*
−0.368786*i*	0.014272*i*	0.272689*i*	0.194100*i*	0.039512*i*	−0.021077*i*	−0.035185*i*	−0.023124*i*	−0.006136*i*
−0.341705*i*	−0.368894*i*	0.014235*i*	0.303678*i*	0.319648*i*	0.209410*i*	0.097794*i*	0.028928*i*	0.003883*i*
0.215986*i*	−0.341407*i*	−0.367518*i*	−0.005749*i*	0.223550*i*	0.211885*i*	0.134258*i*	0.057431*i*	0.011933*i*
0.662133*i*	0.215263*i*	−0.339291*i*	−0.426558*i*	−0.249368*i*	−0.106480*i*	−0.019029*i*	0.009775*i*	0.004985*i*
0.275492*i*	0.657294*i*	0.213079*i*	−0.361458*i*	−0.524275*i*	−0.384450*i*	−0.201652*i*	−0.070746*i*	−0.012142*i*

TABLE IX (Continued)

Real Part of Idempotent Number 5, Z_5								
0.032066	0.014665	−0.008589	−0.007478	−0.010876	−0.019162	−0.015463	−0.005803	−0.000840
−0.046773	0.032066	0.014623	−0.004251	0.009742	0.012383	−0.002904	−0.006722	−0.002125
−0.118637	−0.046907	0.032066	0.025583	0.039397	0.068755	0.053709	0.019382	0.002648
0.147746	−0.118914	−0.046882	0.018427	−0.028756	−0.033983	0.017567	0.026239	0.007816
0.435726	0.147934	−0.118725	−0.087057	−0.141908	−0.245430	−0.185667	−0.064269	−0.008198
−0.456561	0.435854	0.147554	−0.076335	0.081029	0.085097	−0.086856	−0.100510	−0.028440
−1.581993	−0.456162	0.434228	0.293124	0.505955	0.867513	0.635553	0.210348	0.024692
1.370128	−1.576700	−0.453335	0.305281	−0.212395	−0.177123	0.388629	0.377964	0.102188
5.647684	1.360115	−1.560702	−0.970342	−1.775516	−3.018849	−2.141096	−0.674697	−0.071332

Imaginary Part of Idempotent Number 5, Z_5								
0.026234i	−0.016640i	−0.008149i	0.002003i	−0.005619i	−0.007337i	0.000988i	0.003388i	0.001107i
0.061656i	0.026234i	−0.016592i	−0.013822i	−0.020687i	−0.036272i	−0.028797i	−0.010600i	−0.001492i
−0.083321i	0.061832i	0.026234i	−0.008909i	0.016816i	0.020688i	−0.007417i	−0.013312i	−0.004080i
−0.227700i	−0.083516i	0.061800i	0.047255i	0.074869i	0.130064i	0.099982i	0.035349i	0.004673i
0.260515i	−0.227990i	−0.083383i	0.037685i	−0.048607i	−0.054588i	0.039698i	0.051468i	0.014930i
0.831465i	0.260591i	−0.227405i	−0.159970i	−0.268307i	−0.462017i	−0.343960i	−0.116475i	−0.014284i
−0.794556i	0.830738i	0.259619i	−0.153326i	0.132746i	0.127065i	−0.185551i	−0.195434i	−0.054020i
−2.997480i	−0.791897i	0.825591i	0.534861i	0.950365i	1.622631i	1.169725i	0.377985i	0.042248i
2.334907i	−2.975574i	−0.783863i	0.602097i	−0.328900i	−0.217789i	0.799749i	0.726105i	0.192233i

Real Part of Idempotent Number 7, Z_7								
0.104966	−0.068151	0.021811	−0.005655	−0.042614	−0.022050	−0.013519	−0.007658	−0.000978
−0.054450	0.104966	−0.067956	0.026787	0.014388	−0.015529	0.003124	−0.003328	−0.003383
−0.188881	−0.054606	0.104966	−0.050555	0.096332	0.108338	0.050134	0.032367	0.011665
0.650940	−0.189323	−0.054577	0.044874	−0.290136	−0.227123	0.117808	−0.071959	−0.019103
−1.064913	0.651769	−0.189022	0.043789	0.436730	0.239062	0.142926	0.081852	0.012138
0.675950	−1.065225	0.650096	−0.251037	−0.204984	0.100643	0.004144	0.016367	0.028705
1.596714	0.675360	−1.061251	0.500610	−0.838042	−0.998642	−0.454934	−0.296033	−0.110552
−6.134331	1.591371	0.671175	−0.488937	2.755553	2.215039	1.138272	0.698339	0.190797
10.544899	−6.089500	1.575224	−0.308743	−4.364958	−2.508383	−1.468328	−0.848726	−0.140551

TABLE IX (Continued)

Imaginary Part of Idempotent Number 7, Z_7

−0.064336*i*	−0.015070*i*	0.032881*i*	−0.016010*i*	0.032421*i*	0.035534*i*	0.016562*i*	0.010655*i*	0.003776*i*
0.210220*i*	−0.064336*i*	−0.015027*i*	0.013397*i*	−0.093345*i*	−0.072112*i*	−0.037581*i*	−0.022904*i*	−0.005991*i*
−0.334493*i*	0.210823*i*	−0.064336*i*	0.015844*i*	0.136502*i*	0.072727*i*	0.044005*i*	0.025070*i*	0.003472*i*
0.193729*i*	0.335275*i*	0.210712*i*	−0.082220*i*	−0.055425*i*	0.040458*i*	0.005553*i*	0.007835*i*	0.009902*i*
0.551997*i*	0.193976*i*	−0.334743*i*	0.159530*i*	−0.285339*i*	−0.329936*i*	−0.151523*i*	−0.098199*i*	−0.036011*i*
−2.005497*i*	0.552159*i*	0.193478*i*	−0.148921*i*	0.897338*i*	0.711808*i*	0.367492*i*	0.224964*i*	0.060595*i*
3.370653*i*	−2.003745*i*	0.550099*i*	−0.118306*i*	−1.388902*i*	−0.779621*i*	−0.461012*i*	−0.265290*i*	−0.041728*i*
−2.315384*i*	3.359374*i*	−1.991330*i*	0.760725*i*	0.734233*i*	−0.232623*i*	0.027947*i*	−0.025712*i*	−0.082150*i*
−4.540232*i*	−2.298463*i*	3.325289*i*	−1.553030*i*	2.424876*i*	2.986088*i*	1.348650*i*	0.881413*i*	0.335482*i*

Real Part of Idempotent Number 9, Z_9

0.082517	−0.035394	0.012938	−0.011084	−0.027023	−0.028421	−0.016312	−0.007832	−0.003082
−0.171572	0.082517	−0.035293	0.028747	0.052064	0.058292	0.031253	0.015889	0.005759
0.321569	−0.172064	0.082517	−0.064998	−0.089536	−0.107689	−0.053380	−0.029008	−0.009530
−0.531825	0.322322	−0.171974	0.131613	0.130629	0.174708	0.077013	0.046300	0.013019
0.725744	−0.532502	0.321809	−0.238788	−0.135439	−0.230088	−0.077439	−0.059072	−0.011036
−0.614597	0.725956	−0.531135	0.377889	−0.012401	0.170143	−0.016507	0.037824	−0.010630
−0.591309	−0.614060	0.723248	−0.474846	0.594663	0.281616	0.375889	0.094620	0.085111
4.722622	−0.589330	−0.610255	0.283174	−2.210440	−1.755621	−1.362319	−0.511979	−0.280707
−15.513976	4.688109	−0.583350	0.827801	5.989431	5.520013	3.653779	1.559771	0.722222

Imaginary Part of Idempotent Number 9, Z_9

0.022267*i*	−0.023528*i*	0.016635*i*	−0.011919*i*	−0.000833*i*	−0.006414*i*	−0.000230*i*	−0.001493*i*	0.000184*i*
0.010224*i*	0.022267*i*	−0.023461*i*	0.015652*i*	−0.015666*i*	−0.005916*i*	−0.009977*i*	−0.002153*i*	−0.002318*i*
−0.129413*i*	0.010253*i*	0.022267*i*	−0.011526*i*	0.063297*i*	0.048639*i*	0.039091*i*	0.014291*i*	0.008120*i*
0.453158*i*	−0.129716*i*	0.010248*i*	−0.019566*i*	−0.178287*i*	−0.161887*i*	−0.108881*i*	−0.045881*i*	−0.021620*i*
−1.205240*i*	0.453735*i*	−0.129510*i*	0.121499*i*	0.423926*i*	0.420792*i*	0.257129*i*	0.117203*i*	0.049604*i*
2.762504*i*	−1.205593*i*	0.452571*i*	−0.384268*i*	−0.895139*i*	−0.949610*i*	−0.539931*i*	−0.261291*i*	−0.101668*i*
−5.655385*i*	2.760090*i*	−1.201096*i*	0.973200*i*	1.697571*i*	1.917802*i*	1.018184*i*	0.521959*i*	0.186925*i*
10.372094*i*	−5.636461*i*	2.742989*i*	−2.151790*i*	−2.847914*i*	−3.465682*i*	−1.695918*i*	−0.931801*i*	−0.301100*i*
−16.641079*i*	10.296292*i*	−5.579271*i*	4.252820*i*	3.990078*i*	5.447788*i*	2.346907*i*	1.439397*i*	0.392065*i*

Representation of any arbitrary age distribution in terms of the several stable vectors is now readily carried through. The c_j which are the weights on the several vertical vectors when the age distribution of mid-1963 $\{K'\}$, is expanded as

$$\{K'\} = c_1\{K_1\} + c_2\{K_2\} + \cdots + c_9\{K_9\} \tag{4.1}$$

are given as table X; much the heaviest weight is on the $\{K_1\}$, with the next

TABLE X

UNITED STATES FEMALES 1963
COEFFICIENTS c_j OF THE VERTICAL STABLE VECTORS IN THE EXPANSION OF THE AGE DISTRIBUTION

$$\{K'\} = \sum_1^9 c_j K_j$$

c_1	10563.038818	0.
c_2	−55.489507	321.165070i
c_3	−55.489507	−321.165070i
c_4	51.007514	34.245224i
c_5	51.007514	−34.245224i
c_6	9.147722	59.698857i
c_7	9.147722	−59.698857i
c_8	27.375117	29.359058i
c_9	27.375117	−29.359058i

heaviest on $\{K_2\}$ and $\{K_3\}$. Table XI shows the comparison of the actual age distribution with the stable age distribution taken from the dominant root alone; the high birth rates of the 1920's, the low ones of the 1930's, and the high ones after World War II account for the divergencies.

TABLE XI

UNITED STATES FEMALES 0–44 1963
DECOMPOSITION OF AGE DISTRIBUTION
(in thousands)

Age	$\{K'\}$	$c_1\{K_1\}$	$\sum_2^9 c_1\{K_i\}$
0–4	10168	10069	99
5–9	9841	9204	+637
10–14	8848	8437	+411
15–19	7663	7730	−67
20–24	6284	7075	−791
25–29	5522	6469	−947
30–34	5760	5908	−148
35–39	6289	5382	907
40–44	6271	4883	1388

The projection of the same set of ages through one, two, and so forth, cycles of five years each is given in table XII. For each of these points of time the age distribution is broken down into its several stable components; underneath $\{K'\}$ are shown the several $\lambda_i^t c_i\{K_i\}$ for odd i, the values for each even i being the conjugates of the next following odd component. The conjugate terms which

TABLE XIIa

UNITED STATES FEMALES 1963
EXPANSION $\lambda_j^t c_j\,\{K_j\}$ OF THE PROJECTED AGE DISTRIBUTION IN TERMS OF THE VERTICAL STABLE VECTORS
AGE DISTRIBUTION FOR POWER NUMBER 1
(in thousands)

$M\{K'\}$		$\lambda_5 c_5\{K_5\}$	
10332.		2.94867	$-16.99166i$
10125.		31.92012	$4.33936i$
9827.		-5.92887	$60.05378i$
8831.		-112.77073	$-6.96657i$
7641.		5.28009	$-211.25302i$
6260.		394.81927	$-4.62754i$
5494.		35.67371	$736.03989i$
5716.		-1366.91512	$116.65806i$
6216.		-309.19679	$-2525.00507i$
$\lambda_1 c_1\{K_1\}$		$\lambda_7 c_7\{K_7\}$	
10968.866		-28.83569	$9.96298i$
10026.240		24.40195	$-47.96229i$
9190.897		27.84397	$91.02839i$
8420.713		-147.17172	$-81.64017i$
7706.868		287.16903	$-76.70719i$
7046.633		-270.66454	$449.20200i$
6435.442		-206.38340	$-900.91016i$
5862.719		1360.76651	$887.86477i$
5319.733		-2794.14963	$535.55664i$
$\lambda_3 c_3\{K_3\}$		$\lambda_9 c_9\{K_9\}$	
-277.93436	$-75.93741i$	-14.44050	$-5.09432i$
-37.91038	$-338.06706i$	30.78978	$0.49448i$
355.40239	$-189.59271i$	-59.19912	$18.76960i$
363.88443	$307.96510i$	101.14082	$-73.76244i$
-179.40571	$534.25397i$	-145.95839	$205.49875i$
-664.62184	$36.19712i$	147.16822	$-485.20786i$
-329.31755	$-712.80093i$	29.13303	$1018.33508i$
636.29776	$-669.97761i$	-703.38704	$-1916.86337i$
1005.22473	$402.94263i$	2546.45999	$3182.80139i$

are omitted are taken into account by doubling the real parts of the terms which are shown, after the first, and omitting the imaginary parts. When this is done the decomposition accounts exactly for the number of individuals projected, age by age. For example, the projection through one period, $M\{K'\}$, in table XIIa shows at 0–4, the first age group, 10,332. This is equal to $10{,}969 + 2(-277.9 +$

2.9 − 28.8 − 14.4), obtained from the top row of the expansion. The point of the exhibit is that the several complex roots have considerable importance at the outset, since the actual age distribution is far from the real stable vector;

TABLE XIIb

AGE DISTRIBUTION FOR POWER NUMBER 2

$M^2\{K'\}$		$\lambda_5^2 c_5\{K_5\}$	
11682.		−8.99390	−1.90321i
10288.		2.93610	−16.91920i
10110.		31.87514	4.33325i
9808.		−5.91739	59.93758i
8804.		−112.43287	−6.94570i
7611.		5.25912	−210.41391i
6228.		392.79226	−4.60378i
5452.		35.40273	730.44892i
5650.		−1351.13950	115.31241i
$\lambda_1^2 c_1\{K_1\}$		$\lambda_7^2 c_7\{K_7\}$	
11948.945		15.61981	7.25487i
10922.094		−28.71273	9.92050i
10012.113		24.36756	−47.89471i
9173.112		27.79009	90.85225i
8395.484		−146.73078	−81.39558i
7676.256		286.02839	−76.40250i
7010.455		−269.27493	446.89577i
6386.558		−204.81568	−894.06682i
5795.057		1345.06183	877.61790i
$\lambda_3^2 c_3\{K_3\}$		$\lambda_9^2 c_9\{K_9\}$	
−146.10042	194.15527i	5.97935	4.66210i
−276.74924	−75.61361i	−14.37893	−5.07260i
−37.85696	−337.59072i	30.74639	0.49378i
354.71467	−189.22584i	−59.08457	18.73328i
362.79423	307.04242i	100.83780	−73.54144i
−178.69310	532.13190i	−145.37864	204.68250i
−661.20966	36.01128i	146.41267	−482.71680i
−326.81605	−707.38648i	28.91172	1010.59979i
628.95423	−662.24538i	−695.26923	−1894.74080i

first the stable vectors beyond the third lose their importance, and by the end of 15 five year cycles or 75 years only the first vector is of consequence.

The latent roots beyond the first are responsible for waves in the trajectory, and it is easier to portray the waves in terms of the logarithms of the λ's. If r be defined by the equation $e^{5r} = \lambda$, or $r = 0.2 \ln \lambda$, and $r = x + iy$, then the equation (2.29) becomes

$$(4.2) \qquad \begin{aligned} M^t\{K'\} &= \lambda_1^t c_1\{K_1\} + \lambda_2^t c_2\{K_2\} + \cdots \\ &= e^{5r_1t} c_1\{K_1\} + e^{5r_2t} c_2\{K_2\} + \cdots , \end{aligned}$$

and we can see what happens with the increase of t by breaking up λ^t in terms of $r = x + iy$

$$\lambda^t = e^{5rt} = e^{5xt}(\cos 5yt + i \sin 5yt). \tag{4.3}$$

The real part x is negative and determines the rate of attenuation of the waves,

TABLE XIIc

AGE DISTRIBUTION FOR POWER NUMBER 5

$M^5\{K'\}$	$\lambda_5^5 c_5\{K_5\}$	
15514.	0.43628	$-1.32243i$
14460.	2.49846	$0.72373i$
13141.	−1.18307	$4.72695i$
11593.	−8.92563	$-1.88876i$
10223.	2.91752	$-16.81215i$
10021.	31.59216	$4.29478i$
9690.	−5.84606	$59.21506i$
8658.	−110.56506	$-6.83031i$
7427.	5.13245	$-205.34596i$

$\lambda_1^5 c_1\{K_1\}$	$\lambda_7^5 c_7\{K_7\}$	
15446.606	0.05585	$-0.91874i$
14119.178	0.48010	$1.78763i$
12942.829	−2.06279	$-3.11126i$
11858.239	5.93396	$4.62671i$
10852.987	−14.28795	$-5.04051i$
9923.228	30.47344	$0.48940i$
9062.536	−58.37234	$18.50746i$
8256.013	99.16262	$-72.31972i$
7491.369	−141.87711	$199.75260i$

$\lambda_3^5 c_3\{K_3\}$		$\lambda_9^5 c_9\{K_9\}$	
30.30301	−142.55841	2.84847	$-1.21883i$
169.63020	−28.93227	−2.12311	$5.03573i$
106.03543	173.99350	−3.48222	$-9.01827i$
−144.99136	192.68142	15.50124	$7.19979i$
−274.99816	−75.13518	−28.53106	$9.85773i$
−37.52087	−334.59368	24.15124	$-47.46951i$
350.43877	−186.94482	27.45509	$89.75707i$
356.76724	301.94164	−144.29319	$-80.04338i$
−174.38918	519.31517	279.13922	$-74.56229i$

and y determines their period, in time units of five years; r is on an annual basis and λ on a five year basis. Table XIII shows eleven of the roots $r = x + iy$ corresponding to each of the λ.

The record for the United States female population from 1920 to 1963 is pulled together in table XIV. The real root r_1 is known as the intrinsic rate of natural increase; from its definition it may be shown to be the rate, compounded momently, at which the female population would increase if the age specific rates of mortality and fertility of the year in question were maintained long enough for the initial age distribution to wear off. The absolute value of x, the (negative) real part of the first pair of complex roots r_2, r_3, shows a steady fall from 1940

TABLE XIId

Age Distribution for Power Number 10

$M^{10}\{K'\}$		$\lambda_5^{10}c_5\{K_5\}$	
23759.		−0.05254	−0.02885i
21804.		0.05026	−0.10005i
19894.		0.19075	0.08720i
18023.		−0.15005	0.36290i
16442.		−0.68858	−0.25555i
15289.		0.42995	−1.30326i
14238.		2.46009	0.71261i
12859.		−1.15768	4.62553i
11235.		−8.65006	−1.83044i
$\lambda_1^{10}c_1\{K_1\}$		$\lambda_7^{10}c_7\{K_7\}$	
23695.846		0.17013	0.05105i
21659.508		−0.27848	0.14355i
19854.932		0.17820	−0.52479i
18191.117		0.41748	0.88647i
16649.011		−1.61767	−0.61476i
15222.716		2.80719	−1.20116i
13902.372		−2.09051	4.95840i
12665.127		−3.40750	−8.82478i
11492.126		15.02265	6.97751i
$\lambda_3^{10}c_3\{K_3\}$		$\lambda_9^{10}c_9\{K_9\}$	
31.46132	−53.62598	−0.02724	−0.00288i
72.56017	11.12862	0.05379	−0.01190i
19.49444	84.70806	−0.09525	0.05720i
−84.63487	58.47288	0.14545	−0.17029i
−101.19973	−67.44146	−0.16885	0.41813i
29.86378	−140.49211	0.05504	−0.90542i
167.02544	−28.48800	0.47272	1.76018i
103.76033	170.26028	−2.01853	−3.04451i
−140.51487	186.73253	5.75075	4.48387i

to 1960; this corresponds to the narrowing of the ages within which reproduction takes place; the narrower this range of ages, the less rapid the attenuation of waves arising from disturbances in the age distribution. The complex part y has tended to increase, on the other hand. The period of the waves caused in later generations by a disturbance in the age distributions, which is equal to $2\pi/y$ years, tends to diminish. The upward trend in y shown in table XIV, from 0.2088 in 1920 to 0.2420 in 1960, corresponds to a decline in the wave length from $2\pi/0.2088 = 30.09$ years to $2\pi/0.2420 = 25.96$ years. These periods are related to the mean age of women at childbearing. The mean length of generation T may be defined as the length of time that the population would increase in the ratio of the net reproduction rate R_0 when subject to the intrinsic rate r.

TABLE XIIe

AGE DISTRIBUTION FOR POWER NUMBER 15

$M^{15}\{K'\}$		$\lambda_5^{15}c_5\{K_5\}$	
36391.		−0.00163	0.00200i
33282.		−0.00384	−0.00291i
30451.		0.00519	−0.00739i
27822.		0.01418	0.00920i
25477.		−0.01622	0.02713i
23415.		−0.01578	−0.02843i
21469.		0.04949	−0.09851i
19468.		0.18666	0.08533i
17466.		−0.14542	0.35170i
$\lambda_1^{15}c_1\{K_1\}$		$\lambda_7^{15}c_7\{K_7\}$	
36350.584		0.00561	0.00850i
33226.742		−0.01753	−0.00392i
30458.434		0.02713	−0.01654i
27906.062		−0.01421	0.05435i
25540.395		−0.04859	−0.08650i
23352.389		0.16766	0.05031i
21326.916		−0.27421	0.14135i
19428.923		0.17438	−0.51353i
17629.482		0.40459	0.85910i
$\lambda_3^{15}c_3\{K_3\}$		$\lambda_9^{15}c_9\{K_9\}$	
19.95576	−17.47109	−0.00012	0.00081i
27.87318	14.27472	−0.00028	−0.00162i
−3.51449	36.91380	0.00157	0.00291i
−42.13248	12.27343	−0.00487	−0.00455i
−31.88385	−40.92579	0.01221	0.00558i
31.00531	−52.84870	−0.02685	−0.00284i
71.44598	10.95773	0.05296	−0.01172i
19.07617	82.89056	−0.09320	0.05597i
−82.02183	56.66757	0.14096	−0.16503i

The reproduction rate R_0 is defined as the number of girl children expected to be born to a girl aged 0, at the given age specific rates of birth m_x and survivorship L_x/ℓ_0,

$$R_0 = \sum_x \frac{L_x m_x}{\ell_0}. \tag{4.4}$$

If $e^{r}t = R_0$, then $T = \ln R_0/r$ and works out to 28.99 years for 1920 and 25.90 for 1960.

TABLE XIII

United States Females 1963
Values of r, the Intrinsic Rate of Natural Increase,
Obtained from the λ by $r = 0.2\,(\ln \lambda + m\pi i)$, for $m = 0, \pm 1, \cdots, 5$

Values of r from				
λ_1	λ_2, λ_3	λ_4, λ_5	λ_6, λ_7	λ_8, λ_9
$0.0171 + 0i$	$-0.0341 \mp 0.2384i$	$-0.1258 \mp 0.3068i$	$-0.1144 \mp 0.4748i$	$-0.1406 \mp 0.5637i$
$0.0171 \pm 1.2566i$	$-0.0341 \pm 1.0182i$	$-0.1258 \pm 0.9498i$	$-0.1144 \pm 0.7818i$	$-0.1406 \pm 0.6929i$
$0.0171 \pm 2.5133i$	$-0.0341 \pm 2.2748i$	$-0.1258 \pm 2.2065i$	$-0.1144 \pm 2.0385i$	$-0.1406 \pm 1.9496i$
$0.0171 \pm 3.7699i$	$-0.0341 \pm 3.5314i$	$-0.1258 \pm 3.4631i$	$-0.1144 \pm 3.2951i$	$-0.1406 \pm 3.2062i$
$0.0171 + 5.0265i$	$-0.0341 \pm 4.7881i$	$-0.1258 \pm 4.7197i$	$-0.1144 \pm 4.5517i$	$-0.1406 \pm 4.4628i$
$0.0171 \pm 6.2832i$	$-0.0341 \pm 6.0447i$	$-0.1258 \pm 5.9764i$	$-0.1144 \pm 5.8084i$	$-0.1406 \pm 5.7195i$
	$-0.0341 \pm 1.4950i$	$-0.1258 \pm 1.5634i$	$-0.1144 \pm 1.7314i$	$-0.1406 \pm 1.8203i$
	$-0.0341 \pm 2.7516i$	$-0.1258 \pm 2.8201i$	$-0.1144 \pm 2.9881i$	$-0.1406 \pm 3.0770i$
	$-0.0341 \pm 4.0082i$	$-0.1258 \pm 4.0767i$	$-0.1144 \pm 4.2447i$	$-0.1406 \pm 4.3336i$
	$-0.0341 \pm 5.2648i$	$-0.1258 \pm 5.3334i$	$-0.1144 \pm 5.5013i$	$-0.1406 \pm 5.5902i$
	$-0.0341 \pm 6.5214i$	$-0.1258 \pm 6.5900i$	$-0.1144 \pm 6.7580i$	$-0.1406 \pm 6.8469i$

TABLE XIV

United States Females, 1920–63
Results of Matrix Computation

	λ_1	λ_2, λ_3	r_1	r_2, r_3	c_1	c_2, c_3
Total 1963	1.08935	$0.3115 \pm 0.7837i$	0.01711	$-0.03408 \pm 0.2385i$	10,563	$-55.49 \pm 321.17i$
White 1963	1.08110	$0.3151 \pm 0.7835i$	0.01559	$-0.03380 \pm 0.2377i$	8,945	$-62.40 \pm 272.32i$
Nonwhite 1963	1.14282	$0.2771 \pm 0.7750i$	0.02670	$-0.03895 \pm 0.2455i$	1,622	$4.31 \pm 48.50i$
Total 1960	1.10983	$0.3001 \pm 0.7953i$	0.02084	$-0.03250 \pm 0.2420i$	10,253	$67.58 \pm 214.65i$
Total 1950	1.06653	$0.3279 \pm 0.7448i$	0.01288	$-0.04121 \pm 0.2312i$	7,636	$335.45 \pm 165.92i$
Total 1940	0.9899	$0.3603 \pm 0.6814i$	−0.0020	$-0.05210 \pm 0.2169i$	5,955	$-138.83 \pm 92.56i$
Total 1930	1.01792	$0.3948 \pm 0.6799i$	0.00355	$-0.04809 \pm 0.2089i$	6,690	$-129.26 \pm 44.27i$
Total 1920	1.05441	$0.4089 \pm 0.7028i$	0.01060	$-0.04137 \pm 0.2088i$	6,874	$-35.48 \pm 103.59i$

5. Bibliographical note

The population projection on which this argument is based goes back to A. L. Bowley [2] and P. K. Whelpton [9]. The suggestion for representing the operation of projection as a matrix seems first to have been made in print by H. Bernardelli [1] and E. G. Lewis [6]. An elegant and complete exposition of the theory and its application to a population of rats is due to P. H. Leslie [3], [4], [5]. A. J. Coale and A. Lopez [7] extended the theory to prove ergodicity when the projection matrix varies. E. M. Murphy [8] has set up and analyzed matrices which recognize parity and incorporate the two sexes.

The Fortran program which delivered the "matrix package" of tables VI to XIV was worked out by Susan Borker.

REFERENCES

[1] H. Bernardelli, "Population waves," *J. Burma Res. Soc.*, Vol. 31 (1941), pp. 1–18.
[2] A. L. Bowley, "Births and population of Great Britain," *J. Roy. Econom. Soc.*, Vol. 34 (1924), pp. 188–192.
[3] P. H. Leslie, "On the use of matrices in certain population mathematics," *Biometrika*, Vol. 33 (1945), pp. 183–212.
[4] ———, "Some further notes on the use of matrices in population mathematics," *Biometrika*, Vol. 35 (1948), pp. 213–245.
[5] ———, "On the distribution in time of the births in successive generation," *J. Roy. Statist. Soc. Ser. A*, Vol. 111 (1948), pp. 44–53.
[6] E. G. Lewis, "On the generation and growth of a population," *Sankhyā*, Vol. 6 (1942), pp. 93–96.
[7] A. Lopez, *Problems in Stable Population Theory*, Princeton, Office of Population Research, 1961.
[8] E. M. Murphy, "A generalization of stable population techniques," unpublished Ph.D. thesis, Department of Sociology, University of Chicago, 1965.
[9] P. K. Whelpton, "An empirical method of calculating future population" *J. Amer. Statist. Assoc.*, Vol. 31 (1936), pp. 457–473.